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ON OPTIMAL LINEAR ARRANGEMENTS OF TREES

F. R. K. CHUNG
Bell Laboratories, Murray Hill, NJ 07974, U.S.A.

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Abstract—In this paper we investigate the problem of finding a labeling of the vertices of an undirected tree with distinct integers such that the sum of weights of all edges is minimized. (The weight of an edge is the absolute value of the difference between the labelings of its end points). We give an $O(n^{\lambda})$ algorithm where λ is approximately log 3/log2 ~ 1.6 for solving this problem.

1. INTRODUCTION

Let G be an undirected graph with vertex set $V(G)$ and edge set $E(G)$ (see [2] for undefined graph theory terminology). *A linear arrangement* π of G is a one-to-one mapping from $V(G)$ to the set of positive integers. The *weight* of an edge $e = \{u, v\}$ in a linear arrangement π is defined to be the absolute value of $\pi(u) - \pi(v)$. The *cost* of the *linear arrangement* π of G, denoted by $f_\pi(G)$, is defined to be:

$$f_\pi(G) = \sum_{\{u,v\} \in E(G)} |\pi(u) - \pi(v)|.$$

The *cost* of G, denoted by $f(G)$, is the minimum value of $f_\pi(G)$ as π ranges over all possible linear arrangements of G. Finally, a linear arrangement π is said to be *optimal* if $f_\pi(G) = f(G)$. We will often abbreviate the term optimal linear arrangement by OLA.

During the past 20 yr, a number of researchers have investigated properties of optimal linear arrangements for a variety of classes of graphs. The first work in this subject appears to originate in the 1964 paper of Harper[12] who determined the OLA's for the class of n-cubes (which came up in connection with a minimization problem in error-correcting codes). This was followed by the papers of Seidvasser[16], who dealt with the costs of trees with bounded degrees, and of Iordanskii[13], who improved some of Seidvasser's bounds. For *complete* k-level binary trees, T_k the value of $f(T_k)$ was given by Chung[4], settling an earlier question of Cahit[3]. Other properties and applications of OLA's (in the context of location and assignment problems) can be found in [1, 5, 11–13, 16, 17], for example.

From the algorithmic point of view, it was shown by Garey *et al.*[6] that the computational problem of determining an optimal linear arrangement for a general G is NP-complete (see [7] for a complete discussion of this concept). However, in the special case that G is a tree T, Goldberg and Klipker[10] gave an $O(n^3)$ algorithm for determining an OLA for T on n vertices. This was recently improved to $O(n^{2.2})$ in an interesting paper of Shiloach[17]. In this paper we improve the upper bound even further by presenting an $O(n^{\lambda})$ algorithm for finding an OLA of any tree T on n vertices, where λ can be chosen to be any real number satisfying

$$\lambda > \frac{\log 3}{\log 2} = 1.585\ldots.$$

An outline of the remainder of the paper is as follows. In Section 2 we list a number of useful known properties we will need for our analysis. In Section 3, we give a variety of preliminary lemmas. In Section 4, we give a detailed analysis of a simplified version of our algorithm (whose running time can be bounded by $O(n^2)$ elementary operations) which illustrates the basic structure of the more complex $O(n^{\lambda})$ algorithm described in Section 5. Finally, in Section 6, we list possible extensions and related questions.

2. PROPERTIES OF OPTIMAL LINEAR ARRANGEMENTS

We summarize here some known properties of OLA's which will be needed later.

Property 1 (*see* [16]). An OLA, of a tree T maps $V(T)$ onto a set of consecutive integers.

We may in fact assume that an OLA π of T maps $V(T)$ onto $\{1, \ldots, n\}$ where $n = |V(T)|$ unless otherwise specified.

Property 2 (*see* [16]). The vertices u and v with $\pi(u) = 1$ and $\pi(v) = n$ both have degree 1.

Property 3 (*see* [13]). Let P denote the path in T connecting the two vertices labeled by 1 and n in P. Suppose P has vertices $v_0, v_1, \ldots, v_t$. Then the labelings of the vertices of P are monotone, i.e.

$$\pi(v_i) < \pi(v_{i+1}) \text{ for } i = 0, \ldots, t-1$$

or

$$\pi(v_i) > \pi(v_{i+1}) \text{ for } i = 0, \ldots, t-1.$$

P is said to be the *basic path* in π.

Property 4 (*see* [13]). Suppose we remove all edges of a basic path P in T. The remaining graph is a union of vertex disjoint subtrees. Let $\bar{T}_i$ denote the subtree which contains the vertex v_i, $i = 0, \ldots, t$. Then for a fixed i, the vertices in $\bar{T}_i$ are labeled by consecutive integers. Moreover, the restricted linear arrangement $\pi_i = \pi / V(\bar{T}_i)$ is optimal.

Let T^* be a *rooted* tree with root r. For a linear arrangement π of T^*, we *define*

$$g_\pi(T^*) := f_\pi(T^*) + \pi(r) - 1.$$

We remark that the additional quantity $\pi(r) - 1$ will eventually contribute to the weight of the edge joining the root r and some other vertex in a tree containing T^* as a subtree. The *cost* of T^*, denoted by $g(T^*)$, is a minimum value of $g_\pi(T^*)$ as π ranges over all linear arrangements of T^*. A linear arrangement π with $g_\pi(T^*) = g(T^*)$ is said to be optimal.

Property 5. An OLA of a rooted tree T^* maps $V(T^*)$ into integers $1, \ldots, |V(T^*)|$, and furthermore $2\pi(r) \le |V(T)| + 1$ where r is the root of T^*.

Proof. It is easy to see that the OLA π maps $V(T^*)$ onto integers $1, \ldots, n$ where $n = |V(T^*)|$. Consider π' with $\pi'(v) = n - \pi(v) + 1$ for all $v \in V(T^*)$. Then we have $g_{\pi'}(T^*) = f_\pi(T^*) + n - \pi(r) \ge g_\pi(T^*)$. This implies $n - \pi(r) \ge \pi(r) - 1$ as required. ∎

Property 6. Let π be an OLA of T. Let $\{u, v\}$ be an edge in the basic path. Suppose we remove the edge $\{u, v\}$. The remaining graph can be viewed as two rooted trees T_1^*, T_2^*, with roots u, v, respectively. Then $f(T) = g(T_1^*) + g(T_2^*) + 1$.

Proof. Suppose π_1 and π_2 are OLA's of T_1^* and T_2^*, respectively. We define π' as follows:

$$\pi'(w) = |V(T_1^*)| - \pi_1(w) + 1 \quad \text{if } w \in V(T_1^*),$$
$$\pi'(w) = \pi_2(w) + |V(T_1^*)| \quad \text{if } w \in V(T_2^*).$$

Then

$$\begin{aligned} f_{\pi'}(T) &= f_{\pi_1}(T_1^*) + f_{\pi_2}(T_2^*) + \pi'(v) - \pi'(u) \\ &= f_{\pi_1}(T_1^*) + \pi_1(u) - |V(T_1^*)| - 1 \\ &\quad + f_{\pi_2}(T_2^*) + \pi_2(v) + |V(T_1^*)| \\ &= g(T_1^*) + g(T_2^*) + 1 \\ &\le f_\pi(T). \end{aligned}$$

From Properties 3 and 4 we may assume that π maps $V(T_1^*)$ onto $1, \ldots, |V(T_1^*)|$ and we have

$$\begin{aligned} f_\pi(T) &= f_\pi(T_1^*) + |V(T_1^*)| - \pi(u) \\ &\quad + f_\pi(T_2^*) + \pi(v) - |V(T_1^*)| \\ &\geqslant g(T_1^*) + g(T_2^*) + 1. \end{aligned}$$

This proves that $f(T) = g(T_1^*) + g(T_2^*) + 1$. ■

Let v be a vertex in T. A substree formed by removing v and its incident edges is called a *branch* of $T - v$. A *rooted* branch of $T - v$ is a branch with the vertex adjacent to v as the root. A vertex is said to be a *center* of T if all branches $T - v$ have no more than $|V(T)|/2$ vertices.

Property 7 (*see* [2]). In any tree there exist one or two centers.

Property 8 (*see* [10, 17]). Any center of a tree is contained in the basic path of any OLA.

For a rooted tree T_i^*, we mean by T_i the unrooted version of T_i^*.

Property 9 (*see* [17]). Let r be the root of a tree T^*. Let $T_0, T_1, T_2, \ldots$ denote the branches in $T - \{r\}$ with $|V(T_i)| = t_i$, ordered so that $t_0 \geq t_1 \geq \ldots$. Let $p = p(T^*)$ be the largest integer satisfying

$$t_{2p+1} \geq \left\lfloor \frac{t_0 + 2}{2} \right\rfloor + \left\lfloor \frac{y + 2}{2} \right\rfloor$$

where

$$y = y(T^*) := n - \sum_{i=0}^{2p+1} t_i$$

and

$$n = |V(T)|.$$

Then in this case an OLA π is either of type $(: T_0)$ or of type $(T_2, \ldots, T_{2p} : T_{2p+1}, \ldots, T_1)$ where by type $(T_{i_1}, \ldots, T_{i_s} : T_{i_{s+1}}, \ldots, T_{i_t})$ we mean the set of linear arrangements in which $V(T_{i_1}, \ldots, V(T_{i_s}), V(T - \bigcup_{k=1}^{t} T_{i_k})), V(T_{i_{s+1}}), \ldots, V(T_{i_t})$ are labeled by consecutive integers in this order. If no such p exists, we set $p = -1$ and the OLA's are all of type $(: T_0)$.

