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Equilibrium problems in Hadamard manifolds

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ABSTRACT

An equilibrium theory is developed in Hadamard manifolds. The existence of equilibrium points for a bifunction is proved under suitable conditions, and applications to variational inequality, fixed point and Nash equilibrium problems are provided. The convergence of Picard iteration for firmly nonexpansive mappings along with the definition of resolvents for bifunctions in this setting is used to devise an algorithm to approximate equilibrium points.

1. Introduction

Let H be a Hilbert space, K a nonempty subset of H and $F : K \times K \to \mathbb{R}$ a bifunction. In [4,36] it was shown that a broad class of problems in optimization, such as variational inequality, convex minimization, fixed point and Nash equilibrium problems can be formulated as the equilibrium problem associated to the bifunction F and the set K

$$\text{find } x \in K \text{ such that } F(x, y) \geqslant 0, \ \forall y \in K. \tag{1.1}$$

A point $x \in K$ solving this problem is said to be an equilibrium point (called equilibria as well). Bearing in mind the numerous applications in physics, optimization and economic (e.g., see [4,3,7,18]) many techniques and algorithms have been devised to analyze the existence and approximation of a solution to equilibrium problems; see [18,21,22,10,9].

In most of the known results in equilibrium theory it is assumed that the set K is closed and convex and the bifunction F is convex in the second variable. Convexity seems to be an essential property, nevertheless, it happens that in certain problems in optimization and other applied areas convexity is a sufficient but not necessary condition to obtain significant results.

Inspired by the concept of convexity on a linear vector space the notion of geodesic convexity on some nonlinear metric spaces has become a successful tool in optimization; see [43] and [38]. Udriste in [43] introduced the theory of convex functions on Riemannian manifolds motivated by the fact that some constrained optimization problems can be seen as unconstrained ones from the Riemannian geometry point of view. In addition, another advantage is that optimization problems

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with nonconvex objective functions can be written as convex optimization problems by endowing the space with an appropriate Riemannian metric. To illustrate this, let us consider the following problem (see [38, p. 169]):

$$\min_{x\in K} f(x) \tag{1.2}$$

where K is the positive orthant $R^n_{++} = \{x \in \mathbb{R}^n\colon\ x_i > 0,\ i = 1, \ldots, n\}$ and $f : K \to \mathbb{R}$ is a nonconvex function defined as

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{m} c_i \prod_{j=1}^{n} x_j^{b_{ij}}, \tag{1.3}$$

where $c_i \in K$ and $b_{ij} \in \mathbb{R}$ for any i, j. By endowing K with the so-called affine metric (also called Dikin metric) defined by

$$G(x) = \begin{pmatrix} 1/x_1^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1/x_n^2 \end{pmatrix},$$

that is, for any $x \in K$, $u, v \in T_x K$,

$$\langle u, v\rangle_x = \big\langle G(x)u, v\big\rangle = \sum_{i=1}^{n} \frac{u_i v_i}{x_i^2},$$

it is well known that K with the affine metric is a Hadamard manifold with null sectional curvature whose tangent space at a point x is $\mathbb{R}^n$. The geodesic joining $x \in K$ to $y \in K$ is the curve $\gamma : [0, 1] \to K$ defined by

$$\gamma(t) = \big(x_1^{1-t} y_1^t, \ldots, x_n^{1-t} y_n^t\big),$$

with

$$\gamma'(t) = \left(x_1^{1-t} y_1^t \ln \frac{y_1}{x_1}, \ldots, x_n^{1-t} y_n^t \ln \frac{y_n}{x_n}\right).$$

Thus, the distance between x and y is

$$\mathrm{d}(x, y) = \left(\sum_{i=1}^{n} \left(\ln \frac{y_i}{x_i}\right)^2\right)^{1/2}.$$

It is easy to check that the function f is (geodesic) convex on K with respect to the affine metric. Then solving the nonconvex constrained problem (1.2) in $\mathbb{R}^n$ with the Euclidean metric is equivalent to solving the unconstrained convex minimization problem for f in the Hadamard manifold K with the affine metric.

These ideas have opened a new way to solve other related problems in optimization. In the study of these problems several classes of monotone vector fields have been introduced, along with some convergent iterative methods; see [43,38,17,23,11,39,37] and reference therein. For instance, in [11] examples of non-monotone vector fields which can be transformed into monotone by choosing an appropriate Riemannian metric were given. All this has been one of the motivations for researchers from different areas to extend concepts and techniques which fit in Euclidean spaces to Riemannian manifolds; see, for example, [23,34,1,29,44,28].

