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Integer symmetric matrices having all their eigenvalues in the interval $[-2, 2]$

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Abstract

We completely describe all integer symmetric matrices that have all their eigenvalues in the interval $[-2, 2]$. Along the way we classify all signed graphs, and then all charged signed graphs, having all their eigenvalues in this same interval. We then classify subsets of the above for which the integer symmetric matrices, signed graphs and charged signed graphs have all their eigenvalues in the open interval $(-2, 2)$.

Keywords: Integer symmetric matrices; Cyclotomic matrices; Signed graphs; Charged graphs

1. Introduction

Let A be an $n \times n$ integer symmetric matrix with characteristic polynomial $\chi_A(x) = \det(xI - A)$. The aim of this paper is to describe all such matrices A that have the maximum modulus of their eigenvalues at most 2. The significance of the bound 2 is that, by a result of Kronecker [K], every eigenvalue of such a matrix A is then of the form $\omega + \omega^{-1}$, for some root of unity ω. Thus $z^n \chi_A(z + 1/z)$ is a cyclotomic polynomial. For this reason we call such integer symmetric matrices *cyclotomic* matrices.

In 1970 J.H. Smith [Smi] classified all cyclotomic $\{0, 1\}$-matrices with zeros on the diagonal, regarding them as adjacency matrices of graphs (see Fig. 9). Such graphs were called *cyclotomic*

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graphs in [MS]. It turns out that a full description of cyclotomic matrices is conveniently stated using more general graphs. So if we allow the off-diagonal elements of our matrix to be chosen from the set $\{-1, 0, 1\}$, we obtain a *signed* graph (see [CST,Z2]), a non-zero (i, j)th entry denoting a 'sign' of -1 or 1 on the edge between vertices i and j. Further, for a general symmetric $\{-1, 0, 1\}$ matrix, where now the diagonal entries may be non-zero, we obtain what we call a *charged* signed graph; we regard a non-zero (i, i)th entry of A as corresponding to a 'charge' on its ith vertex. If none of the edges of a charged signed graph in fact have sign -1, then we have a charged (unsigned) graph. However, a graph is also a signed graph, and a signed graph is also a charged signed graph. The notion of a charged signed graph is a convenient device for picturing and discussing symmetric integer matrices with entries in $\{-1, 0, 1\}$. These are the most important matrices in our description of general cyclotomic matrices.

In this paper we extend Smith's result to cyclotomic charged signed graphs (Theorem 2), and then, with little further work, to all cyclotomic matrices (Theorem 3). Along the way we find all cyclotomic signed graphs (Theorem 1). As a consequence, we can also describe all cyclotomic charged graphs (Theorem 7) and all cyclotomic matrices whose entries are non-negative (Theorem 9).

Having obtained our results for the closed interval $[-2, 2]$, it is then very natural to consider restricting the eigenvalues to the open interval $(-2, 2)$. We give a complete classification of symmetric integer matrices with eigenvalues in this restricted set (Theorem 6). As in the case of the closed interval, there are corresponding results for cyclotomic signed graphs (Theorem 4), cyclotomic charged signed graphs (Theorem 5), cyclotomic charged graphs (Theorem 8) and cyclotomic matrices whose entries are non-negative (Theorem 10). Having dealt with the general cyclotomic case, this is a relatively straightforward problem. There is a connection here with the theory of finite reflection groups and their Coxeter graphs, and we conclude with a discussion of this.

In [MS], cyclotomic graphs were used to construct Salem numbers and Pisot numbers. The original motivation for this current work was that it provides one of the ingredients necessary to extend the work in [MS]. But we think that our results may be of independent interest.

Throughout the paper, a *subgraph* of the (charged, signed) graph under consideration will always mean a vertex-deleted subgraph, that is, an induced subgraph on a subset of the vertices.

2. Interlacing, and reduction to maximal indecomposable matrices

In order to state our results, we need some preliminaries. The matrix A will be called *indecomposable* if and only if the underlying graph is *connected*. (In the underlying graph, vertices i and j are adjacent if and only if the (i, j)th entry of A is non-zero.) If A is not indecomposable, then there is a reordering of the rows (and columns) such that the matrix has block diagonal form with more than one block, and its list of eigenvalues is found by pooling the lists of the eigenvalues of the blocks. For our classification of cyclotomic matrices, it is clearly sufficient to consider indecomposable ones.

A repeatedly useful tool for us is Cauchy's interlacing theorem (for a short proof, see [Fis]).

Lemma 1 *(Interlacing theorem). Let A be a real symmetric matrix, with eigenvalues $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$. Pick any row i, and let B be the matrix formed by deleting row i and column i from A. Then the eigenvalues of B interlace with those of A: if B has eigenvalues $\mu_1 \leqslant \cdots \leqslant \mu_{n-1}$, then*

$$\lambda_1 \leqslant \mu_1 \leqslant \lambda_2 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_{n-1} \leqslant \lambda_n.$$

In view of this Lemma, if A is cyclotomic, then so is any matrix obtained by deleting from A any number of its rows, along with the corresponding columns: we then speak of the smaller matrix as being *contained in* the larger one (the smaller graph is an induced subgraph of the larger graph). We call an indecomposable cyclotomic matrix (or its graph) *maximal* if it is not contained in a strictly-larger indecomposable cyclotomic matrix: the corresponding cyclotomic graph is not an induced subgraph of a strictly larger connected cyclotomic graph. We shall see that every non-maximal indecomposable cyclotomic matrix is contained in a maximal one. It is therefore enough for us to classify all maximal indecomposable cyclotomic matrices.

When we consider matrices that have all their eigenvalues in the open interval $(-2, 2)$, we shall see that it is no longer always true that every such matrix is contained in a maximal one: there is an infinite family of indecomposable exceptions.

3. Equivalence, strong equivalence and switching

Denote by $O_n(\mathbb{Z})$ the orthogonal group of $n \times n$ signed permutation matrices. Then conjugation of a cyclotomic matrix by a matrix in $O_n(\mathbb{Z})$ gives a cyclotomic matrix with the same eigenvalues. We say that two $n \times n$ cyclotomic matrices are *strongly equivalent* if they are related in this way. Further, we say that two indecomposable cyclotomic matrices A and A' are merely *equivalent* if A' is strongly equivalent to A or $-A$. This notion then extends easily to decomposable cyclotomic matrices. Both of these notions are equivalence relations on the set of all cyclotomic matrices. For indecomposable cyclotomic matrices, the equivalence classes for the weaker notion are the union of one or two strong equivalence classes, depending on whether or not $-A$ is in the same strong equivalence class as A. It is clearly sufficient to classify all cyclotomic matrices up to equivalence.

For a charged signed graph, the notions of strong equivalence and equivalence of course carry over via the adjacency matrix. Now $O_n(\mathbb{Z})$ is generated by diagonal matrices of the form $\mathrm{diag}(1, 1, \ldots, 1, -1, 1, \ldots, 1)$ and by permutation matrices. Conjugation by these diagonal matrices corresponds to reversing the signs of all edges incident at a certain vertex v; we call this *switching at* v. Conjugation by a permutation matrix merely means that we can ignore vertex labels; we therefore do not label the vertices of our graphs. Thus for unlabelled charged signed graphs, strong equivalence classes are generated only by such switching operations. The concept of switching, and signed switching classes, appeared earlier for signed graphs in [CST].

Equivalence of charged signed graphs is generated both by switching, and by the operation of reversing all the edge signs and vertex charges of a component of a graph.

Since most of our graphs will in fact be signed graphs, we avoid clutter by drawing edges with sign 1 as unbroken lines ——, and edges with sign -1 as dashed lines - - - - - -. For vertices, those of charge $1, 0, -1$ will be drawn $\oplus$, $\bullet$, $\ominus$ respectively, with the vertices $\bullet$ without a charge being called *neutral* vertices.

4. Main results

Theorem 1 *("Uncharged, signed,* $[-2, 2]$*"). Every maximal connected cyclotomic signed graph is equivalent to one of the following*:

(i) *For some* $k = 3, 4, \ldots,$ *the* $2k$*-vertex toral tessellation* T_{2k} *shown in Fig.* 1;
(ii) *The* 14*-vertex signed graph* S_{14} *shown in Fig.* 3;
(iii) *The* 16*-vertex signed hypercube* S_{16} *shown in Fig.* 4.

Further, every connected cyclotomic signed graph is contained in a maximal one.

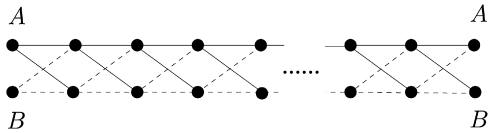


Fig. 1. The family T_{2k} of $2k$-vertex maximal connected cyclotomic toral tessellations, for $k \geqslant 3$. (The two copies of vertices A and B should be identified, as in Fig. 2 below.)

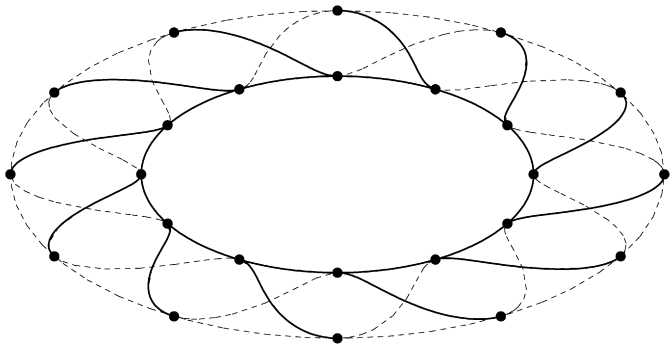


Fig. 2. A typical toral tessellation T_{2k}: the signed graph T_{24}.

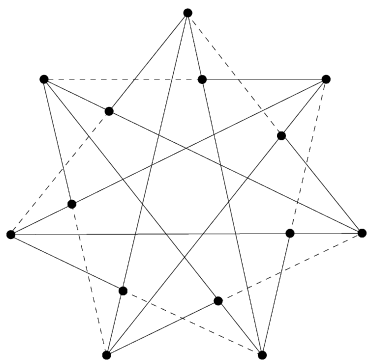


Fig. 3. The 14-vertex sporadic maximal connected cyclotomic signed graph S_{14}. See also Section 14.2.

In particular, $k = 3$ of case (i) gives an octahedron T_6, shown in Fig. 5, while a more typical example T_{24} is shown in Fig. 2.

Theorem 2 *("Charged, signed,* $[-2, 2]$*"). Every maximal connected cyclotomic charged signed graph not included in Theorem* 1 *is equivalent to one of the following*:

(i) *For some* $k = 2, 3, 4, \ldots$, *one of the two* $2k$*-vertex cylindrical tessellations* C_{2k}^{++}, C_{2k}^{+-} *shown in Fig.* 6;
(ii) *One of the three sporadic charged signed graphs* S_7, S_8, S_8' *shown in Fig.* 7;

Further, every connected cyclotomic charged signed graph is contained in a maximal one.

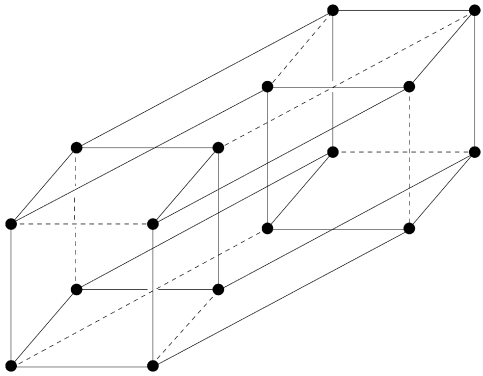


Fig. 4. The hypercube sporadic maximal connected cyclotomic signed graph S_{16}.

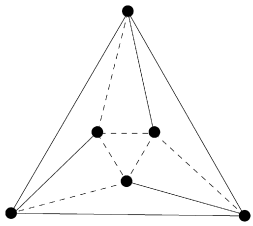


Fig. 5. The octahedral maximal connected cyclotomic signed graph T_6.

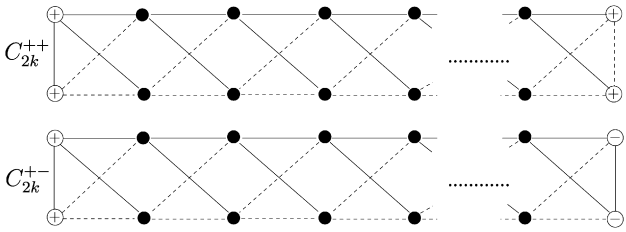


Fig. 6. The families of $2k$-vertex maximal connected cyclotomic cylindrical tessellations C_{2k}^{++} and C_{2k}^{+-}, for $k \geqslant 2$.