Property 10 (*see* [15]). Suppose u is a center of T. Let $T_0, T_1, \ldots$ denote the branches in $T - \{u\}$ with $|V(T_i)| = t_i$ and $t_0 \geq t_1 \geq \ldots$. Let $q = q(T)$ be the largest positive integer satisfying

$$t_{2q} \geq \left\lceil \frac{t_0 + 2}{2} \right\rceil + \left\lceil \frac{z + 2}{2} \right\rceil$$

where

$$z = z(T) = n - \sum_{i=0}^{2q} t_i.$$

Then there is an OLA π either of type $(T_0 :)$ or of type $(T_2, \ldots, T_{2q} : T_{2q-1}, \ldots, T_1)$. If no such q exists we set $q = -1$ and there is always an OLA of type $(T_0 :)$.

Property 11 (*see* [15]). Suppose there is an OLA for T^* or T of type $(T_{i_1}, T_{i_2}, \ldots, T_{i_s} : T_{i_{s+1}}, \ldots, T_{i_t})$. Then a linear arrangement π of type $(T_{i_1}, T_{i_2}, \ldots, T_{i_s} : T_{i_{s+1}}, \ldots, T_{i_t})$ is optimal if π satisfies the following conditions:

(i) For $j > s$, the restrictions of π to $T_{i_j}^*$, denoted by π_{i_j}, is optimal for $T_{i_j}^*$

(ii) For $m \leq s$, suppose the restriction of π to $T_{i_m}^*$, is denoted by π_{i_m}. Then π'_{i_m}, as defined in Prop. 5 is optimal for $T_{i_m}^*$

(iii) The restriction of π to $T - \bigcup_{j=1}^{t} T_{i_j}$ is optimal for $T - \bigcup_{j=1}^{t} T_{i_j}$.

We denote the preceding linear arrangements π by $\pi(T_{i_1}, T_{i_2}, T_{i_s}, : T_{i_{s+1}}, \ldots, T_{i_t})$. We note that such a π is not necessarily unique. We also observe that for an unrooted tree T, $\pi(T_{i_1}, T_{i_2}, \ldots, T_{i_s} : T_{i_{s-1}}, \ldots, T_{i_t})$ is optimal if and only if $\pi(T_{i_t}, \ldots, T_{i_{s+1}} : T_{i_s}, \ldots, T_{i_2}, T_{i_1})$ is optimal. However, this is *not* true for rooted trees.

By combining the preceding properties we have:

Property 12. Define

$$C(T_{i_1}, T_{i_2}, \ldots, T_{i_s} : T_{i_{s+1}}, \ldots, T_{i_{2s}}) := \sum_{j=1}^{2s} g(T_{i_j}^*) + f\left(T - \bigcup_{j=1}^{2s} T_{i_j}\right) + ns - \sum_{j=1}^{s} (s-j+1)(t_{i_j} + t_{i_{2s-j+1}})$$

where $|V(T_{i_j})| = t_{i_j}$ and $|V(T)| = n$.

If $\pi(T_{i_1}, T_{i_2}, \ldots, T_{i_s} : T_{i_{s+1}}, \ldots, T_{i_{2s}})$ is optimal for T, then the cost of T is $C(T_{i_1}, T_{i_2}, \ldots, T_{i_s} : T_{i_{s+1}}, \ldots, T_{i_{s2}})$.

Property 13. We define

$$C(T_{i_1}, T_{i_2}, \ldots, T_{i_{s+1}} : T_{i_{s+1}}, \ldots, T_{i_{2s+1}}) = \sum_{j=1}^{2s+1} g(T_{i_j}^*) + f\left(T - \bigcup_{j=1}^{2s+1} T_{i_j}\right) + n(s+1) - \sum_{j=1}^{s} (s-j+1)(t_{i_j} + t_{i_{2s-j+1}}) - (s+1)t_{i_{2s+1}}$$

where $|V(T_{i_j})| = t_{i_j}$ and $|V(T)| = n$.

If $\pi(T_{i_1}, T_{i_2}, \ldots, T_{i_{s+1}} : T_{i_{s+1}}, \ldots, T_{i_{2s+1}})$ is optimal for T^*, then the cost of T^* is $C(T_{i_1}, T_{i_2}, \ldots, T_{i_s} : T_{i_{s+1}}, \ldots, T_{i_{2s+1}})$.

We next establish a series of facts which are used in Section IV to prove optimality of the generated linear arrangements.

LEMMA 1

For a rooted tree T^*, $g(T^*) = \min\{C(: T_0), C(: T_1)\}$.

Proof. It follows from Property 9 that the OLA is either of type $(: T_0)$ or of type $(T_2, \ldots, T_{2p} : T_{2p+1}, \ldots, T_1)$. Since type $(T_2, \ldots, T_{2p} : T_{2p+1}, \ldots, T_1)$ is a subset of $(: T_1)$, Lemma 1 is proved.

Lemmas 2 and 3 are immediate consequences of Property 13.

LEMMA 2

If $p(T^*) = -1$, then $g(T^*) = C(: T_0)$.

LEMMA 3

If $p = p(T^*) \geq 0$, then we have

$$g(T^*) = \min\{C(: T_0), C(T_2, \ldots, T_{2p} : T_{2p+1}, \ldots, T_1)\}.$$

LEMMA 4

For a tree T as defined in Property 10, we have

$$f(T) = \min\{C(T_0 : T_1), C(T_0 : T_2), C(T_1 : T_2)\}.$$

Proof. It follows from Property 10 that there is an OLA π of type $(T_0 :)$ or of type $(T_2 : T_1)$. If π is of type $(T_0 :)$, then by Property 6 the induced map π' of π on $T^* - T_0$ is optimal where $\pi'(v) = \pi(v) - |V(T_0)|$. It follows from Property 9 that π' is of type $(: T_1)$ or $(: T_2)$. Thus π on T is of type $(T_0 : T_1)$, $(T_0 : T_2)$ or $(T_2 : T_1)$.

LEMMA 5

If $q = q(T) > 0$, then we have

$$f(T) = \min\{C(T_2, \ldots, T_{2q} : T_{2q-1}, \ldots, T_1)\}.$$

Proof. This follows immediately for Property 10. ∎

LEMMA 6

If $q = -1$, then $f(T) = \min\{C(T_0 : T_1), C(T_0 : T_2)\}$.

Proof. It follows from Property 10 that there exists an OLA π of type (T_0:). From Property 6, we have $f(T) = g(T_0^*) + g(T^* - T_0) + 1$ where $T^* - T_0$ has a center u as the root. It follows from Lemma 3 that the OLA of $T^* - T_0$ is of type $(: T_1)$ or $(: T_2)$. Thus the OLA $\pi(T)$ of T is of type $(T_0 : T_1)$ or $(T_0 : T_2)$. ■

LEMMA 7

Suppose $q \geq 1$. We define $Q_i = \{0, 1, \ldots, 2q\} - \{i\}$ and we define i_j to be the jth smallest integer in Q_i. Then

$$f(T) = \min\{C(T_{i_2}, T_{i_4}, \ldots, T_{i_{2q}} : T_{i_{2q-1}}, \ldots, T_{i_1}):\quad i = 0, 1, \ldots, 2a\}.$$

Proof. It sufficies to show that there exists an OLA of type $W_i = (T_{i_2}, T_{i_4}, \ldots, T_{i_{2q}} : T_{i_{2q-1}}, T_{i_1})$ for some i. We will prove this by induction on q. From Lemma 4, it holds for $q = 1$. Suppose it is true for all trees T' with $q(T') < q$ and $q \geq 2$. From Lemmas 5 and 6 we know that there exists an OLA of type $(T_0 : T_1)$, $(T_0 : T_2)$ or W_0. Suppose $\pi = \pi(T_1 : T_0)$ is optimal. Consider $T' = T - T_0 - T_1$. Then $q(T') \geq q(T) - 1 = q - 1$. By the induction hypothesis there exists an OLA for T' of type $(T_{i_4}, \ldots, T_{i_{2q}} : T_{i_{2q-1}}, \ldots, T_{i_3})$ where $\{i_3, \ldots, u_{2q}\} = \{3, \ldots, 2q\} - \{i\}$ for some i. This implies there exists $\pi = \pi(T)$ of type W_i. Suppose $\pi(T_0 : T_2)$ is optimal. Then the rooted tree $T^* - T_0$ has $p(T^* - T_0) \geq q(T) - 1$ since the branch T_{2q} in $T^* - T_0$ has at least $\lfloor t_1/2 + 1 \rfloor + \lfloor y'/2 + 1 \rfloor$ vertices where $y' = y(T^* - T_0$. Thus there exists an OLA of $T^* - T_0$ of type $(T_3, \ldots, T_{2q-1} : T_{2q}, \ldots, T_2)$ or of type $(: T_1)$. Since we assumed π is of type $(T_0 : T_2)$, we conclude that there exists an OLA for T of type W_1. This completes the proof.

LEMMA 8

Suppose $p = p(T^*) \geq 0$. For $i \leq 2p + 1$, we define $P_i = \{0, \ldots, 2p + 1\} - \{i\}$ and i_j to be the jth smallest integer in P_i. Then $g(T^*) = \min\{C(T_{i_2}, T_{i_4}, \ldots, T_{i_{2p}} : T_{i_{2p+1}}, \ldots, T_{i_1}):$ $i = 0, 1, \ldots, 2p + 1\}$.

