Abstract

This research paper is devoted to establish the coincidence between Choquet boundaries and a new type of approximate efficient points sets in ordered Hausdorff locally convex spaces, being based on the first result established by us concerning such a property as this for Pareto-type efficient points sets and the corresponding Choquet boundaries of non-empty compact sets, with respect to appropriate convex cones of real, increasing and continuous functions. Thus, the main result represents a strong connection between two great fields of mathematics: The Axiomatic Theory of Potential and Vector Optimization. The present study contains also important relationships concerning strong optimization and approximate efficiency, interesting examples, pertinent remarks and some open problems.

Keywords: Choquet boundary; Altomare projection; Approximate efficiency; Full nuclear cone

1. Introduction

The Choquet boundary is an important concept in the Axiomatic Theory of Potentials, while efficiency is a fundamental notion in Vector Optimization. The main aim of this work is to generalize our coincidence result given in [7] between the set of all Pareto-type minimum points of any non-empty, compact set in an ordered Hausdorff locally convex space and the Choquet boundary of the same set with respect to the convex cone of all real, increasing and continuous functions defined on the set, using an appropriate and new concept of approximate efficiency. The paper is organized as follows: Section 2 is devoted to the Choquet boundary and its immediate properties presented in an original manner; it contains some important links between this notion and Altomare projections and relevant examples on Choquet boundaries. In Section 3, we define the approximate efficiency, we show how this notion is related to: Pareto-type efficiency, the fixed points of multifunctions and the strong optimisation, and we give the principal theorem. Both these sections are completed with significant remarks. Section 4 concludes the paper with some open problems. All the elements of ordered topological vector spaces used in this work are in accordance with [28].

2. Preliminaries

In this section, the Choquet boundary and its immediate properties are revised. Let us consider an arbitrary Hausdorff locally space \((E, \tau)\), where \(\tau\) denotes its topology, and let \(K\) be any closed, convex, pointed cone in \(E\).
The usual order relation \( \leq_k \) associated with \( K \) is defined by \( x \leq_k y \) (\( x, y \in E \)) if there exists \( k \in K \) with \( y = x + k \). Clearly, this order relation on \( E \) is closed, that is, the set \( G_k \) given by \( G_k = \{(x, y) \in E \times E : x \leq_k y \} \) is a closed subset of \( E \times E \) endowed with the product topology.

Following the main considerations given in [6,9–12], for every non-empty and compact subset \( X \) of \( E \), we recall some basic concepts and results in Potential Theory concerning the Choquet boundary of \( X \) with respect to any convex cone of lower semicontinuous and bounded from below real functions defined on \( X \). Thus, we remember that, if \( S \) is any convex cone satisfying the properties:

(a) \( \forall x \in X, \exists s \in S, s > 0 \) and \( s(x) < +\infty \);
(b) \( S \) linearly separates \( X_1 = \{x \in X : \exists s \in S \text{ with } s(x) < 0\} \), that is, for every \( x, y \in X_1 \), \( x \neq y \), there exists \( s, t \in S \) with real values in \( x \) and \( y \) such that \( s(x)t(y) \neq s(y)t(x) \);

then, on the set \( M_+(X) \) of all positive Radon measures defined on \( X \), one associates the following natural pre-order relation: if \( \mu, \upsilon \in M_+(X) \), then \( \mu \leq_S \upsilon \) means that \( \mu(s) \leq \upsilon(s) \) for all \( s \in S \).

Let \( S_1 \) be the convex cone of all lower semicontinuous and bounded from below real functions \( s \) on \( X \) having the next property:

If \( x \in X \) and \( \mu \leq_S \varepsilon_x \), where \( \varepsilon_x(f) = f(x) \) for every real continuous function \( f \) on \( X \) denotes the Dirac measure, implies that \( \mu(s) \leq s(x) \). Any non-empty subset \( T \subseteq X \) will be called an \( S \)-boundary if, whenever \( s \in S_1 \) and its restriction on \( T \) denoted by \( s/T \) is positive, it follows that \( s \geq 0 \). The small closed \( S \)-boundary is usually called the Silov boundary of \( X \) with respect to \( S \). A closed set \( A \subseteq X \) is called \( S \)-absorbent if \( x \in A \) and \( \mu \leq_S \varepsilon_x \) implies that \( \mu(X \setminus A) = 0 \). The set \( \partial_S X = \{x \in X_1 : \{x\} \text{ is } S \text{-absorbent}\} \) is named the Choquet boundary of \( X \) with respect to \( S \). The trace on \( \partial_S X \) of the topology on \( X \) in which the closed sets coincide with \( X \) or with any of the \( S \)-absorbent subsets of \( X \) contained in \( X_1 \) is usually called the Choquet topology of \( \partial_S X \).

Theorem 2.1. \( \partial_S X \) is the smallest \( S \)-boundary subset of \( X \) with respect to the inclusion relation which is non-empty if and only if \( X_1 \neq \emptyset \).

Proof. Since it is clear that \( T \subseteq X \) is an \( S \)-boundary if and only if \( A \cap T \neq \emptyset \) for every non-empty and \( S \)-absorbent set \( A \subseteq X_1 \), it is sufficient to prove this for \( \partial_S X \). Indeed, let \( A \) be an arbitrary non-empty and \( S \)-absorbent subset of \( X_1 \). Then, because the class of all non-empty \( S \)-absorbent subsets of \( A \) is inductively ordered with respect to the inclusion relation, there exists a minimal \( S \)-absorbent subset \( A_0 \subseteq A \). But \( S \) separates linearly \( X_1 \). Therefore, \( A_0 \) is a singleton set, and obviously, \( A_0 \cap \partial_S X \neq \emptyset \). Moreover, if \( T \) is an arbitrary \( S \)-boundary and \( x \in \partial_S X \), then \( \{x\} \cap T \neq \emptyset \); that is, \( x \in T \) and the proof is completed. \( \square \)

Corollary 2.1.1. (i) (The minimum principle). For every \( s \in S_1 \) it follows that \( s/\partial_S X \geq 0 \Rightarrow s \geq 0 \) and \( s/\partial_S X > 0 \Rightarrow s/\partial_S X > 0 \);

(ii) Silov’s boundary of \( X \) with respect to \( S \) coincides with the closure of the Choquet boundary;

(iii) if \( A \) is any non-empty \( S \)-absorbent subset of \( X \) and one considers the convex cone \( S_A = \{s/A : s \in S\} \), then \( \partial_S A = A \cap \partial_S X \);

(iv) \( x \in \partial_S X \) iff the Dirac measure \( \varepsilon_x \) is minimal with respect to \( S_1 \); if, in addition, \( \inf(s, 0) \in S \) for every \( s \in S \), then \( x \in \partial_S X \) iff \( \varepsilon_x \) is minimal with respect to \( S \).