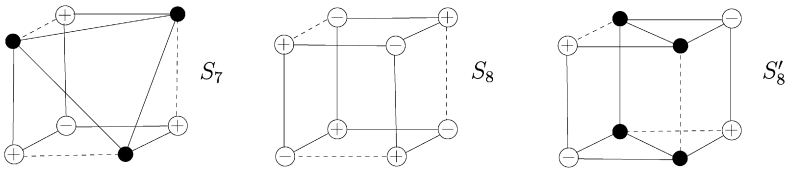


Fig. 7. The three sporadic maximal connected cyclotomic charged signed graphs S_7, S_8, S_8'.

In particular, $k = 2$ of case (i) gives two charged tetrahedra C_4^{++}, C_4^{+-}, shown in Fig. 8.

We remark that all the maximal cyclotomic graphs of Theorems 1 and 2 are 'visibly' cyclotomic: their adjacency matrices A all satisfy $A^2 = 4I$, so all their eigenvalues are ± 2. The exact multiplicity of these eigenvalues is given in Table 1 at the end of the paper.

Our most general result is readily deduced from the previous two theorems.

Fig. 8. The two maximal connected cyclotomic charged signed tetrahedra C_4^{++} and C_4^{+-}.

Theorem 3 *("Integer matrix, $[-2, 2]$"). Every maximal indecomposable cyclotomic matrix is equivalent to one of the following*:

(i) *The adjacency matrix of a maximal connected charged cyclotomic signed graph* (*given by Theorems* 1 *and* 2);
(ii) *The* 1×1 *matrix* (2) *or the matrix* $\left(\begin{smallmatrix} 0 & 2 \\ 2 & 0 \end{smallmatrix}\right)$.

Further, every indecomposable cyclotomic matrix is contained in a maximal one.

5. Simplifications

A signed graph G is called *bipartite* if its vertices can be split into two disjoint parts such that every edge of G joins a vertex in one part to a vertex in the other [Z3]. The eigenvalues of G are then symmetric about 0, counted with multiplicity; we record this fact as a lemma.

Lemma 2. *Let G be a bipartite signed graph with n vertices. Then*

$$\chi_G(-x) = (-1)^n \chi_G(x).$$

Proof. One can mimic a standard proof for graphs (as in [Big, p. 11]; this result first appeared in a Chemistry paper [CoR]), or simply note that if one changes the signs of all edges incident with vertices in one part then χ_G is unchanged, yet every edge has then changed sign so that $\chi_G(x)$ is changed to $(-1)^n \chi_G(-x)$. □

It will be convenient to extend the definition of bipartite to cover any charged signed graph such that changing the sign of every edge and charge produces a graph that is strongly equivalent to the original. For (neutral) signed graphs, this captures the usual definition of being bipartite. The extension of Lemma 2 holds true for this larger class of bipartite charged signed graphs, with the same proof.

A *cycle* of length r in a charged signed graph G is a list of distinct vertices $v_1, \ldots, v_r$ such that there is an edge in G between v_i and v_{i+1} $(1 \leqslant i < r)$ and between v_1 and v_r. A charged signed graph without cycles is called a (charged signed) *forest*. A connected forest is called a *tree*.

Lemma 3. *(See [CST, Theorem 2.2].) Any charged signed forest is equivalent to one for which all the edges are positive.*

Proof. An easy induction on the number of vertices: for the inductive step consider removing a leaf (a vertex with exactly one neighbour), unless there are no edges. □

For detecting non-cyclotomic integer symmetric matrices, the following trivial and obvious sufficient condition can be useful.

Lemma 4. *Let A be an $n \times n$ integer symmetric matrix. If either $\chi_A(2) < 0$ or $(-1)^n \chi_A(-2) < 0$, then A is not cyclotomic.*

Lemma 5. *Up to equivalence, the only indecomposable* 1*-by-*1 *or* 2*-by-*2 *cyclotomic matrices are*

$$(0), (1), (2), \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}.$$

Of these, the only maximal ones are (2) *and* $\left(\begin{smallmatrix} 0 & 2 \\ 2 & 0 \end{smallmatrix}\right)$.

Proof. This is an easy computation, using Lemma 4 to constrain the matrix entries. For example, to show that $\left(\begin{smallmatrix} 0 & 2 \\ 2 & 0 \end{smallmatrix}\right)$ is maximal, suppose that

$$A = \begin{pmatrix} 0 & 2 & a \\ 2 & 0 & b \\ a & b & c \end{pmatrix}$$

is cyclotomic. To achieve $\chi(2) \geqslant 0$ and $\chi(-2) \leqslant 0$ requires both $-2(a+b)^2 \geqslant 0$ and $2(b-a)^2 \leqslant 0$, giving $a = b = 0$, so that A is not indecomposable. □

Lemma 6. *Apart from matrices equivalent to either* (2) *or* $\left(\begin{smallmatrix} 0 & 2 \\ 2 & 0 \end{smallmatrix}\right)$*, any indecomposable cyclotomic matrix has all entries from the set* $\{0, 1, -1\}$*. In other words, it is the adjacency matrix of a cyclotomic charged signed graph.*

Proof. Let $A = (a_{ij})$ be an indecomposable cyclotomic matrix, not equivalent to either (2) or $\left(\begin{smallmatrix} 0 & 2 \\ 2 & 0 \end{smallmatrix}\right)$. Suppose first that some diagonal entry of A had modulus at least 2, say $|a_{ii}| \geqslant 2$. By interlacing (Lemma 1), the 1-by-1 matrix (a_{ii}) is cyclotomic, and then by Lemma 5 it equals $\pm(2)$ and is maximal, so equals A, giving a contradiction.

Next suppose that some off-diagonal entry a_{ij} had modulus at least 2. By interlacing, the 2-by-2 matrix $\left(\begin{smallmatrix} a_{ii} & a_{ij} \\ a_{ij} & a_{jj} \end{smallmatrix}\right)$ is cyclotomic, and by Lemma 5 this must equal $\pm\left(\begin{smallmatrix} 0 & 2 \\ 2 & 0 \end{smallmatrix}\right)$, and is maximal, so equals A. Again we have a contradiction.

Thus no entry of A has modulus greater than 1. □

We conclude that, apart from two (up to equivalence) trivial examples, all indecomposable cyclotomic matrices are the adjacency matrices of connected cyclotomic charged signed graphs. Thus Theorem 3 follows from Theorems 1 and 2, and we can restrict our attention to charged signed graphs.

6. Representation via Gram matrices

6.1. Gram matrices and line systems

Let A be the adjacency matrix of a cyclotomic charged signed graph with n vertices. In particular, A has all eigenvalues at least -2. Hence $A + 2I$ is positive semi-definite. This implies that we can find vectors $\mathbf{w}_1, \ldots, \mathbf{w}_n$ in real n-dimensional space such that $A + 2I$ is their Gram

matrix: the (i, j)-entry of $A + 2I$ is the dot product of $\mathbf{w}_i$ and $\mathbf{w}_j$. The dimension of the space spanned by the $\mathbf{w}_i$ might of course be smaller than n.

A particularly simple case is that of a signed graph, where there are no charges. Then the diagonal entries of $A + 2I$ all equal 2, so that the vectors $\mathbf{w}_i$ all have length $\sqrt{2}$. Moreover the lines spanned by the $\mathbf{w}_i$ meet each other with angles $\pi/3$ or $\pi/2$. In the language of [CvL] we have represented our signed graph in a line system, and if the graph is connected then the line system is indecomposable. If we change the sign of one of our Gram vectors, then the line that it spans is unchanged, and the new Gram matrix is equivalent to the old one: we have just changed the sign of all edges incident with the vertex that corresponds to our Gram vector. Since we are working up to equivalence, we can fix (at our discretion) the direction of each line in our system.

Indecomposable line systems have been classified. Every such line system is contained in a maximal one. It follows that every cyclotomic connected signed graph is contained in a maximal one. Moreover we can hunt for these by looking inside the maximal indecomposable line systems. These are $\mathcal{D}_n$ $(n \geqslant 4)$ and $\mathcal{E}_8$, which we now describe.

6.2. *The line system and signed graph $\mathcal{D}_n$*

Fix $n \geqslant 2$, and let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be an orthonormal basis for $\mathbb{R}^n$. The signed graph $\mathcal{D}_n$ has $n(n-1)$ vertices, represented by the vectors

$$\mathbf{e}_i \pm \mathbf{e}_j \quad (1 \leqslant i < j \leqslant n).$$

Adjacency of unequal vertices is given by the dot product of the corresponding vectors, which always equals one of 0, 1, -1. If A is the adjacency matrix of $\mathcal{D}_n$, then $A + 2I$ is the Gram matrix of the set of vectors.

6.3. *The line system and signed graph $\mathcal{E}_8$*

Let $\mathbf{e}_1, \ldots, \mathbf{e}_8$ be an orthogonal basis for $\mathbb{R}^8$, where, in contrast to the previous subsection, each $\mathbf{e}_i$ has length $\sqrt{2}$. The signed graph $\mathcal{E}_8$ has 120 vertices, represented by the vectors $\mathbf{e}_1, \ldots, \mathbf{e}_8$ and 112 vectors of the form

$$\frac{1}{2}(\mathbf{e}_i \pm \mathbf{e}_j \pm \mathbf{e}_k \pm \mathbf{e}_\ell),$$

where $ijk\ell$ is one of the 14 strings

$$1234,\ 1256,\ 1278,\ 1357,\ 1368,\ 1458,\ 1467,$$
$$2358,\ 2367,\ 2457,\ 2468,\ 3456,\ 3478,\ 5678.$$

(The referee has pointed out that these strings are the supports of the non-trivial words in the extended binary Hamming code of length 8.)

As for $\mathcal{D}_n$, adjacency of unequal vertices is given by the dot product (one of 0, 1, -1).

As a notational convenience, the vertices of $\mathcal{E}_8$ will be written as strings of digits, some of them overlined. Single digits $1, \ldots, 8$ refer to the basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_8$. Strings of four digits, with any of the last three overlined, refer to the vectors $(\mathbf{e}_i \pm \mathbf{e}_j \pm \mathbf{e}_k \pm \mathbf{e}_\ell)/2$, with overlining indicating a minus sign. For example, $14\bar{6}\bar{7}$ indicates the vector $(\mathbf{e}_1 + \mathbf{e}_4 - \mathbf{e}_6 - \mathbf{e}_7)/2$.

We sum up this discussion with the following result. For the proof one trivially adapts to signed graphs the argument for graphs in Chapter 3 of [CvL], noting that the fact that we can have negative edges makes the argument significantly easier.

Proposition 7. *Up to equivalence, the only* (*neutral*) *connected signed graphs that have all their eigenvalues in* $[-2, \infty)$ *are the connected subgraphs of* $\mathcal{D}_n$ $(n \geqslant 2)$ *and of* $\mathcal{E}_8$.

Signed graphs with all their eigenvalues in $[-2, \infty)$ have been studied earlier by Vijayakumar [V], Singhi and Vijayakumar [VS] and Ray-Chaudhuri, Singhi and Vijayakumar [RSV].

7. Cyclotomic signed graphs

In this section we prove Theorem 1, and so classify all cyclotomic signed graphs. The plan is as follows. First we find all the connected cyclotomic signed graphs that contain triangles (triples of vertices with each pair being adjacent). Then, in view of Proposition 7, it suffices to consider triangle-free subgraphs of $\mathcal{D}_n$ and $\mathcal{E}_8$. We find all maximal triangle-free subgraphs of $\mathcal{D}_n$, and observe the remarkable fact that they are all cyclotomic. We then find all maximal triangle-free subgraphs of $\mathcal{E}_8$: these are not all cyclotomic, and so we need to search among their subgraphs for any new maximal connected cyclotomic signed graphs that had not already been found as subgraphs of some $\mathcal{D}_n$.

7.1. Reduction to triangle-free graphs

Lemma 8. *Suppose that G is a cyclotomic signed graph that contains a triangle on vertices v, w, x (the signs of the three edges being arbitrary). If z is a fourth vertex in G then z is a neighbour of an even number of v, w, x.*

Proof. Direct computation of the small number of cases. One finds that if z is a neighbour of one or three of v, w, x then the subgraph induced by v, w, x, z is not cyclotomic, contradicting G being cyclotomic, by interlacing. □

If z is a neighbour of exactly two of v, w, x, then the subgraph induced by v, w, x, z is not always cyclotomic, and the next lemma describes the extra condition on the signs of the edges that is required for a cyclotomic graph.

Lemma 9. *If G is a cyclotomic signed graph containing two triangles that share an edge, then one triangle has an even number of negative edges, and the other has an odd number of negative edges.*

Proof. If two triangles share an edge and the parities of the numbers of negative edges in the two triangles are equal, then one quickly checks that a suitable equivalence will make all the edges on both triangles positive. But then the subgraph induced by the two triangles is not cyclotomic (it has $(1 + \sqrt{17})/2$ as an eigenvalue), and by interlacing neither is G. □

Corollary 10. *If G is a cyclotomic signed graph, then no three triangles can share a single edge.*

Corollary 11. *If G is a cyclotomic signed graph, then it does not contain a tetrahedron as an induced subgraph.*

This latter corollary also follows from Lemma 8.