Proof. It suffices to show that there exists an OLA of type $U_i = (T_{i_2}, T_{i_4}, \ldots, T_{i_{2p}} : T_{i_{2p+1}}, \ldots, T_{i_1})$ for some i. From Lemma 1, we know that it is true for T^* with $p(T^*) = 0$. Suppose it is true for all trees $\bar{T}^*$ with $p(\bar{T}^*) < p(T^*) = p$ for a fixed $p > 0$. From Lemma 3, there exists an OLA of type $(: T_0)$ or of type U_0. Suppose there exists a linear arrangement π of type $(: T_0)$. Then the induced map of π on $T - T_0$ is optimal. Moreover, $q(T - T_0) \geq p(T^*)$. From Lemma 7 there exists an OLA of type $(T_{i_2}, \ldots, T_{i_{2p}} : T_{i_{2p+1}}, \ldots, T_{i_3})$ for $T - T_0$ where $\{i_2, i_3, \ldots, i_{2p+1}\} = \{1, 2, \ldots, 2p + 1\} - \{i\}$ for some i. This implies that there exists an OLA for T^* of type U_i.

The following two lemmas follow immediately from Lemmas 7 and 8.

LEMMA 9

Suppose $q = q(T) > q' \geq 1$. We define $Q_i' = \{0, 1, \ldots, 2q'\} - \{i\}$ and we define i_j to be the jth smallest integer in Q_i'. Then $f(T) = \min\{C(T_{i_2}, T_{i_4}, \ldots, T_{i_{2q'}} : T_{i_{2q'-1}}, \ldots, T_{i_1}):$ $i = 0, 1, \ldots, 2q'\}$.

LEMMA 10

Suppose $p = p(T^*) > p' \geq 1$. We define $P_i' = \{0, 1, \ldots, 2p' + 1\} - \{i\}$, for $i \leq 2p' + 1$ and we define i_j to be the jth smallest integer in P_i'. Then $g(T^*) = \min\{C(T_{i_2}, T_{i_4}, \ldots, T_{i_{2p'}} : T_{i_{2p'+1}}, \ldots, T_{i_1}): i = 0, 1, \ldots, 2p' + 1\}$.

3. AN $O(n^2)$ ALGORITHM

Here we will give a simplified recursion algorithm to find OLA for a tree or a rooted tree. Our analysis will show that the running time can be bounded by $O(n^2)$ elementary operations. We want to point out that this algorithm is quite similar to the $O(n^{2.2})$ algorithm of Shiloach[17], though the complexity analysis is done more carefully. Some of the arguments given here will be used in the next section to improve the running time from $O(n^2)$ to $O(n^\lambda)$ where $\lambda > \log 3/\log 2$.

Algorithm 1

Goal. For a given tree T or a rooted tree T^*, we want to determine the cost of $f(T)$ or $g(T^*)$ and the OLA $\pi(T)$ or $\pi(T^*)$.

Step 0. If the tree has a root r, go to step 5.

Step 1. Find a center u of T.

Step 2. Determine branches $T_0, T^1, \ldots,$ of $T - \{u\}$ where $|V(T_i)| = t_i$ and $t_0 \geq t_1 \geq \ldots$. Find the greatest positive integer q satisfying $t_{2q} \geq \lfloor t_0/2 + 1 \rfloor + \lfloor z/2 + 1 \rfloor$ where $z = n - \sum_{i=0}^{2q} t_i$. Set $q = -1$, if no such q exists.

Step 3. If $q \neq -1$, go the Step 4. Find $g(T_0^*)$, $g(T^* - T_0)$ and the corresponding OLA's $\pi(T_0^*)$ and $\pi(T^* - T_0)$. Then set $g(T^*) = C(T_0 :) = g(T_0^*) + g(T^* - T_0) + 1$ and combine $\pi(T_0^*)$ and $\pi(T^* - T_0)$ to form $\pi(T) = \pi(T_0 :)$ (see Property 11). Stop.

Step 4. Find the costs and the OLA's of T_i^* and $T_i \cup Z$ where $Z = T - \bigcup_{i=0}^{2q} T_i$ for $i = 0, 1, \ldots, 2q$. Define $Q_i = \{0, 1, \ldots, 2q\} - \{i\}$ and define i_j to be the jth smallest integer in Q_i. Determine $f(T) = \min\{C(T_{i_2}, T_{i_4}, \ldots, T_{i_{2q}} : T_{i_{2q-1}}, \ldots, T_{i_1})$: $i = 0, 1, \ldots, 2q\}$ and form $\pi(T)$ accordingly. Stop.

Step 5. Determine branches $T_0, T_1, \ldots,$ of $T^* - r$ where $|V(T_i)| = t_i$ and $t_0 \geq t_1 \geq \ldots$. Find the greatest integer p satisfying $t_{2p+1} \geq \lfloor t_0/2 + 1 \rfloor + \lfloor y/2 + 1 \rfloor$ where $y = n - \sum_{i=0}^{2p+1} t_i$. Set $p = -1$ if no such p exists.

Step 6. If $p \neq -1$, go to Step 7. Find $g(T_0^*)$, $f(T - T_0)$, $\pi(T_0^*)$ and $\pi(T - T_0)$. Determine $C(: T_0)$. Set $g(T^*) = C(: T_0)$ and $\pi(T^*) = \pi(: T_0)$. Stop.

Step 7. Find $g(T_i^*)$ and $f(T_i \cup Y)$, for $i = 0, \ldots, 2p + 1$, where $Y = T - \bigcup_{i=0}^{2p+1} T_i$. Define $P_i = \{0, 1, \ldots, 2p + 1\} - \{i\}$ and define i_j to be the jth smallest integer in P_i. Determine $g(T^*) = \min\{C(T_{1_2}, T_{i_4}, \ldots, T_{1_{2p}} : T_{1_{2p+1}}, \ldots, T_0)$: $i = 0, 1, \ldots, 2p + 1\}$ and set $\pi(T^*)$ accordingly. Stop.

We note that the optimality of the linear arrangement generated by algorithm 1 follows immediately from Lemmas 7 and 8. We let $F(T)$ and $G(T^*)$ denote the number of elementary computational operations required to find $f(T)$ and $g(T^*)$ and the corresponding OLA's in algorithm 1. Let $F(n)$ and $G(n)$ denote the maximum value of $F(T)$ and $G(T^*)$, respectively, over all trees T on n vertices and rooted trees T^* on n vertices. We will prove the following:

THEOREM 1

$F(n) < 0.8\,\mathrm{cn}^2$ and $G(n) < \mathrm{cn}^2$ for a suitable absolute constant $c > 0$.

Proof. We will prove by induction the following stronger statements:

(i) $$F(T) < 0.8\, cn^2$$

(ii) $$G(T^*) < cn^2 \text{ if } p = p(T^*) = -1$$

(iii) $$G(T^*) < 4.3c\,\frac{(2p + 2)}{(2p + 3)^2}\, n^2 \text{ if } p \geq 0$$

Note that

$$4.3\,\frac{(2p + 2)}{(2p + 3)^2} \leq \frac{8.6}{3^2} < 1.$$

Suppose (i), (ii) and (iii) hold for trees with fewer than n vertices.

Proof of (i). For a tree T *with* n vertices we consider the following possibilities:

Case 1: $q = q(T) = -1$.

$F(T) \leq G(T_0^*) + G(T^* - T_0) + c'n$ where $c'n$ steps are required to perform Steps 1, 2 and 6 of the algorithms.

If $p(T^* - T_0) = -1$, we have

$$G(T^* - T_0) \le G(T_1^*) + F(T - T_0 - T_1) + c'n.$$

Thus

$$F(T) \le c(t_0^2 + t_1^2 + 0.8(n - t_0 - t_1)^2) + 2c'n < 0.8\,\mathrm{cn}^2.$$

Since the function $b(x) = x^2 + a(n - x^2)$ has $(db(x)/dx^2) > 0$ for a fixed a, the maximum of b is achieved at the boundary. Note that $1 \le t_1 \le t_0 \le (n/2)$. It can be easily verified that the maximum is at $t_0 = t_1 = 1$. If $p' = p(T^* - T_0) \ne 1$, we have, by induction hypothesis (iii) that

$$G(T^* - T_0) < 4.3c\frac{2p'+2}{(2p'+3)^2}(n - t_0)^2.$$

Thus

$$F(T) < c\left(t_0^2 + 4.3\frac{2p'+2}{(2p'+3)^2}(n - t_0)^2\right) < 0.8\,\mathrm{cn}^2$$

since

$$\frac{n}{2} \ge t_0 \ge \frac{n}{2p'+4}$$

and

$$p' \ge 0$$

Case 2: $q \ge 1$.

We have

$$F(T) \le \sum_{i=0}^{2q} (G(T_i^*) + F(T_i \cup Z)) + c'n$$

where

$$Z = \bigcup_{i<2q} T_i$$

and

$$t_{2q} \ge \frac{t_0 + z}{2}.$$

We then have

$$F(T) < \left(\sum_{i=0}^{2q} (t_i^2 + 0.8(t_i + z)^2)\right) + c'n$$

Here we will use the following fact which will be verified in the Appendix.

LEMMA 11

The following function

$$H(\alpha_0, \alpha_1, \ldots, \alpha_m) = \sum_{i=0}^{m-1} (\alpha_i^2 + 0.8(\alpha_i + \alpha_m)^2)$$

with $\sum_{i=0}^{m} \alpha_i = 1$, $(1/2) \ge \alpha_1 \ge \ldots \ge \alpha_{m-1} \ge (\alpha_0 + \alpha_m)/2 \ge \alpha_m \ge 0$, $m \ge 2$ has a maximum at the point with $\alpha_0 = \alpha_1 = \ldots = \alpha_m = 1/(m+1)$.