Remark 2.1. Consequently, if \( S \) is any convex cone of real and continuous functions on \( X \), and one denotes by \( C(X) \) the usual Banach space of all real continuous functions on \( X \), then a measure \( \mu \in M_+(X) \) is minimal [24] with respect to the previous pre-order relation when \( \mu(Q_S f) = \mu(f) \), \( \forall f \in C(X) \), where \( Q_S f = \inf \{s \in S : f \leq s\} \). Hence, if \( x \in X \), then the corresponding Dirac measure \( \varepsilon_x \) is minimal iff \( \varepsilon_x(Q_S f) = \varepsilon_x(f) \), that is, \( Q_S f(x) = f(x) \), \( \forall f \in C(X) \). Following (iv) in the above corollary, we have the coincidence \( \partial_S X = \{x \in X : Q_S f(x) = f(x), \forall f \in C(X)\} \).

Theorem 2.2. If for every upper semicontinuous and bounded from above real function \( f \) on \( X \), the function \( Q_S f \) is upper semicontinuous on \( \partial_S X \), in particular, if \( S \) is an arbitrary convex cone of real continuous functions, then \( \partial_S X \) is a Baire subset of \( X \) endowed with the corresponding trace topology.

Proof. Clearly, \( Q_S f(x) = f(x) \) for every \( x \in \partial_S X \) and any upper semicontinuous and bounded from above real function \( f \) on \( X \). Let \( (G_n) \) be a decreasing sequence of open sets in \( X \) such that \( G_n \cap \partial_S X \) (\( n \in N \)) is dense in \( \partial_S X \). We
shall show that \((\bigcap_{n \in N} G_n) \cap \partial_S X\) is also dense in \(\partial_S X\). Indeed, if \(G\) is an arbitrary open set such that \(G \cap \partial_S X \neq \emptyset\) then, by considering the function \(\varphi : X \to \{-1, 1\}\) defined as
\[
\varphi(x) = \begin{cases} 
-1, & x \in G \\
1, & x \in X \setminus G,
\end{cases}
\]
one obtains \(Q_S \varphi(x) = \varphi(x) = -1, \forall x \in G \cap \partial_S X, Q_S \varphi(x) \geq 1, \forall x \in X \setminus G\) and the set \(A_0 = \{x \in X : Q_S \varphi(x) < -\frac{1}{2}\}\) is a neighbourhood for \(G \cap \partial_S X\) with \(A_0 \subseteq G \cap X_1\). Therefore, there exists an open set \(G_0\) so that \(G_0 \cap \partial_S X \neq \emptyset\) and its closure \(\bar{G}_0 \subseteq A_0\). Let \(\psi_1\) be the characteristic function of \(X \setminus (G_0 \cap G_1)\), and \(\psi_{n+1}\) be the characteristic function for \(X \setminus \left[\text{int}(K_n) \cap G_{n+1}\right]\). Since the function \(u_n = Q_S \psi_n\) is upper semicontinuous on \(\partial_S X\) and \(u_{n+1}(x) = Q_S \psi_{n+1}(x) = \psi_{n+1}(x) = 0\), for every \(x \in \text{int}(K_n) \cap G_{n+1} \cap \partial_S X\), it follows that the set \(\{x \in X : u_{n+1}(x) < \frac{1}{2n+1}\}\) is a neighbourhood for \(\text{int}(K_n) \cap G_{n+1} \cap \partial_S X\). Therefore, there exists a compact denoted by
\[
K_{n+1} = \left\{x \in X : u_{n+1}(x) < \frac{1}{2n+1}\right\} \subset \text{int}(K_n) \cap G_{n+1} \text{ and } \text{int}(K_{n+1}) \cap \partial_S X \neq \emptyset, \ n \in N.
\]
Because \(\text{int}(K_{n+1}) \cap G_{n+1} \subseteq \text{int}(K_n) \cap G_{n+1}\) for every \(n \in N\), we have \(u_{n+1} \leq u_{n+2}, \forall n \in N\).

Now let \(u = \sup_{n \in N} u_n\) and \(K = \bigcap_{n \in N} K_n\). It is clear that
\[
K = \{x \in X : u(x) = 0\} \neq \emptyset, \ \forall x \in X \text{ and } u(x) \geq 1, \ \forall x \in X \setminus K.
\]
Moreover, \(\mu(u_n) \leq u_n(x)\), \(\mu(u) \leq u(x)\) if \(x \in X\) and \(\mu \in \{v \in M_+(X) : v \leq \varepsilon x\}\). On the other hand, the relations
\[
\mu(X \setminus K) \leq \mu(u) \leq u(x) = 0, \ \forall x \in K \text{ and } \mu \in \{v \in M_+(X) : v \leq \varepsilon x\}
\]
show that \(K\) is a \(S\)-absorbing set in \(X\), and \(K \subset G_0 \subset X\) implies that \(K \cap \partial_S X \neq \emptyset\). Hence,
\[
K \cap \partial_S X \subseteq \bigcap_{n \in N} (G_n \cap G_0) \cap \partial_S X \subseteq G \cap \left(\bigcap_{n \in N} G_n\right) \cap \partial_S X.
\]
This ends the proof. \(\square\)

**Definition 2.1.** A real function \(s\) on \(X\) is called strictly \(S\)-concave in \(x \in X\) if it has the next properties:

(i) \(\mu(s) \leq s(x)\) whenever \(\mu \leq \varepsilon x\);

(ii) if \(\mu \leq \varepsilon x\) and \(\mu = s(x)\), then \(\mu = \varepsilon x\).

**Theorem 2.3.** If there exists a lower semicontinuous function \(s\), bounded from below and strictly \(S\)-concave in any \(x \in X_1\), then
\[
\partial_S X = \{x \in X_1 : Q_S(-s)(x) = -s(x)\}.
\]
Whenever \(X\) is metrizable and \(S\) is any convex cone of real, continuous functions on \(X\), then there exists at least a real, continuous and strictly \(S\)-concave function in every \(x \in X_1\). In all these cases, \(\partial_S X\) is a \(G_S\)-set.