Lemma 12. *If G is a connected cyclotomic signed graph that contains a triangle, then it is equivalent to a subgraph of the signed octahedron T_6 of Fig.* 5.

Proof. Suppose that G is a connected cyclotomic signed graph that contains a triangle, on vertices v_1, v_2, v_3. By a suitable equivalence, we may suppose that the three edges of this triangle are all positive. If G contains no other vertices then we are done.

Otherwise suppose that v_4 is another vertex of G, joined to v_1, say. By Lemma 8, v_4 is adjacent to exactly one other of the v_i. Relabelling if necessary, we suppose that v_4 is adjacent to v_1 and v_2. If G contains no other vertices then we are done.

Otherwise G contains a fifth vertex v_5, adjacent to at least one of v_1, v_2, v_3, v_4. By Lemma 8, v_5 is adjacent to two vertices on one of the triangles $v_1v_2v_3$, $v_1v_2v_4$, and hence is adjacent to one of v_1 or v_2. By Corollary 10, v_5 cannot be adjacent to both v_1 and v_2. Without loss of generality, v_5 is adjacent to v_1. By Lemma 8 (using triangles $v_1v_2v_3$ and $v_1v_2v_4$), v_5 is also adjacent to both v_3 and v_4. If G contains no other vertices then we are done.

Otherwise G contains a sixth vertex, v_6, adjacent to one of v_2, v_3, v_4, v_5 (it cannot be adjacent to v_1, or else by Lemma 8 it would be adjacent to one of the others, producing three triangles sharing an edge, contrary to Corollary 10). Applying Lemma 8 repeatedly, we see that v_6 must be adjacent to all of v_2, v_3, v_4, v_5.

We now have a subgraph of G that is equivalent to the signed octahedron pictured in the Lemma (by Lemma 9 the parity of the number of negative edges on faces sharing an edge must differ, and up to equivalence one sees that there is just one choice of signs).

Finally, G can have no more vertices, as each existing triangle shares each of its edges with another: we cannot adjoin a new vertex in a way that is compatible with both Lemma 8 and Corollary 10. □

Corollary 13. *In a cyclotomic signed graph G, each vertex has degree at most* 4.

Proof. If G contains a triangle then it is equivalent to a subgraph of the signed octahedron, and hence has maximal degree at most 4. We may therefore assume that G is triangle-free.

If G has a vertex v of degree at least 5, then v has neighbours $v_1, \ldots, v_5$ say (and possibly others), and since G is triangle-free there are no edges between any pair of $v_1, \ldots, v_5$. By computation the starlike subgraph induced by $v, v_1, \ldots, v_5$ is not cyclotomic (up to equivalence all the edges are positive, so there is only one case to compute). This contradicts G being cyclotomic, by interlacing. □

7.2. The maximal triangle-free subgraphs of $\mathcal{D}_n$

After Lemma 12, our search for connected cyclotomic signed graphs can be restricted to triangle-free connected cyclotomic signed graphs. After Proposition 7 we can hunt for these triangle-frees as subgraphs of one of the $\mathcal{D}_n$, or of $\mathcal{E}_8$. Here we deal with the $\mathcal{D}_n$, classifying all the maximal triangle-free subgraphs. Fortunately for us (in view of our ultimate goal) these subgraphs are all cyclotomic.

For $v = \mathbf{e}_i \pm \mathbf{e}_j \in \mathcal{D}_n$—so that $i < j$—define the *conjugate* vertex v^* to be $\mathbf{e}_i \mp \mathbf{e}_j$. If $v = \mathbf{e}_i \pm \mathbf{e}_j$, then we say that v *includes* $\mathbf{e}_i$ and $\mathbf{e}_j$. Note that v and v^* have the same neighbours in $\mathcal{D}_n$.

Lemma 14. *Let G be a maximal triangle-free subgraph of $\mathcal{D}_n$. If v is a vertex of G, then so is v^*.*

Proof. If v^*ab is a triangle in G, then so is vab. Hence if G contained v but not v^* we could add v^* to the vertex set and get a larger triangle-free signed graph, contradicting the maximality of G. □

Lemma 15. *Let G be a maximal triangle-free subgraph of $\mathcal{D}_n$. Each $\mathbf{e}_i$ is included in at most four vertices of G.*

Proof. If $\mathbf{e}_i$ is included at all, then take a vertex v including $\mathbf{e}_i$ and $\mathbf{e}_j$ ($j \neq i$).

Suppose first that there exists a vertex w in G that includes $\mathbf{e}_i$ and $\mathbf{e}_k$ for some other k ($k \neq i$, $k \neq j$). Then if x is a vertex of G that includes $\mathbf{e}_i$ and $\mathbf{e}_\ell$ ($\ell \neq i$) we must have either $\ell = j$ or $\ell = k$, or else vwx would be a triangle. Hence $\mathbf{e}_i$ is included exactly four times, in v, w, v^*, w^*.

If no such w exists, then $\mathbf{e}_i$ is included in exactly two vertices, v and v^*. □

Lemma 16. *Let G be a maximal triangle-free subgraph of $\mathcal{D}_n$. The maximum degree of G is at most 4. Moreover if a vertex v in G has distinct neighbours a and b with $a \neq b^*$, then v has four neighbours, a, b, a^*, b^*.*

Proof. Take any vertex v in G. By relabelling, and moving to v^* if necessary, we can suppose that $v = \mathbf{e}_1 + \mathbf{e}_2$. Let w be a neighbour of v. Again after relabelling, and so on, we can suppose that $w = \mathbf{e}_2 + \mathbf{e}_3$. Then w^* is also a neighbour of v. If v has a third neighbour x, then, in the same way, we can suppose that $x = \mathbf{e}_1 + \mathbf{e}_4$. Note that x cannot include $\mathbf{e}_2$, by Lemma 15. Then x^* is a fourth neighbour of v. By Lemma 15 again, there can be no more neighbours, as these would have to include either $\mathbf{e}_1$ or $\mathbf{e}_2$, both of which have been included four times already (in v, v^*, x, x^* and in v, v^*, w, w^* respectively). □

Recall that a path $v_1 v_2 \dots v_m$ in G is a sequence of distinct vertices v_i in G with v_i adjacent to v_{i+1} for $i = 1, \dots, m-1$.

Lemma 17. *Let G be a maximal connected triangle-free subgraph of $\mathcal{D}_n$, where $n \geqslant 4$. Let $P = v_1 v_2 \dots v_m$ be a path in G, maximal subject to no v_i equalling any v_j^*. Then*

- *v_1 and v_m are adjacent, so that the induced subgraph on the vertices of P is a cycle.*
- *$P^* := v_1^* \dots v_m^*$ is a path in G disjoint from P, and G is the subgraph spanned by P and P^*.*

Proof. First suppose that v_1 and v_m are not adjacent. By Lemma 14, P^* is a subgraph of G. No vertex in P can have more than two neighbours in P, else together with its neighbours in P^* it would have more than four neighbours in G, contradicting Lemma 16. Without loss of generality, $v_1 = \mathbf{e}_1 + \mathbf{e}_2$, $v_2 = \mathbf{e}_2 + \mathbf{e}_3, \dots, v_{m-1} = \mathbf{e}_{m-1} + \mathbf{e}_m$, $v_m = \mathbf{e}_m + \mathbf{e}_{m+1}$.

Now for $2 \leqslant i \leqslant m-1$, v_i has neighbours v_{i-1}, v_{i+1}, v_{i-1}^*, v_{i+1}^*, so has no other neighbours in G, by Lemma 16. By maximality of P, v_1 and v_m have no neighbours in G that are not in P or P^*, so P and P^* span a component of G, and hence span G. But then we could add $\mathbf{e}_1 + \mathbf{e}_{m+1}$ to G without introducing triangles, contradicting maximality of G.

Thus v_1 and v_m are adjacent, and without loss of generality $v_1 = \mathbf{e}_1 + \mathbf{e}_2$, $v_2 = \mathbf{e}_2 + \mathbf{e}_3, \dots, v_{m-1} = \mathbf{e}_{m-1} + \mathbf{e}_m$, $v_m = \mathbf{e}_1 + \mathbf{e}_m$. Now each element of $P \cup P^*$ has four neighbours in $P \cup P^*$, so no others, and again P and P^* span the whole of G. □

The proof of Lemma 17 establishes the first sentence of the next result.

Proposition 18. *Every maximal connected triangle-free signed graph that is a subgraph of some $\mathcal{D}_n$ $(n \geqslant 4)$ is equivalent to one with vertex set of the form*

$$\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \ldots, \mathbf{e}_{m-1} + \mathbf{e}_m, \mathbf{e}_1 + \mathbf{e}_m, \mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_2 - \mathbf{e}_3, \ldots, \mathbf{e}_{m-1} - \mathbf{e}_m, \mathbf{e}_1 - \mathbf{e}_m,$$

for some m in the range $4 \leqslant m \leqslant n$. Moreover, every such graph is cyclotomic, and is a maximal connected cyclotomic signed graph.

Proof. It remains to prove that such a graph G is cyclotomic (maximality as a connected cyclotomic signed graph follows from Corollary 13). For $n = 4$ one gets this by computation (or an easy adaptation of the following argument). For $n > 4$, note that if v and $w \neq v^*$ are distance 2 apart in G, then there are exactly two 2-paths from v to w, one along edges of the same sign, and one along edges of opposite sign. There are four 2-paths from v to v^*, two along edges of the same sign, and two along edges of opposite sign. Hence (with A the adjacency matrix of G) all off-diagonal entries of A^2 are zero. Since each vertex has degree 4, we deduce that $A^2 = 4I$. Hence all the eigenvalues are either 2 or -2, so G is cyclotomic. □

A nice representative of the equivalence class of the maximal cyclotomic signed graph, denoted T_{2n} in the theorem, described in Proposition 18 is obtained by replacing the vertex $\mathbf{e}_1 - \mathbf{e}_n$ by $\mathbf{e}_n - \mathbf{e}_1$. Then one of the n-cycles (say $v_1 v_2 \cdots v_n$) has all positive edges, and the other $(v_1^* v_2^* \cdots v_n^*)$ has all negative edges. The linking edges of the form $v_i v_{i+1}^*$ (interpreted cyclically) are all positive, and those of the form $v_i v_{i-1}^*$ are all negative. One gets a nice picture if the two cycles are viewed as the ends of a cylinder. Alternatively, the graph can be drawn on a torus without crossings, wrapping each cycle round the torus in such a way that it cannot be shrunk to a point (as in Fig. 2).

7.3. The maximal triangle-free subgraphs of $\mathcal{E}_8$

The search for triangle-free subgraphs of $\mathcal{E}_8$ (up to equivalence) was done by computer, using moderately intelligent backtracking. A lexicographical ordering was given to the 120 vertices, and a set of equivalences of $\mathcal{E}_8$ was precomputed (each as an explicit permutation of the 120 vertices), as follows.

We can change the sign of any $\mathbf{e}_i$: for any $i \in \{1, 2, 3, 4, 5, 6, 7, 8\}$, we can swap the roles of i and $\bar{i}$, which preserves all dot products. Then flip the sign of any vector that is no longer a vertex of $\mathcal{E}_8$ to induce an equivalence of $\mathcal{E}_8$. If G is a signed subgraph of $\mathcal{E}_8$, then applying this process gives an equivalent (but perhaps different) subgraph.

Some, but not all, permutations of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ induce a permutation of the lines spanned by the vertices of $\mathcal{E}_8$ (and hence induce a permutation of the vertices of $\mathcal{E}_8$). For any string $ijk\ell$ that appears as a vertex, we can apply elements of the Klein 4-group acting on $\{i, j, k, \ell\}$ to induce a permutation of the vertices of $\mathcal{E}_8$. For example, if we apply (12)(56) to the vertex $1\bar{2}3\bar{4}$ we get the vector $\bar{1}23\bar{4}$, which spans the same line as the vertex $1\bar{2}\bar{3}4$, so the image of $1\bar{2}3\bar{4}$ under (12)(56) is $1\bar{2}\bar{3}4$. Note that such a transformation might not be an isomorphism of signed graphs (since some of the vertices may be switched) but will be an equivalence. Again, applying this to a subgraph of $\mathcal{E}_8$ will give an equivalent subgraph.

Also, if $ijk\ell$ is a vertex of $\mathcal{E}_8$, then we can perform a change of basis by the following four swaps: $i \leftrightarrow ijk\ell$, $j \leftrightarrow ij\bar{k}\bar{\ell}$, $k \leftrightarrow i\bar{j}k\bar{\ell}$, $l \leftrightarrow i\bar{j}\bar{k}\ell$. This induces an equivalence on E. (It is enough to check that this works for $ijk\ell = 1234$, and then use the previous symmetries to reduce to this case.)