Note we use Lemma 11 and consider $H(t_0/n, \ldots, t_{2q}/n, z/n)$. We then obtain

$$F(t) \le c\left(\frac{4.2(2q+1)}{(2q+2)^2}\right)n^2 + c'n < 0.8\,\mathrm{cn}^2.$$

Proof of (ii). For a rooted tree T^* with $p = p(T^*) = -1$ we have

$$\begin{aligned} G(T^*) &\le G(T_0^*) + F(T - T_0) + c'n \\ &< c(t_0^2 + 0.8(n - t_0)^2 + c'n \\ &< cn^2. \end{aligned}$$

Proof of (iii). For $p \ge 0$ we have

$$G(T^*) \le \sum_{i=0}^{2p+1} (G(T_i^*) + F(T_i \cup Y)) + c'n$$

where

$$Y = \bigcup_{i > 2p+1} T_i$$

and

$$t_{2p+1} \ge \frac{t_0 + y}{2}$$

By Lemma 11, we have

$$\begin{aligned} G(T^*) &< c\left(\sum_{i=0}^{2p+1} (t_i^2 + 0.8(t_i + y)^2)\right) + c'n \\ &< 4.2 \frac{(2p+2)}{(2p+3)^2} \mathrm{cn}^2 + c'n \\ &< 4.3 \frac{(2p+2)}{(2p+3)^2} \mathrm{cn}^2. \end{aligned}$$

We remark that the bound $O(n^2)$ can be improved to $O(n^{1.99})$ by a more careful analysis of the above proof.

4. AN n^λ ALGORITHM WHERE $\lambda > \mathrm{LOG}\ 3/\mathrm{LOG}\ 2$

The $O(n^\lambda)$ algorithm that we will give is a refined version of algorithm 1. The main idea here is to make use of the OLA's of subtrees more efficiently in the recursive process. The algorithm consists of three parts: Algorithm 2a for determining the cost and the OLA's for a tree T: Algorithm 2b for determining the cost and the OLA for a rooted tree T^*; and Algorithm 2c which, for two trees T^* and $\bar{T}^*$ with $|V(\bar{T}) \le |V(T)|$, determines the cost and the OLA's of T^*, and $T \cup \bar{T}$, which is the tree formed by combining T and $\bar{T}$ with an edge joining the roots of T^* and $\bar{T}^*$. The computational complexity of finding a pair $(g(T^*), f(T \cup \bar{T}))$, denoted by $h(T^*, \bar{T}^*)$, is in general much less that that of finding $g(T^*)$ and $f(T \cup \bar{T})$ separately.

Algorithm 2

Algorithm 2*a*

Step 1. Find a center u of T.

Step 2. Determine branches $T_0, T_1, \ldots$ of $T - u$ where $|V(T_i)| = t_i$, $t_0 \ge t_1 \ge \ldots$. Find the greatest positive integer $q = q(T)$ satisfying

$$t_{2q} \ge \left\lfloor \frac{t_0 + 2}{2} \right\rfloor + \left\lfloor \frac{z + 2}{2} \right\rfloor$$

where

$$z = n - \sum_{i=0}^{2q} t_i.$$

If no such q exists, set $q = -1$.

Step 3. If $q \neq -1$, go to Step 4. Find $g(T_0^*)$, $g(T^* - T_0)$ and the corresponding OLA's $\pi(T_0^*)$, $\pi(T^* - T_0)$. Determine $C(T_0 :) = g(T_0^*) + g(T^* - T_0) + 1$. Set $f(T) = C(T_0 :)$ and $\pi(T) = \pi(T_0 :)$. Stop.

Step 4. Find $h(T_i^*, Z^*)$ for $i = 0, 1, \ldots, 2q$, where $Z^* = T^* - \bigcup_{i=0}^{2q} T_i$. Define $Q_i = \{0, 1, \ldots, 2q\} - \{i\}$ and define i_j to be the jth smallest integer in Q_i. Determine $f(T) = \min\{C(T_{i_2}, T_{i_4}, \ldots, T_{i_{2q}} : T_{i_{2q-1}}, \ldots, T_{i_1})$: $i = 0, 1, \ldots, 2q\}$ and the corresponding $\pi(T)$. Stop.

Algorithm 2b

Step 1. Determine branches T_0, T_1, $T_2, \ldots$ of $T^* - r$, where $|V(T_i)| = t_i$ and $t_0 \geq t_1 \geq \ldots$. Find the greatest integer $p = p(T^*)$ satisfying

$$t_{2p+1} \geq \left\lfloor \frac{t_0 + 2}{2} \right\rfloor + \left\lfloor \frac{y+2}{2} \right\rfloor \text{ where } y = n - \sum_{i=0}^{2p+1} t_i.$$

If no such P exists, set $p = -1$.

Step 2. If $p \neq -1$, go to Step 3.

Find $g(T_0^*)$, $f(T - T_0)$, $\pi(T_0^*)$ and $\pi(T - T_0)$. Determine $C(: T_0)$. Set $g(T^*) = C(: T_0)$ and $\pi(T^*) = \pi(: T_0)$. Stop.

Step 3. Find $h(T_i^*, Y^*)$ for $i = 0, 1, \ldots, 2p+1$, where $Y^* = T^* - \bigcup_{i=0}^{2p+1} T_i$. Define $P_i = \{0, 1, \ldots, 2p+1\} - \{i\}$, and i_j to be the jth smallest integer in P_i. Determine $g(T^*) = \min\{C(T_{i_2}, T_{i_4}, \ldots, T_{i_{2p}} : T_{i_{2p+1}}, \ldots, T_{i1})$: $i = 0, 1, \ldots, 2p+1\}$ and $\pi(T^*)$ accordingly.

Algorithm 2c

Step 1. Find a center u of the tree $T \cup \bar{T}$ in T.

Step 2. Determine branches X, T_1, $T_2, \ldots$ of $T \cup \bar{T} - u$ where $|V(T_i)| = t_i$, $t_1 \geq t_2 \geq \ldots$. $|V(X)| = x$, $|V(T)| = n$, $|V(T')| = n' \leq n$ and X is the branch which contains $\bar{T}$.

Step 3. Let $\bar{P}$ denote the path joining u and the root r of T^*. Suppose $\bar{P}$ contains $u = v_0$, $v_1, \ldots, v_s = r$. Let X_i denote the branch of $T \cup \bar{T} - v_i$ which contains $\bar{T}$. Consider the tree $R_i^* = (T \cup \bar{T} - X_i)^*$ with v_i as the root. Determine $q = q(T \cup \bar{T})$, $p_i = p(R_i^*)$, $p' = p((T \cup \bar{T} - T_1)^*)$.

Step 4. If $n' > (n/3)$, go to Step 8. If $p_i = -1$ for all $0 \leq i \leq s$, go to Step 5. If $q = 1$, go to Step 8. If $x < t_1$, $q = -1$, $p' = 0$, go to Step 8. Go to Step 7.

Step 5. If $x < t_2$, $q = -1$, $p' = -1$ go to Step 6. Go to Step 8.

Step 6. If $p_0 \geq 0$, go to Step 7. If $p_0 = -1$ and $t_2 \geq \sum_{i>2} t_i$, go to Step 9. Go to Step 10.

Step 7. Determine $g(T^*)$, $\pi(T^*)$ and remember the cost and OLA's of the following trees, if available. (1) R_0^*; (2) S_i^*, the second largest branch of R_i^*; (3) T_i^*; (4) $T - T_1$. Determine $f(T \cup \bar{T})$ using the above data.

Step 8. Determine $f(T \cup \bar{T})$, $\pi(T \cup \bar{T})$ and remember the cost and OLA's of the following trees, if available: (1) T_i^*; (2) $T_i \cup W$ where W is a subtree of $T \cup \bar{T}$ not containing vertices in any of the $2q+1$ largest branch of $T \cup \bar{T} - u$; (3) R_0^*; (4) $R_0 - T_1$; (5) S_i, $R_i - R_{i-1}$. Determine $g(T^*)$ using the above data.

Step 9. Determine $g(T_1^*)$, $h(T_2^*, (R_0 - T_1 - T_2)^*)$, $f(T \cup \bar{T} - T_1 - T_2)$, $f(X - R_0 - \bar{P})$. Set $g(T^*) = C(: T_1)$; $f(T \cup \bar{T}) = C(T_1 : T_2)$.

Step 10. Determine $g(T_1^*)$, $g(T_2^*)$, $h(T_0^* - T_1 - T_2, X^*)$, $f(X - R_0 - \bar{P})$ if $q(R_0 - T_1) = -1$. Otherwise determine $g(T_1^*)$, $h(R_0^* - T_1 - T_2, X^*)$, $h(T_2^*, (W'')^*)$,

$f(X - R_0 - \bar{P})$ where $W'' = R_0 - \bigcup_{i=1}^{2p''+2} T_i$ and $p'' = p(R_0^* - T_1 - T_2)$. Set $g(T^*) = C(: T_1)$; $f(T \cup \bar{T}) = C(T_1 : T_2)$. Stop.

To see that Algorithm 2 gives the OLA's and cost for trees, the case-by-case analysis is included in the proof of Theorem 2.

We remark that in our algorithm, when the cost of a rooted tree Q^* is determined, the cost and the OLA's of the largest branch Q_0^* and $Q - Q_0$ are always found. Thus in Step 8 of Algorithm 2c, the cost and the OLA's of R_0^*, R_i^*, T_1^* and $T - T_1$ are found in determining $g(T^*)$. This is not true in general for S_i^*.