**Proof.** If \(x \in X_1\) and \(Q_S(-s)(x) = -s(x)\), then \(\mu \leq \varepsilon x\) implies that \(\mu(-s) \leq Q_S(-s)(x) = -s(x)\). Hence \(s(x) \leq \mu(s)\), and because \(\mu \leq \varepsilon x\) it follows that \(\mu(s) = s(x)\), that is, \(\mu = \varepsilon x\). Let now \(X\) be metrizable and \(S\) be any convex cone of real continuous functions on \(X\). If one considers again \(C(X)\) being the usual Banach space of all real and continuous functions on \(X\) equipped with the topology induced by the supremum norm \(\|\cdot\|\), then there exists a countable set \(A = \{s_n : n \in N\} \subset S_1 \cap C(X)\) and the function \(s : X \to R\) defined by \(s = \sum_{n \in N} \frac{1}{2^n} \cdot \frac{s_n}{\|s_n\|}\) is strictly \(S\)-concave on \(X\).

Since \(\partial_S X = \{x \in X_1 : Q_S(-s)(x) = -s(x)\}\) and the function \(Q(-s)\) is upper semicontinuous, one concludes that
\[
\partial_S X = X_1 \cap \bigcap_{n=1}^{\infty} \left\{x \in X : Q_S(-s)(x) < -s(x) + \frac{1}{n}\right\}
\]
as claimed. \(\square\)
Theorem 2.4. (i) $\partial S X$ is a compact topological space with respect to Choquet’s topology; (ii) the set $\partial S X \cap \{x \in X : s (x) \leq 0\}$ is compact with respect to the Choquet topology, for each $s \in S$; (iii) $\partial S X$ is closed if and only if the Choquet topology is separated.

Proof. (i) It is straightforward because the family of all non-empty $S$-absorbent subsets of $X_1$ is closed with respect to the intersection of the decreasing nets containing such as theses sets; (ii) Let $(A_\alpha)_{\alpha \in I}$ be any decreasing net of non-empty $S$-absorbent subsets in $X$ such that $A_\alpha \cap \partial S X \cap \{x \in X : s(x) \leq 0\} \neq \emptyset$, with arbitrary $s \in S$. Since every set $\{x \in X : s(x) \leq 0\} (s \in S)$ is closed, it follows that

$$K = \bigcap_{\alpha \in I} A_\alpha \cap \{x \in X : s(x) \leq 0\} \neq \emptyset.$$  

If one assumes that $K \cap \partial S X = \emptyset$, then $K \cap \partial S_A = \emptyset$, where $S_A = \{s/A : s \in S\}$. Consequently, $s(x) > 0, \forall s \in \partial S_A A$, that is, $s(x) > 0, \forall x \in \bigcap_{\alpha \in I} A_\alpha$, in contradiction with the definition of $K$. The result follows; (iii) The family of all the sets $\{x \in X : s(x) > 0\} (s \in S_1)$ is a base for the topology on $X$, because $X$ is compact. If $\partial S X$ is closed, then it is obvious that the Choquet topology is Hausdorff separated, since it coincides with the trace of the topology of $X$ on $\partial S X$. Conversely, if Choquet’s topology is Hausdorff separated then, by virtue of the above two theorems, one obtains that any set $\partial S X \cap \{x \in X : s(x) \leq 0\}, (s \in S)$ is closed in this topology and because $\partial S X$ is compact with respect to the Choquet topology, it follows that $\partial S X$ is closed. This completes the proof. □

Corollary 2.4.1. If $X$ is an arbitrary, non-empty, compact and convex set in any Hausdorff locally convex space, and $S$ is the convex cone of all real continuous and concave functions on $X$, then $\partial S X$ is closed if and only if the Choquet topology is separated.

Some examples of Choquet boundaries accompanied by adjusted details and comments are indicated below.

Example 2.1. If $X$ is any non-empty, compact and convex subset of every Hausdorff locally convex space and $S = \{f : X \to R/f$ is continuous and concave\}, then its Choquet boundary with respect to $S$ coincides with the set ex($X$) of all extreme points $x$ of $X$, that is, if $y, z \in X$ and there exists $\alpha \in (0, 1)$ with $\alpha y + (1 - \alpha) z = x$, then $y = z = x$.

Example 2.2. Let $X$ be an arbitrary, compact, convex subset of any Hausdorff locally convex space, and $C(X)$ be the Banach space of all real-valued continuous functions on $X$, endowed with the natural supremum norm topology and the usual order.

If $T : C(X) \to C(X)$ is the natural positive projection and $Y = T [C(X)]$, then $T$ is called an Altomare projection [2] iff the space of all continuous affine functions on $X$ is contained in $Y$ and $f_{i, \alpha} \in Y$ whenever $f \in Y, t \in X$ and $\alpha \in [0, 1]$, where $f_{i, \alpha} (x) = f [\alpha x + (1 - \alpha) t], x \in X$. In this context, we have $\partial Y X = \{x \in X : T f (x) = f (x), \forall f \in C(X)\}$ and, geometrically, $\partial Y X$ can be viewed as the union of the corresponding faces for $X$. Therefore, ex($X$) $\subseteq \partial Y X \subseteq \text{Frt}(X)$, where Frt($X$) represents the usual topological boundary of $X$, and $T f$ is the unique function of $Y$ which coincides with $f$ on $\partial Y X$ for every $f \in C(X)$. With respect to every Altomare projection $T$, any continuous function $\varphi : X \to [0, 1]$ and the probability Radon measure $\mu_\varphi x$ on $X$ given by $\mu_\varphi x (f) = \varphi (x) T f (x) + [1 - \varphi (x)] f (x) (x \in X, f \in C(X))$, the Lototsky–Schnabl operators considered in [1–5] were defined as

$$L_{n, \varphi} f (x) = \int_X f \left( \sum_{i=1}^{n} t_i \right) \text{d} \mu_\varphi x (t_1) \ldots \text{d} \mu_\varphi x (t_n), \quad \forall f \in C(X), x \in X, n \in \mathbb{N}^*.$$  

Hence, $L_{n, \varphi} Y = y, \forall y \in Y, L_{n, \varphi} f = f, \forall f \in \partial Y X$ and $\lim_{n \to \infty} L_{n, \varphi} f = f, \forall f \in C(X)$.

