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On some maximal non-selfadjoint operator algebras[☆]

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Abstract

We show that the reflexive algebra given by the lattice generated by a maximal nest and a rank one projection is maximal with respect to its diagonal.

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1. Introduction

The study of non-selfadjoint operator algebras on a Hilbert space is closely related to the invariant subspace problem which asks whether every operator acting on a separable Hilbert space admits a non-trivial invariant subspace. The well known classes of non-selfadjoint operator algebras include transitive algebras, triangular algebras, nest algebras, reflexive algebras, etc.

Let $\mathscr{H}$ be a Hilbert space (over the field of complex numbers) and $\mathscr{B}(\mathscr{H})$ the algebra of all bounded linear operators on $\mathscr{H}$. Similar to upper triangular matrix algebras, Kadison and Singer [9] introduced and studied triangular (operator) algebras on an infinite-dimensional Hilbert space. Recall that a subalgebra $\mathscr{A}$ of $\mathscr{B}(\mathscr{H})$ is *triangular* if its

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diagonal subalgebra $\mathscr{A} \cap \mathscr{A}^*$ is a maximal abelian selfadjoint subalgebra (MASA) of $\mathscr{B}(\mathscr{H})$. It follows easily from Zorn's lemma that every triangular algebra is contained in a maximal triangular algebra. A special class of maximal triangular algebras is nest algebras introduced by Ringrose [12]. Considerable effort has gone into the study of nest algebras (see [1] and references therein). Reflexive (operator) algebras (see [5]) are more general than nest algebras.

Recently, motivated by the notion of maximal triangular algebras, Ge and Yuan [3,4] introduced a new class of non-selfadjoint algebras which they call Kadison–Singer algebras or KS-algebras for simplicity, by combining triangularity, reflexivity and the von Neumann algebra property into one consideration. These algebras are reflexive and maximal with respect to the reflexivity and their "diagonal subalgebras". More importantly, KS-algebras are closely related to the von Neumann algebras generated by their lattices of invariant projections. The lattice of invariant projections of a KS-algebra is reflexive and "minimally generating" in the sense that it generates the commutant of the diagonal as a von Neumann algebra. Nest algebras are KS-algebras with "abelian cores" and whose lattices of invariant projections are commutative. Many examples of Kadison–Singer algebras are given in [3, 4,6,7,13,2]. Surprisingly, Kadison–Singer lattices often carry nice geometrical structures.

Note that a triangular subalgebra of $\mathscr{B}(\mathscr{H})$ does not have to be reflexive or closed. Even a maximal triangular algebra may not be norm closed (see [11]). While in the definition of KS-algebras, we require that the algebras be maximal in the class of all *reflexive* operator algebras with the same diagonal subalgebras. In the article [8], Kadison and Singer suggested that one can study the subalgebras of $\mathscr{B}(\mathscr{H})$ with the maximality with respect to the property of having a given selfadjoint operator algebra as its diagonal. Nest algebras are KS-algebras. KS-algebras can be viewed as maximal triangular algebras with a non-commutative core. In [4], Ge and Yuan gave some examples of weakly closed algebras which are maximal in the class of all *weakly closed* subalgebras with the same diagonal subalgebra. Motivated by these ideas and results, we give the following definition of algebraic maximality.

Definition. A subalgebra $\mathscr{A}$ of $\mathscr{B}(\mathscr{H})$ is called to be maximal with respect to its diagonal subalgebra (or diagonally maximal), if $\mathscr{B}$ is an arbitrary subalgebra of $\mathscr{B}(\mathscr{H})$ such that $\mathscr{A} \subseteq \mathscr{B}$ and $\mathscr{B} \cap \mathscr{B}^* = \mathscr{A} \cap \mathscr{A}^*$, then $\mathscr{B} = \mathscr{A}$.

Remark. In the definition, we do not require that the diagonal be norm closed. This definition fails to rule out the possibility that there may exist a dense selfadjoint subalgebra of $\mathscr{B}(\mathscr{H})$ so that it is diagonally maximal. So we do not know whether a proper selfadjoint subalgebra of $\mathscr{B}(\mathscr{H})$ may be maximal with respect to its diagonal.

Clearly, maximal triangular algebras in $\mathscr{B}(\mathscr{H})$ have the algebraic maximality with respect to its diagonal. Thus our above definition is a generalization of maximal triangular algebras with a non-commutative diagonal subalgebra. From Zorn's lemma, every subalgebra of $\mathscr{B}(\mathscr{H})$ is contained in a subalgebra which is maximal with respect to its diagonal subalgebra. It is easy to verify that each nest algebra $\operatorname{Alg}\mathscr{N}$, for a given nest $\mathscr{N}$, also has the algebraic maximality by noting that $PTP^{\perp}$ belongs to $\operatorname{Alg}\mathscr{N}$ for each $T \in \mathscr{B}(\mathscr{H})$ and each $P \in \mathscr{N}$. The reflexive operator algebra determined by two free projections with trace half, given in [2] by the first author of this paper, is maximal with respect to its diagonal. This is an example of maximal non-selfadjoint algebra with a non-commutative diagonal and a non-commutative core.