Let $F(T)$ and $G(T^*)$ denote the number of elementary computational operations required in Algorithm 2 to find the cost and the OLA's for T and T^*, respectively. Let $F(n)$ and $G(n)$ denote the maximum values of $F(T)$ and $G(T^*)$ over all trees T and rooted trees T^* on n vertices. Let $H(T^*, \bar{T}^*)$ denote the number of operations required for finding the cost and the OLA's of T^* and $T \cup T$ and let $H(n, n')$, $n > n'$, denote the maximum value of $H(T^*, \bar{T}^*)$ over all trees T^* on n vertices and $\bar{T}^*$ on n' vertices.

We will prove the following

THEOREM 2

$$F(n) < 0.7cn^{\lambda}$$

$$G(n) < cn^{\lambda}$$

where $\lambda = \log 3/\log 2 + \epsilon$ for any $\epsilon > 0$ and some constant $c > 0$.

Proof. We will establish by induction on n and n' the following stronger statements for a tree T^* on n vertices and a tree $\bar{T}^*$ on n' vertices, $n' \leq n$. (1) $F(T) \leq 0.7\,cn^{\lambda}$ if $q = q(T) = -1$; (2) $F(T) \leq 0.58\,cn^{\lambda}$ if $p((T - T_0)^*) \geq 0$ and $q = -1$; (3) $F(T) \leq 2.1c\,((2q+1)/(2q+2)^{\lambda})n^{\lambda}$ if $q = q(T) \geq 1$; (4) $G(T^*) \leq cn^{\lambda}$ if $p = p(T^*) = -1$; (5) $G(T^*) \leq 2.1c\,((2p+2)/(2p+3)^{\lambda})n^{\lambda}$ if $p \geq 0$; (6) $H(T^*, \bar{T}^*) \leq 0.7c\,((n+n')^{\lambda} + ((n-n')/2)^{\lambda})$ if $n' > (n/3)$; (7) $H(T^*, \bar{T}^*) \leq c(n^{\lambda} + 0.7((n+n')/2)^{\lambda})$ if $n' \leq (n/3)$.

Note that $2.1\,((2q+1)/(2q+2)^{\lambda}) \leq 0.7$ for $q \geq 1$ and $2.1\,((2p+2)/(2p+3)^{\lambda}) < 1$ for $p \geq 0$. Thus Theorem 2 will be proved after we establish these seven inequalities by induction on n and n'.

We need the following auxiliary lemma whose proof will be given in the appendix.

LEMMA 12

Consider the function

$$K_k(\alpha_0, \alpha_1, \ldots, \alpha_m) = \sum_{i=0}^{k-1}\left(\alpha_i^{\lambda} + 0.7\left(\frac{\alpha_i + \alpha_m}{2}\right)^{\lambda}\right) + 0.7\sum_{i=k}^{m-1}\left((\alpha_i + \alpha_m)^{\lambda} + \left(\frac{\alpha_i - \alpha_m}{2}\right)^{\lambda}\right).$$

The maximum of K_k over all points, satisfying

$$\sum_{i=0}^{m} \alpha_i = 1,\ \frac{1}{2} \geq \alpha_0 \geq \alpha_i \geq \ldots \geq \alpha_{k-1} \geq 3\alpha_m \geq \alpha_k \geq \ldots \geq \alpha_{m-1} \geq \frac{\alpha_0 + \alpha_m}{2} \geq \alpha_m \geq 0,$$

and $m \geq 2$, is no more than

$$K_0\left(\frac{1}{m+1}, \frac{1}{m+1}, \ldots, \frac{1}{m+1}\right) = \frac{2.1m}{(m+1)^{\lambda}}.$$

Proof of (1) *and* (2). Since $q(T) = -1$, we have

$$F(T) \leq G(T_0^*) + G(T^* - T_0) + c'n..$$

If $p(T^* - T_0) = -1$, we have

$$G(T^* - T_0) \leq G(T_1^*) + F(T - T_0 - T_1) + c'n.$$

This implies that

$$F(T) < c(t_0^\lambda + t_1^\lambda + 0.7(n - t_0 - t_1)^\lambda) + c'n.$$

Note that for the function $b(x) = x^\lambda + a(n - x)^\lambda$, where a is a constant, we have $(d^2b/dx^2) > 0$. The maximum of b is at a boundary point. Since $1 \le t_1 \le t_0 \le n/2$, it can be easily checked that

$$F(T) < c(2 + 0.7(n - 2)^\lambda) + c'n < 0.07n^\lambda$$

Note that $c'n$ is in fact much smaller than n^λ and can often be neglected. In the remaining part of the proof the term $c'n$ will sometimes be omitted to simplify the proof.

If $p(T^* - T_0) = p' \ge 0$, by induction hypothesis (5), we have

$$G(T^* - T_0) < 2.1c\frac{(2p' + 2)}{(2p' + 3)^\lambda}(n - t_0)^\lambda.$$

Therefore

$$F(T) < c\left(t_0^\lambda + 2.1\frac{(2p' + 2)}{(2p' + 3)^\lambda}(n - t_0)^\lambda\right)$$

where

$$n/2 \ge t_0 \ge \frac{n}{2p' + 4}.$$

The maximum of the above expression is achieved at $t_0 = (n/2)$ or $t_0 = n/(2p' + 4)$ and we have

$$F(T) < c\left(\frac{n}{2}\right)^\lambda + \frac{4.2c}{3^\lambda}\left(\frac{n}{2}\right)^\lambda < 0.58\,\mathrm{c}n^\lambda.$$

Proof of (3). In this case $q(T) = q \ge 1$, and we have

$$F(T) \le \sum_{i=0}^{2q} H(T_i^*, Z^*) = K_k(t_0, \ldots, t_{2q}, z)$$

where k satisfies

$$n/2 \ge t_0 > \ldots \ge t_{k-1} \ge 3z \ge t_k \ge \ldots \ge t_{2q} \ge \frac{t_0 + z}{2} \ge z \ge 0.$$

By Lemma 12, we have

$$F(T) \le 2.1c\frac{2q + 1}{(2q + 2)^\lambda}n^\lambda.$$

Proof of (4). In this case $p(T^*) = -1$. Therefore

$$\begin{aligned} G(T^*) &\le G(T_0^*) + F(T^* - T_0) \\ &< c(t_0^\lambda + 0.7(t - t_0)^\lambda) \\ &< cn^\lambda. \end{aligned}$$

Proof of (5). Since $p = p(T^*) \ne -1$, we have, by Lemma 12,

$$\begin{aligned} G(T^*) &\le \sum_{i=0}^{2p+1} H(T_i^*, Y^*) \\ &\le 2.1c\frac{(2p + 2)}{(2p + 3)^\lambda}n^\lambda. \end{aligned}$$

Proof of (6). We use the definition in algorithm 2c. If $p_i = p(R_i^*) \neq -1$ for some $i > 0$, then we have $x = |V(X) \geq n' + |V(S_i)|$ where S_i is the second largest branch of R_i.

If $p_i = 0$, we have

$$s = |V(S_i)| > \frac{|V(R_i)| - s}{2}.$$

Therefore

$$x \geq n' + s \geq \frac{n}{3} + s > |V(R_i)| - s \geq |V(R_0)| \geq \frac{n+n'}{2}.$$

This contradicts the fact that u is a center. If $p_i > 0$, then

$$|V(R_{i-1})| \leq \frac{|V(R_i)|}{1+p_i} < |V(R_i)|/2 \leq \frac{n}{2}.$$

This again contradicts the fact that u is a center and $|V(R_0)| > (n+n')/2$. Thus, $p_i = -1$ for all $i > 0$ and we determine $f(T \cup \bar{T})$ first (except for one case) while the cost and the OLA's of T_i^*, $T_i \cup W$, R_0^*, $R_0 - T_1$ are saved if available. Since $x \geq n' > (n/3)$, we have $x > t_3$.

We consider the following possibilities:

Case 1: $x \geq t_2$

Since $p_i = -1$ for all $i > 0$, $g(T^*)$ can be determined by finding $g(R_0^*)$ and $f(T - R_0\bar{P})$.

If $q = -1$ and $x \geq t_1$, then $g(R_0^*)$ are found in determining $f(T \cup \bar{T})$. Thus we have

$$H(T^*, \bar{T}) \leq F(T \cup \bar{T}) + F(T - R_0 - \bar{P}) < 0.7c\left((n+n')^\lambda + \left(\frac{n-n'}{2}\right)^\lambda\right)$$

since

$$|V(T - R_0 - \bar{P})| \leq n - |V(R_0)| \leq n - \frac{n+n'}{2} \leq \frac{n-n'}{2}$$

and $T - R_0 - \bar{P}$ denote the forest formed by removing the edges of $\bar{P}$ and vertices and edges of R_0.

If $q = -1$ and $t_1 > x \geq t_2$, then it follows from the definition that $p_0 = q - 1$. Thus $g(R_0^*)$ can be determined by combining $g(T_1^*)$ and $f(R_0 - T_1)$ which are found in determining $f(T \cup \bar{T})$. Therefore we have

$$H(T^*, \bar{T}^*) \leq F(T \cup \bar{T}) + F(T - R_0 - \bar{P}) \leq 0.7c\left((n+n')^\lambda + \left(\frac{n-n'}{2}\right)^\lambda\right).$$

If $q \geq 1$, then it follows from the definition that $p_0 \geq q - 1$ and $g(R_0^*)$ can be determined by using $g(T_i^*)$, $f(T_i \cup W)$ (see Lemma 10). This again implies

$$H(T^*, \bar{T}^*) \leq F(\cup \bar{T}) + F(T - R_0 - \bar{P}) \leq 0.7c\left((n+n')^\lambda + \left(\frac{n-n'}{2}\right)^\lambda\right).$$

Case 2: $t_3 < x < t_2$

If $q \geq 1$, then $p_0 \geq 0$. The proof is similar to that in Case 1. It remains to consider the case that $q = -1$. By definition we have

$$x < \frac{t_1+4}{2} + \frac{n - t_1 - t_2 - x}{2} = \frac{n + 4 - t_2 - x}{2}$$

i.e.

$$3x + t_2 \leq n + 2.$$

Since $t_2 > x \geq (n+1)/3$, we have a contradiction. This completes the proof of (6).