Riemannian manifolds constitute a broad and fruitful framework for the development of different fields. Actually, in the last decades concepts and techniques which fit in Euclidean spaces have extended to this nonlinear framework. Most of the extended methods, however, require the Riemannian manifold to have nonpositive sectional curvature. This is an important property which is enjoyed by a large class of Riemannian manifolds and it is strong enough to imply tight topological restrictions and rigidity phenomena (cf. [39]). Particularly, Hadamard manifolds, which are complete simply connected and finite-dimensional Riemannian manifolds of nonpositive sectional curvature, have turned out to be a suitable setting for diverse disciplines. Hadamard manifolds are examples of hyperbolic spaces and geodesic spaces, more precisely, a Busemann nonpositive curvature (NPC) space and a CAT(0) space; see [8,24,26].

Motivated by what we have mentioned previously, the purpose of this article is to develop an equilibrium theory in the nonlinear framework of Hadamard manifolds. The organization of the paper is as follows. In Section 2 some notations, concepts and results in Riemannian manifolds are presented. Although this preliminary section is similar to the one appearing in other recent papers dealing with the extension of some results from the setting of Hilbert spaces to Hadamard manifolds (see for instance [29,31]), its inclusion in this paper facilitates the reading of the present work. In Section 3 the existence of equilibrium points is proved under similar conditions required in the case of Euclidean spaces, where a counterpart of KKM Lemma is provided. Applications to mixed variational inequality, fixed point and Nash equilibrium problems are studied. In particular, the counterpart of Kakutani Fixed Point Theorem for set-valued mappings defined on a Hadamard manifold is proved. Section 4 is devoted to the approximation of equilibrium points. To this end, the convergence of Picard iteration for the class of firmly nonexpansive mappings is proved and the definition of the resolvent of a bifunction on a Hadamard manifold is introduced.

2. Preliminaries

The fundamental and basic knowledge needed for a comprehensive reading of this paper and included in this section can be found in the books on Riemannian geometry [12,41,43].

Let M be a simply connected m-dimensional manifold. Given $x \in M$, the tangent space of M at x is denoted by T_xM and the tangent bundle of M by $TM = \bigcup_{x\in M} T_xM$, which is naturally a manifold. A vector field A on M is a mapping of M into TM which associates to each point $x \in M$ a vector $A(x) \in T_xM$. We always assume that M can be endowed with a Riemannian metric to become a *Riemannian* manifold. We denote by $\langle\cdot,\cdot\rangle_x$ the scalar product on T_xM with the associated norm $\|\cdot\|_x$, where the subscript x will be omitted. Given a piecewise smooth curve $\gamma : [a,b] \to M$ joining x to y (i.e., $\gamma(a) = x$ and $\gamma(b) = y$), by using the metric, we can define the length of γ as $L(\gamma) = \int_a^b \|\gamma'(t)\|\,dt$. Then, for any $x, y \in M$ the Riemannian distance $\mathrm{d}(x, y)$, which induces the original topology on M, is defined by minimizing this length over the set of all such curves joining x to y.

Let ∇ be the Levi-Civita connection associated with $(M, \langle\,,\rangle)$. Let γ be a smooth curve in M. A vector field A is said to be parallel along γ if $\nabla_{\gamma'} A = 0$. If γ' itself is parallel along γ, we say that γ is a *geodesic*, and in this case $\|\gamma'\|$ is constant. When $\|\gamma'\| = 1$, γ is said to be normalized. A geodesic joining x to y in M is said to be minimal if its length equals $\mathrm{d}(x, y)$.

A Riemannian manifold is *complete* if for any $x \in M$ all geodesics emanating from x are defined for all $t \in \mathbb{R}$. By the Hopf–Rinow Theorem, we know that if M is complete then any pair of points in M can be joined by a minimal geodesic. Moreover, (M, d) is a complete metric space and bounded closed subsets are compact.

Assuming that M is complete, the *exponential map* $\exp_x : T_xM \to M$ at x is defined by $\exp_x v = \gamma_v(1, x)$ for each $v \in T_xM$, where $\gamma(\cdot) = \gamma_v(\cdot, x)$ is the geodesic starting at x with velocity v (i.e., $\gamma(0) = x$ and $\gamma'(0) = v$). Then $\exp_x tv = \gamma_v(t, x)$ for each real number t.

A complete simply connected Riemannian manifold of nonpositive sectional curvature is called a *Hadamard manifold*. Throughout the remainder of the paper, we always assume that M is an m-dimensional Hadamard manifold. The following result is well known and essential for our work.

Proposition 2.1. *(See [41].) Let $x \in M$. Then $\exp_x : T_xM \to M$ is a diffeomorphism, and for any two points $x, y \in M$ there exists a unique normalized geodesic joining x to y, $\gamma_{x,y}$, which is minimal.*

So from now on, when referring to the geodesic joining two points we mean the unique minimal normalized one. This proposition also says that M is diffeomorphic to the Euclidean space $\mathbb{R}^m$. Thus M has the same topology and differential structure as $\mathbb{R}^m$. Moreover, Hadamard manifolds and Euclidean spaces have similar geometrical properties. Some of them are described next.

Recall that a *geodesic triangle* $\Delta(x_1, x_2, x_3)$ of a Riemannian manifold is a set consisting of three points x_1, x_2, x_3 and three minimal geodesics joining these points.