In this paper, we show that the reflexive algebra given by the lattice generated by a nest and a rank one projection (introduced in [6]) is maximal with respect to its diagonal. In [6], we show that such an algebra is a KS-algebra and our proof depends heavily on the reflexivity and on properties of the KS-lattice. Our proof of maximality in this paper is purely algebraic. Some examples of non-selfadjoint algebras which are diagonally maximal are also given.

2. Main results

Before we state our main result, we recall some notation and preliminary results. Let $\mathscr{H}$ be a separable complex Hilbert space, $\mathscr{B}(\mathscr{H})$ the algebra of all bounded linear operators acting on $\mathscr{H}$. For a set $\mathscr{L}$ of orthogonal projections in $\mathscr{B}(\mathscr{H})$, we denote by $\operatorname{Alg}\mathscr{L}$ the set of all bounded linear operators on $\mathscr{H}$ leaving each element in $\mathscr{L}$ invariant, i.e., $\operatorname{Alg}\mathscr{L} = \{T \in \mathscr{B}(\mathscr{H}) : PTP = TP \text{ for all } P \in \mathscr{L}\}$. Then $\operatorname{Alg}\mathscr{L}$ is a unital weak-operator closed subalgebra of $\mathscr{B}(\mathscr{H})$. A nest $\mathscr{N}$ is a totally ordered family of projections on $\mathscr{H}$ which contains zero and the identity operator I on $\mathscr{H}$ and is closed in strong-operator topology and $\operatorname{Alg}\mathscr{N}$ is called a nest algebra.

For convenience, we shall not distinguish an orthogonal projection P from its range $P(\mathscr{H})$. Hence we write $\gamma \in P$ for $\gamma \in P(\mathscr{H})$. Let $P^\perp$ denote the orthogonal complement $I - P$ of P.

Throughout the paper, we use $\mathscr{N}$ to denote a nontrivial nest of projections on $\mathscr{H}$, $\operatorname{Alg}\mathscr{N}$ the nest algebra given by $\mathscr{N}$. Since $\mathscr{N}''$ (the von Neumann algebra generated by $\mathscr{N}$, or the double commutant of $\mathscr{N}$) is abelian, it has a separating vector, say ξ. Then the mapping: $T \to T\xi$, from $\mathscr{N}''$ into $\mathscr{H}$, is injective [8]. We assume that $\|\xi\| = 1$. Let P_ξ be the orthogonal projection from $\mathscr{H}$ onto the one dimensional subspace of $\mathscr{H}$ generated by ξ. Then for each projection $P \in \mathscr{N}$ with $P \neq 0, I$, we have $\xi \notin P$ and $\xi \notin P^\perp$. Hence $P \wedge P_\xi = 0$. Obviously, $P \vee P_\xi$ is the orthogonal projection from $\mathscr{H}$ onto the closed subspace $P(\mathscr{H}) + \mathbb{C}\xi$.

The lattice $\mathscr{L}$ generated by $\mathscr{N}$ and P_ξ is called a *one point extension* of $\mathscr{N}$ by P_ξ in [6]. It is not difficult to show that $\mathscr{L} = \{0, I, P, P_\xi, P \vee P_\xi : P \in \mathscr{N}, P \neq 0, I\}$, and $\operatorname{Alg}\mathscr{L} = \{T \in \operatorname{Alg}\mathscr{N} : T\xi = \lambda\xi \text{ for some } \lambda \in \mathbb{C}\}$.

Theorem 2.1 ([6]). *A one point extension $\mathscr{L}$ of $\mathscr{N}$ by P_ξ is a KS-lattice, and the reflexive algebra $\operatorname{Alg}\mathscr{L}$ is a KS-algebra. Moreover, if $\mathscr{N}''$ is a MASA in $\mathscr{B}(\mathscr{H})$, then $\operatorname{Alg}\mathscr{L}$ is a KS-algebra with the trivial diagonal $\mathbb{C}I$.*

For $P \in \mathscr{N}$, we let $P_- = \vee\{Q \in \mathscr{N} : Q < P\}$ for $P \neq 0$ and define $0_- = 0$. We call P_- the *immediate predecessor* of P in $\mathscr{N}$. Similarly, for $P \in \mathscr{N}$, we define $P_+ = \wedge\{Q \in \mathscr{N} : P < Q\}$ for $P \neq I$ and $I_+ = I$. We call P_+ the *immediate successor* of P in $\mathscr{N}$.

For nonzero vectors γ and η in $\mathscr{H}$, denote by $\gamma \otimes \eta$ the rank one operator given by $(\gamma \otimes \eta)(z) = \langle z, \eta\rangle\gamma$ for all $z \in \mathscr{H}$. Clearly, for any A, B in $\mathscr{B}(\mathscr{H})$, we have $A(\gamma \otimes \eta)B = (A\gamma) \otimes (B^*\eta)$, where B^* is the adjoint of B. It is well-known that a rank one operator $\gamma \otimes \eta \in \operatorname{Alg}\mathscr{N}$ if and only if there exists $P \in \mathscr{N}$ such that $\gamma \in P$ and $\eta \in P_-^\perp$ [12]. The following properties are not difficult to show, some of which are also listed in [6].