Example 2.3. In the environment of the final part for Example 2.2, following [2,3], let us consider the infinitesimal generator $A_\varphi : D (A_\varphi) \to C(X)$ of the attached Feller semigroup $(T_\varphi (t))_{t \geq 0}$, which coincides with the closure of the operator $W_\varphi : D (W_\varphi) \to C(X)$ defined by

$$W_\varphi f = \lim_{n \to \infty} n \left( L_{n, \varphi} f - f \right), \quad \forall f \in D (W_\varphi) = \left\{ f \in C(X) : \exists \lim_{n \to \infty} n \left( L_{n, \varphi} f - f \right) \text{ in } C(X) \right\}.$$
Then, \( \partial_Y X = \{ x \in X : A_{\varphi} f (x) = 0, \forall f \in D(A_{\varphi}) \} \) [5]; that is, from the Markov processes point of view [20], \( \partial_Y X \) describes the set of all trop points. Moreover, in all finite \( p \)-dimensional cases, using [4] one obtains

\[
\partial_Y X = \{ x \in X : T e (x) = e (x) \} = \bigcap_{i=1}^{p} \{ x \in X : T h_i^2 (x) = h_i^2 (x) \}
\]

with \( T = \lim_{t \to \infty} T_{\varphi} (t) \), \( e = \sum_{i=1}^{p} h_i^2 \) and \( h_i (x_1, x_2, \ldots, x_p) = x_i, \forall i \in \{1,2,\ldots,p\}, (x_1, x_2, \ldots, x_p) \in X \).

In accordance with [37], if \( X = \{ x \in R^p : \| x \| \leq 1 \} \) \( T : C(X) \to C(X) \) is the Altomare projection with \( T f (f \in C(X)) \) being the unique harmonic function on \( \text{int}(X) \), which coincides with \( f \) on \( \text{Fr}(X) \) and \( \Delta \) is the Laplace operator, then

\[
A_{\varphi} f (x) = \varphi (x) \frac{1 - \| x \|^2}{2p} \Delta f (x), \quad \forall f \in C^2 (X).
\]

Consequently,

\[
\partial_Y X = \{ x \in X : \varphi (x) \frac{1 - \| x \|^2}{2p} \Delta f (x) = 0, \forall f \in C^2 (X) \}.
\]

When \( X = [0,1], \) \( T f (x) = (1 - x) f (0) + xf (1), \forall x \in [0,1], f \in C ([0,1]) \) and \( \varphi : [0,1] \to [0,1] \) is an arbitrary continuous function, then the following definition introduces a new concept of approximate efficiency, which generalizes the well-known notion of Pareto efficiency.

**Definition 3.1.** Let \( E \) be a vector space ordered by a convex cone, \( K, K_1 \) a non-void subset of \( K \), and \( A \) a non-empty subset of \( E \). The following definition introduces a new concept of approximate efficiency, which generalizes the well-known notion of Pareto efficiency.

\[
L_{n,\varphi} f (x) = \sum_{k=0}^{n} \sum_{i=0}^{n-k} \left( \begin{array}{c} n \\ k \\ i \end{array} \right) \varphi^k (x) [1 - \varphi (x)]^{n-k} x^i (1 - x)^{k-i} f \left[ \frac{i}{n} + \left( 1 - \frac{k}{n} \right) x \right], \quad n \in N^*.
\]

\[
f \in C(X), \quad A_{\varphi} f (x) = \varphi (x) \frac{x (1 - x)}{2} f'' (x) \quad \text{and}
\]

\[
D(A_{\varphi}) = \left\{ g \in C^2 ([0,1]) : \exists \lim_{x \to 0,1} x (1 - x) g'' (x) = 0 \right\}.
\]

Therefore,

\[
\partial_Y X = \{ x \in X : \varphi (x) \frac{x (1 - x)}{2} f'' (x) = 0, \forall f \in D(A_{\varphi}) \}.
\]

### 3. Efficiency and recent related topics

Let \( E \) be a vector space ordered by a convex cone, \( K, K_1 \) a non-void subset of \( K \), and \( A \) a non-empty subset of \( E \). The following definition introduces a new concept of approximate efficiency, which generalizes the well-known notion of Pareto efficiency.

**Definition 3.1.** We say that \( a_0 \in A \) is a \( K_1 \)-Pareto (minimal) efficient point of \( A \), in notation, \( a_0 \in \text{eff} (A, K, K_1) \) (or \( a_0 \in \text{MIN}_{K+K_1} (A) \)) if it satisfies one of the following equivalent conditions:

(i) \( A \cap (a_0 - K - K_1) \subseteq a_0 + K + K_1 \);

(ii) \( (K + K_1) \cap (a_0 - A) \subseteq -(K + K_1) \).

In a similar manner, one defines the Pareto (maximal) efficient points by replacing \( K + K_1 \) with \( -(K + K_1) \).

Clearly,

\[
A \cap (a_0 - K) \subseteq a_0 + K_1 \Rightarrow A \cap (a_0 - K - K_1) \subseteq a_0 + K + K_1 \Rightarrow A \cap (a_0 - K_1) \subseteq a_0 + K,
\]

which suggests other possible kinds of concepts for the approximate efficiency in ordered linear spaces.

**Remark 3.1.** \( a_0 \in \text{eff} (A, K, K_1) \) iff it is a fixed point for the multifunction \( F : A \to A \) defined by \( F (t) = \{ a \in A : A \cap (a - K - K_1) \subseteq t + K + K_1 \} \).
Consequently, for the existence of Pareto type efficient points, appropriate fixed point theorems concerning the multi-functions can be applied (see, for instance, [8,42] and any other appropriate scientific paper).

**Remark 3.2.** In [26], it is shown that whenever \( K_1 \subset K \setminus \{ 0 \} \), the existence of this new type of efficient points for bounded from below sets characterizes the semi-Archimedian ordered vector spaces and the regular ordered locally convex spaces.

**Remark 3.3.** When \( K \) is pointed, that is, \( K \cap (-K) = \{ 0 \} \), then \( a_0 \in \text{eff}(A, K, K_1) \) means that \( A \cap (a_0 - K - K_1) = \emptyset \) or, equivalently, \( (K + K_1) \cap (a_0 - A) = \emptyset \) for \( 0 \notin K_1 \) and \( A \cap (a_0 - K - K_1) = \{ a_0 \} \), respectively, if \( 0 \in K_1 \).

Whenever \( K_1 = \{ 0 \} \), from **Definition 3.1**, one obtains the usual concept of efficient (Pareto minimal, optimal or admissible) point: \( a_0 \in \text{eff}(A, K) \) (or \( a_0 \in \text{MIN}_K(A) \)) if it fulfils (i), (ii) or any of the next equivalent properties:

(iii) \( (A - K) \cap (a_0 - K) \subseteq a_0 + K \);

(iv) \( K \cap (a_0 - A - K) \subseteq -K \).