Proposition 2.2 *(Comparison theorem for triangles). (See [41].) Let $\Delta(x_1, x_2, x_3)$ be a geodesic triangle. Denote, for each $i = 1, 2, 3 \pmod 3$, by $\gamma_i : [0, l_i] \to M$ the geodesic joining x_i to x_{i+1}, and set $\alpha_i := \angle(\gamma_i'(0), -\gamma_{i-1}'(l_{i-1}))$, the angle between the vectors $\gamma_i'(0)$ and $-\gamma_{i-1}'(l_{i-1})$, and $l_i := L(\gamma_i)$. Then*

$$\alpha_1 + \alpha_2 + \alpha_3 \leqslant \pi, \tag{2.1}$$

$$l_i^2 + l_{i+1}^2 - 2l_i l_{i+1} \cos\alpha_{i+1} \leqslant l_{i-1}^2. \tag{2.2}$$

In terms of the distance and the exponential map, inequality (2.2) can be rewritten as

$$\mathrm{d}^2(x_i, x_{i+1}) + \mathrm{d}^2(x_{i+1}, x_{i+2}) - 2\big\langle \exp_{x_{i+1}}^{-1} x_i, \exp_{x_{i+1}}^{-1} x_{i+2}\big\rangle \leqslant \mathrm{d}^2(x_{i-1}, x_i), \tag{2.3}$$

since

$$\big\langle \exp_{x_{i+1}}^{-1} x_i, \exp_{x_{i+1}}^{-1} x_{i+2}\big\rangle = \mathrm{d}(x_i, x_{i+1})\,\mathrm{d}(x_{i+1}, x_{i+2}) \cos\alpha_{i+1}.$$

Lemma 2.3. *(See [8].) Let $\Delta(x, y, z)$ be a geodesic triangle in a Hadamard manifold M. Then, there exist $x', y', z' \in \mathbb{R}^2$ such that*

$$\mathrm{d}(x, y) = \big\|x' - y'\big\|, \qquad \mathrm{d}(y, z) = \big\|y' - z'\big\|, \qquad \mathrm{d}(z, x) = \big\|z' - x'\big\|.$$

The triangle $\Delta(x', y', z')$ is called the *comparison triangle* of the geodesic triangle $\Delta(x, y, z)$, which is unique up to isometry of M. The next result shows the relation between a geodesic triangle and its comparison triangle involving angles and distances between points. This relation expresses the geometric idea of a manifold having nonpositive sectional curvature.

Lemma 2.4. *(See [30].) Let $\Delta(x, y, z)$ be a geodesic triangle in a Hadamard manifold M and $\Delta(x', y', z')$ be its comparison triangle.*

(1) *Let α, β, γ (resp. α', β', γ') be the angles of $\Delta(x, y, z)$ (resp. $\Delta(x', y', z')$) at the vertices x, y, z (resp. x', y', z'). Then*

$$\alpha' \geqslant \alpha, \qquad \beta' \geqslant \beta, \qquad \gamma' \geqslant \gamma. \tag{2.4}$$

(2) *Given any point q belonging to the geodesic which joins x to y, its comparison point is the point q' in the interval $[x', y']$ such that $\mathrm{d}(q, x) = \|q' - x'\|$ and $\mathrm{d}(q, y) = \|q' - y'\|$. Then*

$$\mathrm{d}(z, q) \leqslant \|z' - q'\|. \tag{2.5}$$

The following lemma is a consequence of inequality (2.5) and the parallelogram identity in Euclidean spaces.

Lemma 2.5. *For all $x, y, z \in M$ and $q \in M$ with $\mathrm{d}(x, q) = \mathrm{d}(y, q) = \mathrm{d}(x, y)/2$, one has*

$$\mathrm{d}^2(z, q) \leqslant \frac{1}{2}\mathrm{d}^2(z, x) + \frac{1}{2}\mathrm{d}^2(z, y) - \frac{1}{4}\mathrm{d}^2(x, y). \tag{2.6}$$

From the "law of cosines" in inequality (2.3) it readily follows the following inequality which is a general characteristic of the spaces with nonpositive curvature (see [8]):

$$\langle \exp_x^{-1} y, \exp_x^{-1} z\rangle + \langle \exp_y^{-1} x, \exp_y^{-1} z\rangle \geqslant \mathrm{d}^2(x, y), \quad \forall x, y, z \in M. \tag{2.7}$$

Using the properties of the exponential map the following lemma was proved in [29].