Lemma 2.2. *Let $\mathscr{L}$ be a one-point extension of a nest $\mathscr{N}$ by P_ξ given as above.*

(i) *If $P, Q \in \mathscr{N}$ with $0 < P < Q < I$, then $(P \vee P_\xi) < (Q \vee P_\xi)$ and $(P \vee P_\xi) \not\geq Q$.*
(ii) *If ξ is also a separating vector for $\mathscr{N}'$, then $I_- \vee P_\xi = I$.*
(iii) *If $\{P_n\}_n$ is a strictly decreasing sequence of projections in $\mathscr{N}$ such that $\wedge_n P_n = P$, then $\wedge_n(P_n \vee P_\xi) = P \vee P_\xi$.*

Using Longstaff's characterization of rank one operators in reflexive operator algebras (see [10]), we have the following lemma.

Lemma 2.3 (*[6]*). *For nonzero vectors x and y in $\mathscr{H}$, the rank one operator $x \otimes y$ is in* $\operatorname{Alg}\mathscr{L}$ *if one of the following statements holds:*

(i) *there exists a $P \in \mathscr{N}$ with $P \neq 0$ and $P_- \vee P_\xi \neq I$, such that $x \in P$ and $y \in (P_- \vee P_\xi)^\perp$.*
(ii) *$I_- < I$, $x \in \mathbb{C}\xi$ and $y \in (I_-)^\perp$.*
(iii) *$(I_- \vee P_\xi) < I$, and $x \in \mathscr{H}$, $y \in (I_- \vee P_\xi)^\perp$.*

Before we state our main theorem, we prove some preliminary results.

Lemma 2.4. *For P and Q in $\mathscr{N}$ with $P < Q$, if* $\dim(Q-P) \geq 2$, *then $Q \wedge (P \vee P_\xi)^\perp \neq 0$, where* $\dim(A)$ *is the dimension of the range of a linear operator A.*

Proof. Clearly, $(Q - P)\xi$ is a nonzero vector in $Q - P$. From our assumption that dim $(Q - P) \geq 2$, we choose a nonzero vector η in $Q - P$ such that η and $(Q - P)\xi$ are orthogonal. Then η is a nonzero vector in $Q \wedge (P \vee P_\xi)^\perp$. This completes our proof.

In order to show that $\operatorname{Alg}\mathscr{L}$ is maximal non-selfadjoint with respect to its diagonal subalgebra, we shall consider a bigger algebra $\mathscr{B}$ generated by T and $\operatorname{Alg}\mathscr{L}$ for some T in $\mathscr{B}(\mathscr{H})$. We assume that $T \neq 0$. □

Lemma 2.5. *With the above notation, assume that $\mathscr{B} \cap \mathscr{B}^* = \mathscr{L}'$. Then $TP(\mathscr{H}) \subseteq (P \vee P_\xi)$ for each P in $\mathscr{N}$.*

Proof. Suppose $P \in \mathscr{N}$ such that $0 < P < I$. We consider two cases.

Case 1: $P_+ = P$. In this case, there exists a strictly decreasing sequence $\{P_n\}_n$ of projections in $\mathscr{N}$ such that $\dim(P_n - P_{n+1}) \geq 2$ for each $n \geq 1$ and $\wedge_{n\geq 1} P_n = P$. For any given $n \geq 1$, by Lemma 2.4, we can choose nonzero vectors $\sigma_1 \in P_n \wedge (P_{n+1} \vee P_\xi)^\perp$ and $\sigma_2 \in P_{n+1} \wedge (P_{n+2} \vee P_\xi)^\perp$.

Let $\sigma = \sigma_1 + \sigma_2$. Then σ is a nonzero vector in $P_n \wedge (P \vee P_\xi)^\perp$. Hence, by Lemma 2.3, for each nonzero vector η in P and x in $(P_n \vee P_\xi)^\perp$, we have $\sigma \otimes x$ and $\eta \otimes \sigma$ are in $\operatorname{Alg}\mathscr{L}$, and therefore $(\sigma \otimes x)T(\eta \otimes \sigma)$ is in $\mathscr{B}$. Thus $\langle T\eta, x\rangle \sigma \otimes \sigma \in \mathscr{B}$. Assume that there are η in P and x in $(P_n \vee P_\xi)^\perp$ such that $\langle T\eta, x\rangle \neq 0$. Then $\sigma \otimes \sigma$ is a selfadjoint operator in $\mathscr{B}$ and thus in the diagonal of $\mathscr{B}$. By the choice of σ_1 and σ_2, we note that $P_{n+1}(\sigma \otimes \sigma) = \sigma_2 \otimes \sigma$ and $(\sigma \otimes \sigma)P_{n+1} = \sigma \otimes \sigma_2$. Hence $P_{n+1}(\sigma \otimes \sigma) \neq (\sigma \otimes \sigma)P_{n+1}$, so that $\sigma \otimes \sigma \notin \mathscr{L}'$, which is a contradiction. Thus we have $\langle T\eta, x\rangle = 0$ for each η in P and x in $(P_n \vee P_\xi)^\perp$. This follows that $T\eta \in (P_n \vee P_\xi)$ for each $n \geq 1$. Since $\wedge_{n\geq 1}(P_n \vee P_\xi) = P \vee P_\xi$, we have $T\eta \in P \vee P_\xi$ for each η in P. This completes the proof of our lemma in this case.

