A symbolic test for \((i, j)\)-uniformity in reduced zero-dimensional schemes

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Abstract

Let \(Z\) denote a finite collection of reduced points in projective \(n\)-space and let \(I\) denote the homogeneous ideal of \(Z\). The points in \(Z\) are said to be in \((i, j)\)-uniform position if every cardinality \(i\) subset of \(Z\) imposes the same number of conditions on forms of degree \(j\). The points are in uniform position if they are in \((i, j)\)-uniform position for all values of \(i\) and \(j\). We present a symbolic algorithm that, given \(I\), can be used to determine whether the points in \(Z\) are in \((i, j)\)-uniform position. In addition it can be used to determine whether the points in \(Z\) are in uniform position, in linearly general position and in general position. The algorithm uses the Chow form of various \(d\)-uple embeddings of \(Z\) and derivatives of these forms. The existence of the algorithm provides an answer to a question of Kreuzer.

Keywords: Uniform position; Chow variety; Chow form; General position; Zero-dimensional scheme; Points

1. Introduction

Let \(k\) be a field, either of characteristic zero or of sufficiently large characteristic (see below). Let \(R = k[x_0, x_1, \ldots, x_n]\) and let \(\mathbb{P}^n = \text{Proj}(R)\). The set of all degree 1 monomials in \(R\) forms a basis for the \(k\)-vector space of all forms of degree 1 in \(R\). In terms of this basis, with each linear form (up to scalar multiples), \(L = a_0x_0 + a_1x_1 + a_2x_2 + \cdots + a_nx_n \in R\), we can associate the point \([a_0 : a_1 : \cdots : a_n] \in \mathbb{P}^n\). In addition, we can associate with \(L\) the hyperplane in \(\mathbb{P}^n\) determined by \(L = 0\). This gives the well-known bijection between hyperplanes in \(\mathbb{P}^n\) and points in \(\mathbb{P}^{n\ast}\) (similarly, between points in \(\mathbb{P}^{n\ast}\) and hyperplanes...
in $\mathbb{P}^n$). In a similar way, letting $N_d = \binom{n+d}{d} - 1$, we have the well-known bijection between degree $d$ hypersurfaces in $\mathbb{P}^n$ and points in $\mathbb{P}^{N_d}$.

In this paper we are concerned with the question of what can be said about the uniformity of a set of points in projective space. The notion of uniformity appears in a variety of settings. For instance, on the theoretical side, it is central to the study of curves in projective space, via hyperplane sections and Castelnuovo theory (cf. for instance Harris and Eisenbud, 1982). On the applied side, it has been shown to be important in coding theory (Hansen, 1994; Kreuzer, 2002).

Given a homogeneous ideal, it is possible to determine whether the ideal corresponds to a zero-dimensional object. In this case, by taking the radical, it is possible to form the largest ideal that defines this zero-dimensional object set theoretically. However, it is in general not possible to determine by symbolic methods the precise coordinates of the points (although there exist numerical methods for approximating them)—for instance, even the case of finding the coordinates of five generally chosen points on a line would amount to solving a quintic equation, which Galois theory shows to be impossible. Without this information it would seem at first glance to be difficult to determine (symbolically) what kind of uniformity properties the set of points enjoys. We were thus somewhat surprised at the level of detail that can be extracted by symbolic methods and by the simplicity of the approach.

Let $P$ be a point in $\mathbb{P}^n$. For each fixed value of $d$, the collection of all degree $d$ hypersurfaces that contain $P$ determines a hyperplane, $H_{P,d} \subset \mathbb{P}^{N_d}$ (the degree $d$ Chow form of $P$). Thus, with a collection of $r$ distinct points $Z = \bigcup_{i=1}^r P_i \subset \mathbb{P}^n$, we can associate an arrangement of $r$ distinct hyperplanes $\Gamma_i = \bigcup_{i=1}^r H_{i,d} \subset \mathbb{P}^{N_d}$ (one for each point in $Z$). In addition, $Z$ can be recovered from $\Gamma_i$ (for a good reference, see (Shafarevich, 1977, Section I.6.5) or Gelfand et al. (1994)). With this correspondence, questions about finite collections of points in $\mathbb{P}^n$ can be rephrased as questions about hyperplane arrangements in $\mathbb{P}^{N_d}$ for various choices of $d$. In this paper we will utilize this correspondence to obtain a symbolic algorithm for determining when a finite collection of reduced points satisfies various types of uniformity condition. Because of the use of derivatives in some of the algorithms, we make the further assumption that $\text{char}(k) > r$. Good general references for the background material of this paper are Eisenbud (1995) and Hartshorne (1977).