Lemma 2.6. *Let $x_0 \in M$ and $\{x_n\} \subset M$ such that $x_n \to x_0$. Then the following assertions hold.*

(i) *For any $y \in M$,*

$$\exp_{x_n}^{-1} y \to \exp_{x_0}^{-1} y \quad \textit{and} \quad \exp_y^{-1} x_n \to \exp_y^{-1} x_0.$$

(ii) *If $\{v_n\}$ is a sequence such that $v_n \in T_{x_n}M$ and $v_n \to v_0$, then $v_0 \in T_{x_0}M$.*
(iii) *Given the sequences $\{u_n\}$ and $\{v_n\}$ satisfying $u_n, v_n \in T_{x_n}M$, if $u_n \to u_0$ and $v_n \to v_0$ with $u_0, v_0 \in T_{x_0}M$, then*

$$\langle u_n, v_n\rangle \to \langle u_0, v_0\rangle.$$

A subset $K \subseteq M$ is said to be *convex* if for any two points x and y in K, the geodesic joining x to y is contained in K; that is, if $\gamma : [a, b] \to M$ is a geodesic such that $x = \gamma(a)$ and $y = \gamma(b)$, then $\gamma((1-t)a + tb) \in K$ for all $t \in [0, 1]$. From now on $K \subseteq M$ will denote a nonempty closed convex set, unless explicitly stated otherwise.

For an arbitrary subset $C \subseteq M$ the minimal convex subset which contains C is called the *convex hull* of C and denoted by $\mathrm{co}(C)$. It is not difficult to check that $\mathrm{co}(C) = \bigcup_{n=1}^{\infty} C_n$, where $C_0 = C$ and $C_n = \{z \in \gamma_{x,y}\colon x, y \in C_{n-1}\}$.

A real-valued function f defined on M is said to be *convex* if for any geodesic γ of M, the composition function $f \circ \gamma : \mathbb{R} \to \mathbb{R}$ is convex; that is,

$$(f \circ \gamma)\big(ta + (1-t)b\big) \leqslant t(f \circ \gamma)(a) + (1-t)(f \circ \gamma)(b)$$

for any $a, b \in \mathbb{R}$, and $0 \leqslant t \leqslant 1$.

The *subdifferential* of a function $f : M \to \mathbb{R}$ is the set-valued mapping $\partial f : M \to 2^{TM}$ defined by

$$\partial f(x) = \big\{u \in T_xM\colon \langle u, \exp_x^{-1} y\rangle \leqslant f(y) - f(x),\ \forall y \in M\big\}, \quad \forall x \in M,$$

and its elements are called *subgradients*. The subdifferential $\partial f(x)$ at a point $x \in M$ is a closed convex (possibly empty) set. Let $\mathcal{D}(\partial f)$ denote the domain of ∂f defined by

$$\mathcal{D}(\partial f) = \big\{x \in M \bigm| \partial f(x) \neq \emptyset\big\}.$$

The existence of subgradients for convex functions is guaranteed by the following proposition.

Proposition 2.7. *(See [17].) Let M be a Hadamard manifold and $f : M \to \mathbb{R}$ be convex. Then, for any $x \in M$, the subdifferential $\partial f(x)$ of f at x is nonempty. That is, $\mathcal{D}(\partial f) = M$.*

The following proposition describes the convexity property of the distance function.

Proposition 2.8. *(See [41, p. 222].) Let* $\mathrm{d} : M \times M \to \mathbb{R}$ *be the distance function. Then* d *is a convex function with respect to the product Riemannian metric; that is, given any pair of geodesics* $\gamma_1 : [0,1] \to M$ *and* $\gamma_2 : [0,1] \to M$ *the following inequality holds for all* $t \in [0,1]$:

$$\mathrm{d}\big(\gamma_1(t), \gamma_2(t)\big) \leqslant (1-t)\,\mathrm{d}\big(\gamma_1(0), \gamma_2(0)\big) + t\,\mathrm{d}\big(\gamma_1(1), \gamma_2(1)\big).$$

In particular, for each $y \in M$, *the function* $\mathrm{d}(\cdot, y) : M \to \mathbb{R}$ *is a convex function.*

The next proposition generalizes the result stated in [37, Proposition 3.4(ii)]. Some ideas of the proof, following the same argument in [37], are provided here for the sake of completeness.

Proposition 2.9. *Let* $x \in K$ *and* $u \in T_xM$. *Define the function* $g : M \to \mathbb{R}$ *by*

$$g(y) = \langle u, \exp_x^{-1} y \rangle.$$

Then both g *are affine, in other words,* g *and* $-g$ *are convex functions.*

Proof. The proof, involving technical notions in differential geometry, is based on the fact that the function g is convex if and only if the covariant derivative of any vector field by the gradient of g is positive (cf. [43]). Then, by using variations of geodesics, one is able to see that the gradient of g is the parallel transport of the vector u, whose covariant derivative is zero. Therefore, g happens to be convex and so is $-g$ as well. □

3. Equilibrium problem

3.1. Existence of equilibrium points

An equilibrium theory in Euclidean spaces was first introduced by Ky Fan in [14,15] and then developed by Brezis, Nirenberg and Stampacchia [7], Blum and Oettli [4] among others. The main theorem of this section is an existence result for equilibrium problems in Hadamard manifolds, closely related to the results contained in [7]. Moreover, applications to solve other related problems are provided.