Case 2: $P_+ > P$. Clearly, $TP(\mathscr{H}) \subseteq (P \vee P_\xi)$ if $P \vee P_\xi = I$. Hence we may assume that $P \vee P_\xi < I$. We shall consider P_- for the following two cases.

First, when $P_- < P$, let $e_1 = (P_+ - P)\xi$ and $e_2 = (P - P_-)\xi$. Then $e_i \neq 0$ and $\langle e_i, \xi\rangle \neq 0$ for $i = 1, 2$. Let $\eta = e_2 - \frac{\langle e_2,\xi\rangle}{\langle e_1,\xi\rangle} e_1$. A simple computation shows that η is a nonzero vector in $(P_- \vee P_\xi)^\perp$. For each nonzero vectors x in P and y in $(P \vee P_\xi)^\perp$, $e_1 \otimes y$ and $x \otimes \eta$ are in $\operatorname{Alg}\mathscr{L}$, which yield that $(e_1 \otimes y)T(x \otimes \eta)$ and hence $\langle Tx, y\rangle e_1 \otimes \eta$ are in $\mathscr{B}$. Assume that there are x_0 in P and y_0 in $(P \vee P_\xi)^\perp$ such that $\langle Tx_0, y_0\rangle \neq 0$. Then $e_1 \otimes \eta \in \mathscr{B}$. We remark that $e_2 \otimes \eta$ is in $\operatorname{Alg}\mathscr{L}$. Then $e_2 \otimes \eta - \frac{\langle e_2,\xi\rangle}{\langle e_1,\xi\rangle} e_1 \otimes \eta$ and, hence, $\eta \otimes \eta$ are in $\mathscr{B}$. It is easy to check that $P(\eta \otimes \eta) \neq (\eta \otimes \eta)P$. Thus the operator $\eta \otimes \eta$ belongs to the diagonal of $\mathscr{B}$, but not in $\mathscr{L}'$. This contradiction shows that $\langle Tx, y\rangle = 0$ for each x in P and y in $(P \vee P_\xi)^\perp$. This proves that $TP(\mathscr{H}) \subseteq (P \vee P_\xi)$.

Now we assume $P_- = P$. In this case, our argument is similar to Case 1. Choose a strictly increasing sequence $\{P_n\}$ of projections in $\mathscr{N}$ such that $P_1 > 0$, $\dim(P_{n+1} - P_n) \geq 2$ for each $n \geq 1$ and $\vee_{n\geq 1} P_n = P$. For each $n \geq 1$, by Lemma 2.4, choose nonzero vectors $\sigma_1 \in P_{n+1} \wedge (P_n \vee P_\xi)^\perp$ and $\sigma_2 \in P_{n+2} \wedge (P_{n+1} \vee P_\xi)^\perp$. Let $\sigma = \sigma_1 + \sigma_2$. Then σ is a nonzero vector in $P \wedge (P_n \vee P_\xi)^\perp$. Hence for each nonzero vectors x in P_n and y in $(P \vee P_\xi)^\perp$, the fact that $\sigma \otimes y$ and $x \otimes \sigma$ are in $\operatorname{Alg}\mathscr{L}$ induces that $(\sigma \otimes y)T(x \otimes \sigma)$ is in $\mathscr{B}$. Hence $\langle Tx, y\rangle\sigma \otimes \sigma \in \mathscr{B}$. Similarly we can show that $\langle Tx, y\rangle = 0$ for each x in P_n and y in $(P \vee P_\xi)^\perp$. Thus $Tx \in P \vee P_\xi$ for each x in P_n and $n \geq 1$. Also since $\vee_{n\geq 1} P_n = P$, we have $TP(\mathscr{H}) \subseteq (P \vee P_\xi)$. This completes the proof of our above lemma. □

Lemma 2.6. *With the above notation, assume that $\mathscr{B} \cap \mathscr{B}^* = \mathscr{L}'$. Then $T \in \operatorname{Alg}\mathscr{N}$.*

Proof. We only need to show that $TP(\mathscr{H}) \subseteq P$ for each $P \in \mathscr{N}$ with $0 < P < I$. With a given P in $\mathscr{N}$ such that $0 < P < I$, we have the following two cases.