Starting with S being empty, the search grew S by adding the smallest possible vertices (with respect to the chosen ordering) whilst (i) maintaining triangle-freeness, and (ii) checking that none of the above equivalences of $\mathcal{E}_8$ would map the enlarged S to a lexicographically earlier set. The use of equivalences was hugely powerful in cutting down on the number of sets S considered by rejecting most sets at an early stage. When no more vertices could be added, the set S was tested for maximality, and maximal triangle-frees were written to a file. Then backtracking was done to find the next candidate for S.

The following twenty inequivalent maximal triangle-free subgraphs were found.

G1 $1, 2, 3, 4, 5, 6, 7, 8, 1234, 12\bar{3}\bar{4}, 1\bar{2}3\bar{4}, 1\bar{2}\bar{3}4, 5678, 56\bar{7}\bar{8}, 5\bar{6}7\bar{8}, 5\bar{6}\bar{7}8$.
This is two copies of the toral tessellation T_8.

G2 $1, 2, 3, 4, 5, 6, 7, 8, 1234, 12\bar{3}\bar{4}, 1\bar{2}5\bar{6}, 1\bar{2}\bar{5}6, 3\bar{4}56, 3\bar{4}\bar{5}\bar{6}$.
This comprises two isolated vertices plus T_{12}.

G3 $1, 2, 3, 4, 5, 6, 7, 8, 1234, 12\bar{3}\bar{4}, 1\bar{2}5\bar{6}, 1\bar{2}\bar{5}6, 3\bar{4}7\bar{8}, 3\bar{4}\bar{7}8, 5678, 56\bar{7}\bar{8}$.
This is T_{16}.

G4 $1, 2, 3, 4, 5, 6, 7, 8, 1234, 1\bar{2}5\bar{6}, 1\bar{3}\bar{5}7, 1\bar{4}6\bar{7}, 2\bar{3}5\bar{8}, 2\bar{4}\bar{6}8, 3\bar{4}7\bar{8}, 5678$.
This is the hypercube S_{16}.

G5 $1, 2, 3, 4, 5, 6, 7, 8, 1234, 1\bar{2}5\bar{6}, 1\bar{3}\bar{5}7, 1\bar{4}6\bar{7}, 2\bar{3}\bar{6}\bar{7}, 2\bar{4}57, 3\bar{4}\bar{5}\bar{6}$.
This is an isolated vertex plus S_{14}.

G6 $1, 2, 3, 4, 5, 6, 1234, 12\bar{3}\bar{4}, 1\bar{2}5\bar{6}, 1\bar{2}\bar{5}6, 3\bar{4}7\bar{8}, 3\bar{4}\bar{7}8, 567\bar{8}, 56\bar{7}8$.
This is T_{14}.

G7 $1, 2, 3, 4, 5, 6, 1234, 12\bar{3}\bar{4}, 1\bar{2}7\bar{8}, 1\bar{2}\bar{7}8, 3\bar{4}7\bar{8}, 3\bar{4}\bar{7}8, 5678, 56\bar{7}\bar{8}$.
This is a square plus T_{10}.

G8 $1, 2, 3, 5, 1278, 14\bar{6}\bar{7}, 2\bar{4}6\bar{8}, 3456, 3\bar{4}\bar{7}8, 5\bar{6}7\bar{8}$.
10 vertices, 2 cyclotomic components (both are 5-cycles).

By Corollary 13, S_{14} and S_{16} are maximal.

In the remaining cases, the larger component was non-cyclotomic.

G9 $1, 2, 3, 4, 5, 6, 1234, 1\bar{2}5\bar{6}, 1\bar{3}\bar{5}7, 1\bar{4}6\bar{7}, 3\bar{4}7\bar{8}, 567\bar{8}$.
12 vertices, 1 component, 29 maximal cyclotomic subgraphs (maximal in the sense that no larger subgraph of **G9** is cyclotomic).

G10 $1, 2, 3, 4, 5, 6, 1234, 1\bar{2}7\bar{8}, 1\bar{3}5\bar{7}, 1\bar{4}\bar{5}8, 3\bar{4}\bar{7}\bar{8}, 5678$.
12 vertices, 1 component, 13 maximal cyclotomic subgraphs.

G11 $1, 2, 3, 4, 5, 6, 1234, 1\bar{2}7\bar{8}, 1\bar{3}5\bar{7}, 2\bar{4}57, 3\bar{4}\bar{7}8$.
11 vertices, 2 components (one being a single vertex), 15 maximal cyclotomic subgraphs.

G12 $1, 2, 3, 4, 5, 6, 1234, 1\bar{2}7\bar{8}, 1\bar{3}5\bar{7}, 2\bar{4}6\bar{8}, 3\bar{4}\bar{7}8, 5678$.
12 vertices, 1 component, 15 maximal cyclotomic subgraphs.

G13 $1, 2, 3, 4, 5, 6, 1234, 1\bar{2}7\bar{8}, 1\bar{3}5\bar{7}, 2\bar{4}6\bar{8}, 5678, 5\bar{6}7\bar{8}$.
12 vertices, 1 component, 19 maximal cyclotomic subgraphs.

G14 $1, 2, 3, 4, 5, 6, 1278, 1\bar{2}7\bar{8}, 13\bar{5}\bar{7}, 235\bar{8}, 3478, 5\bar{6}\bar{7}8$.
12 vertices, 1 component, 17 maximal cyclotomic subgraphs.

G15 $1, 2, 3, 4, 5, 6, 1278, 1\bar{2}7\bar{8}, 13\bar{5}\bar{7}, 24\bar{6}\bar{8}, 3478, 5\bar{6}\bar{7}8$.
12 vertices, 1 component, 37 maximal cyclotomic subgraphs.

G16 $1, 2, 3, 4, 5, 1234, 1\bar{2}5\bar{6}, 1\bar{3}6\bar{8}, 2\bar{4}\bar{6}8, 3\bar{4}7\bar{8}, 56\bar{7}8$.
11 vertices, 1 component, 44 maximal cyclotomic subgraphs.

G17 $1, 2, 3, 4, 5, 1234, 1\bar{2}7\bar{8}, 1\bar{3}\bar{6}8, 2\bar{4}6\bar{8}, 3\bar{4}\bar{7}8, 5678$.
11 vertices, 2 components: K_2 plus a 9-vertex component; 36 maximal cyclotomic subgraphs.

G18 $1, 2, 3, 4, 5, 1256, 1\bar{2}7\bar{8}, 13\bar{5}\bar{7}, 24\bar{6}\bar{8}, 3478, 3\bar{4}7\bar{8}, 5\bar{6}\bar{7}\bar{8}$.
12 vertices, 1 component, 3-regular, 45 maximal cyclotomic subgraphs.

G19 $1, 2, 3, 4, 5, 1278, 13\bar{6}\bar{8}, 1\bar{4}6\bar{7}, 236\bar{7}, 2\bar{4}\bar{6}\bar{8}, 3\bar{4}78, 5\bar{6}\bar{7}8$.
12 vertices, 2 components: K_2 plus a 3-regular 10-vertex component, equivalent to the Petersen graph (switch at vertex 4 to get it) 57 maximal cyclotomic subgraphs.

G20 $1, 2, 3, 5, 1234, 1\bar{2}5\bar{6}, 1\bar{3}6\bar{8}, 2\bar{4}5\bar{7}, 3\bar{4}\bar{7}\bar{8}, 567\bar{8}$.
10 vertices, 1 component, 23 maximal cyclotomic subgraphs.

For each of the non-cyclotomic components listed above, it was checked by computer that none of their cyclotomic subgraphs are maximal cyclotomic graphs.

This completes the proof of Theorem 1.

7.4. An alternative view of the cyclotomic subgraphs of $\mathcal{E}_8$

Let G be a cyclotomic subgraph of $\mathcal{E}_8$ that is not equivalent to a subgraph of any $\mathcal{D}_r$, with vertices given by vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$, contained in 8-dimensional real space. Then $-G$, obtained from G by changing the signs of all edges and charges, is also cyclotomic. Now $-G$ is equivalent to G, so cannot be represented in any line system $\mathcal{D}_r$, so must be represented in the line system $\mathcal{E}_8$, and hence the vertices of $-G$ can be represented as vectors $\mathbf{w}_1, \ldots, \mathbf{w}_n$, where for each i either $\mathbf{w}_i$ or $-\mathbf{w}_i$ is in the signed graph $\mathcal{E}_8$.

We can view the concatenated vectors $[\mathbf{v}_1, \mathbf{w}_1], \ldots, [\mathbf{v}_n, \mathbf{w}_n]$ as elements of 16-dimensional real space, a subset of the 28 800 vectors $[\mathbf{v}, \mathbf{w}]$ where $\mathbf{v} \in \mathcal{E}_8$, $\pm\mathbf{w} \in \mathcal{E}_8$. Moreover, since the $\mathbf{w}_i$ represent $-G$, we have

$$\mathbf{w}_i \cdot \mathbf{w}_j = -\mathbf{v}_i \cdot \mathbf{v}_j$$

for all $i \neq j$. This implies that

$$[\mathbf{v}_i, \mathbf{w}_i] \cdot [\mathbf{v}_j, \mathbf{w}_j] = 0$$

for all $i \neq j$: our concatenated vectors $[\mathbf{v}_1, \mathbf{w}_1], \ldots, [\mathbf{v}_n, \mathbf{w}_n]$ are pairwise orthogonal. Since these vectors lie in 16-dimensional space, we must have $n \leqslant 16$, as is confirmed by the examples computed in Section 7.3.

Conversely, suppose that we take any orthogonal subset $[\mathbf{v}_1, \mathbf{w}_1], \ldots, [\mathbf{v}_n, \mathbf{w}_n]$ of the 28 800 vectors considered above, with the constraint that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are distinct. Then the signed graph G with vertices $\mathbf{v}_1, \ldots, \mathbf{v}_n$ (and adjacency of unequal vertices given by the dot product) is cyclotomic, for both G and $-G$ are represented in the line system $\mathcal{E}_8$, with $\mathbf{w}_1, \ldots, \mathbf{w}_n$ spanning the lines that represent $-G$.

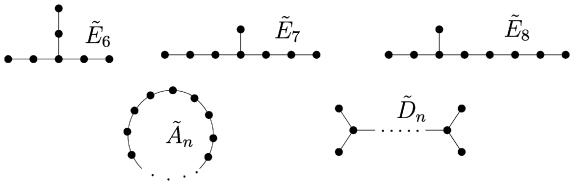


Fig. 9. The maximal connected cyclotomic graphs $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$, $\tilde{A}_n$ $(n \geqslant 2)$ and $\tilde{D}_n$ $(n \geqslant 4)$. The number of vertices is 1 more than the index. (From [MS].)

7.5. Remark on maximal cyclotomic (unsigned) graphs

The maximal cyclotomic graphs classified by Smith are shown in Fig. 9. The n-cycle $\tilde{A}_{n-1}$ and the graph $\tilde{D}_n$ are subgraphs of T_{2n}, while the sporadic examples are all subgraphs of the hypercube S_{16}. Unlike in the signed case, however, the maximal unsigned graphs are not visibly cyclotomic.

We can deduce Smith's classification as a corollary of Theorem 1, by checking that these graphs are the only maximal (unsigned) subgraphs of the signed graphs of the theorem. A useful fact to use in this check is that the graphs $\tilde{D}_4$ and $\tilde{D}_5$, since they have 2 as an eigenvalue, cannot be a proper subgraph of any such graph. This is because otherwise the graph would have an eigenvalue greater than 2—see [CvR, p. 4].

We also note in passing that the classification of all graphs having all their eigenvalues in the open interval $(-2, 2)$ follows from Smith's result. Such a graph is either a subgraph of E_8 or of some D_n for $n \geqslant 8$ (Fig. 18). Here, E_8 is $\tilde{E}_8$ (Fig. 9) with its rightmost vertex removed (same as U_5 in Fig. 12), and D_n is $\tilde{D}_n$ with a leaf removed. See also Theorem 10 below for a generalisation of this result.

8. Cyclotomic charged signed graphs

We now embark upon the trickier task of proving Theorem 2, and so classifying all cyclotomic charged signed graphs. The addition of charges means that we can no longer appeal to Proposition 7, although the Gram matrix approach will still prove extremely powerful.

8.1. Excluded subgraphs I

By interlacing, every subgraph of a cyclotomic charged signed graph is cyclotomic. We can therefore exclude as subgraphs any that are not cyclotomic. In particular, the following eight non-cyclotomic charged signed graphs $X_1, \ldots, X_8$ of Fig. 10 (or anything equivalent to any of them) cannot be subgraphs of any cyclotomic charged signed graph.