2. Ingredients

In what follows, $Z$ will denote a finite set of reduced points in $\mathbb{P}^n$, $I_Z$ will denote the homogeneous ideal of $Z$ and $\mathcal{I}_Z$ will denote the associated ideal sheaf of $Z$. There exists a symbolic algorithm for finding the radical of an ideal (Eisenbud et al., 1992). In the light of this, if $I \subset k[x_0, x_1, \ldots, x_n]$ is an ideal which defines $Z$ set theoretically, we can determine the unique radical ideal $I_Z \subset k[x_0, x_1, \ldots, x_n]$ which defines the reduced zero-dimensional scheme $Z$.

2.1. Uniform position and general position

If $\mathcal{E}$ is a sheaf on $\mathbb{P}^n$ then $H^0(\mathbb{P}^n, \mathcal{E})$ will denote the (finite-dimensional) vector space of all global sections of $\mathcal{E}$ and $h^0(\mathbb{P}^n, \mathcal{E})$ will denote the dimension of $H^0(\mathbb{P}^n, \mathcal{E})$. 

Hence, $H^0(X, I_Z(t))$ represents the vector space of all forms of degree $t$ whose associated hypersurfaces contain the scheme $Z$, and $h^0(X, I_Z(t))$ represents the dimension of this vector space. This information is equivalent to knowing the Hilbert function of $Z$,

$$h_Z(t) = \dim(R/I_Z)_t = \binom{n + t}{t} - h^0(P^n, I_Z(t)).$$

**Definition 1.** Let $Z$ denote a finite collection of reduced points in $P^n$ and let $|Z|$ denote the number of points in $Z$. Then $Z$ is said to be in (i, j)-uniform position if for every pair of subsets $X, Y \subseteq Z$ with $|X| = |Y| = i$ it holds that $h^0(P^n, I_X(j)) = h^0(P^n, I_Y(j))$. Furthermore, $Z$ is said to be in uniform position if it is (i, j)-uniform for all integers $i$ and $j$.

**Remark 2.** In Kreuzer (1998, 2000, 2002) Kreuzer introduced the concept of (i, j)-uniformity for a set of points. In his definition, $Z$ is (i, j)-uniform if whenever $X$ and $Y$ are two sets of points obtained from $Z$ by removing $i$ points, then the Hilbert functions of the corresponding ideals agree in the $j$th spot. Our definition states that $Z$ is (i, j)-uniform if whenever $X$ and $Y$ are two sets of points obtained by selecting $i$ points from $Z$, then the Hilbert functions of the corresponding ideals agree in the $j$th spot. The difference is superficial in that knowing (i, j)-uniformity for all values of $i$ and $j$ provides exactly the same information regardless of which definition is used. The definition that we have chosen provides a notational advantage in the course of the paper.

**Definition 3.** Let $Z$ denote a finite collection of reduced points in $P^n$ and let $|Z|$ denote the number of points in $Z$. Then $Z$ is said to be in d-general position if for every subset $X \subseteq Z$, it holds that $h^0(P^n, I_X(d)) = \max(0, \binom{n+d}{d} - |X|)$. $Z$ is said to be in general position if it is in d-general position for all $d > 0$. $Z$ will be said to be in linearly general position if it is in 1-general position.

**Remark 4.** Let $Z$ be a finite set of reduced points in $P^n$. Then $Z$ has regularity $r$ if and only if $r - 1 = \min\{t \mid h_Z(t) = |Z|\}$ (cf. Iarrobino and Kanev, 1999, Theorem 1.69). That is, $r - 1$ is the first degree, $t$, in which $Z$ imposes $|Z|$ independent conditions on forms of degree $t$. Hence, any subset $X$ of $Z$ has regularity $\leq r$. If $d \geq r - 1$, then we get that $h_X(d) = |X|$. Thus $Z$ is in $d$-general position. As a consequence, to determine whether such a set of points is in general position, it is only necessary to determine whether it is in $d$-general position for $0 < d < r - 1$. There is a symbolic algorithm for determining the regularity of a scheme (regularity can be determined from a minimal free resolution, and there exists a symbolic algorithm for computing minimal free resolutions (Eisenbud, 1995)). Thus, if we have a finite step algorithm for determining whether such a scheme is in $d$-general position for any given $d$, then we have a finite step algorithm to determine whether it is in general position. Similarly, if we have a finite step algorithm to determine whether such a scheme is in (i, j)-uniform position then we have a finite step algorithm to determine whether a scheme is in uniform position.