Case 1. *There is a projection Q in $\mathscr{N}$ such that $P < Q < I$.* For any nonzero vector x in P, by Lemma 2.5 we have that $Tx \in P \vee P_\xi$. It is easy to see that P and $P^\perp\xi$ span $P \vee P_\xi$. Then we may write $Tx = y_0 + \lambda P^\perp\xi$ for some y_0 in P and $\lambda \in \mathbb{C}$. Let $\alpha = -\frac{\|(Q-P)\xi\|^2}{\|Q^\perp\xi\|^2}$. Then $\eta = (Q - P)\xi + \alpha Q^\perp\xi$ is a nonzero vector in $(P \vee P_\xi)^\perp$. From Lemma 2.3, we have that $x \otimes \eta$ and $y_0 \otimes \eta$ are in $\operatorname{Alg}\mathscr{L}$ and $T(x \otimes \eta) - y_0 \otimes \eta$ is in $\mathscr{B}$. Thus $\lambda P^\perp\xi \otimes \eta$ is in $\mathscr{B}$.

We may assume that $\lambda \neq 0$ (otherwise $Tx_0 = y_0 \in P$). Then $P^\perp\xi \otimes \eta \in \mathscr{B}$. Write $P^\perp\xi = (Q - P)\xi + Q^\perp\xi$. Hence

$$\begin{aligned} P^\perp\xi \otimes \eta &= [(Q - P)\xi + Q^\perp\xi] \otimes [(Q - P)\xi + \alpha Q^\perp\xi] \\ &= (Q - P)\xi \otimes (Q - P)\xi + \alpha Q^\perp\xi \otimes Q^\perp\xi \\ &\quad + \alpha(Q - P)\xi \otimes Q^\perp\xi + Q^\perp\xi \otimes (Q - P)\xi. \end{aligned}$$

Let $A = (Q - P)\xi \otimes (Q - P)\xi + \alpha Q^\perp\xi \otimes Q^\perp\xi$. Then A is a selfadjoint operator and $P^\perp\xi \otimes \eta = A + \alpha(Q - P)\xi \otimes Q^\perp\xi + Q^\perp\xi \otimes (Q - P)\xi$. Also let $B = (Q - P)\xi \otimes Q^\perp\xi - \|Q^\perp\xi\|^2(Q - P)$. Then $B \in \operatorname{Alg}\mathscr{N}$ and $B\xi = 0$. Thus $B \in \operatorname{Alg}\mathscr{L}$. We also have $(Q-P)\xi \otimes Q^\perp\xi = B + \|Q^\perp\xi\|^2(Q-P)$ and $Q^\perp\xi \otimes (Q-P)\xi = B^* + \|Q^\perp\xi\|^2(Q-P)$. Now consider the operator $P^\perp\xi \otimes \eta + (1 - \alpha)B$:

$$P^\perp\xi \otimes \eta + (1 - \alpha)B = A + (B + B^*) + (\alpha + 1)\|Q^\perp\xi\|^2(Q - P).$$

Then it is a selfadjoint operator in $\mathscr{B}$.

It is easy to check that $Q[P^\perp\xi \otimes \eta + (1 - \alpha)B] \neq [P^\perp\xi \otimes \eta + (1 - \alpha)B]Q$. Hence the operator $P^\perp\xi \otimes \eta + (1 - \alpha)B$ is not in $\mathscr{L}'$. This contradicts our assumption in the lemma. This shows that $\lambda = 0$, and then $Tx = y_0 \in P$ for each x in P. It follows that $TP(\mathscr{H}) \subset P$.

Case 2. $P = I_- < I$. First we assume that $P_- < P$. For any nonzero vector x in P, again by Lemma 2.5 and similar to Case 1, we may assume that $Tx = y_0 + \lambda P^\perp \xi$ for y_0 in P and $\lambda \in \mathbb{C}$. Let $\eta = (P - P_-)\xi + \alpha P^\perp \xi$ for $\alpha = -\frac{\|(P-P_-)\xi\|^2}{\|P^\perp \xi\|^2}$. Then $\eta \neq 0$, $\eta \in (P_- \vee P_\xi)^\perp$. It is easy to see that $x \otimes \eta$ and $y_0 \otimes \eta$ are in $\mathrm{Alg}\,\mathscr{L}$. Thus $T(x \otimes \eta) - y_0 \otimes \eta$ and then $\lambda P^\perp \xi \otimes \eta$ are in $\mathscr{B}$. If $\lambda \neq 0$, then $P^\perp \xi \otimes \eta$ is in $\mathscr{B}$. Also since $(P - P_-)\xi \otimes \eta$ is in $\mathrm{Alg}\,\mathscr{L}$, we have $\alpha P^\perp \xi \otimes \eta + (P - P_-)\xi \otimes \eta$ is in $\mathscr{B}$. This implies that $\eta \otimes \eta \in \mathscr{B}$. Since $P\eta$ and η are linearly independent and $P\eta = (P - P_-)\xi$, we have $P(\eta \otimes \eta) \neq (\eta \otimes \eta)P$. Hence $\eta \otimes \eta$ is an operator in the diagonal of $\mathscr{B}$, but not in $\mathscr{L}'$. This contradicts our assumption. So $\lambda = 0$ and $Tx = y_0 \in P$ for each x in P.