Let $F : K \times K \to \mathbb{R}$ be a bifunction with $K \subseteq M$ a closed convex subset. Consider the equilibrium problem (1.1) and denote the equilibrium point set of F by $\mathrm{EP}(F)$. In order to get an existence result for this equilibrium problem we first provide an analogous to KKM Lemma [25] in the setting of Hadamard manifolds.

Lemma 3.1. *Let* $G : K \to 2^K$ *be a mapping such that, for each* $x \in K$, $G(x)$ *is closed. Suppose that*

(i) *there exists* $x_0 \in K$ *such that* $G(x_0)$ *is compact*;
(ii) $\forall x_1, \dots, x_m \in K$, $\mathrm{co}(\{x_1, \dots, x_m\}) \subset \bigcup_{i=1}^m G(x_i)$.

Then $\bigcap_{x \in K} G(x) \neq \emptyset$.

Proof. Fix $x_1, \dots, x_m \in K$ and define the subset of K

$$D\big(\{x_1, \dots, x_m\}\big) := \bigcup_{i=1}^m D_i$$

where $D_1 = \{x_1\}$ and, for any $2 \leqslant j \leqslant n$,

$$D_j = \{z \in \gamma_{x_j, y}\colon\ y \in D_{j-1}\}.$$

Then $D(\{x_1, \dots, x_m\})$ is a closed subset of $\mathrm{co}(\{x_1, \dots, x_m\})$. Moreover any element $y_k \in D_k \subseteq D(\{x_1, \dots, x_m\})$ can be written in the form

$$y_k = \gamma(t_k) \tag{3.1}$$

where $t_k \in [0,1]$ and γ is the geodesic joining x_k to some $y_{k-1} \in D_{k-1}$. To each x_i we associate a corresponding vertex e_i of the simplex $\sigma = \langle e_1, \dots, e_m \rangle \subset \mathbb{R}^{m+1}$. Let $T : \sigma \to D(\{x_1, \dots, x_m\})$ be the mapping defined by induction as follows: if $\lambda_1 \in \langle e_1, e_2 \rangle$, then

$$T(\lambda_1) := \gamma_1(t_1),$$

where t_1 is the unique element in $[0,1]$ such that $\lambda_1 = t_1 e_2 + (1-t_1)e_1$ and γ_1 is the geodesic joining x_1 to x_2. Given $1 < k \leqslant m$, if $\lambda_k \in \langle e_1, \ldots, e_k\rangle \setminus \langle e_1, \ldots, e_{k-1}\rangle$, then $\lambda_k = t_k e_k + (1-t_k)\lambda_{k-1}$ for some $t_k \in (0,1]$ and $\lambda_{k-1} \in \langle e_1, \ldots, e_{k-1}\rangle$. Hence we define

$$T(\lambda_k) := \gamma_k(t_k),$$

where γ_k is the geodesic joining x_k to $T(\lambda_{k-1})$.

As a matter of fact, $T(\sigma)$ coincides with $D(\{x_1, \ldots, x_m\})$ by equality (3.1). Moreover T is continuous. To prove this, for any $j = 1, 2$, let

$$\lambda^j = \sum_{i=1}^m t_i^j e_i \in \sigma,$$

for some sequences $\{t_i^j\}_{i=1}^m \subset [0,1]$ satisfying $\sum_{i=1}^m t_i^j = 1$. By definition we have that $T(\lambda^j) = \gamma_m^j(t_m^j)$, where γ_m^j joins x_m to $T(\sum_{i=1}^{m-1} t_i^j e_i)$. If we denote the diameter $L := diam(D(\{x_1, \ldots, x_m\}))$, by the convexity of the distance it follows that

$$\begin{aligned} \mathrm{d}\big(T(\lambda^1), T(\lambda^2)\big) &\leqslant \mathrm{d}\big(\gamma_m^1(t_m^1), \gamma_m^1(t_m^2)\big) + \mathrm{d}\big(\gamma_m^1(t_m^2), \gamma_m^2(t_m^2)\big) \\ &\leqslant \big|t_m^1 - t_m^2\big| \, \mathrm{d}\left(x_m, T\left(\sum_{i=1}^{m-1} t_i^1 e_i\right)\right) + \mathrm{d}\left(T\left(\sum_{i=1}^{m-1} t_i^1 e_i\right), T\left(\sum_{i=1}^{m-1} t_i^2 e_i\right)\right) \\ &\leqslant L\big|t_m^1 - t_m^2\big| + \mathrm{d}\left(T\left(\sum_{i=1}^{m-1} t_i^1 e_i\right), T\left(\sum_{i=1}^{m-1} t_i^2 e_i\right)\right). \end{aligned}$$

Therefore by recursion we obtain that

$$\mathrm{d}\big(T(\lambda^1), T(\lambda^2)\big) \leqslant L \sum_{i=1}^m \big|t_i^1 - t_i^2\big|.$$

This is sufficient to prove the continuity of T.