8.2. Excluded subgraphs II

Certain cyclotomic charged signed graphs have the property that if one tries to grow them to give larger connected cyclotomic graphs then one always stays inside one of the maximal examples on the following list: S_7, S_8, S'_8, C_4^{++}, C_4^{+-}, C_6^{++}, C_6^{+-}, T_6. The process of proving that a cyclotomic graph has this property is in principle simple, although perhaps tedious, to carry out. Starting from the given graph, one considers all possible ways of adding a vertex (up to equivalence) such that the graph remains connected and cyclotomic. Check that the resulting

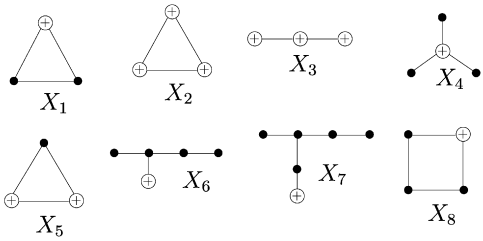


Fig. 10. Excluded subgraphs I: some non-cyclotomic charged signed graphs.

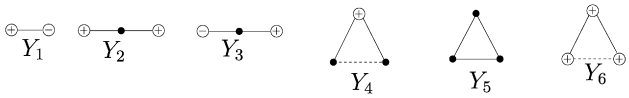


Fig. 11. Excluded subgraphs II: some cyclotomic charged signed graphs that are contained as subgraphs of a maximal connected cyclotomic charged signed graph only in one of the maximal graphs S_7, S_8, S_8', C_4^{++}, C_4^{+-}, C_6^{++}, C_6^{+-}, T_6.

graphs are (equivalent to) subgraphs of one of graphs on this list. Repeat with all the larger graphs found. If the checks in this process are always valid, then, since the process terminates, the original graph is suitable for exclusion.

By this technique, the six cyclotomic graphs $Y_1, \ldots, Y_6$ of Fig. 11 (and anything equivalent to them) can be excluded from future consideration.

8.3. Charged and neutral components

Let G be a charged signed graph. We define the *charged subgraph* of G to be the subgraph induced by all its charged vertices, and the *neutral subgraph* of G to be the subgraph induced by all its neutral vertices. The components of the charged (respectively neutral) subgraph of G will be called the *charged components* of G (respectively the *neutral components* of G).

Our next task will be to show that the charged components of a cyclotomic charged signed graph are tiny, provided that G does not contain Y_1, Y_6, or any equivalent subgraph.

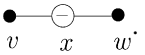
Lemma 19. *Let G be a cyclotomic charged signed graph that does not contain any subgraph equivalent to Y_1 or Y_6 of Section* 8.2. *Then each charged component of G contains at most two vertices, necessarily of the same charge.*

Proof. The last phrase is clear, since Y_1 is excluded as a subgraph. Moreover the exclusion of Y_1 forces every charged component to have all charges of the same sign, which by equivalence we may assume to be all positive. Since graphs X_2 and X_3 of Section 8.1 are not cyclotomic, and Y_6 is excluded by assumption, no charged component of G can have as many as three vertices. □

8.4. Local geometric constraints

Lemma 20. *Let G be a cyclotomic charged signed graph. Suppose that G contains two non-adjacent neutral vertices v and w that have a charged vertex x as a common neighbour. Then v and w have the same neighbours.*

Proof. Adjacency being unchanged by equivalence, we may suppose that the charge on x has negative sign, and that the edges joining v and w to x are positive. The subgraph induced by v, w, x is then



Since G is cyclotomic, all eigenvalues of its adjacency matrix A are in $[-2, \infty)$, so $A + 2I$ is the Gram matrix of some set of vectors. Let $\mathbf{v}$, $\mathbf{w}$, $\mathbf{x}$ be the Gram vectors corresponding to v, w, x. Since v and w are neutral, $\mathbf{v}$ and $\mathbf{w}$ have length $\sqrt{2}$. Since x has a negative charge, $\mathbf{x}$ has length 1. The angle between $\mathbf{v}$ and $\mathbf{x}$ is $\pi/4$, the angle between $\mathbf{w}$ and $\mathbf{x}$ is $\pi/4$, and the angle between $\mathbf{v}$ and $\mathbf{w}$ is $\pi/2$. Hence $\mathbf{v}$, $\mathbf{w}$, $\mathbf{x}$ are coplanar, with $\mathbf{x}$ in the direction of $\mathbf{v} + \mathbf{w}$. By consideration of their lengths we have

$$2\mathbf{x} = \mathbf{v} + \mathbf{w}. \tag{1}$$

Now let y be any other vertex of G, with corresponding Gram vector $\mathbf{y}$. Taking dot products with (1) gives

$$2\mathbf{y} \cdot \mathbf{x} = \mathbf{y} \cdot \mathbf{v} + \mathbf{y} \cdot \mathbf{w}. \tag{2}$$

The left-hand side of (2) is an even integer, hence the parities of the two integers on the right must agree. Hence y is adjacent to v if and only if it is adjacent to w. □

Lemma 21. *Let G be a cyclotomic charged signed graph containing adjacent charged vertices v and w, where the signs on the charges for v and w agree. Then v and w have the same neighbours.*

Proof. Adjacency is preserved by equivalence, so we may suppose that the charges on v and w are both negative, and that the edge between v and w is positive. In the usual way, let $\mathbf{v}$ and $\mathbf{w}$ be Gram vectors corresponding to v, w. These have length 1, and the angle between them is zero, so $\mathbf{v} = \mathbf{w}$, although $v \neq w$. Hence v and w have the same neighbours. □

8.5. Removing charged components I

We now show that if a connected cyclotomic charged signed graph does not contain a subgraph equivalent to any of the excluded subgraphs of Section 8.2, then it has a single neutral component. As a first step, we show that certain charged vertices can be deleted without disconnecting the graph.

Lemma 22. *Suppose that a connected cyclotomic charged signed graph G has two adjacent charged vertices v and w, with the charges on v and w having the same sign. Then the vertex w can be deleted without disconnecting G.*

Proof. By Lemma 21 every neighbour of v is a neighbour of w (and vice versa). Let x and y be any distinct vertices in G, with neither of them being w. We must show that there is a walk in G from x to y that does not pass through w. Certainly there is a path $v_1 v_2 \ldots v_r$ in G from x to y ($v_1 = x$, $v_r = y$). Suppose that this path contains w, say $v_i = w$. If either v_{i-1} or v_{i+1} is v, then

we can simply remove w from the path, since v shares its neighbours. Otherwise we can replace w by v in the path (producing a walk, but perhaps no longer a path), again since v and w share their neighbours. □

Lemma 23. *Let G be a connected cyclotomic charged signed graph, with more than three vertices, that contains no subgraph equivalent to either Y_1 or Y_4 of Section* 8.2. *Suppose that G contains two non-adjacent neutral vertices v and w that share a common charged neighbour x* (*as in Lemma* 20). *Then x can be deleted from G without disconnecting the graph.*

Proof. By the hypothesis on the number of vertices in G, there is some fourth vertex y in G that is adjacent to one of v, w, x.

First we dispose of the cases where y is adjacent to x. If y has a charge, then since Y_1 is an excluded subgraph, y and x have charges of the same sign. Then Lemma 22 shows that x can be removed without disconnecting G. If y is neutral, then since subgraphs equivalent to Y_4 are excluded, and a subgraph equivalent to X_1 of Section 8.1 is impossible, y cannot be adjacent to either v or w. But then G would contain a subgraph equivalent to X_4 of Section 8.1, which is not possible.

We may now suppose that y is not adjacent to x, and more strongly may suppose that v and w are the only neighbours of x. By Lemma 20, v and w share all their neighbours. In particular, y is adjacent to both v and w.

Let z_1 and z_2 be any vertices in G other than x. It is enough to show that there is a walk in G from z_1 to z_2 that does not pass through x. Certainly there is a path $v_1 v_2 \dots v_r$ from z_1 to z_2 ($v_1 = z_1$, $v_r = z_2$). Suppose that x is on this path: say $x = v_i$. We know that v_{i-1} and v_{i+1} each equal one of v and w. We can therefore replace x by y in our path to produce the desired walk. □

The requirement that G has more than three vertices is clearly necessary: if v, w, x are the only vertices in G then deleting x disconnects G.

8.6. Removing charged components II

Lemma 24. *Let G be a connected cyclotomic charged signed graph that does not contain a subgraph equivalent to Y_1, Y_4 or Y_6 of Section* 8.2. *Suppose further that G has at least four vertices. Then G contains a single neutral component*: *all charged vertices can be deleted without disconnecting G.*

Proof. By Lemma 19, all charged components have at most two vertices, and do not equal Y_1. By Lemma 22, we can remove a charged vertex from any charged component that has two vertices, without disconnecting G. We are thus reduced to charged components containing only one vertex.

If a charged vertex is a leaf, it can be removed without disconnecting G.

If a charged vertex has two neutral neighbours, then since subgraphs equivalent to X_1 and Y_4 are excluded we can appeal to Lemma 23 to see that this vertex can be removed without disconnecting G.

No charged vertex can have three or more neutral neighbours, or G would contain a subgraph equivalent to one of X_1, X_4 or Y_4. □

8.7. Growing the neutral component

Let G be a connected cyclotomic charged signed graph that contains at least four vertices, at least one of which is charged, but does not contain any of the excluded subgraphs $Y_1, \ldots, Y_6$. Then Lemma 24 tells us that G has a single neutral component, H say. By interlacing, H is cyclotomic, and from the classification of all cyclotomic signed graphs we know that H is (equivalent to) a subgraph of one of $\mathcal{D}_r$ (for some r), S_{14} or S_{16}. We treat each of these cases in turn.

8.7.1. H is equivalent to a subgraph of $\mathcal{D}_r$

We may suppose that r is minimal such that $\mathcal{D}_r$ contains a subgraph equivalent to H. Cases with $r \leqslant 4$ can be dealt with exhaustively by growing each possible H in all possible ways, adding only charged vertices, and checking that each maximal connected cyclotomic charged signed graph (maximal subject to the neutral component being H) is contained in some C_{2k}^{++} or C_{2k}^{+-}. We may therefore suppose that $r \geqslant 5$.

Working up to equivalence, we identify H with some subgraph of $\mathcal{D}_r$ (which has vertices $\mathbf{e}_i \pm \mathbf{e}_j$ for $1 \leqslant i < j \leqslant r$, where $\mathbf{e}_1, \ldots, \mathbf{e}_r$ is an orthonormal set of vectors). From our knowledge of the structure of cyclotomic signed graphs, we see that by relabelling and changing signs of basis vectors (thereby inducing an equivalence), we can suppose that H contains $\mathbf{e}_1 + \mathbf{e}_2$, $\mathbf{e}_2 + \mathbf{e}_3, \ldots, \mathbf{e}_{r-1} + \mathbf{e}_r$, and that all other vertices of H are of the form $\mathbf{e}_i - \mathbf{e}_{i+1}$ (for some i in the range $1 \leqslant i \leqslant r-1$), or $\mathbf{e}_1 \pm \mathbf{e}_r$.

Now suppose that w is a charged vertex in G. Since (i) G is connected, (ii) Y_1 is an excluded subgraph, and (iii) adjacent charged vertices that have the same charge share all their neighbours (Lemma 21), we deduce that w is adjacent to one or more vertices in H. We treat first the case where w has charge -1. We have represented (a graph equivalent to) H by a set of Gram vectors, where adjacency of unequal vertices is given by the dot product, and we can extend this to (a graph equivalent to) $H \cup \{w\}$, where w is represented by the Gram vector $\mathbf{w} = \sum_{i=1}^{r+1} \lambda_i \mathbf{e}_i$. If $\mathbf{w}$ is in the span of $\mathbf{e}_1, \ldots, \mathbf{e}_r$, then we may set $\mathbf{e}_{r+1} = \mathbf{0}$; otherwise we need an extra dimension for $\mathbf{w}$, and take $\mathbf{e}_{r+1}$ of length 1 and orthogonal to all of $\mathbf{e}_1, \ldots, \mathbf{e}_r$. Since w has charge -1, $\mathbf{w}$ has length 1.

We consider two subcases.

Case 1 (which we shall prove to be impossible): H contains one or both of $\mathbf{e}_1 \pm \mathbf{e}_r$, so that H contains a cycle of length r containing no pair of conjugate vertices.

Case 2: H contains neither of the vertices $\mathbf{e}_1 \pm \mathbf{e}_r$.

In Case 1, H contains at least one cycle of length r containing no pair of conjugate vertices. Suppose that w were adjacent to at least two vertices on such a cycle, say x and y (and perhaps others). Since subgraphs of G equivalent to Y_4 have been excluded, and G cannot contain a subgraph equivalent to X_1, the vertices x and y are not adjacent. By Lemma 20, every neighbour of x is a neighbour of y. But in a cycle of length at least 5 containing unadjacent vertices x and y and containing no pair of conjugate vertices, there will be a neighbour of x that is not a neighbour of y.