This shows that \( a_0 \) is a fixed point for at least one of the following multifunctions:

\[
F_1 : A \to A, \quad F_1(t) = \{ \alpha \in A : A \cap (\alpha - K) \subseteq t + K \},
\]

\[
F_2 : A \to A, \quad F_2(t) = \{ \alpha \in A : A \cap (t - K) \subseteq \alpha + K \},
\]

\[
F_3 : A \to A, \quad F_3(t) = \{ \alpha \in A : (A + K) \cap (\alpha - K) \subseteq t + K \},
\]

\[
F_4 : A \to A, \quad F_4(t) = \{ \alpha \in A : (A + K) \cap (t - K) \subseteq \alpha + K \}.
\]

That is, \( a_0 \in F_i(a_0) \) for some \( i = 1, 4 \). If, in addition, \( K \) is pointed, then \( a_0 \in A \) is an efficient point of \( A \) with respect to \( K \) if and only if one of the following equivalent relations holds:

(v) \( A \cap (a_0 - K) = \{ a_0 \} \)

(vi) \( A \cap (a_0 - K \setminus \{ 0 \}) = \emptyset \);

(vii) \( K \cap (a_0 - A) = \{ 0 \} \);

(viii) \( (K \setminus \{ 0 \}) \cap (a_0 - A) = \emptyset \);

(ix) \( (A + K) \cap (a_0 - K \setminus \{ 0 \}) = \emptyset \)

and we notice that \( \text{eff}(A, K) = \bigcap_{0 \neq K_2 \subseteq K} \text{eff}(A, K, K_2) \). Moreover, \( a_0 \in \text{eff}(A, K) \) iff it is a critical (equilibrium) point [13,14] for the generalized dynamical system \( \Gamma : A \to 2^A \) defined by \( \Gamma(a) = A \cap (a - K), a \in A \).

Thus, \( \text{eff}(A, K) \) describes a state of equilibrium for \( \Gamma \), and the ideal equilibria are contained in this set. Taking \( K_1 = \{ \varepsilon \} (\varepsilon \in K \setminus \{ 0 \}) \), it follows that \( a_0 \in \text{eff}(A, K, K_1) \) iff \( A \cap (a_0 - \varepsilon - K) = \emptyset \). In all these cases, the set \( \text{eff}(A, K, K_1) \) is denoted by \( \varepsilon \text{-eff}(A, K) \), and it is obvious that \( \text{eff}(A, K) = \bigcap_{\varepsilon \in K \setminus \{ 0 \}} [\varepsilon \text{-eff}(A, K)] \). Concerning existence results on the efficient points and significant properties for the efficient points sets, we suggest [7,13–19,22,23,26,27,30–34,38–41].

The following theorem offers the first immediate connection between strong optimization and this kind of approximate efficiency, in the environment of ordered vector spaces.

**Theorem 3.1.** If we denote by \( S(A, K, K_1) = \{ a_1 \in A : A \subseteq a_1 + K + K_1 \} \) and \( S(A, K, K_1) \neq \emptyset \), then \( S(A, K, K_1) = \text{eff}(A, K, K_1) \).

**Proof.** Clearly, \( S(A, K, K_1) \subseteq \text{eff}(A, K, K_1) \).

Indeed, if \( a_0 \in S(A, K, K_1) \) and \( a \in A \cap (a_0 - K - K_1) \) are arbitrary elements, then \( a \in a_0 + K + K_1 \); that is, \( a_0 \in \text{eff}(A, K, K_1) \), by virtue of the point (i) contained in the **Definition 3.1**. Suppose now that \( \bar{a} \in S(A, K, K_1) \neq \emptyset \), and there exists \( a_0 \in \text{eff}(A, K, K_1) \) \( \setminus S(A, K, K_1) \). Out of the fact \( \bar{a} \in S(A, K, K_1) \), it follows that \( a_0 \in \bar{a} + K + K_1 \), that is, \( a_0 \in a_0 - K - K_1 \), from which, since \( \bar{a} \in A \) and \( a_0 \in \text{eff}(A, K, K_1) \), we conclude that \( \bar{a} \in a_0 + K + K_1 \).

Therefore, \( A \subseteq \bar{a} + K + K_1 \subseteq a_0 + K + K_1 \), in contradiction with \( a_0 \notin S(A, K, K_1) \), as claimed. \( \square \)

**Remark 3.4.** We shall denote by \( S(A, K) \) the set \( S(A, K, \{ 0 \}) \). If \( S(A, K, K_1) \neq \emptyset \), then \( K + K_1 = K \), and hence \( \text{eff}(A, K, K_1) = \text{eff}(A, K) \). Indeed, let \( a \in S(A, K, K_1) \). Then, \( a \in a + K + K_1 \) which implies that \( 0 \in K + K_1 \). Therefore, \( K \subseteq K_1 + K + K = K_1 + K \subseteq K \).
The above theorem shows that, for any non-empty subset of an arbitrary vector space, the set of all strong minimal elements with respect to any convex cone through the agency of every non-void subset of it coincides with the corresponding set of the efficient points, whenever there exists at least a strong minimal element. Obviously, the result remains valid for the strong maximal elements and the corresponding efficient points, respectively.

Using this conclusion and our abstract construction given in [29] for splines in the \( H \)-locally convex spaces (introduced by Precupanu in [35] as separated locally convex spaces with any seminorm satisfying the parallelogram law, also studied in [21]), it follows that the only best simultaneous and vectorial approximation for each element in the direct sum of any (closed) linear subspace and its orthogonal, with respect to a linear (continuous) operator between two arbitrary \( H \)-locally convex spaces, is its spline function. We also note that it is possible to have \( S(A, K) = \emptyset \) and \( \text{eff}(A, K) = A \). Thus, for example, if one considers \( X = R^2 \) endowed with the separated locally convex topology generated by the seminorms \( p_1, p_2 : X \to R_+ , p_1(x, y) = |x| , p_2(x, y) = |y| , K = R^2_+ = \{(x, y) \in R^2 : x, y \geq 0 \} \), \( K_1 = \{(0, 0)\) and \( A = \{ (\lambda , 1 - \lambda ) : 0 \leq \lambda \leq 1 \} \), then it is clear that \( S(A, K) = \emptyset \) and \( \text{eff}(A, K) = A \).

In all our further considerations, we suppose that \( X \) is a Hausdorff locally convex space having the topology induced by family \( P = \{ p_\alpha : \alpha \in I \} \) of seminorms, ordered by a convex cone \( K \) and its topological dual space \( X^* \).

In this framework, the next theorem contains a significant criterion for the existence of the approximate efficient points; in particular, for the usual efficient points, taking into account that the dual cone of \( K \) is defined by \( K^* = \{ x^* \in X^* : x^*(x) \geq 0, \forall x \in X \} \) and its attached polar cone is \( K^0 = -K^* \).