### 2.2. Chow forms

Let $Z$ be an equidimensional variety in $P^n$ of codimension $d$. A general linear space of dimension $d - 1$ will not intersect $Z$. Consider the collection of all linear spaces
of dimension $d - 1$ which do intersect $Z$. These form a codimension 1 subvariety of \( \text{GR}(n, d - 1) \) (the Grassmannian of \((d - 1)\)-dimensional linear spaces in \( \mathbb{P}^n \)). This codimension 1 subvariety is called the Chow variety of $Z$. If $Z$ decomposes into $s$ irreducible components then so will the Chow variety of $Z$.

Points in \( \mathbb{P}^n \) have codimension $n$. To compute the Chow variety of a set of points in \( \mathbb{P}^n \) we will use linear spaces of dimension $n - 1$ (i.e. hyperplanes). The Grassmann variety of hyperplanes in \( \mathbb{P}^n \) is the dual projective space \( \mathbb{P}^{ns} \). The correspondence is simply

$$A_0x_0 + A_1x_1 + \cdots + A_nx_n \iff \{A_0 : A_1 : \cdots : A_n\}.$$ 

The Chow variety of a set, $Z$, of $s$ distinct points in \( \mathbb{P}^n \) will be a hypersurface of degree $s$ in \( \mathbb{P}^{ns} \). This hypersurface will be the union of $s$ distinct hyperplanes (the hyperplanes corresponding to the points of $Z$ in the bijection described in the introduction) and will be defined by a single equation. This equation is the Chow form of the set of points. The points can be recovered from the Chow form. Thus, in some sense, the Chow form contains all of the information of the original set of distinct points. There are a number of ways to establish the following, somewhat classical, result. The method chosen below mixes some ideas from Shafarevich (1977) and Gelfand et al. (1994).

**Proposition 5.** The Chow form of a set of points, $Z$, can be computed symbolically from the ideal $I_Z$, without knowing the coordinates of the points.

**Proof.** We first introduce a polynomial, $H = A_0x_0 + A_1x_1 + \cdots + A_nx_n$, representing a general hyperplane. We would like to determine algebraic conditions on the coefficients, $A_0, \ldots, A_n$, that force $H$ to intersect $Z$. To do this, we need to form the ideal $J = I_Z + H \subseteq k[x_0, \ldots, x_n, A_0, \ldots, A_n]$ and determine conditions on $A_0, \ldots, A_n$ such that the resulting ideal (viewed as a subset of $k[x_0, \ldots, x_n]$) defines a non-empty scheme.

Let us denote by $P^n_A$ the projective space with coordinate ring $R = k[x_0, \ldots, x_n]$, by $P^n_P$ the projective space with coordinate ring $S := k[A_0, \ldots, A_n]$ and by $P^{2n+1}$ the projective space with coordinate ring $T := k[x_0, \ldots, x_n, A_0, \ldots, A_n]$. Let $Q \subset \mathbb{P}^{2n+1}$ be the quadric hypersurface defined by $H$. Note that in $\mathbb{P}^{2n+1}$, $P^n_P$ is defined by $A_0 = \cdots = A_n = 0$ and $P^n_A$ is defined by $x_0 = \cdots = x_n = 0$. Also,

$$Z \subset P^n_P \subset Q \subset P^{2n+1} \quad \text{and} \quad P^n_A \subset Q \subset \mathbb{P}^{2n+1}.$$ 

The generators of $I_Z \subset R$ can also be viewed in $T$. Let $\tilde{Z}$ denote the subscheme of $\mathbb{P}^{2n+1}$ defined by $T \cdot I_Z \subset T$. $\tilde{Z}$ is determined geometrically as follows: for each point $P$ of $Z$, let $A_P$ be the linear space spanned by $P$ and $P^n_P$. Then $\tilde{Z} = \bigcup_{P \in Z} A_P$. Note that $\dim \tilde{Z} = n + 1$.