The proof of (7) is basically similar to that of (6) though more cases have to be considered. A detailed proof will be included in the Appendix.

Theorem 2 can be rewritten as follows:

THEOREM 3

$$F(n) < 0.7n^{\lambda}$$

$$G(n) < n^{\lambda}$$

for any $\lambda > \log 3/\log 2$ and n sufficiently large.

5. CONCLUDING REMARKS

The worse-case complexity for the optimal linear arrangement problem has a lower bound $O(n \log n)$ since the values of OLA's for subtrees have to be sorted. In this paper we obtain an upper bound $O(n^{\lambda})$ where $\lambda > \log 3/\log 2$. It is natural to ask the problem of further narrowing the gap between the upper bound and the lower bound. Here we will mention a few other related problems.

(1) A linear arrangement π for a directed graph is required to satisfy the additional condition that $\pi(u) < \pi(v)$ if there is an edge from u to v. The optimal linear arrangement problem for directed graph can then be viewed as a job sequencing problem[1]. Even and Shiloach[5] proved that the optimal linear arrangement problem for a cyclic directed graphs is also NP-complete. Adolphson and Hu[1] solved the problem for sorted directed trees (in which all edges are directed toward the root) with an $O(n \log n)$ algorithm. We can consider another type of trees, namely, *directed trees* in which each edge has certain assigned orientation. What is the algorithmic complexity for determining the optimal linear arrangements for directed trees?

(2) Meir[3] suggested the corresponding problem for the case that the weight function for an edge $\{u, v\}$ is $(\pi(u) - \pi(v))^2$. In general we may consider the problem of determining the linear arrangement to minimize the generalized cost of the graph. For example, for fixed value m, what is

$$\min_{\pi} \sum_{\{u,v\} \in E(G)} |\pi(u) - \pi(v)|^m?$$

We note that when m approaches infinity that is equivalent to the bandwidth problem of determining the linear arrangement which minimizes

$$\max_{\{u,v\} \in E(G)} |\pi(u) - \pi(v)|$$

Papadimitriou[15] proved that the bandwidth problem for graphs is NP-complete. It remains NP-complete for trees with no vertex degree exceeding 3 (see [8]). However, for general m the problem is far from being answered.

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APPENDIX

Proof of Lemma 11. Suppose H has a maximum at $(\beta_0, \beta_1, \ldots, \beta_m)$ with

$$\frac{1}{2} \geq \beta_0 \geq \ldots \geq \beta_{m-1} \geq \frac{\beta_0 + \beta_m}{m} \geqslant \beta_0 \geq 0.$$

If $\beta_0 = \beta_m$, then, then $\beta_i = (1/m+1)$, $i = 0, \ldots, m$. Suppose $\beta_0 > \beta_m$. Since $(\alpha + \epsilon)^2 + (b - \epsilon)^2 > a^2 + b^2$ for $a > b$ and ϵ small, we have

$$H(\beta_0, \ldots, \beta_m) = H\left(\beta_0, \beta_0, \ldots, \beta_0, x, \frac{\beta_0 + \beta_m}{2}, \ldots, \frac{\beta_0 + \beta_m}{2}, \beta_m\right)$$

where

$$\beta_0 > x \geq \frac{\beta_0 + \beta_m}{m}.$$

$$\bar{H}(\beta_0, \beta_m) = H\left(\beta_0, \ldots, \beta_m\right) = H\left(\beta_0, \beta_0, \ldots, x, \frac{\beta_0 + \beta_m}{2}, \ldots, \frac{\beta_0 + \beta_m}{m}, \beta_m\right) = i\beta_0^2 + x^2 + (m - i - 1)\left(\frac{\beta_0 + \beta_m}{2}\right)^2$$
$$+ 0.8i(\beta_0 + \beta_m)^2 + 0.8(x + \beta_m)^2 + 0.8(m - i - 1)\left(\frac{\beta_0 + 3\beta_m}{2}\right)^2$$

where

$$i\beta_0 + x + (m - i - 1)\frac{\beta_0 + \beta_m}{2} + \beta_m = 1, \ \beta_0 > x \geq \frac{\beta_0 + \beta_m}{2}.$$

By straightforward calculations, it can be shown that

$$\frac{\partial^2 \bar{H}(\alpha_0, \alpha_m)}{\partial \alpha_m^2} > 0.$$

Thus the maximum of $\bar{H}$ is attained at a boundary point. This implies that

$$x = \frac{\beta_0 + \beta_m}{2}$$

or $\beta_m = 0$. Note that for $\beta_m = 0$ we have, for $m \geq 3$,

$$H(\beta_0, \ldots, \beta_{m-1}, 0) \leq H\left(\frac{2}{m+1}, \frac{1}{m+1}, \ldots, \frac{1}{m+1}, 0\right) = \frac{1.8(m+3)}{(m+1)^2} < H\left(\frac{1}{m+1}, \frac{1}{m+1}, \ldots, \frac{1}{m+1}\right) = \frac{4.2m}{(m+1)^2}$$

and for $m = 2$ we have

$$H(\beta_0, \beta_1, 0) \leq H\left(\frac{1}{2}, \frac{1}{2}, 0\right) = 0.225 < H\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

Thus we have

$$x = \frac{\beta_0 + \beta_m}{2}.$$

Therefore

$$\bar{\bar{H}}(\beta_0, \ldots, \beta_m) = j\beta_0^2 + (m - j)\left(\frac{\beta_0 + \beta_m}{2}\right)^2 + 0.8j(\beta_0 + \beta_m)^2 + 0.8(m - j)\left(\frac{\beta_0 + 3\beta_m}{2}\right)^2$$

where

$$j\beta_0 + (m - j)\left(\frac{\beta_0 + \beta_m}{2}\right) + \beta_m = 1, \ \beta_0 \geq \beta_m.$$

Since

$$\frac{\partial^2 \bar{\bar{H}}(\alpha_0)}{\partial \alpha_0^2} > 0,$$

the maximum of $\bar{\bar{H}}$ is achieved on the boundary. Note that

$$\frac{1}{m+1} \leq \beta_0 \leq \min\left(\frac{1}{2}, \frac{2}{m+1}\right).$$

If $\beta_0 \neq \beta_m$, then $\beta_0 \neq 1/(m+1)$. We may assume the maximum of $\bar{\bar{H}}$ is achieved at $\beta_0 = \min((1/2), 2/(m+1))$. For $m \geq 3$, we have

$$\bar{\bar{H}}\left(\frac{2}{m+1}\right) < \bar{\bar{H}}\left(\frac{1}{m+1}\right).$$

For $m = 2$, we have $\beta_0 = (1/2)$, $\beta_1 \geq (1 + 2\beta_2)/2$, and $\bar{\bar{H}}(1/2) = H((1/2), (1/3), (1/6)) < \bar{\bar{H}}(1/3)$, which is a contradiction. Therefore we have $\beta_0 = \beta_m$. Lemma 9 is proved.

The proof of Lemma 12. We will first consider $K_0(\alpha_0, \alpha_1, \ldots, \alpha_m)$ with $\alpha_0 \geq \alpha_1 \geq \ldots \geq (\alpha_0 + \alpha_m)/2 \geq \alpha_m \geq 0$. We will show that $K_0(\alpha_0, \alpha_1, \ldots, \alpha_m) \leq K_0(1/(m+1), 1/(m+1), \ldots, 1/(m+1))$. Suppose K_0 has a maximum at $(\beta_0, \beta_1, \ldots, \beta_m)$, and $\beta_0 > \beta_m$. Since $(\alpha + \epsilon)^\lambda + (b - \epsilon)^\lambda > a^2 + b^2$ for $a > b$ and ϵ small we can assume

$$K_0(\beta_0, \beta_1, \ldots, \beta_m) = K_0\left(\beta_0, \beta_0, \ldots, \beta_0, x, \frac{\beta_0 + \beta_m}{2}, \ldots, \frac{\beta_0 + \beta_m}{2}, \beta_m\right) = \bar{K}(\beta_0, \beta_m)$$

where $i\beta_0 + x + (m - i - 1)(\beta_0 + \beta_m/2) + \beta_m = 1$, $\beta_0 > x \geq (\beta_0 + \beta_m)/2$. Similar to the proof of Lemma 11, we note that $(\partial^2 \bar{K}/\partial \beta_m{}^2) > 0$; Thus we have $x = (\beta_0 + \beta_m)/2$ or $\beta_m = 0$. If $\beta_m = 0$, then $K_0(\beta_0, 0) \leq K_0(2/(m+1), 0) < K_0(1/(m+1), 1/(m+1))$. We then have $x = (\beta_0 + \beta_m)/2$ and

$$K_0(\beta_0, \beta_1, \ldots, \beta_m) = K_0\left(\beta_0, \beta_0, \ldots, \beta_0, \frac{\beta_0 + \beta_m}{2}, \frac{\beta_0 + \beta_m}{2}, \ldots, \frac{\beta_0 + \beta_m}{2}, \beta_m\right) = \bar{\bar{K}}_0(\beta_0)$$

where $i\beta_0 + (m-1)((\beta_0 + \beta_m)/2) + \beta_m = 1$. Again $(\partial^2 \bar{\bar{H}}/\partial \beta_0{}^2) > 0$, and the maximum of $\bar{\bar{K}}_0$ is attained on the boundary. Since $1/(m+1) \leq \beta_0 \leq 2/(m+1)$ and we assume $\beta_0 \neq (1/m+1)$, we have $\beta_0 = 2/(m+1)$. However,

$$\bar{\bar{K}}_0\left(\frac{2}{m+1}\right) < K_0\left(\frac{1}{m+1}, \frac{1}{m+1}, \ldots, \frac{1}{m+1}\right),$$

which is a contradiction. Thus we prove that K_0 attains its maximum at $(1/(m+1), \ldots, 1/(m+1))$.