Consider the closed sets $\{E_i\}_{i=1}^m$, defined by $E_i := T^{-1}(D(\{x_1, \ldots, x_m\}) \cap G(x_i))$. Let us prove that for every $I \subset \{1, \ldots, m\}$

$$\mathrm{co}\big(\{e_i \colon i \in I\}\big) \subset \bigcup_{i \in I} E_i. \tag{3.2}$$

Indeed, let $\lambda = \sum_{j=1}^k t_{i_j} e_{i_j} \in \mathrm{co}(\{e_{i_1}, \ldots, e_{i_k}\})$, with $\{t_{i_j}\} \subset [0,1]$ such that $\sum_{j=1}^k t_{i_j} = 1$. Since, by hypothesis,

$$T(\lambda) \in D\big(\{x_{i_1}, \ldots, x_{i_k}\}\big) \subseteq \mathrm{co}\big(\{x_{i_1}, \ldots, x_{i_k}\}\big) \subseteq \bigcup_{i=1}^m G(x_i),$$

then there exists $j \in \{1, \ldots, k\}$ for which $T(\lambda) \in G(x_{i_j}) \cap D(\{x_1, \ldots, x_m\})$ and, consequently, $\lambda \in E_{i_j}$. By applying KKM Lemma to the family $\{E_i\}_{i=1}^m$, we get the existence of a point $\hat{\lambda} \in \mathrm{co}(\{e_1, \ldots, e_m\})$ such that $\hat{\lambda} \in \bigcap_{i=1}^m E_i$, so $T(\hat{\lambda}) \in \bigcap_{i=1}^m G(x_i)$. Then we have proved that the family of closed sets $\{G(x) \cap G(x_0)\}_{x \in K}$ has the finite intersection property. Since $G(x_0)$ is compact, it implies that

$$\bigcap_{x \in K} G(x) = \bigcap_{x \in K} \big(G(x_0) \cap G(x)\big) \neq \emptyset. \qquad \square$$

Thanks to the previous lemma, we are able to get existence of solutions to the equilibrium problem (1.1) under mild conditions on the bifunction F.

Theorem 3.2. *Let $F : K \times K \to \mathbb{R}$ be a bifunction such that*

(i) *for any $x \in K$, $F(x, x) \geqslant 0$;*
(ii) *for every $x \in K$, the set $\{y \in K \colon F(x, y) < 0\}$ is convex;*
(iii) *for every $y \in K$, $x \mapsto F(x, y)$ is upper semicontinuous;*
(iv) *there exists a compact set $L \subseteq M$ and a point $y_0 \in L \cap K$ such that*

$$F(x, y_0) < 0, \quad \forall x \in K \setminus L.$$

Then there exists a point $x_0 \in L \cap K$ satisfying

$$F(x_0, y) \geqslant 0, \quad \forall y \in K.$$

Proof. Define the mapping $G : K \to 2^K$ such that for any $y \in K$

$$G(y) := \{x \in K\colon F(x, y) \geqslant 0\}.$$

Since $F(\cdot, y)$ is upper semicontinuous, $G(y)$ is closed for all $y \in K$. Additionally, by condition (iv) there exists a point $y_0 \in K$ for which $G(y_0) \subseteq L$, so $G(y_0)$ is compact. In order to apply Lemma 3.1 we have to prove that for any choice of $y_1, \dots, y_m \in K$,

$$\operatorname{co}(\{y_1, \dots, y_m\}) \subset \bigcup_{i=1}^{m} G(y_i). \tag{3.3}$$

To this end, suppose on the contrary that there exists a point $\hat{x}$, such that $\hat{x} \in \operatorname{co}(\{y_1, \dots, y_m\})$ but $\hat{x} \notin \bigcup_{i=1}^m G(y_i)$; that is,

$$F(\hat{x}, y_i) < 0, \quad \forall i \in \{1, \dots, m\}. \tag{3.4}$$

This implies that for any $i \in \{1, \dots, m\}$, $y_i \in \{y \in K\colon F(\hat{x}, y) < 0\}$. Since the set $\{y \in K\colon F(\hat{x}, y) < 0\}$ is convex by hypothesis (ii),

$$\hat{x} \in \operatorname{co}(\{y_1, \dots, y_m\}) \subseteq \{y \in K\colon F(\hat{x}, y) < 0\},$$

which contradicts the assumption (i).

Then by Lemma 3.1 there exists a point $x_0 \in K$ such that

$$x_0 \in \bigcap_{y \in K} G(y),$$

with $x_0 \in G(y_0) \subseteq L \cap K$. In other words, there exists $x_0 \in L \cap K$ such that

$$F(x_0, y) \geqslant 0, \quad \forall y \in K. \qquad \square$$

By setting $L = K$ in the previous theorem, the following corollary is obtained. This fact might be deduced from results in more general settings, but this would imply the introduction of some abstract concepts in homology theory such as Lefschetz number or Cech cycle on acyclic absolute neighborhood retract spaces (cf. [13,2,32]).