Still in Case 1, suppose next that w is adjacent to exactly one vertex in some cycle of length r containing no pair of conjugate vertices. Then G would contain a subgraph equivalent to X_6, giving a contradiction.

To kill off Case 1, we now consider the remaining subcase where w is adjacent to none of the vertices in the cycle $\mathbf{e}_1 + \mathbf{e}_2$, $\mathbf{e}_2 + \mathbf{e}_3, \ldots, \mathbf{e}_{r-1} + \mathbf{e}_r$, $\mathbf{e}_1 \pm \mathbf{e}_r$. Then H must contain at least

one more vertex, and after some relabelling and equivalence we can assume that w is adjacent to $\mathbf{e}_1 - \mathbf{e}_2$, with a positive edge. Then

$$\lambda_1 - \lambda_2 = 1, \qquad \lambda_1 + \lambda_2 = \lambda_2 + \lambda_3 = \lambda_3 + \lambda_4 = \lambda_4 \pm \lambda_5 = 0,$$

where the '$\pm$' might be '$-$' if $r = 5$. This gives

$$\lambda_1 = 1/2, \qquad \lambda_2 = -1/2, \qquad \lambda_3 = 1/2, \qquad \lambda_4 = -1/2, \qquad \lambda_5 = \pm 1/2,$$

and hence $|\mathbf{w}| > 1$, giving a contradiction.

We now move to Case 2, where H contains the path formed by the vertices $\mathbf{e}_1 + \mathbf{e}_2$, $\mathbf{e}_2 + \mathbf{e}_3, \ldots, \mathbf{e}_{r-1} + \mathbf{e}_r$, and all other vertices in H are of the form $\mathbf{e}_i - \mathbf{e}_{i+1}$ (for some i in the range $1 \leqslant i \leqslant r-1$).

If w were adjacent to more than one vertex in our path, say x and y, then as in Case 1 we would have x and y unadjacent, implying that they share all their neighbours, giving a contradiction.

If w were not adjacent to any vertex in our path, then it would be adjacent to some $\mathbf{e}_i - \mathbf{e}_{i+1}$, and from

$$\lambda_i - \lambda_{i+1} = \pm 1, \qquad \lambda_1 + \lambda_2 = \lambda_2 + \lambda_3 = \cdots = \lambda_{r-1} + \lambda_r = 0,$$

we would get at least five distinct j such that $|\lambda_j| = 1/2$, contradicting $|\mathbf{w}| = 1$.

We are reduced to the case where w is adjacent to exactly one vertex in our path. Since X_6 is excluded as a subgraph, this neighbour of w must be an endvertex of our path. Relabelling, we can suppose that w is attached to $\mathbf{e}_1 + \mathbf{e}_2$ by a positive edge, but to none of $\mathbf{e}_2 + \mathbf{e}_3, \ldots, \mathbf{e}_{r-1} + \mathbf{e}_r$. If H also contained $\mathbf{e}_1 - \mathbf{e}_2$, then w would necessarily be adjacent to it, or else G would contain a subgraph equivalent to X_7. Moreover, as $\mathbf{e}_2 + \mathbf{e}_3$ is joined to $\mathbf{e}_1 - \mathbf{e}_2$ by a negative edge, exclusion of subgraphs equivalent to X_8 implies that w must then be connected to $\mathbf{e}_1 - \mathbf{e}_2$ by a positive edge.

To sum up, if the minimal value of r is at least 5, then we can assume that H contains the vertices $\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \ldots, \mathbf{e}_{r-1} + \mathbf{e}_r$, and that all other vertices are of the form $\mathbf{e}_i - \mathbf{e}_{i+1}$. Any negatively charged vertex w in G is adjacent to one of $\mathbf{e}_1 \pm \mathbf{e}_2$ or $\mathbf{e}_{r-1} \pm \mathbf{e}_r$. If both of $\mathbf{e}_1 \pm \mathbf{e}_2$ are in H and w is adjacent to one of them, then it is adjacent to both; similarly for $\mathbf{e}_{r-1} \pm \mathbf{e}_r$. The excluded graph X_8 constrains the signs of the edges that connect w to H. In short, $H \cup \{w\}$ is equivalent to a subgraph of one of the C_{2k}^{++} or C_{2k}^{+-}.

By equivalence, similar remarks hold for positively-charged vertices in G.

If more than one charged vertex in G is adjacent to the same vertex in H, then the exclusion of subgraphs equivalent to Y_2 and Y_3 implies that these charged vertices are adjacent to each other; the exclusion of subgraph Y_1 implies that they all have the same sign; Lemma 21 implies that there are at most two such. We conclude that G is equivalent to a subgraph of one of the C_{2k}^{++} or C_{2k}^{+-}.

8.7.2. H is equivalent to a subgraph of S_{16}

We shall show that H is in fact equivalent to a subgraph of $\mathcal{D}_r$ for some r, so that we are reduced to the previous case.

Recalling previous notation, the vertices of S_{16} are labelled 1, 2, 3, 4, 5, 6, 7, 8, 1234, $1\bar{2}56$, $1\bar{3}\bar{5}7$, $1\bar{4}\bar{6}\bar{7}$, $2\bar{3}58$, $2\bar{4}6\bar{8}$, $3\bar{4}78$, $5\bar{6}7\bar{8}$ (a trivial relabelling of **G4**). These are vectors in

8-dimensional real space, with adjacency of unequal vectors given by the dot product. Each vector has length $\sqrt{2}$. Our restrictions on G imply that it has no triangles except perhaps involving two charged vertices and one neutral vertex.

Note that S_{16} is bipartite, with parts $\mathcal{V}_1 = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $\mathcal{V}_2 = \{1234, 1\bar{2}56, 1\bar{3}\bar{5}7, 1\bar{4}\bar{6}\bar{7}, 2\bar{3}58, 2\bar{4}6\bar{8}, 3\bar{4}78, 5\bar{6}7\bar{8}\}$. There is an equivalence of S_{16} that interchanges these two parts, induced by the orthogonal map with matrix

$$\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 & 0 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \end{pmatrix}$$

with respect to $\mathbf{e}_1, \ldots, \mathbf{e}_8$.

Let w be a charged vertex in G. Arguing as before, w is adjacent to at least one vertex in H. First we treat the case where w is adjacent to at least two vertices in H, say x and y. Now x and y cannot be adjacent in G, or we would have a forbidden triangle equivalent to X_1 or Y_4. Then by Lemma 20 the vertices x and y share all their neighbours (and they must have at least one neighbour in H or H would not be connected). It follows that x and y are either both in $\mathcal{V}_1$ or both in $\mathcal{V}_2$. Working up to equivalence, and swapping $\mathcal{V}_1$ and $\mathcal{V}_2$ as above if necessary, we may suppose that x and y are both in $\mathcal{V}_1$. We may also suppose that w is negatively charged, so that if we extend our set of Gram vectors representing H (some subset of the vectors/vertices in $\mathcal{V}_1 \cup \mathcal{V}_2$) to a set of Gram vectors representing $H \cup \{w\}$, the vector $\mathbf{w}$ representing w will have length 1. We may write $\mathbf{w} = \sum_{i=1}^{9} \lambda_i \mathbf{e}_i$, where $\mathbf{e}_9$ (length $\sqrt{2}$, orthogonal to $\mathbf{e}_1, \ldots, \mathbf{e}_8$) is included in case we need an extra dimension to make room for $\mathbf{w}$. Since $|\mathbf{w}| = 1$, we have $\sum_{i=1}^{9} \lambda_i^2 = 1/2$.

If x and y correspond to i and j in our labelling of the vertices of S_{16}, then from

$$\mathbf{w}.\mathbf{e}_i = \pm 1, \qquad \mathbf{w}.\mathbf{e}_j = \pm 1, \tag{3}$$

we have $\lambda_i, \lambda_j \in \{1/2, -1/2\}$, and hence all other λ_k are zero.

There are now essentially two cases (up to equivalence): $\{i, j\} = \{1, 2\}$ and $\{i, j\} = \{1, 8\}$. Indeed there are self-equivalences of S_{16} induced by elements of the Klein 4-group acting on any of the six 'missing' quartets $\{1, 2, 7, 8\}$, $\{1, 3, 6, 8\}$, $\{1, 4, 5, 8\}$, $\{2, 3, 6, 7\}$, $\{2, 4, 5, 7\}$, $\{3, 4, 5, 6\}$ (one then needs to apply appropriate further transpositions of the form $(i\bar{i})$, and some changes of signs of certain vertices, to map S_{16} to itself). We see that any vertex in $\mathcal{V}_1$ can be mapped to 1 by a self-equivalence of S_{16}, and that with 1 fixed, any vertex in $\mathcal{V}_1 \backslash \{1, 8\}$ can be mapped to 2.

In the case $\{i, j\} = \{1, 2\}$, since 1 and 2 are not adjacent, Lemma 20 implies that they have the same neighbours in G, and hence also in H, whence $1\bar{3}\bar{5}7$, $1\bar{4}\bar{6}\bar{7}$, $2\bar{3}58$, $2\bar{4}6\bar{8} \notin H$. One of 1234 and $1\bar{2}56$ has dot product ± 1 with $\mathbf{w}$, and hence must be excluded from H (or else together with w and 1 (or 2) we would have a forbidden triangle). Hence the vertices in H are a subset of $\mathcal{W}_1 \cup \mathcal{W}_2$, where

$$\mathcal{W}_1 = \{1, 2, 3, 4, 5, 6, 7, 8, 3\bar{4}78, 5\bar{6}7\bar{8}\}, \qquad \mathcal{W}_2 = \{1234\} \quad \text{or} \quad \{1\bar{2}56\},$$

depending on the signs of λ_1 and λ_2. Then H is readily seen to be equivalent to a subgraph of $\mathcal{D}_8$.

For the other essentially distinct case, $\{i, j\} = \{1, 8\}$, similar reasoning shows that H is a subset of $\mathcal{V}_1$, contradicting the connectedness of H.

We are left with the possibility that w is adjacent to exactly one vertex in H. Let us temporarily call a signed charged graph K *friendly* if it is cyclotomic, contains exactly one charged vertex w, the vertex w is joined to exactly one neutral vertex, and the neutral vertices in K form a single component. In our current case, $H \cup \{w\}$ is friendly. It will be enough to show that any friendly graph with neutral component equivalent to a subgraph of either S_{16} or S_{14} is contained in a larger friendly graph (where the neutral component of the larger friendly graph might or might not be equivalent to a subgraph of either S_{16} or S_{14}). For then we can grow our friendly graph $H \cup \{w\}$ to a larger friendly graph $H' \cup \{w\}$ with H' not equivalent to a subgraph of either S_{16} or S_{14}. Then H' must be equivalent to a subgraph of some $\mathcal{D}_r$, and hence the same is true for H.

A computer search checked that all friendly graphs with up to 14 neutral vertices are contained in larger friendly graphs. As an indication of the work involved, some 377 friendly graphs with 15 vertices (14 neutral vertices) were considered; these would not all have been inequivalent, as it proved more efficient to perform a fast but imperfect weeding out of equivalent graphs, allowing some repeats through. The search could have been pushed further, but it was easier simply to check that there are no friendly graphs with 15 or 16 neutral vertices for which the neutral component is equivalent to a subgraph of S_{16}.

8.7.3. H is equivalent to a subgraph of S_{14}

The argument here is very similar to that for S_{16}, but in fact slightly simpler, as S_{14} has fewer vertices. Analogously, we have $\mathcal{V}_1 = \{1, 2, 3, 4, 5, 6, 7\}$, $\mathcal{V}_2 = \{1234, 1\bar{2}5\bar{6}, 1\bar{3}\bar{5}7, 1\bar{4}6\bar{7}, 2\bar{3}\bar{6}\bar{7}, 2\bar{4}57, 3\bar{4}\bar{5}\bar{6}\}$. In the 'unfriendly' case we find that the vertices in H are (after a suitable equivalence) a subset of $\mathcal{W}_1 \cup \mathcal{W}_2$ where

$$\mathcal{W}_1 = \{1, 2, 3, 4, 5, 6\}, \qquad \mathcal{W}_2 = \{1234\} \quad \text{or} \quad \{1\bar{2}5\bar{6}\}.$$

Then H is equivalent to a subgraph of $\mathcal{D}_6$.

This completes the proof of Theorem 2.

9. Eigenvalues in the open interval $(-2, 2)$

9.1. Introduction to the next three sections

Sections 9–11 are devoted to results for matrices and graphs under further restrictions. These follow more or less straightforwardly from Theorems 1 and 2. We consider first restricting to eigenvalues in the open interval $(-2, 2)$ (Section 9), deferring the proofs to Section 12. Then we consider charged (unsigned) graphs, treating both the open and closed intervals (Section 10). Finally we treat symmetric matrices that have non-negative integer entries (Section 11).