Now when $P_- = P$, there is a strictly increasing sequence $\{P_n\}_{n \geq 1}$ of projections in $\mathscr{N}$ such that $\vee_{n=1}^\infty P_n = P$. For each $n \geq 1$, by Case 1, we have $TP_n \leq P_n$. Since the linear span of $\{x \in P_n : n \geq 1\}$ is dense in the range of P, we have $TP(\mathscr{H}) \leq P$. This completes the proof of our lemma. □

Lemma 2.7. *With the above notation, assume that* $\mathscr{B} \cap \mathscr{B}^* = \mathscr{L}'$. *Then* $T\xi \in \mathbb{C}\xi$.

Proof. From Lemma 2.6, we have that $T \in \mathrm{Alg}\,\mathscr{N}$. Again, we separate the argument into two cases.

Case 1: $0_+ > 0$. It is enough to show that $\langle \eta, T\xi \rangle = 0$ for each η in $(\mathbb{C}\xi)^\perp$. Denote $Q = 0_+$.

Suppose on the contrary that there is a vector η in $(\mathbb{C}\xi)^\perp$ such that $\langle \eta, T\xi \rangle \neq 0$. Then $Q\xi \otimes \eta \in \mathrm{Alg}\,\mathscr{L}$ and $(Q\xi \otimes \eta)T \in \mathscr{B}$. Let $\lambda = \frac{\langle \eta, T\xi \rangle}{\|Q\xi\|^2}$, and set $A = (Q\xi \otimes \eta)T - Q\xi \otimes (\lambda Q\xi)$. Then $A = Q\xi \otimes (T^*\eta - \lambda Q\xi) \in \mathrm{Alg}\,\mathscr{N}$ and $A\xi = \langle \xi, T^*\eta - \lambda Q\xi \rangle Q\xi = 0$. So $A \in \mathrm{Alg}\,\mathscr{L}$, and thus $\overline{\lambda} Q\xi \otimes Q\xi = Q\xi \otimes (\lambda Q\xi) = (Q\xi \otimes \eta)T - A \in \mathscr{B}$. Since $\lambda \neq 0$, it follows that $Q\xi \otimes Q\xi$ is a selfadjoint element in $\mathscr{B}$. Clearly, $P_\xi(Q\xi \otimes Q\xi) \neq (Q\xi \otimes Q\xi)P_\xi$. This implies that $(Q\xi \otimes Q\xi)$ is not in $\mathscr{L}'$. This contradiction shows that $\langle \eta, T\xi \rangle = 0$ for each vector η in $(\mathbb{C}\xi)^\perp$. Consequently, $T\xi \in \mathbb{C}\xi$.

Case 2. $0_+ = 0$. In this case, there exists a strictly decreasing sequence $\{P_n\}_{n \geq 1}$ of projections in $\mathscr{N}$ such that $P_1 < I$ and $\wedge_{n=1}^\infty P_n = 0$. For any given $n \geq 1$ and for each nonzero vector η in $(P_n \vee P_\xi)^\perp$, we have $P_n\xi \otimes \eta \in \mathrm{Alg}\,\mathscr{L}$, and hence $(P_n\xi \otimes \eta)T \in \mathscr{B}$. We define $\lambda_\eta = \frac{\langle \eta, T\xi \rangle}{\|P_n^\perp \xi\|^2}$ for each nonzero vector η in $(P_n \vee P_\xi)^\perp$.

Suppose that $\lambda_{\eta_0} \neq 0$ for some nonzero vector η_0 in $(P_n \vee P_\xi)^\perp$. By Lemma 2.6, P_n is invariant under T. Then we have $T^*\eta_0 - \lambda_{\eta_0} P_n^\perp \xi$ is in $(P_n \vee P_\xi)^\perp$, so that $P_n\xi \otimes (T^*\eta_0 - \lambda_{\eta_0} P_n^\perp \xi)$ is in $\mathrm{Alg}\,\mathscr{L}$. Also by Lemma 2.3, we have $P_n\xi \otimes \eta_0$ is in $\mathrm{Alg}\,\mathscr{L}$. It follows that $P_n\xi \otimes T^*\eta_0 = (P_n\xi \otimes \eta_0)T$ is in $\mathscr{B}$. Hence $P_n\xi \otimes \lambda_{\eta_0} P_n^\perp \xi$ is in $\mathscr{B}$. This implies that $P_n\xi \otimes P_n^\perp \xi$ is in $\mathscr{B}$. Let $D = P_n\xi \otimes P_n^\perp \xi - \|P_n^\perp \xi\|^2 P_n$. Then $D \in \mathrm{Alg}\,\mathscr{N}$ and $D\xi = 0$, and hence, $D \in \mathrm{Alg}\,\mathscr{L}$. By the equation $\|P_n^\perp \xi\|^2 P_n = P_n\xi \otimes P_n^\perp \xi - D$, we have P_n is in $\mathscr{B}$. Clearly, $P_n \notin \mathscr{L}'$. Now the selfadjoint operator P_n is in $\mathscr{B}$, but not in $\mathscr{L}'$. This contradiction tells us that $\lambda_\eta = 0$ and $\langle \eta, T\xi \rangle = 0$ for each nonzero vector η in $(P_n \vee P_\xi)^\perp$. Consequently, $T\xi \in P_n \vee P_\xi$ for each $n \geq 1$. Also since $\wedge_{n \geq 1}(P_n \vee P_\xi) = P_\xi$, we have $T\xi \in \mathbb{C}\xi$. This completes the proof of our lemma. □