**Theorem 3.2.** If \( A \) is any non-empty subset of \( X \) and \( K_1 \) is every non-void subset of \( K \), then \( a_0 \in \text{eff}(A, K, K_1) \) whenever for each \( p_\alpha \in P \) and \( \eta \in (0, 1) \) there exists \( x^* \) in the polar cone \( K^0 \) of \( K \) such that \( p_\alpha (a_0 - a) \leq x^*(a_0 - a) + \eta , \forall a \in A \).

**Proof.** Let us suppose that, under the above hypotheses, \( (K + K_1) \cap (a_0 - A) \not\subset -(K + K_1) \); that is, there exists \( a \in A \) so that \( a_0 - a \in K + K_1 \setminus (-K - K_1) \). Then, \( a_0 - a \neq 0 \) and, because \( X \) is separated in Hausdorff’s sense, there exists \( p_\alpha \in P \) such that \( p_\alpha (a_0 - a) > 0 \). On the other hand, there exists \( n \in N^* \) sufficiently large with \( p_\alpha (a_0 - a)/n \in (0, 1) \), and the relation given by the hypothesis of the theorem leads to \( p_\alpha (a_0 - a) \leq x^*(a_0 - a) + p_\alpha (a_0 - a)/n \) with \( x^* \in K^0 \) and \( n \to \infty \), which implies that \( p_\alpha (a_0 - a) \leq 0 \), a contradiction, and the proof is completed. \( \square \)

**Remark 3.5.** The above theorem represents an immediate extension of Precupanu’s result given in Proposition 1.2 of [36]. In general, the converse of this theorem is not valid, at least in (partially) ordered separated locally convex spaces, as we can see from the example considered in Remark 3.4. Indeed, if one assumes the contrary in the corresponding mathematical background then, taking \( \eta = \frac{1}{4} \), it follows that for each \( \lambda_0 \in [0, 1] \), there exists \( c_1, c_2 \leq 0 \) such that \( |\lambda_0 - \lambda| \leq (c_1 - c_2) (\lambda_0 - \lambda) + \frac{1}{4} , \forall \lambda \in [0, 1] \).

Taking \( \lambda_0 = \frac{1}{4} \), one obtains \( |1 - 4\lambda| \leq (c_1 - c_2) (1 - 4\lambda) + 1 , \forall \lambda \in [0, 1] \), which for \( \lambda = 0 \) implies that \( c_2 \leq c_1 \) and for \( \lambda = \frac{1}{2} \) leads to \( c_1 \leq c_2 \); that is, \( |1 - 4\lambda| \leq 1 , \forall \lambda \in [0, 1] \), a contradiction.

**Remark 3.6.** If \( a_0 \in A \) and for every \( p_\alpha \in P , \eta \in (0, 1) \) there exists \( x^* \in K^0 \) such that \( p_\alpha (a_0 - a) \leq x^*(a_0 - a) + \eta , \forall a \in A \), then \( K \cap (a_0 - A) = \{ 0 \} \) even if \( K \) is not pointed. Indeed, if \( x \in K \cap (a_0 - A) \), then \( a_0 - x \in A \), and for each \( p_\alpha \in P \) and \( \eta \in (0, 1) \) there exists \( x^* \in K^0 \) with \( p_\alpha (x) = p_\alpha (a_0 - (a_0 - x)) \leq x^*(x) + \eta \leq \eta \). Because \( \eta \) is arbitrarily chosen in \( (0, 1) \), we obtain \( p_\alpha (x) = 0 \), and since \( X \) is separated, it follows that \( x = 0 \). If \( 0 \in K + K_1 \), then \( K + K_1 = K \) and \( 0 \not\in K + K_1 \) implies that \( (K + K_1) \cap (a_0 - A) = \emptyset \). Consequently, \( a_0 \in \text{eff}(A, K, K_1) \) in both cases, and in this way we indicate also another proof of the theorem.

The beginning and the considerations of Section 4 in [19] suggested us to consider, for each function \( \varphi : P \to K^* \setminus \{ 0 \} \), the full nucleus cone \( K_\varphi = \{ x \in X : p(x) \leq \varphi (p)(x) , \forall p \in P \} \) in order to give the next generalization of Theorem 7 [19].

**Theorem 3.3.** If \( 0 \in K_1 \) and there exists \( \varphi : P \to K^* \setminus \{ 0 \} \) with \( K \subseteq K_\varphi \), then

\[
\text{eff}(A, K, K_1) = \bigcup_{a \in A \cap (a - K - K_1) \neq \emptyset} S(A \cap (a - K - K_1) , K_\varphi)
\]

for any non-empty subset \( K_1 \) of \( K \).
Proof. If $a_0 \in \text{eff}(A, K, K_1)$ is an arbitrary element, then, in accordance with the first point (i) of Definition 3.1 and the hypothesis of the above theorem, we have $A \cap (a_0 - K - K_1) - a_0 \subseteq K + K_1 \subseteq K \subseteq K_{\varphi}$ for some $\varphi : P \to K^* \setminus \{0\}$. Therefore, $a_0 \in S(A \cap (a_0 - K - K_1), K_{\varphi})$. Hence, $\text{eff}(A, K, K_1) \subseteq \bigcup_{a \in A, \varphi : P \to K^* \setminus \{0\}} S(A \cap (a_0 - K - K_1), K_{\varphi})$.

Conversely, let now $a_1 \in S(A \cap (a_0 - K - K_1), K_{\varphi})$ for at least two elements $a_0 \in A$ and $\varphi : P \to K^* \setminus \{0\}$. Then, $a_1 \in A \cap (a_0 - K - K_1)$, and $A \cap (a_0 - K - K_1) - a_1 \subseteq K_{\varphi}$, that is, $p(a_1) \leq p(a_1 - a) + \eta, \forall a \in A \cap (a_0 - K - K_1), p \in P$, which implies immediately that $p(a_1 - a) \leq -\varphi(p)(a_1 - a) + \eta, \forall a \in A \cap (a_0 - K - K_1), p \in P, \eta \in (0, 1)$, and, by virtue of Theorem 3.2, one obtains that $a_1 \in \text{eff}(A \cap (a_0 - K - K_1), K_{\varphi})$. But $\text{eff}(A \cap (a_0 - K - K_1), K, K_1) \subseteq \text{eff}(A, K, K_1)$.