We first claim that the hyperquadric $Q$ does not contain any component $A_P$ of $\tilde{Z}$. Indeed, if it did, this would say that every hyperplane $H$ of $P^n_P$ vanishes at the point $P$ (choose arbitrary values for the $A_i$), which is clearly nonsense. So $J$ corresponds to a proper hypersurface section of $\tilde{Z}$. $\tilde{Z}$ is arithmetically Cohen–Macaulay since $Z$ is arithmetically Cohen–Macaulay. As a result, $J$ is unmixed.

Fix again a component $A_P$ of $\tilde{Z}$. The intersection of $A_P$ and $Q$ will be an arithmetically Cohen–Macaulay scheme of degree 2 and of codimension 1 in $A_P$. Since $A_P$ contains $P^n_A$, which is also contained in $Q$, on this component, $Q$ cuts out the union of $P^n_A$ and
some linear variety $\lambda_P$ of dimension $n$. We have that $\lambda_P$ meets $P_X^n$ at $P$ and meets $P_A^n$ in an $(n - 1)$-dimensional linear space $V_P$ (since $P_A^n$ and $\lambda_P$ are both hyperplanes in the $(n + 1)$-dimensional linear space $A_P$). Note that $V_P$ is precisely the hyperplane in $P_A^n$ dual to $P$. As such, $V_P \neq V_{P'}$ if and only if $P \neq P'$. $V_P$ is defined by a linear form $L_P$ in $k[A_0, \ldots, A_n]$. The product of the $L_P$ is the Chow form of $Z$.

From this discussion it follows that the ideal $J = I_Z + \langle H \rangle$ defines a subscheme of $P^{2n+1}$ that consists of the union of the $\lambda_P$ and some subscheme supported on $P_A^n$. Removing this latter component and restricting the result to $P_A^n$ gives the Chow form.

This suggests two algorithms for determining the Chow form of $Z$. In the first algorithm, we isolate the union of the $\lambda_P$, and then we intersect with the plane defined by $(x_0, \ldots, x_n)$ to determine the Chow form. Note again that $I_{P_A^n} = \langle x_0, \ldots, x_n \rangle$.

Algorithm 1 for computing the Chow form of a reduced zero-dimensional scheme:

- Start with an ideal $I$ which defines $Z$ set theoretically.
- Compute the radical of $I$ to obtain the ideal $I_Z$.
- Form the ideal $J := I_Z + \langle H \rangle$ where $H = A_0x_0 + A_1x_1 + \cdots + A_nx_n$.
- For each $i$, saturate $J$ with respect to the ideal $(x_i)$ to obtain a new ideal $L_i$. In other words, $L_i = J : \langle x_i \rangle^\infty$.
- Form the ideal $C_i = L_i + \langle x_0, \ldots, x_n \rangle$ and view the result in $k[A_0, \ldots, A_n]$. This yields a principal ideal $C_i' = \langle F_i \rangle$ ($F_i$ is the Chow form of all the points in $Z$ which do not lie on the hyperplane defined by $x_i$).
- Let $(F) = (F_0) \cap (F_1) \cap \cdots \cap (F_n)$.
- $F$ is the Chow form of $Z$.

**Remark 6.** If no point of $Z$ lies in the hyperplane in $P_X^n$ defined by $x_i$ then we do not need to iterate the above algorithm, we only need to compute $(F_i)$ and then $(F) = (F_i)$. In general, if no point of $Z$ lies on the intersection of the hyperplanes defined by $x_{i_1}, x_{i_2}, \ldots, x_{i_{\ell}}$ then $(F) = (F_{i_1}) \cap (F_{i_2}) \cap \cdots \cap (F_{i_{\ell}})$.

In the second algorithm we project the entire scheme defined by $J$ to the projective space $\text{Proj}(k[x_1, A_0, \ldots, A_n])$. Note that this projective space is the $(n + 1)$-dimensional linear space spanned by $P_A^n$ and the point in $P_X^n$ defined by $x_0 = \cdots = \hat{x}_1 = \cdots = x_n = 0$.