Now we consider $K_m(\alpha_0, \alpha_1, \ldots, \alpha_m)$ with $\alpha_0 \geq \alpha_1 \geq \ldots \geq \alpha_{m-1} \geq 3\alpha_m(\alpha_0 + \alpha_m/2) \geq \alpha_m \geq 0$. It can be proved in a similar manner that

$$K_m(\alpha_0, \alpha_1, \ldots, \alpha_m) \leq K_m\left(\frac{3}{3m+1}, \frac{3}{3m+1}, \ldots, \frac{3}{3m+1}, \frac{1}{3m+1}\right) = \frac{8.4m}{(3m+1)^\lambda}$$

For $m \geq 4$ and $2 \leq k \leq 2$, we consider

$$K_k(\alpha_0, \ldots, \alpha_m) = K_k(\alpha_0, \ldots, \alpha_{k-1}, \alpha_m) + K_0(\alpha_k, \ldots, \alpha_{m-1}, \alpha_m) \leq \frac{8.4k}{(3k+1)^\lambda}\left(\sum_{i=0}^{k-1} \alpha_i + \alpha_m\right)^\lambda + \frac{2.1(m-k)}{(m-k+1)^\lambda}\left(\sum_{i=k}^{m} \alpha_i\right)^\lambda$$

where

$$\sum_{i=k}^{m} \alpha_i \leq \frac{m-k+1}{m+1} \text{ and } \alpha_m \leq \frac{1}{m+1}.$$

Therefore K_k is bounded above by the maximum of the following function

$$w(b) = \frac{8.4k}{(3k+1)^\lambda}\left(a + \frac{a}{3k}\right)^\lambda + \frac{2.1(m-k)}{(m-k+1)^\lambda} b^\lambda$$

where $a + b = 1$, $(m - k + 1)/(m+1) \geq b \geq 3(m-k)/(3m+2k)$.

$$\left(\text{Note that we choose } a, b \text{ so that } \alpha_0 k \geq a, 3(m-k)\alpha_m \geq b \geq (m-k)\frac{(\alpha_0 + \alpha_m)}{2}\right).$$

Since $(d_2 w/db^2) > 0$, the maximum of w is attained at a boundary point. If

$$b = \frac{m-k+1}{m+1},$$

then

$$w(b) \leq K_0\left(\frac{1}{m+1}, \frac{1}{m+1}, \ldots, \frac{1}{m+1}\right).$$

It suffices to consider the case that $b = 3(m-k)/(3m+2k)$. Then

$$8.4k\left(\frac{5}{3(3m+2k)}\right)^\lambda + 2.1(m-k)\left(\frac{3(m-k)}{(m-k+1)(3m+2k)}\right)^\lambda \leq \frac{19k + (2.1)3^\lambda(m-k-0.9)}{(3m+2k)^\lambda} \leq \frac{(2.1)3^\lambda m + 7.1k - 10}{(3m+4)^\lambda}$$

$$\leq \frac{(2.1)3^\lambda m + 4.2}{(3m+4)^\lambda} \leq \frac{2.1m}{(m+1)^\lambda} \text{ for } 2 \leq k \leq m-2.$$

(Note that $x^{1-\lambda} \leq (x+1)^\lambda(x - 0.9)$ for $x \geq 2$.)

The case for $2 \leq m \leq 3$ can be proved in a similar manner by going through the cases that $\beta_0 \geq \ldots \geq \beta_i \geq 3\beta_m \geq \beta_{i-1} \geq (\beta_0 + \beta_m)/2 \geq \beta_m \geq 0$ and show that the maximum is achieved at a boundary point, in fact, at

$$\left(\frac{1}{m+1}, \frac{1}{m+1}, \ldots, \frac{1}{m+1}\right).$$

Proof of (7). In this case we have $n' \leq (n/3)$. We consider the following two cases: *Case* 1: $p_i = -1$ for $0 < i \leq s$. If $x \geq t_2$, similar to the proof of Case 1 in (6) we have $H(T^*, \bar{T}^*) \leq F(T \cup \bar{T}) + F(E) + c'n$ where $E = T - R_0 - \bar{P}$. If $q(T \cup \bar{T}) \geq 1$, then it follows from the definitions that $p(R_0^*) \geq q(T \cup \bar{T}) - 1$. Thus $g(T_0^*)$ can be determined by using the data available and we have

$$H(T^*, \bar{T}^*) \leq F(T \cup \bar{T}) + F(E) < 0.7c\left((n+n')^\lambda + \left(\frac{n-n'}{2}\right)^\lambda\right) < c\left(n^\lambda + \left(\frac{n+n'}{2}\right)^\lambda\right)$$

since $n' \leq n/3$.

We may assume that $q(T \cup \bar{T}) = -1$ and $x < t_2$.

Subcase a. $p' = p((T \cup \bar{T} - T_1)^*) \geq 0$. Where $(T \cup T - T_1)^*$ has u as the root.

We then have

$$H(T^*, \bar{T}^*) \leq G(T_1^*) + G((T \cup T - T_1)^*) + F(E) + F(R_0 - T_1) + c'n.$$

Note that in this case we can find $g(T^*)$ by using $g(T_1^*)$, $f(E)$ and $f(R_0 - T_1)$.

Since $p' \geq 0$ and $|V(E)| + |V(R_0 - T_1)| = n - t_1$, we have

$$H(T^*, \bar{T}^*) < c\left(t_1^\lambda + 2.1\frac{(2p'+2)}{(2p'+3)^\lambda}(n+n'-t_1)^\lambda + 0.7(n-t_1)^\lambda\right) + c'n.$$

Since the function $b(x) = x^\lambda + a(n-x)^\lambda$ has maximum on the boundary and $(n+n')/(2p'+4) \leq t_1 \leq (n+n')/2$, it suffices to check the following:

(i) $$\left(\left(\frac{1}{2}\right)^\lambda + 0.74\left(\frac{1}{2}\right)^\lambda\right)(n+n')^\lambda + 0.7\left(\left(\frac{n-n'}{2}\right)\right)^\lambda < 0.58(n+n')^\lambda + 0.24(n-n')^\lambda < n^\lambda + 0.7\left(\left(\frac{n+n'}{2}\right)\right)^\lambda$$

(ii) $$\left(\frac{1}{(2p'+4)^\lambda} + 2.1\frac{(2p'+2)}{(2p'+4)^\lambda}\right)(n+n')^\lambda + 0.7\left(n - \frac{n+n'}{2p'+4}\right)^\lambda < \left(\frac{1}{4^\lambda} + \frac{4.2}{4^\lambda}\right)(n+n')^\lambda + 0.7\left(\frac{3n-n'}{4}\right)^\lambda$$
$$< 0.58(n+n')^\lambda + 0.45n^\lambda < n^\lambda + 0.7\left(\frac{n+n'}{2}\right)^\lambda.$$

Subcase b. $p' = p((T \cup \bar{T} - T_1)^*) = -1$

(i) $p_0 = p(R_0^*) \geq 0$.

If $x \geq t_{2p0+2}$, tnen it follows from the definitions that $p' \geq p_0$ which contradicts $p' = -1$. We have $x < t_{2p_0+2}$, and $n' \leq x \leq (n+n')/(2p_0+3)$. Note that in this case we determine $g(T^*)$ first. Then $f(T \cup \bar{T})$ can be determined by finding $f(T \cup \bar{T} - T_0 - T_1)$ since $p' = -1$ and $g(T_2^*)$ is already found.

Thus we have

$$H(T^*, \bar{T}^*) \leq G(T^*) + F(T \cup \bar{T} - T_0 - T_1) \leq \frac{c2.1(2p_0+2)}{(2p_0+3)^\lambda}(n-x')^\lambda + 0.7c(n+n'-t_0-t_1)^\lambda + x'^\lambda$$
$$\leq \frac{2.1(2p_0+2)c}{(2p_0+3^\lambda)}n^\lambda + 0.7c\left(\frac{2p_0+1}{2p_0+3}\right)^\lambda(n+n') \leq \text{cn}^\lambda + c0.7\left(\frac{n+n'}{2}\right)^\lambda.$$

Since $n' + x' \leq x \leq n/(2p_0+2)$.

(ii) $p_0 = -1$.