Indeed, for any $t \in (A \cap (a_0 - K - K_1), K, K_1)$, and $h \in A \cap (t - K - K_1)$, we have $h \in A \cap (a_0 - K - K_1) \cap (t - K - K_1) \subseteq t + K + K_1$; that is, $A \cap (t - K - K_1) \subseteq t + K + K_1$, and by the point (i) of Definition 3.1, it follows that $t \in \text{eff}(A, K, K_1)$. This completes the proof. □

Remark 3.7. If $0 \notin K_1$, then $a_0 \in \text{eff}(A, K, K_1)$ implies that $A \cap (a_0 - K - K_1) = \emptyset$. Therefore, it is not possible to have $a_0 \in S(\varphi, K_{\varphi})$. In case of $0 \in K_1$, then $\text{eff}(A, K, K_1) = \text{eff}(A, K)$ and $a_0 \in \text{eff}(A, K)$ iff $A \cap (a_0 - K) = \{a_0\}$, so in the right member of the first proved inclusion, any convex cone can be selected, not just necessarily $K_{\varphi}$. The hypothesis $K \subseteq K_{\varphi}$ imposed upon the convex cone $K$ is automatically satisfied whenever $K$ is a supernormal (nuclear) cone [13–19], and it was used only to prove the inclusion $\text{eff}(A, K, K_1) \subseteq \bigcup_{a \in A, \varphi : P \to K^* \setminus \{0\}} S(A \cap (a - K - K_1), K_{\varphi})$. Moreover, $K$ is supernormal if and only if there exists $\varphi : P \to K^* \setminus \{0\}$ such that $K \subseteq K_{\varphi}$. Indeed, Lemma 5 in [19] ensures the necessity of the above inclusion condition. Conversely, since for every seminorm $p \in P$ there exists $\varphi(p) \in K^* \setminus \{0\}$ and for any $x \in K \subseteq K_{\varphi}$, it follows that $p(x) \leq \varphi(p)(x)$; we conclude the nuclearity of $K$. When $K$ is an arbitrary pointed convex cone, $A$ is a non-empty subset of $X$ and $a_0 \in \text{eff}(A, K)$, then, by virtue of (v) in Remark 3.3, we have $A \cap (a_0 - K) = \{a_0\}$; that is, $A \cap (a_0 - K) - a_0 = \{0\} \subset K_{\varphi}$.

Corollary 3.3.1. For every non-empty subset $A$ of any Hausdorff locally convex space ordered by an arbitrary, pointed convex cone $K$ with its dual cone $K^*$ we have

$$\text{eff}(A, K) = \bigcup_{a \in A, \varphi : P \to K^* \setminus \{0\}} S(A \cap (a - K), K_{\varphi}).$$

Remark 3.8. The hypothesis of Theorem 3.3, together with Lemma 3 in [19], involves that $K$ is pointed. Consequently, $0 \notin K_1$ iff $0 \in K + K_1$. If $a_0 \in S(A \cap (a - K - K_1), K_{\varphi})$ for some $\varphi : P \to K^*$ and $a \in A$ with $a_0 = a - k - l - k_1$, $k, k_1 \in K_1$, then $K \cap (a - A) = \{a_0\}$ because $A \cap (a - K - K_1) \subseteq a_0 + K_{\varphi}$ in any such a case as this. Indeed, let $x \in K \cap (a - A)$ be an arbitrary element. Then, $a_0 - x \in A$ and $a_0 - x = a - k - l - x \in a - K - K_1$. Therefore, $a_0 - x \in a_0 + K_{\varphi}$; that is, $-x \in K_{\varphi}$. For every $p_a \in P$, we have $p_a(-x) \leq \varphi(p_a)(-x) = -\varphi(p_a)(x) \leq 0$. Since $p_a$ was arbitrary chosen in $P$ and $X$ is a Hausdorff locally convex space, it follows that $x = 0$.

Remark 3.9. Clearly, the announced theorem represents a significant result concerning the possibilities of scalarization for the study of some Pareto efficiency programs in separated locally convex spaces, as we can see also in the final comments of [19] for the particular cases of Hausdorff locally convex spaces ordered by closed, pointed and normal cones.

Remark 3.10. As an open problem, it is interesting to replace $K_1$ with any non-empty subset of an ordered linear space $X$, under proper hypotheses.

Definition 3.2. A real function $f : X \to R$ is called ($K + K_1$)-increasing if $f(x_1) \geq f(x_2)$ whenever $x_1, x_2 \in X$ and $x_1 \in x_2 + K + K_1$.

Every real increasing function defined on any linear space ordered by an arbitrary convex cone $K$ is $K + K_1$-increasing, for each non-empty subset $K_1$ of $K$. 

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Finally, we present the next coincidence of the approximate efficient points sets and the Choquet boundaries, which generalizes our main result given in [7], and cannot be obtained as a consequence of the Axiomatic Potential Theory.

**Theorem 3.4.** If $A$ is any non-void, compact subset of $X$ and:

(i) $K$ is an arbitrary, closed, convex, pointed cone in $X$;
(ii) $K_1$ is a non-empty subset of $K$ such that $K + K_1$ is closed with respect to the Hausdorff separated locally convex topology on $X$.

Then, $\text{eff}(A, K, K_1)$ coincides with the Choquet boundary of $A$ with respect to $K + K_1$-increasing real continuous functions on $A$. Consequently, the set $\text{eff}(A, K, K_1)$ endowed with the corresponding trace topology is a Baire space and, if $(A, \tau_A)$ is metrizable, then $\text{eff}(A, K, K_1)$ is a $G_δ$-subset of $X$.

**Proof.** The Theorems 2.2 and 2.3 in Section 2 shows that the last part of the above conclusion is a consequence of the usual properties for Choquet boundaries, under the specified conditions.