We now show how to recover the Chow form from this projection, and we will give the precise algorithm shortly. The discussion in the proof of Proposition 5 shows that the projection gives an ideal defining the union of two $n$-dimensional (hence hypersurface) subschemes of $\text{Proj}(k[x_1, A_0, \ldots, A_n])$. One is supported on $P_A^n$, hence is defined by $x_i^\ell$ for some $\ell$. The second subscheme is a union of $n$-dimensional linear spaces, coming as the image under the projection of those $\lambda_P$ for which $P$ does not lie in the hyperplane of $P_X^n$ defined by $x_i = 0$. But this projection leaves fixed the intersection of $\lambda_P$ and $P_A^n$. Therefore the second scheme is defined by a form $F_i$ only in the variables $A_0, \ldots, A_{n - i}$, which is the Chow form of those points of $Z$ that do not lie in the hyperplane of $P_X^n$ defined by $x_i = 0$.

Since there are no points of $Z$ that lie on $x_i = 0$ for every $0 \leq i \leq n$, we can determine the Chow form of $Z$ by finding the least common multiple of the Chow forms that show up in each projection.
Algorithm 2 for computing the Chow form of a reduced zero-dimensional scheme:

- Start with an ideal $I$ which defines $Z$ set theoretically.
- Compute the radical of $I$ to obtain the ideal $I_Z$.
- Form the ideal $J := I_Z + (H)$ where $H = A_0x_0 + A_1x_1 + \cdots + A_nx_n$.
- For each $i$, compute $P_i := J \cap k[x_i, A_0, \ldots, A_n]$.
- Let $F_i = (x_i^{a_i} F_i)$ where $F_i \in k[A_0, \ldots, A_n]$. Determine $F_i$.
- Let $(F) = (F_0 \cap (F_1) \cap \cdots \cap (F_n))$.
- $F$ is the Chow form of $Z$.

**Remark 7.** If there exists an $i$ such that no points of $Z$ lie on the hyperplane $x_i = 0$ then $F = F_i$. This simple observation can save a substantial amount of time in the two algorithms.

**Example 8.** Let $I = (x^2 + y^2 + z^2, x^3 + y^3 + z^3) \subseteq \mathbb{C}[x, y, z]$. $I$ is a complete intersection and defines six reduced points in $\mathbb{P}^2$. Form the ideal $J = (x^2 + y^2 + z^2, x^3 + y^3 + z^3, Ax + By + Cz) \subseteq \mathbb{C}[x, y, z, A, B, C]$. A quick check shows that none of the six points lies on the hyperplane $z = 0$. If we were to use the second algorithm, we would compute $P_z = J \cap \mathbb{C}[z, A, B, C]$ (and determine that $P_z = (z^4 F)$). Alternatively, we can compute $C' = (F)$ by the first algorithm. In either case we find

$$F = A^6 + B^6 + C^6 + \frac{1}{2}(A^4 B^2 + A^2 C^4 + B^4 C^2 + A^2 B^4 + A^4 C^2 + B^2 C^4)$$
$$+ 3(A^4 BC + A B^4 C + ABC^4)$$
$$- (A^3 B^3 + A^3 C^3 + B^3 C^3).$$

$F$ is the Chow form of the zero-dimensional scheme defined by $I$. $F$ factors as a product of six linear forms in $\mathbb{C}[A, B, C]$ (corresponding to the Chow forms of the six individual points). The six linear forms determine six hyperplanes (lines) in $\mathbb{P}^2$. These hyperplanes are the Chow varieties of the six individual points.

### 2.3. Degree $d$ Chow forms and $d$-uple embeddings

The Chow form describes precisely which hyperplanes contain at least one of the points of $Z$. In the following we will be very interested in the analogous problem for hypersurfaces of degree $d$. The algorithm to achieve this is strongly related to the one in the previous section.

Let $R = k[x_0, \ldots, x_n]$ and let $P^n = \text{Proj}(R)$. For a given $d > 0$, let $v_0, \ldots, v_{N_d}$ be a monomial basis for the vector space of forms of degree $d$ in $R$ (where $N_d = \binom{n+d}{d} - 1$). The $d$-uple embedding of $P^n$ into $P^{N_d}$ is the image of the map which sends $a = [a_0 : a_1 : \cdots : a_n]$ to $[v_0(a) : v_1(a) : \cdots : v_{N_d}(a)]$ (i.e. evaluate each of the monomials at $a$).