Corollary 3.3. *Let $K \subseteq M$ be convex and compact and $F : K \times K \to \mathbb{R}$ such that*

(i) *for any $x \in K$, $F(x, x) \geqslant 0$;*
(ii) *for every $x \in K$, the set $\{y \in K\colon F(x, y) < 0\}$ is convex;*
(iii) *for every $y \in K$, $x \mapsto F(x, y)$ is upper semicontinuous.*

Then there exists a point $x_0 \in K$ satisfying

$$F(x_0, y) \geqslant 0, \quad \forall y \in K.$$

Example 3.4. We present an example of an equilibrium problem defined in a Euclidean space whose set K is not convex so it cannot be solved by using the classical results known in vector spaces. However, if we rewrite the problem in a Riemannian manifold then it turns out to satisfy the conditions required in the previous corollary. Let

$$K = \{(x, y, z) \in \mathbb{R}^3\colon 0 \leqslant x \leqslant 1,\ y^2 - z^2 = -1,\ z \geqslant 0\} \tag{3.5}$$

and $F : K \times K \to \mathbb{R}$ the bifunction defined by

$$F(x_1, y_1, z_1, x_2, y_2, z_2) = 4(x_2 - x_1) + (1 - x_1)\big(\big(y_2^2 + z_2^2\big) - \big(y_1^2 + z_1^2\big)\big). \tag{3.6}$$

Note that K is indeed not convex in $\mathbb{R}^3$.

Given a natural number $m \geqslant 1$, let $\mathbb{E}^{m,1}$ denote the vector space $\mathbb{R}^{m+1}$ endowed with the symmetric bilinear form (which is called the Lorentz metric) defined by

$$\langle x, y\rangle = \sum_{i=1}^{m} x_i y_i - x_{m+1} y_{m+1}, \quad \forall x = (x_i),\ y = (y_i) \in \mathbb{R}^{m+1}.$$

The *hyperbolic m-space* $\mathbb{H}^m$ is defined by

$$\left\{x = (x_1, \ldots, x_{m+1}) \in \mathbb{E}^{m,1}\colon\ \langle x, x\rangle = -1,\ x_{m+1} > 0\right\},$$

that is the upper sheet of the hyperboloid $\{x \in \mathbb{E}^{m,1}\colon\ \langle x, x\rangle = -1\}$. Note that $x_{m+1} \geqslant 1$ for any $x \in \mathbb{H}^m$, with equality if and only if $x_i = 0$ for all $i = 1, \ldots, m$. The metric of $\mathbb{H}^m$ is induced from the Lorentz metric $\langle\cdot,\cdot\rangle$ and it will be denoted by the same symbol. Then $\mathbb{H}^m$ is a Hadamard manifold with sectional curvature -1 (cf. [8] and [16]). Furthermore, the normalized geodesic $\gamma : \mathbb{R} \to \mathbb{H}^m$ starting from $x \in \mathbb{H}^m$ is given by

$$\gamma(t) = (\cosh t)x + (\sinh t)v, \quad \forall t \in \mathbb{R}, \tag{3.7}$$

where $v \in T_x\mathbb{H}^m$ is a unit vector.

Considering the set K immersed in the space $M = \mathbb{R} \times \mathbb{H}^1$ which is a Hadamard manifold for being the product space of Hadamard manifolds (cf. [8]), it is readily seen that K is convex and compact in M. On the other hand, conditions (i) and (iii) in Corollary 3.3 hold, and the fact that F is convex in the second variable can be deduced from Example 3.14 and the results which will be given Section 3.2.3. So Corollary 3.3 implies the existence of an equilibrium point for F.

3.2. Applications

3.2.1. Mixed variational inequalities

Given a single-valued vector field $A : K \to TM$ and a real-valued function $f : K \to \mathbb{R}$, the mixed variational inequality problem associated to A and f, MVIP(A, f), is formulated as follows:

$$\text{find } x_0 \in K \text{ such that } \left\langle Ax_0, \exp_{x_0}^{-1} y\right\rangle + f(y) - f(x_0) \geqslant 0,\ \forall y \in K.$$

This problem has extensively been studied in the linear setting; see, for instance, [45,19]. Our approach to the problem is to turn it into an equilibrium problem for a particular bifunction $F_{A,f}$.