Let $S_1 = \{ f \in C(A) : f = K + K_1\text{-increasing} \}$. Clearly, $S_1$ is a convex cone which contains the constant functions on $A$, it is min-stable, and it separates the points of $A$. If $a^* \in \partial S_1 A$ and $a' \in A$ with $a^* \in a' + K_1 + K$, then $s(a^*) \geq s(a')$ for all $s \in S_1$; therefore $\varepsilon_a(a') \leq \varepsilon_a(a^*)$. Because $\varepsilon_a(a^*)$ is minimal with respect to “$\leq_{S_1}$”, one deduces $\varepsilon_a(a') = \varepsilon_a(a^*)$; that is, $a^* = a'$ (if $a^* \neq a'$ then, from the fact that $S_1$ separates the points of $A$, it would follow that there exists $f \in S_1$ with $f(a^*) \neq f(a')$ in contradiction with the equality $\varepsilon_a(a^*) = \varepsilon_a(a')$, which means that $f(a^*) = f(a')$, $\forall f \in C(A))$. This is possible if $0 \in K_1$ and a contradiction when $0 \notin K_1$. Therefore, $a^* \in \text{eff}(A, K, K_1)$ and the inclusion $\partial S_1 A \subseteq \text{eff}(A, K, K_1)$ is proved. □

For the converse inclusion, let $f \in C(A)$ be arbitrary and $\widetilde{f} : A \to R$ be defined by $\widetilde{f}(a) = \sup \{ f(a') : a' = a \text{ or } a' \in A \text{ and } a \in a' + K + K_1 \}$. It is obvious that $\widetilde{f}$ is $K + K_1$-increasing, and since for any $a \in A$ the set $\{ a' \in A : a' = a \text{ or } a' \in A \text{ and } a \in a' + K + K_1 \}$ is compact, it follows that there exists $a_0 \in A$ with $a \in a_0 + K + K_1$ such that $\widetilde{f}(a) = f(a_0)$. Moreover, $\widetilde{f}$ is upper semicontinuous (see also, for example, Lemma 4 in the Appendix of [25]). We have $f \leq \widetilde{f}$ and $f \leq g$ for any function $g \in S_1$ with $f \leq g$, because

$$\widetilde{f}(a) = \sup \{ f(a') : a' = a \text{ or } a' \in A \text{ and } a \in a' + K + K_1 \} \leq \sup \{ g(a') : a' = a \text{ or } a' \in A \text{ and } a \in a' + K + K_1 \} = g(a), \quad \forall a \in A.$$

Particularly, for any $s \in S_1$ with $s \geq f$, we have $s \geq \widetilde{f}$ and $Q_{S_1} f \geq \widetilde{f} \geq f$. On the other hand, by virtue of Nachbin’s Theorem 3 in the Appendix of [25], one deduces $Q_{S_1} \widetilde{f} = \widetilde{f}$.

Since $f \leq \widetilde{f}$, we have $\widetilde{f} \leq Q_{S_1} f \leq Q_{S_1} \widetilde{f} = \widetilde{f}$. Hence, $Q_{S_1} f = \widetilde{f}, \forall f \in C(A)$, which implies that $\partial S_1 A = \{ a \in A : f(a) = \widetilde{f}(a), \forall f \in C(A) \}$.

Let $a_0 \in \text{eff}(A, K, K_1)$, and $f \in C(A)$. We have

$$Q_{S_1} f(a_0) = \widetilde{f}(a_0) = \sup \{ f(a) : a = a_0 \text{ or } a \in A \text{ and } a \in a + K + K_1 \} = f(a_0).$$

Therefore, $a_0 \in \partial S_1 A$ and $\text{eff}(A, K, K_1) \subseteq \partial S_1 A$.

Thus we proved that $\text{eff}(A, K, K_1) = \{ a \in A : f(a) = \widetilde{f}(a), \forall f \in C(A) \} = \partial S_1 A$.

**Corollary 3.4.1.**

(i) $\text{eff}(A, K, K_1) = \{ a \in A : f(a) = \sup \{ f(a') : a' \in A \cap (a - K - K_1) \} \text{ for all } f \in C(A) \}$;
(ii) $\text{eff}(A, K, K_1)$ and $\text{eff}(A, K, K_1) \cap \{ a \in A : s(a) \leq 0 \text{ (} s \in S \text{) are compact sets with respect to Choquet’s topology;}
(iii) $\text{eff}(A, K, K_1)$ is a compact subset of $A$.

**Remark 3.11.** In general, $\text{eff}(A, K, K_1)$ coincides with the Choquet boundary of $A$ only with respect to the convex cone of all real, continuous and $K + K_1$-increasing functions in $A$. Thus, for example, if $A$ is a non-empty, compact and convex subset of $X$, then, taking into account the Example 2.1, the Choquet boundary of $A$ with respect to the convex cone of all real, continuous and concave functions on $A$ coincides with the set of all extreme points for $A$. But, it is easy to see that, even in finite dimensional cases, an extreme point for a compact convex set is not necessarily an efficient point and conversely.
Remark 3.12. As we have already specified before Theorem 3.1, there exist more general conditions than compactness imposed upon a non-empty set $A$ in a separated locally convex space ordered by a convex cone $K$ ensuring that $\text{eff} (A, K) \neq \emptyset$. Perhaps our coincidence result suggests a natural extension of the Choquet boundary, at least in these cases. Anyhow, Theorem 3.4 represents an important link between Vector Optimization and Potential Theory, and a new way for the study of the properties of efficient points sets and the Choquet boundaries. Indeed, one of the main question in Potential Theory is to find the Choquet boundaries. This fact is relatively easy for particular cases but, in general, it is an unsolved problem. Since in a lot of cases the efficient points sets contain dense subsets which can be identified by adequate optimization methods, it is possible to determine the corresponding Choquet boundaries in all these situations. Consequently, our coincidence result has its practical consequences, at first for the Axiomatic Theory of Potentials.

4. Some open problems

The above context of research suggests the following open problems:

3.1. If $\text{eff} (A, K) \neq \emptyset$, there exist a Hausdorff locally convex space $Y$, a supernormal cone [14] $K_0$ in $Y$, and a non-empty set $A_0 \subset Y$ with $\text{eff} (A, K) = \text{eff} (A_0, K_0)$ (or, at least, $\text{eff} (A, K)$ is dense in $\text{eff} (A_0, K_0)$)?

3.2. If $\text{eff} (A, K) \neq \emptyset$, there exist a separated locally convex space $X_1$, a (pointed), convex cone $K_1$ in $X_1$, and a compact set $A_1 \subset X_1$ such that $\text{eff} (A, K) = \text{eff} (A_1, K_1)$ (or, at least, $\text{eff} (A, K)$ to be dense in $\text{eff} (A_1, K_1)$ or conversely)?

3.3. If $T$ is a Hilbert space, $K$ is a closed, convex, pointed cone in $T$, and $A$ is a non-empty closed, convex subset of $T$, then does $\text{eff} (A, K)$ preserve the property of coincidence with the corresponding Choquet boundary, as in the above theorem?

3.4. The same question can be asked in each of the following cases:

(i) $T$ quasi-complete locally convex space, $K$ supernormal, $A$ is a $K$-bounded and $K$-closed set in $T$ [16].

(ii) $T$ quasi-complete locally convex space, the closure $\overline{K}$ of $K$ has the properties given in [40,41] and $A$ is a $K$-bounded and $K$-closed subset in $T$.

References