Let $Z$ denote a scheme in $P^n$. Let $I_Z$ denote its saturated ideal. To determine the ideal of the image of the $d$-uple embedding of $Z$, we can use the following well-known algorithm. Algorithm for computing the $d$-uple embedding of a scheme:

- Start with the ideal $I_Z \subseteq k[x_0, x_1, \ldots, x_n]$.
- Let $v_0, v_1, \ldots, v_{N_d}$ be a monomial basis for the forms of degree $d$. 


Let $S = k[x_0, x_1, \ldots, x_n, X_0, X_1, \ldots, X_{N_d}]$ with the degrees of $x_0, x_1, \ldots, x_n$ set equal to 1 and the degrees of $X_0, X_1, \ldots, X_{N_d}$ set equal to $d$. Let $J = (X_0 - v_0, X_1 - v_1, \ldots, X_{N_d} - v_{N_d}) + I_z \subseteq S$.

Let $L = J \cap k[X_0, X_1, \ldots, X_{N_d}]$.

$L$ is the ideal of the $d$-uple embedding of $Z$.

With this algorithm in hand, we now define:

**Definition 9.** The degree $d$ Chow form of a reduced zero-dimensional scheme, $Z$, is the Chow form of the $d$-uple embedding of $Z$.

The $d$-uple embedding embeds $P^n$ into $P^{N_d}$ as a non-degenerate, smooth projectively normal subvariety $X$ of dimension $n$. A hyperplane section of $X$ corresponds to a hypersurface in $P^n$ of degree $d$. Let $P$ be a point of $P^n$ and let $P' \in P^{N_d}$ be the image of the $d$-uple embedding of $P$. Then $h^0(P^n, I_P(d)) = h^0(P^{N_d}, I_{P'}(1)) = N_d$. Thus $P$ corresponds to a hyperplane in $(P^{N_d})^\ast$. This hyperplane in $(P^{N_d})^\ast$ parametrizes the degree $d$ hypersurfaces of $P^n$ that contain $P$. For any reduced zero-dimensional scheme $Z$ in $P^n$, the degree $d$ Chow form of $Z$ parametrizes the degree $d$ hypersurfaces of $P^n$ that contain at least one point of $Z$.

Thus, to compute the degree $d$ Chow form of a zero-dimensional scheme, we need only compute a $d$-uple embedding of the zero-dimensional scheme and then apply the algorithm for computing the Chow form. Alternatively, we can skip the computation of the $d$-uple embedding if we modify the algorithm for computing the Chow form. All that has to be done is to replace $H = A_0x_0 + A_1x_1 + \cdots + A_nx_n$ with $H = A_0v_0 + A_1v_1 + \cdots + A_{N_d}v_{N_d}$.

By computing the degree $d$ Chow forms of a set of points, we have transformed questions about the set of points into questions about hyperplane arrangements in $P^{N_d}$ for various $d$. We now need to understand how the geometry of these hyperplane arrangements relates to the geometry of the original set of points and we need to understand how to extract this information from the hyperplane arrangement.

### 2.4. Geometry of the hyperplane arrangements

Let $R = k[x_0, \ldots, x_n]$. Let $Z = \bigcup_{i=1}^r P_i \subset P^n$ be a set of $r$ reduced points with saturated ideal $I_Z$. Let $G_d = \bigcap_{i=1}^r L_{i,d}$ be the degree $d$ Chow form of $Z$. Let $\Gamma_d = \bigcup_{i=1}^r H_{i,d}$ be the union of hyperplanes in $(P^{N_d})^\ast$ defined by $G_d$. $H_{i,d}$ is the projective space parametrizing the linear system of hypersurfaces of degree $d$ passing through $P_i$.

Consider two hyperplanes $H_{i,d}$ and $H_{j,d}$ corresponding to two points $P_i$ and $P_j$. The points in the intersection of $H_{i,d}$ and $H_{j,d}$ correspond to the degree $d$ hypersurfaces in $P^n$ which contain both $P_i$ and $P_j$. Considering all pairs of hyperplanes in $\Gamma_d$, we see that the singular locus of $\Gamma_d$ corresponds to the collection of all degree $d$ hypersurfaces in $P^n$ which contain 2 or more points of $Z$.