We summarize the above results in the following theorem.

Theorem 2.8. *Suppose $\mathscr{L}$ is a one point extension of a nest $\mathscr{N}$ by P_ξ, where ξ is a unit vector separating for $\mathscr{N}''$, and T is in $\mathscr{B}(\mathscr{H})$. If* $\operatorname{Alg}\mathscr{L}$ *and the algebra generated by T and* $\operatorname{Alg}\mathscr{L}$ *have the same diagonal, then $T \in \operatorname{Alg}\mathscr{L}$.*

Corollary 2.9. *Let $\mathscr{N}$ be a nest on a separable Hilbert space $\mathscr{H}$, ξ be a separating vector for $\mathscr{N}''$. Let $\mathscr{L}$ be the lattice generated by $\mathscr{N}$ and P_ξ. Then* $\operatorname{Alg}\mathscr{L}$ *is maximal with respect to its diagonal subalgebra.*

Corollary 2.10. *Let $\mathscr{N}$, $\mathscr{L}$ and ξ be as before. Then* $\operatorname{Alg}\mathscr{L}$ *is a KS-algebra.*

In the rest of the paper, we give some other types of examples of non-selfadjoint algebras which are maximal with respect to their diagonal subalgebras.

Example 2.11. Let $\mathcal{D} = \left\{\begin{pmatrix} a & a-b \\ 0 & b\end{pmatrix} : a, b \in \mathbb{C}\right\}$. It is not difficultto show that $\mathcal{D}$ is a subalgebra of $M_2(\mathbb{C})$ which is maximal with respect to the diagonal $\mathbb{C}I$. We consider the subalgebra of $M_4(\mathbb{C})$ as follows:

$$\mathscr{A} = \left\{\begin{pmatrix} A & A-B \\ 0 & B\end{pmatrix} : A \in \mathcal{D}, B \in M_2(\mathbb{C})\right\}.$$

Then $\mathscr{A}$ has diagonal $\mathbb{C}I$. Now we show that $\mathscr{A}$ is maximal with respect to its diagonal subalgebra.

Suppose $\mathscr{B}$ is a subalgebra of $M_4(\mathbb{C})$ containing $\mathscr{A}$ and with diagonal $\mathbb{C}I$. Let $\begin{pmatrix} T_1 & T_2 \\ T_4 & T_3\end{pmatrix}$ be in $\mathscr{B}$, where T_i is in $M_2(\mathbb{C})$ for $i = 1, 2, 3, 4$.

Let $T_4 = UK$ be the polar decomposition of T_4, where U is unitary and K is semi-positive definite in $M_2(\mathbb{C})$. By calculation, we have

$$\begin{pmatrix} 0 & U^* \\ 0 & -U^*\end{pmatrix}\left[\begin{pmatrix} T_1 & T_2 \\ T_4 & T_3\end{pmatrix} - \begin{pmatrix} 0 & -T_3 \\ 0 & T_3\end{pmatrix}\right] + \begin{pmatrix} 0 & -K \\ 0 & K\end{pmatrix} = \begin{pmatrix} K & -K \\ -K & K\end{pmatrix},$$

which is a selfadjoint element in $\mathscr{B}$. By the assumption, we have $K = 0$ and thus $T_4 = 0$.

Let $T_2 - T_1 + T_3 = LV$ be the polar decomposition of $T_2 - T_1 + T_3$, where V is unitary and L is semi-positive definite in $M_2(\mathbb{C})$. We have

$$\left[\begin{pmatrix} T_1 & T_2 \\ 0 & T_3\end{pmatrix}\begin{pmatrix} 0 & -I \\ 0 & I\end{pmatrix} + \begin{pmatrix} 0 & T_3 \\ 0 & -T_3\end{pmatrix}\right]\begin{pmatrix} 0 & -V^* \\ 0 & V^*\end{pmatrix} + \begin{pmatrix} 0 & -L \\ 0 & L\end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & L\end{pmatrix},$$

which is a selfadjoint element in $\mathscr{B}$. Hence $L = 0$ and thus $T_2 = T_1 - T_3$.