9.2. Cyclotomic signed graphs with all eigenvalues in $(-2, 2)$

Having classified all integer symmetric matrices having all their eigenvalues in the interval $[-2, 2]$, a natural question is what happens if we restrict the eigenvalues to the open interval $(-2, 2)$. From our knowledge of the closed interval case, we can immediately restrict to cyclotomic signed graphs and cyclotomic charged signed graphs, and need only consider subgraphs of the maximal ones.

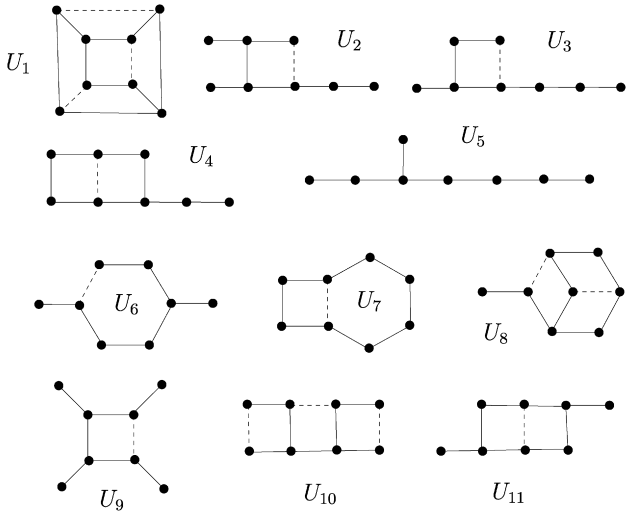


Fig. 12. The sporadic connected cyclotomic signed graphs maximal with respect to having all eigenvalues in $(-2, 2)$.

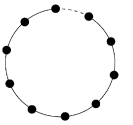


Fig. 13. The $2k$-vertex connected cyclotomic signed graph O_{2k}, maximal with respect to having all eigenvalues in $(-2, 2)$, shown here for $k = 5$.

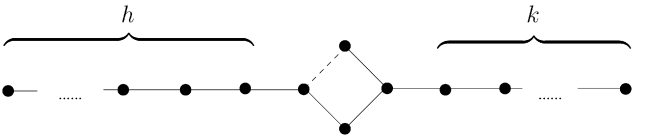


Fig. 14. The doubly infinite family Q_{hk} of $(h+k+4)$-vertex connected cyclotomic signed graphs having all eigenvalues in $(-2, 2)$ but not contained in a maximal one.

Theorem 4 *("Uncharged, signed, $(-2, 2)$"). Up to equivalence, the connected signed graphs maximal with respect to having all their eigenvalues in $(-2, 2)$ are the eleven 8-vertex sporadic examples $U_1, \ldots, U_{11}$ shown in Fig. 12, and the infinite family O_{2k} of $2k$-cycles with one edge of sign -1, for $2k \geqslant 8$, shown in Fig. 13.*

Further, every connected cyclotomic signed graph having all its eigenvalues in $(-2, 2)$ is either contained in a maximal one, or is a subgraph of one of the signed graphs Q_{hk} of Fig. 14 for $h + k \geqslant 4$.

We note in passing that the graphs U_i can all be obtained from the cube U_1 by deleting certain edges. Not every choice of edge-deletion produces a U_i, however. For instance no edge-deleted subgraph of U_1 containing an induced subgraph equivalent to $\tilde{D}_5$ can have all its eigenvalues in $(-2, 2)$.

9.3. *Cyclotomic charged signed graphs with all eigenvalues in* $(-2, 2)$

Next we have a corresponding result for charged signed graphs.

Theorem 5 *("Charged, signed,* $(-2, 2)$*"). Up to equivalence, the connected charged signed graphs maximal with respect to having all their eigenvalues in* $(-2, 2)$*, and not covered by the Theorem* 4 *above, are the eight* 4*-vertex sporadic examples* $V_1, V_2, \ldots, V_8$ *shown in Fig.* 17*, and the infinite family* $P_n^{\pm}$ *of* n*-vertex charged paths of Fig.* 18 *for* $n \geqslant 4$*.*

Further, every connected cyclotomic charged signed graph not covered by the previous theorem is contained in such a maximal one.

9.4. *Cyclotomic matrices with all eigenvalues in* $(-2, 2)$

We can combine the previous two theorems, translated into matrix language, to obtain the following.

Theorem 6 *("Integer matrix,* $(-2, 2)$*"). Every indecomposable cyclotomic matrix maximal with respect to having all its eigenvalues in the open interval* $(-2, 2)$ *is equivalent to the adjacency matrix of one of the graphs* $U_1, U_2, \ldots, U_{11}$*,* O_{2k} $(2k \geqslant 8)$*,* $V_1, V_2, \ldots, V_8$*,* $P_n^{\pm}$ $(n \geqslant 4)$ *(given by Theorems* 4 *and* 5*).*

Further, every indecomposable cyclotomic matrix having all its eigenvalues in $(-2, 2)$ *is either contained in a maximal one, or is contained in the adjacency matrix of one of the signed graphs* Q_{hk} *of Fig.* 14 *for* $h + k \geqslant 4$*.*

10. Maximal cyclotomic charged graphs

10.1. *Cyclotomic charged unsigned graphs*

We now restrict our attention to cyclotomic charged graphs, looking for those that are maximal with respect to having all their eigenvalues in $[-2, 2]$. For such a graph G we need to define $\bar{G}$ as the graph whose edges are the same as those of G, with the same signs, but with the charges on vertices being the opposite of those on G. For example, graphs V_1 and $\bar{V}_1$ are shown in Fig. 17. It is clear that when G is a tree, $\bar{G}$ is equivalent to G.

One difference for this kind of maximality is that it is not a property of equivalence classes of charged graphs: two of them may be equivalent with one of them maximal and the other not. For instance, one consequence of the next result is that for the maximal graphs W_5 and W_6 the graphs $\bar{W}_5$ and $\bar{W}_6$, being subgraphs of W_7, are not maximal.

However, we have the following.

Theorem 7 *("Charged, unsigned,* $[-2, 2]$*"). The maximal connected cyclotomic charged graphs not covered by Smith's result (i.e. not graphs), are the sporadic examples* $W_1, \ldots, W_{13}$ *from Fig.* 15*, along with* $\bar{W}_1$*,* $\bar{W}_{11}$*,* $\bar{W}_{12}$*, and the seven families* F_n $(n \geqslant 5)$*,* G_n $(n \geqslant 5)$*,* H_n $(n \geqslant 3)$*,* I_n $(n \geqslant 3)$*,* $J_n (n \geqslant 2)$ *and* $\bar{I}_n$ $(n \geqslant 3)$*,* $\bar{J}_n$ $(n \geqslant 2)$ *from Fig.* 16*.*

Further, every connected cyclotomic charged graph is contained in such a maximal one.

The proof of this theorem is by inspection of the maximal connected cyclotomic charged signed graphs of Theorem 2 to find their maximal connected charged (unsigned) subgraphs.

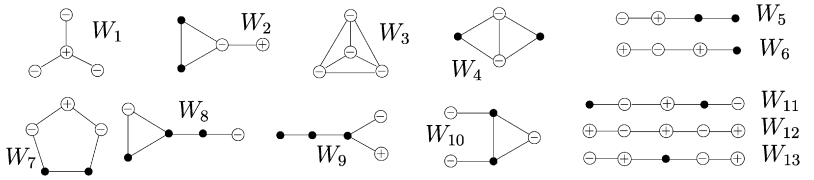


Fig. 15. The sporadic maximal connected cyclotomic charged graphs $W_1, \ldots, W_{13}$.

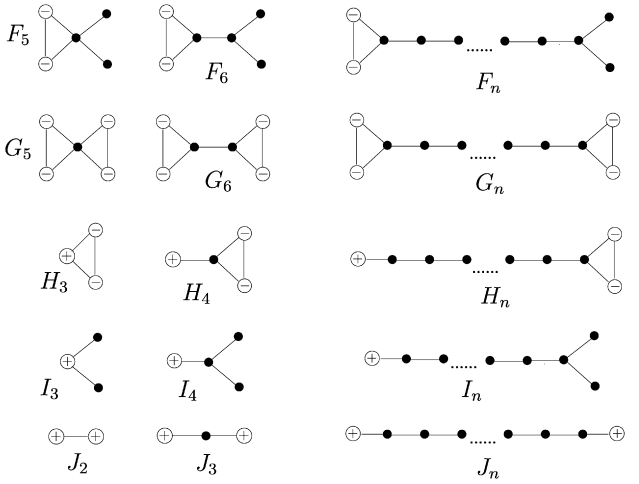


Fig. 16. Five families of maximal connected cyclotomic charged graphs. The smallest two members of each family are also shown.

10.2. Cyclotomic charged unsigned graphs with all eigenvalues in $(-2, 2)$

We next have a corresponding result for eigenvalues in the open interval $(-2, 2)$.

Theorem 8 *("Charged, unsigned,* $(-2, 2)$*"). The connected charged* (*unsigned*) *graphs maximal with respect to having all their eigenvalues in* $(-2, 2)$*, are the graph* U_5*, the charged graphs* $\bar{V}_1$*,* V_1*,* $V_2, \ldots, V_5$*, from Fig.* 17*, and* $P_n^{\pm}$ *of* n*-vertex charged paths of Fig.* 18 *for* $n \geqslant 4$*.*

Further, every connected cyclotomic charged graph not covered by Theorem 4 *is contained in one of the above graphs.*

Note that for $n \geqslant 8$ the graph D_n (Fig. 18) is a subgraph of some Q_{hk}. So it has all its eigenvalues in $(-2, 2)$ and is covered by Theorem 4. It is not contained in any charged graph maximal with respect to having all its eigenvalues in $(-2, 2)$.

11. Maximal cyclotomic symmetric non-negative integer matrices

In this section we record our results for non-negative cyclotomic matrices, i.e., those integer symmetric matrices that are cyclotomic and have only non-negative entries.

Theorem 9 *("Non-negative integer matrix,* $[-2, 2]$*"). Up to conjugation by permutation matrices, the only maximal indecomposable non-negative cyclotomic matrices are the matrices* (2)

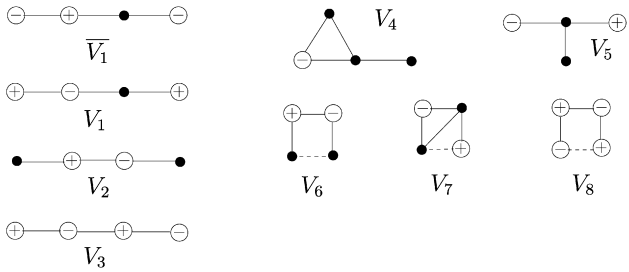


Fig. 17. The sporadic connected cyclotomic charged signed graphs maximal with respect to having all eigenvalues in $(-2, 2)$.

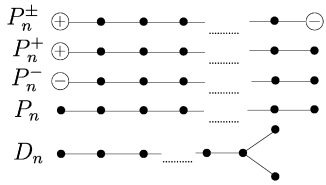


Fig. 18. The n-vertex charged paths $P_n^{\pm}$ (for Theorems 5, 6 and 8), P_n^{+} (Theorem 10), P_n^{-}, P_n (Section 13) and D_n (Theorem 10).

and $\left(\begin{smallmatrix} 0 & 2 \\ 2 & 0 \end{smallmatrix}\right)$*, adjacency matrices of* $\tilde{E}_6$*,* $\tilde{E}_7$*,* $\tilde{E}_8$*,* $\tilde{A}_n$ $(n \geqslant 2)$*,* $\tilde{D}_n$ $(n \geqslant 4)$ (*Fig.* 9) *along with the two families* I_n $(n \geqslant 3)$ *and* J_n $(n \geqslant 2)$ (*Fig.* 16)*.*

Further, every indecomposable non-negative cyclotomic matrix is contained in such a maximal one.

This result is readily deduced from Theorems 3, 7 and Smith's results (Fig. 9).

Theorem 10 *("Non-negative integer matrix,* $(-2, 2)$*"). Up to conjugation by permutation matrices, the only indecomposable non-negative cyclotomic matrix maximal with respect to having all its eigenvalues in* $(-2, 2)$ *is the adjacency matrix of* U_5 (*Fig.* 12)*.*

Further, every indecomposable non-negative cyclotomic matrix is either contained in the adjacency matrix of U_5 *or in the adjacency matrix of either* P_n^{+} *or* D_n (*Fig.* 18) *for some* n*.*

12. Proofs of Theorems 4 and 5

To prove Theorem 4, we first we show that the two infinite families O_{2k}, Q_{hk} have their eigenvalues in the open interval.