Suppose $t_2 \geq \sum_{i>2} t_i = w'$. Let $E = X - R_0 - \bar{P}$, $W' = T - T_1 - T_2 - X$, $g(T^*)$ can be determined by finding $g(T_1^*)$, $f(T_2 \cup W')$ and $f(E)$. $f(\cup \bar{T})$ can be determined by finding $g(T_1^*)$, $g(T_2^*)$ and $f(X \cup W')$. Therefore we have

$$H(T^*, \bar{T}^*) \leq G(T_1^*) + H(T_2^*, (W')^*) + F(X \cup W') + F(E)$$
$$\leq c\left(t_1^\lambda + (t_2+w')^\lambda + 0.7\left(\frac{t_2+w'}{2}\right)^\lambda + 0.7\left(\frac{n+n'}{2}\right)^\lambda + 0.7(|E|)^\lambda\right)$$
$$\leq c\left(\left(\frac{n}{3}\right)^\lambda + \left(\frac{2n}{3}\right)^\lambda + 0.7\left(\frac{n}{3}\right)^\lambda + 0.7\left(\frac{n+n'}{2}\right)^\lambda\right)$$
$$< c\left(n^\lambda + 0.7\left(\frac{n+n'}{2}\right)^\lambda\right).$$

We may assume $t_2 < \sum_{i>2} t_i$. Thus the center of $R_0 - T_1$ is at u. If $q'' = q(R_0 - T_1) = -1$, then $f(R_0 - T_1)$ can be determined by finding $g(T_2^*)$ and $g(R_0^* - T_1 - T_2)$. Thus $h(T^*, \bar{T}^*)$ can be determined by finding $g(T_1^*)$, $g(T_2^*)$,

$g(R_0^* - T_1 - T_2)$, $f(T \cup \bar{T} - T_1 - T_2)$ and $f(E)$. We then have

$$H(T^*, \bar{T}^*) \leq G(T_1^*) + G(T_2^*) + H(R_0^* - T_1 - T_2, X^*) + F(E)$$

If $x \geq (1/3)(\sum_{i>2} t_i)$, we have

$$H(T^*, \bar{T}^*) \leq c\left(t_1^\lambda + t_2^\lambda + 0.7(n + n' - t_1 - t_2)^\lambda + 0.7\left(\frac{n - e - t_1 - t_2 - n'}{2}\right)^\lambda + 0.7e^\lambda\right)$$

for

$$\frac{n + n'}{2} \geq t_1 \geq t_2 \geq e = |V(E)| \geq 0.$$

It can be shown that

$$H(T^*, \bar{T}^*) \leq c\left(0.7(n + n')^\lambda + 0.7(\frac{n - n'}{2})^\lambda\right) < c\left(n^\lambda + 0.7\left(\frac{n + n'}{2}\right)^\lambda\right).$$

If $x < (1/3) \sum_{i>2} t_i$, we have

$$\begin{aligned} H(T^*, \bar{T}^*) &\leq c\left(t_1^\lambda + t_2^\lambda + (n - e - t_1 - t_2)^\lambda + 0.7\left(\frac{n + n' - t_1 - t_2}{2}\right)^\lambda + 0.7e^\lambda\right) \\ &\leq c\left(t_1^\lambda + t_2^\lambda(n - t_1 - t_2)^\lambda + 0.7\left(\frac{n + n' - t_1 - t_2}{2}\right)^\lambda\right) \\ &< c\left(n^\lambda + 0.7\left(\frac{n + n'}{2}\right)^\lambda\right) \end{aligned}$$

since $(n + n')/2 \geq t_1 \geq t_2 \geq 0$.

If $q'' \geq 1$, then $p'' = p(R_0^* - T_1 - T_2) \geq q'' - 1$. $f(R_0 - T_1)$ can be determined by finding $g(T_2^*)$ and $f(T_2 \cup W'')$, where $W'' = R_0 - \bigcup_{i=1}^{2p''+2} T_i$, after $g(R_0^* - T_1 - T_2)$ is determined. Thus $h(T^*, \bar{T}^*)$ can be determined by finding $g(T_1^*)$, $h(R_0^* - T_1 - T_2, X^*)$, $h(T_2^*, (W'')^*)$ and $f(E)$. By completing calculation similar to above, Case 1 is proved.

Case 2. $p_i \neq -1$ for some i, $0 < i < s$. Since the largest branch of R_i^* contains R_0, the second largest branch of R_i^*, denoted by S_i, has at least $(|V(R_{i-1})|/2 \geq (n + n')/4$ vertices. If there are i and j, $i > j > 0$, with $p_i \neq -1$ and $p_i \neq -1$, then we have $x > |V(S_i)| + |V(S_j)| \geq (n + n')/2$, which contradicts the fact that u is the center. Thus there is exactly one i with $p_i \neq -1$, $0 < i \leq s$. If $p_i \geq 1$, then the third and fourth largest branches of R_i^* contain at least $(|V(R_0)|/2)$ vertices. This again implies $x > (n + n')/2$, a contradiction. Thus we have $p_i = 0$. Moreover we have $x > t_2$ since $x > |V(S_i)| \geq (|V(R_0)|/2)$. If $q = q(T \cup \bar{T}) \geq 2$, we have $t_3 \geq t_4 > (|V(R_0)|/4)$ which is impossible. Thus we have $q = -1$ or $q = 1$.

We consider the following subcases:

Subcase a. $q = -1$. Since, for all $j \neq i$, $s_j \leq (n + n')/2 - S_i < R_{i-1} - S_i < (S_i/2)$, $g(X^*)$ can be determined by finding $f(X - S_i)$ when $f(S_i^*)$ is found in determining $g(T^*)$. If $x \geq t_1$, $f(T \cup \bar{T})$ can be determined by finding $g(X^*)$ and $g(R_0^*)$. Thus

$$H(T^*, \bar{T}^*) \leq G(T^*) + F(X - S_i) < c\left(n^\lambda + 0.7\left(\frac{n + n'}{4}\right)^\lambda\right) < c\left(n^\lambda + 0.7\left(\frac{n + n'}{2}\right)^\lambda\right)$$

If $x < t_1$ and $p((T \cup \bar{T} - T_1)^*) = p' = -1$, $f(T \cup \bar{T})$ can be determined by finding $g(T_1^*)$, $g(X^*)$ and $f(R_0 - T_1)$). Thus again we have

$$H(T^*, \bar{T}^*) \leq G(T^*) + F(X - S_i) < c\left(n^\lambda + 0.7\left(\frac{n + n'}{2}\right)^\lambda\right)$$

If $x < t_1$ and $p' \geq 0$, then $p' = 0$ (since $p' \geq 1$ will imply $t_4 > (x/2) > (|V(R_0)|/4)$). We note that in this case $f(T \cup \bar{T})$ is determined first as well as $g(T_1^*)$, $f(R_0 - T_1)$ and $g(S_1^*)$. $g(R_0^*)$ can be determined using $g(T_1^*)$ and $f(R_0 - T_1)$ since it follows from $q(T \cup \bar{T}) = -1$ that $p(R_0^*) = -1$. Thus $g(T^*)$ can be determined by finding $f(T - R_0)$ and $f(T - S_i)$. We have

$$H(T^*, \bar{T}^*) \leq F(T \cup \bar{T}) + F(T - R_0) + F(T - S_i).$$

Note that $t_2 \geq (x + w)/2$ and $t_1 > x \geq |V(S_i)| + w' + n'$ where $w = |V(R_0 - T_1 - T_2)|$, $w' = |V(T - R_0 - S_i)|$. Thus $x - w' - n' \geq |V(S_i)| \geq |V(R_0)|/2 \geq (t_1 + t_2 + w)/2 \geq (3(x + w)/4)$ and $w' + n' < (x - 3w)/4 \leq (n + n')/10$ since $(5/2)x \leq x + t_1 + t_2 \leq n + n'$. Set $s = |V(S_i)|$.

Since $p' \geq 0$ by (2) we have

$$F(T \cup \bar{T}) \leq 0.58c(n + n')^\lambda$$

$$H(T^*, \bar{T}^*) \leq c(0.58(n + n')^\lambda + 0.7(s + w')^\lambda + 0.7(n - s)^\lambda)$$

$$
\begin{aligned}
&\le (0.58(n+n')^{\lambda} + 0.7\left(\frac{n}{4}\right)^{\lambda} + 0.7\left(\frac{3n}{4} + w'\right)^{\lambda}\Big) \\
&\le c\left(0.7\left(\frac{n+n'}{2}\right)^{\lambda} + 0.35(n+n')^{\lambda} + 0.7\left(\frac{n}{4}\right)^{\lambda} + 0.7\left(\frac{3}{4}n + w'\right)^{\lambda}\right) \\
&< c\left(0.7\left(\frac{n+n'}{2}\right)^{\lambda} + 0.35n^{\lambda} + 0.7\left(\frac{n}{4}\right)^{\lambda} + 0.7\left(\frac{3n}{4} + \frac{n}{9}\right)^{\lambda}\right) \\
&< c\left(n^{\lambda} + 0.7\left(\frac{n+n'}{2}\right)^{\lambda}\right).
\end{aligned}
$$

Subcase b: $q = 1$. In this case, $f(T \cup \bar{T})$ is first determined when $g(T_1^*)$, $g(T_2^*)$, $f(T_1 \cup W)$, $f(T_2 \cup W)$, $g(X^*)$, $f(X \cup W)$ are found. Thus $g(R_{i-1}^*)$ can be determined by using $g(T_1^*)$, $g(T_2^*)$, $f(T_1 \cup W)$, $f(T_2 \cup W)$ and $f(R_j - R_{i-1})$, $j \le i - 1$. Since $q = 1$, we have $p' = p(T \cup \bar{T} - T_1)^*) \ge 0$. $g(T^*)$ can be determined by finding $f(T - S_i)$ and $f(T - R_0)$. The proof is then just the same as in *Subcase a* and will be omitted.