Next we only need to show T_1 is in $\mathcal{D}$. We note that $\begin{pmatrix} T_1 & T_1 - T_3 \\ 0 & T_3\end{pmatrix} - \begin{pmatrix} 0 & -T_3 \\ 0 & T_3\end{pmatrix} = \begin{pmatrix} T_1 & T_1 \\ 0 & 0\end{pmatrix}$is in $\mathscr{B}$. Hence $\begin{pmatrix} A & A \\ 0 & 0\end{pmatrix}$is in $\mathscr{B}$ for each A in the algebra $\widetilde{\mathcal{D}}$ generated by T_1 and $\mathcal{D}$.

Suppose the diagonal of $\widetilde{\mathcal{D}}$ is nontrivial. Then there exists a selfadjoint operator H in $\widetilde{\mathcal{D}}$ such that $H \notin \mathbb{C}I$. Thus $\begin{pmatrix} H & H \\ 0 & 0\end{pmatrix} \in \mathscr{B}$. So $\begin{pmatrix} H & H \\ 0 & 0\end{pmatrix} + \begin{pmatrix} 0 & -H \\ 0 & H\end{pmatrix} = \begin{pmatrix} H & 0 \\ 0 & H\end{pmatrix}$is a selfadjoint element in $\mathscr{B}$. We have a contradiction. Hence $\widetilde{\mathcal{D}}$ contains $\mathcal{D}$ and has the same diagonal as $\mathcal{D}$. It follows from the maximality of $\mathcal{D}$ that $\mathcal{D} = \widetilde{\mathcal{D}}$, which yields $T_1 \in \mathcal{D}$. Consequently, $\begin{pmatrix} T_1 & T_2 \\ T_4 & T_3\end{pmatrix}$is in $\mathscr{A}$.

Example 2.12. For $n \geq 2$, we let $\mathscr{A}$ be the subalgebra of $M_n(\mathbb{C})$ consisting of all the upper triangular matrices with the same diagonal entries. Then $\mathscr{A}$ is maximal with respect to the diagonal $\mathbb{C}I$.

Let $\mathscr{H}$ be a separable Hilbert space with an orthogonal basis $\{e_n : n = 1, 2, \ldots\}$. For each $n = 1, 2, \ldots$, let P_n be the orthogonal projection from $\mathscr{H}$ onto the linear span of $\{e_1, e_2, \ldots, e_n\}$, and let $P_0 = 0$. Let $\Phi(T) = \sum_{n=1}(P_n - P_{n-1})T(P_n - P_{n-1})$ for $T \in B(\mathscr{H})$. Then Φ is the (unique) conditional expectation from $B(\mathscr{H})$ onto the von Neumann algebra, denoted by $\mathscr{D}$, generated by $P_n, n = 1, 2, \ldots$. From Zorn's lemma, there is a subalgebra $\mathscr{D}_0$ of $\mathscr{D}$ which is maximal with respect to the diagonal $\mathbb{C}I$. Let $\mathcal{T} = \mathrm{Alg}\{P_n : n = 1, 2, \ldots\}$ and let $\mathscr{A}$ be the algebra generated by $\mathscr{D}_0$ and all the operator T in $\mathcal{T}$ satisfying $\Phi(T) = 0$. Then $\mathscr{A}$ is non-reflexive and has the diagonal $\mathbb{C}I$.

With a given T in $B(\mathscr{H})$, let $\mathscr{B}$ be the algebra generated by T and $\mathscr{A}$. Suppose that $\mathscr{B}$ has the diagonal $\mathbb{C}I$. Then using the fact that $e_i \otimes e_j \in \mathscr{A}$ for $i < j$, by induction, we can show $\langle Te_i, e_j\rangle 0$ for all $i < j$, and thus T is in $\mathcal{T}$. So $T - \Phi(T) \in \mathscr{A}$ and $\Phi(T) \in \mathscr{B}$. The maximality of $\mathscr{D}_0$ shows $\Phi(T) \in \mathscr{D}_0$, which implies that $T \in \mathscr{A}$. Hence $\mathscr{A}$ is maximal with respect to its diagonal.

Example 2.13 (*[4]*)**.** For $n \geq 3$ and $k \geq 2$, suppose $H_1, H_2, \ldots, H_k$ are positive definite matrices in $M_n(\mathbb{C})$ such that $H_1^2, \ldots, H_k^2$ generates $M_n(\mathbb{C})$. We consider the tensor product $M_{k+1} \otimes M_n(\mathbb{C})$. Let $E_{ij},\ i, j = 1, \ldots, k+1$, be the canonical matrix unit system. Define

$$\mathscr{A} = \left\{ \sum_{j=1}^{k+1} E_{jj} \otimes H_{j-1}^{-1} A H_{j-1} + \sum_{i<j} E_{ij} \otimes A_{ij} : A, A_{ij} \in M_n(\mathbb{C}) \right\},$$

where $H_0 = I$. Then $\mathscr{A}$ is maximal (non-selfadjoint) with respect to its diagonal $\mathbb{C}I$ and non-reflexive.

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