Suitable sets of Gram vectors, for the two cases, are:

For O_{2k}, the columns of the $(2k) \times (2k)$ matrix (c_{ij}), where

$$c_{ij} = \begin{cases} 1 & \text{if } i = j \text{ or } i = j + 1, \\ -1 & \text{if } (i, j) = (1, 2k), \\ 0 & \text{otherwise.} \end{cases}$$

For Q_{hk}, the columns of the $(h+k+4)\times(h+k+4)$ matrix (q_{ij}), where

$$q_{ij} = \begin{cases} 1 & \text{if } i=j \ (j=1,\dots,h+k+4) \text{ or } i=j+1 \ (j=1,2,3) \text{ or } (i,j)=(2,k+5) \\ & \quad \text{or } i=j-1 \ (j=5,\dots,k+4,k+6,\dots,h+k+4), \\ -1 & \text{if } (i,j)=(1,4), \\ 0 & \text{otherwise.} \end{cases}$$

Note that both of these sets of columns are easily seen to be linearly independent. Hence, in each case, for the adjacency matrix A of these signed graphs, $A+2I$ is non-singular, so -2 is not an eigenvalue. Since these families comprise bipartite graphs (in the extended sense), 2 is not an eigenvalue. The families are cyclotomic, being subgraphs of T_n for some n, so we are done.

Now we find the remaining graphs.

For subgraphs of the sporadic graphs S_{14} and S_{16}, we know that these are subgraphs of $\mathcal{E}_8$, and so can be embedded in $\mathbb{R}^8$, with $A+2I$ non-singular. Hence such a subgraph can have at most 8 vertices. These can be found by exhaustive search; the maximal ones are $U_1,\dots,U_{11}$ and O_8.

There remain the subgraphs of the infinite families.

We observe that:

- an hour-glass  , equivalent to an unsigned square, has 2 as an eigenvalue;
- the classical $\tilde{D}_n$ graphs (see Fig. 9) have 2 as an eigenvalue.

Hence the subgraph can contain at most one pair of conjugate vertices. So it is either a path, a cycle, some Q_{hk} or Q'_k, defined to be Q_{1k} with its two leaves identified. A path is a subgraph of some Q_{hk}, while a cycle must be equivalent to some O_{2k}, for otherwise it is equivalent to a cycle with all positive edges for which 2 is an eigenvalue. For Q'_k, we can delete one of its pair of conjugate vertices to obtain a graph equivalent to a cycle with all positive edges.

This completes the proof of Theorem 4.

The proof of Theorem 5 is similar. We can assume that the charged graphs we seek do indeed have at least one charged vertex. The relevant subgraphs of S_7, S_8 and S'_8 are found by exhaustive search. For the subgraphs of C_{2k}^{++} and C_{2k}^{+-}, we see by the same argument as above that the neutral component can contain at most one pair of conjugate vertices. Hence the neutral component is a path, or some Q_{hk}.

Two adjacent charges of the same sign have one of ± 2 as an eigenvalue, so each charged component has exactly one charge. Putting charges of the same sign at each end of a path would give one of ± 2 as an eigenvalue, as one can see by writing down an obvious eigenvector.

Putting a charge on either end of some Q_{hk} gives one of ± 2 as an eigenvalue. To see this it suffices to consider adding a negative charge to one end, with corresponding column vector $(0,\dots,0,1)^T$ to add to (q_{ij}), and adding a row of zeroes to (q_{ij}) to make it square, giving a singular matrix, and hence -2 as an eigenvalue. This leaves $P_n^{\pm}$ (and its subgraphs) as the only possibilities.

For $P_n^{\pm}$, the columns of the $n\times n$ matrix (p_{ij}), where

$$p_{ij} = \begin{cases} \sqrt{2} & \text{if } (i,j)=(1,1), \\ 1 & \text{if } i=j \text{ or } i=j+1 \ (i\geqslant 2), \\ 0 & \text{otherwise,} \end{cases}$$

are easily seen to be linearly independent. Again, for the adjacency matrix A of this bipartite charged signed graph, $A + 2I$ is non-singular, so -2 is not an eigenvalue, and hence neither is 2.

This completes the proof of Theorem 5. Theorem 8 then follows easily.

13. The cyclotomic polynomials of charged signed graphs

Table 1 gives the reciprocal polynomials of the maximal connected cyclotomic charged signed graphs that appear in our results. All are maximal in the sense explained where they appear, apart from the Q_{hk} which, as we have seen, do not belong to any connected cyclotomic charged signed graph maximal with respect to having all eigenvalues in $(-2, 2)$. Note, however, that the polynomials associated to C_{2k}^{++} and S_7 will need changes of variable $x \mapsto -x$, $z \mapsto -z$ when going from one equivalent, but not strongly equivalent, graph to another.

Table 2 gives the reciprocal polynomials of the cyclotomic signed graphs of Theorems 4 and 5, shown in Figs. 12, 14, 17, 13 and 18. In the table, Φ_n denotes the nth cyclotomic polynomial.

For a single sporadic graph with adjacency matrix A, the reciprocal polynomial $z^n \chi_A(z + 1/z)$ can be easily calculated. For the infinite families, more work is required. Here, for convenience, we use the same notation for a graph and its associated cyclotomic polynomial.

For computing formulae for families of associated cyclotomic polynomials, a standard tool will be to use induction on the determinant $\det((z + 1/z)I - A)$, where A is the adjacency matrix of the graph under consideration. In this way it is easy first to compute the n-vertex (unsigned) path P_n, giving $P_n = (z^{2n+2} - 1)/(z^2 - 1)$, as in the table (see also [MS]). Then expansion by the first row of the determinant gives $P_n^- = (z^{2n+1} - 1)/(z - 1)$. Also $P_n^{\pm}$ is readily calculated, again expanding in the same way.

For O_{2k}, determinant expansion firstly along the top row, and then down the left rows of the resulting determinants gives $O_{2k} = (z^2 + 1)P_{2k-1} - 2z^2 P_{2k-2} + 2z^{2k}$, and hence the result.

For Q_{hk}, the formulae for Q_{1k} and Q_{2k} can be proved by induction, using the determinant, in a similar way to that for P_n^-. These can then be used as the base cases for an inductive proof of the Q_{hk} formula.

For T_{2k}, label its top vertices $1, 3, 5, \ldots, 2k - 1$ and the bottom vertices $2, 4, 6, \ldots, 2k$, with 2 the conjugate vertex to vertex 1. Then $(-1, 1, 1, 1, 0, \ldots, 0)$ is an eigenvector of T_{2k} with eigenvalue -2, and $(1, -1, 1, 1, 0, \ldots, 0)$ is an eigenvector of T_{2k} with eigenvalue 2, both associated to the hourglass $[1, 2, 3, 4]$. From the symmetry of T_{2k} that acts by $i \mapsto i + 2 \bmod 2k$ on its vertices, we get two eigenvectors, with eigenvalues -2 and 2 for each of the hourglasses $[3, 4, 5, 6], [5, 6, 7, 8], \ldots, [2k - 1, 2k, 1, 2]$. These eigenvectors are independent, so that T_{2k} has characteristic polynomial $(x + 2)^k (x - 2)^k$, which, on putting $x = z + 1/z$, gives the result.

Table 1
The characteristic and cyclotomic polynomials of maximal cyclotomic charged signed graphs

Charged signed graph	Characteristic polynomial	Associated cyclotomic polynomial
T_{2k}	$(x+2)^k(x-2)^k$	$(z^2-1)^{2k}$ $(k \geqslant 3)$
S_{14}	$(x+2)^7(x-2)^7$	$(z^2-1)^{14}$
S_{16}	$(x+2)^8(x-2)^8$	$(z^2-1)^{16}$
C_{2k}^{++}	$(x+2)^{k-1}(x-2)^{k+1}$	$(z-1)^{2k+2}(z+1)^{2k-2}$ $(k \geqslant 2)$
C_{2k}^{+-}	$(x+2)^k(x-2)^k$	$(z^2-1)^{2k}$ $(k \geqslant 2)$
S_7	$(x+2)^3(x-2)^4$	$(z+1)^6(z-1)^8$
S_8, S_8'	$(x+2)^4(x-2)^4$	$(z^2-1)^8$

Table 2
The cyclotomic polynomials of some charged signed graphs having all their eigenvalues in $(-2, 2)$

Charged signed graph	Associated cyclotomic polynomial
O_{2k}	$(z^{2k}+1)^2$
Q_{hk}	$(z^{2h+4}+1)(z^{2k+4}+1)$ $(h+k \geqslant 4)$
U_1	$\Phi_6^4(z^2)$
U_2	$\Phi_{20}(z^2)$
U_3	$\Phi_{24}(z^2)$
U_4	$\Phi_6(z^2)\Phi_{18}(z^2)$
U_5	$\Phi_{30}(z^2)$
U_6, U_9	$\Phi_{12}^2(z^2)$
U_7, U_{11}	$\Phi_{15}(z^2)$
U_8	$\Phi_{12}(z^2)\Phi_6^2(z^2)$
U_{10}	$\Phi_{10}^2(z^2)$
V_1	$\Phi_{15}(z)$
$\bar{V}_1, V_4$	$\Phi_{30}(z)$
V_3, V_6	$\Phi_{20}(z)$
V_2, V_5	$\Phi_{24}(z)$
V_7, V_8	$\Phi_{12}^2(z)$
P_n	$(z^{2n+2}-1)/(z^2-1)$
P_n^-	$(z^{2n+1}-1)/(z-1)$
$P_n^\pm$	$z^{2n}+1$ $(n \geqslant 4)$

For C_{2k}^{++}, label the vertices as for T_{2k}. For the hourglasses $[3,4,5,6]$, $\ldots$, $[2k-5, 2k-4, 2k-3, 2k-2]$ (those without charged vertices), we get the same eigenvectors as for T_{2k}, with the same eigenvalues. The hourglasses $[1,2,3,4]$ and $[2k-3, 2k-2, 2k-1, 2k]$ give the same eigenvectors as for T_{2k} with eigenvalue -2. For the hourglass $[1,2,3,4]$, however, we also get two independent eigenvectors $(1,1,0,\ldots,0)$ and $(2,0,1,1,0,\ldots,0)$ with eigenvalue 2, and from the hourglass $[2k-3, 2k-2, 2k-1, 2k]$ we get two more independent eigenvectors $(0,\ldots,0,-1,1)$ and $(0,\ldots,0,1,-1,0,2)$ with eigenvalue 2. Thus C_{2k}^{++} has characteristic polynomial $(x+2)^{k-1}(x-2)^{k+1}$, giving the result.

For C_{2k}^{+-}, note that this is bipartite in the extended sense, so that the eigenvalues 2 and -2 have equal multiplicities.

14. Final remarks

14.1. Finite reflection groups

Given the root system Φ of a finite reflection group, one classically looks for a subset Δ that is a *simple system*, namely one that is a basis for the $\mathbb{R}$-span of Φ and such that every element of Φ is a linear combination of elements of Δ with all coefficients weakly of the same sign. The Coxeter graph of a simple system is determined by the reflection group, and provides a means of classifying finite reflection groups.

If we have a signed graph with all eigenvalues in $(-2, 2)$ then (as we have seen) its vertices can be associated with a linearly independent set Δ' of vectors, and the reflection group generated by the hyperplanes orthogonal to those vectors is a finite reflection group. The closure of Δ' under this reflection group is a root system Φ: in the language of [CvL] we are taking the star closure of the lines spanned by the elements of Δ'.

Our set Δ' will not generally be a simple system for Φ, but it will be a basis for the $\mathbb{R}$-span of Φ. The unsigned version of our graph (making all edges positive) is the Coxeter graph of Δ'.

The neutral signed graphs of Theorem 4 therefore provide a classification of all Coxeter graphs coming from bases for the $\mathbb{R}$-span of root systems contained in either $\mathcal{D}_n$ $(n \geqslant 4)$ or $\mathcal{E}_8$. For example, one can generate $\mathcal{E}_8$ using eight reflections whose Coxeter graph is the cube U_1 of Fig. 12.

For other connections between signed graphs and Coxeter graphs and roots systems see [CST] and [Z1].

14.2. The graph S_{14}

Robin Chapman has pointed out that, up to equivalence, the signed graph S_{14} of Fig. 3 can be defined as follows: label the vertices $0, 1, \ldots, 6, 0', 1', \ldots, 6'$ and, working modulo 7, for each i join i to each of i', $(i+1)'$ and $(i+3)'$ by positive edges, and join i to $(i-1)'$ by a negative edge. The representation of S_{14} in the figure is based on this observation.

14.3. Chebyshev polynomials and cyclotomic matrices

Let $\mathcal{T}_n(x)$ denote the nth Chebyshev polynomial of the first kind, defined on the interval $[-2, 2]$. So $\mathcal{T}_n(x)$ has integer coefficients and satisfies

$$\mathcal{T}_n\left(z + \frac{1}{z}\right) = z^n + \frac{1}{z^n}. \tag{4}$$

Then for any cyclotomic matrix A, the matrix $\mathcal{T}_n(A)$ is again cyclotomic. This follows from diagonalising A and using (4).

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