The singular locus of $\Gamma_d$ is determined set theoretically by the ideal of all first partial derivatives of $G_d$. More generally, we now show that the locus of all degree $d$ hypersurfaces which contain $t$ or more points of $Z$ is determined set theoretically by the ideal of all $t - 1$ partial derivatives of $G_d$. 
Proposition 10. Let $F = L_1 L_2 \cdots L_r$ be a product of linear forms over a field of characteristic $> r$ or characteristic zero. Let $P \in \mathbb{P}^n$ be a point. Let $L_{i_1}, \ldots, L_{i_k}$ be the factors of $F$ that vanish at $P$. Then

(a) The $i$th derivatives of $F$ vanish at $P$, for $1 \leq i \leq k - 1$.

(b) There is at least one $k$th derivative of $F$ that does not vanish at $P$.

Proof. We have $F = L_1 L_2 \cdots L_r$, where each of the $L_i$ is a linear form. Note that any second or higher derivative of an $L_i$ is 0. Hence any $j$th derivative $\partial^j F / (\partial x_0^{j_0} \cdots x_n^{j_n})$ is a sum of terms of the form $M_1, \ldots, M_r$ where each $M_i$ is either equal to the corresponding $L_i$ or else is a suitable first derivative of the corresponding $L_i$. There are exactly $j$ such first derivatives among the $M_i$ in each term of $\partial^j F / (\partial x_0^{j_0} \cdots x_n^{j_n})$. Then (a) follows since for $j \leq k - 1$ each term must have at least one of $L_{i_1}, \ldots, L_{i_k}$ undifferentiated. Similarly, for (b) we can choose a $k$th derivative that differentiates each of $L_{i_1}, \ldots, L_{i_k}$ yielding a non-zero constant. □

Corollary 11. The locus of all degree $d$ hypersurfaces that contain $t$ or more points of $Z$ is determined set theoretically by the ideal of all $(t - 1)$st partial derivatives of the degree $d$ Chow form.

Proof. The desired locus is precisely the locus of points where $t$ or more of the hyperplanes $H_{i,d}$ meet. By Proposition 10 this locus is determined by the $(t - 1)$st derivatives of $G_d$. □

With this preparation we now turn to the description of the geometry of $\Gamma_d$ and how it relates to geometric properties of $Z$. We will talk about $(s,d)$-uniform position rather than $(i,j)$-uniform position simply for convenience of exposition.

Lemma 12. (a) $\Gamma_d$ is reduced.

(b) If $m \leq d + 1$ then any $m$ components meet in a linear space of codimension $m$.

Proof. The first statement is clear. The second follows from the fact that when $m \leq d + 1$, $m$ points impose independent conditions on forms of degree $d$. □

Proposition 13. Let $V_d = \bigcap_{i=1}^r H_{i,d}$. Then

(a) The value of the Hilbert function in degree $d$ is $h_{R/I_Z}(d) = \text{codim} V_d$.

(b) The points of $Z$ impose independent conditions on forms of degree $d$ if and only if $\text{codim} V_d = r$.

(c) The points of $Z$ are in $(s,d)$-uniform position if and only if for every choice $\{P_1, \ldots, P_s\}$ of $s$ points of $Z$, either

$$\text{codim} \bigcap_{j=1}^s H_{i_j,d} = s \quad \text{or} \quad \text{codim} \bigcap_{j=1}^s H_{i_j,d} = \text{codim} V_d.$$ 

Proof. The number of independent forms of degree $d$ vanishing on $Z \subset \mathbb{P}^n$ is equal to the number of independent linear forms vanishing on the $d$-uple embedding of $Z$. This number is $1 + \dim V_d$. Parts (a) and (b) follow immediately from this observation (see also the following example).
For (c), recall that \((s, d)\)-uniform position means that for every subset \(Y\) of \(Z\) of cardinality \(s \leq r\), we must have

\[ h_{R/I_Y}(d) = \min\{|Y|, h_{R/I_Z}(d)|. \]

In other words, for each \(Y\) with \(|Y| = s\), either \(Y\) imposes independent conditions on forms of degree \(d\) or else \((I_Y)_d = (I_Z)_d\). (Note that it cannot happen that some \(Y\) imposes independent conditions and others do not but satisfy \((I_Y)_d = (I_Z)_d\), so there is no ambiguity in the statement of (c).) This means that every choice of \(s\) of the hyperplanes \(H_i\), \(d\) satisfies one of the two conditions given in (c).

\[ \Box \]

**Example 14.** Let \(Z \subset \mathbb{P}^2\) consist of 20 points on a conic. We consider the 2-uple embedding of \(Z\) in \(\mathbb{P}^5\). This consists of 20 points lying in a single hyperplane.

Then \(V_2\) is a single point in \((\mathbb{P}^5)^*\) (dual to this hyperplane), and \(h_{R/I_Z}(2) = 5\).

In order to implement these ideas, we now give an interpretation in terms of derivatives. Let \(D_i^d\) be the ideal defined by the \(i\)th partial derivatives of \(G_d\). Note that

\[ G_d = D_0^d \subset D_1^d \subset \cdots \subset D_r^d \subset D_r^d = (1) \]

(by Euler’s theorem).

**Remark 15.** \(D_r^{d-1}\) defines the linear variety \(V_d\). If we now combine Corollary 11 and part (a) of Proposition 13 then we see that the Hilbert function \(h_{R/I_Z}(d)\) is just the codimension of the variety defined by \(D_r^{d-1}\). In addition, the points of \(Z\) impose independent conditions on forms of degree \(d\) if and only if the linear variety defined by \(D_r^{d-1}\) has codimension \(r\).

Utilizing Corollary 11, Proposition 13 and the previous remark, it is now apparent how to utilize derivatives to construct algorithms for \((s, d)\)-uniform position and its special cases (e.g. linear general position, uniform position). We summarize our results in the following two propositions.

**Proposition 16** (Tests for General Position). (a) \(Z \subseteq \mathbb{P}^n\) is in linearly general position if and only if one of the following is true:

\[ (i) \quad r \geq n + 1 \text{ and the variety defined by } D_n^d \text{ has codimension } n + 1. \]

\[ (ii) \quad r < n + 1 \text{ and the variety defined by } D_r^{d-1} \text{ has codimension } r. \]

(b) \(Z \subseteq \mathbb{P}^n\) is in \(d\)-general position if and only if one of the following is true:

\[ (i) \quad r \geq \binom{n+d}{d} \text{ and the variety defined by } D_n^{N_d} \text{ has codimension } \binom{n+d}{d}. \]

\[ (ii) \quad r < \binom{n+d}{d} \text{ and the variety defined by } D_r^{r-1} \text{ has codimension } r. \]

**Remark 17.** Let \(Z\) be a finite collection of reduced points. By the definition of regular position and in the light of Remark 4, \(Z\) is in general position if and only if \(Z\) is in \(d\)-general position for all \(1 \leq d \leq t\), where \(t\) is the regularity of \(Z\). To determine whether \(Z\) is in \(d\)-general position, one can use the following algorithm:

- Start with an ideal \(I\) which defines \(Z\) set theoretically in \(\mathbb{P}^n\).
- Compute the radical of \(I\) to obtain the ideal \(I_Z\).
• Let \( r = |Z| \). Let \( N_d = \binom{n+d}{d} - 1 \).

• Compute the ideal of the \( d \)-uple embedding of \( Z \); call it \( I_{Z,d} \).

• Compute the Chow form of the \( d \)-uple embedding of \( Z \); call it \( G_d \).

• Form the ideal generated by all \( (r-1) \)th derivatives of \( G_d \); call it \( D_{r-1,d} \).

• Form the ideal generated by all \( N_d \)th derivatives of \( G_d \); call it \( D_{N_d,d} \).

• Determine the codimensions of the varieties defined by \( D_{N_d,d} \) and \( D_{r-1,d} \) and check that they satisfy the conditions of Proposition 16.

**Proposition 18** (Tests for Uniform Position).

(a) \( Z \) is in \((s, d)\)-uniform position if and only if one of the following is true:

(i) The variety defined by \( D_{r-1,d}^s \) has codimension \( s \).

(ii) The variety defined by \( D_{d}^{s-1} \) has the same codimension as the variety defined by \( D_{r-1,d}^s \).

(b) \( Z \) is in uniform position if and only if for every \( s \leq r \) and for every \( d \) with \( 1 \leq d \leq t \) where \( t \) is the regularity of \( Z \), one of the following is true:

(i) The variety defined by \( D_{r-1,d}^s \) has codimension \( s \).

(ii) The variety defined by \( D_{d}^{s-1} \) has the same codimension as the variety defined by \( D_{r-1,d}^s \).

Using the algorithm found in Remark 17 as a guide, we leave it to the reader to write down an algorithm which tests whether a set of points is in \((s, d)\)-uniform position.

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**References**