Theorem 3.5. *Let $A : K \to TM$ be a continuous vector field and $f : K \to \mathbb{R}$ a convex lower semicontinuous function. Assume that the following condition holds*:

(C) *There exists a compact set $L \subseteq M$ and a point $y_0 \in L \cap K$ such that*

$$\left\langle Ax, \exp_x^{-1} y_0\right\rangle + f(y_0) - f(x) < 0, \quad \forall x \in K \setminus L. \tag{3.8}$$

Then MVIP(A, f) *has a solution in $L \cap K$.*

Proof. We define $F_{A,f} : K \times K \to \mathbb{R}$ as

$$F_{A,f}(x, y) := \left\langle Ax, \exp_x^{-1} y\right\rangle + f(y) - f(x). \tag{3.9}$$

Obviously the solutions to MVIP(A, f) are the equilibrium points of $F_{A,f}$. It is straightforward to see that $F_{A,f}$ satisfies hypotheses (i) and (iii) in Theorem 3.2 thanks to the continuity properties, while condition (C) implies (iv). To prove (ii) we fix $x \in K$. By Proposition 2.9 it follows that the function

$$y \mapsto \left\langle Ax, \exp_x^{-1} y\right\rangle$$

is convex. Then, being the sum of two convex functions, the function $F_{A,f}(x, \cdot)$ is convex as well. This ensures the convexity of the set $\{y \in K\colon\ F_{A,f}(x, y) < 0\}$. As a consequence of Theorem 3.2, there exists a point $x_0 \in L \cap K$ such that

$$F_{A,f}(x_0, y) \geqslant 0, \quad \forall y \in K;$$

that is, $x_0 \in L \cap K$ is a solution to MVIP(A, f). $\square$

Corollary 3.6. *Let $A : K \to TM$ be a continuous vector field and $f : K \to \mathbb{R}$ a convex lower semicontinuous function. If either*

(i) *K is compact, or*
(ii) *there exists $y_0 \in K$ such that the coercivity condition*

$$\frac{\langle Ay_0, \exp_{y_0}^{-1} x\rangle + \langle Ax, \exp_x^{-1} y_0\rangle}{\mathrm{d}(y_0, x)} \to -\infty \quad \text{as } \mathrm{d}(y_0, x) \to \infty \tag{3.10}$$

holds,

then MVIP(A, f) *has a solution.*

Proof. Let us prove that both hypotheses (i) and (ii) imply condition (C). If (i) holds, then the result readily follows by choosing $L = K$.

Suppose now that (ii) holds. Let $y_0 \in K$ satisfying (3.10) and let $u_0 \in \partial f(y_0)$, the subdifferential of f at y_0. Then for any $x \in K$ we have

$$f(y_0) - f(x) \leqslant \|u_0\| \, \mathrm{d}(y_0, x).$$

From this inequality, it follows

$$-\langle Ay_0, \exp_{y_0}^{-1} x\rangle + f(y_0) - f(x) \leqslant \big(\|Ay_0\| + \|u_0\|\big) d(y_0, x). \tag{3.11}$$

Since (3.10) holds, we can choose $R > (\|Ay_0\| + \|u_0\|)$ and $r > 0$ such that for any x which does not belong to the closed ball $\overline{B_r(y_0)} = \{x \in M\colon\ \mathrm{d}(x, y_0) \leqslant r\}$, we have

$$\begin{aligned}\langle Ax, \exp_x^{-1} y_0\rangle + f(y_0) - f(x) &\leqslant -\langle Ay_0, \exp_{y_0}^{-1} x\rangle - R\,\mathrm{d}(y_0, x) + f(y_0) - f(x)\\ &\leqslant \big(\|Ay_0\| - R + \|u_0\|\big)\,\mathrm{d}(y_0, x)\\ &< 0.\end{aligned}$$

By setting $L = \overline{B_r(y_0)}$ this last inequality means that condition (C) is satisfied; therefore Theorem 3.5 implies the existence of solution. □

Remark 3.7. By considering f the function constantly 0, it follows that Corollary 3.6 extends Theorem 1 and Corollary 3 in [34].

3.2.2. Fixed points of set-valued mappings

The problem of existence of fixed points of set-valued mappings defined on a Hadamard manifold can be approached via an equilibrium problem of type (1.1). To this end we need the following lemma, an analogous of Fan's Minimax Theorem, whose proof follows a standard argument in convex analysis (see [4]).

Lemma 3.8. *Let $D, K \subseteq M$ be closed convex sets with D compact. Assume that $\rho : D \times K \to \mathbb{R}$ is upper semicontinuous in the first variable and that for any $x \in D$ and $y \in K$, $-\rho(\cdot, y)$ and $\rho(x, \cdot)$ are convex functions. If*

$$\max_{x\in D} \rho(x, y) \geqslant 0, \quad \forall y \in K, \tag{3.12}$$

then there exists $\bar{x} \in D$ such that $\rho(\bar{x}, y) \geqslant 0$ for any $y \in K$.

Proof. Suppose that the thesis does not hold; that is, for all $x \in D$ there exist $y \in K$ and $\varepsilon > 0$ such that

$$\rho(x, y) < -\varepsilon.$$

Set $S(y, \varepsilon) := \{x \in D\colon\ \rho(x, y) < -\varepsilon\}$. Each $S(y, \varepsilon)$ is open by the upper semicontinuity in the first variable of ρ. Moreover, since the family $\{S(y, \varepsilon)\}$ covers the compact set D there exists $I = \{1, \ldots, m\}$ such that

$$D \subseteq \bigcup_{i\in I} S(y_i, \varepsilon_i),$$

where $y_i \in K$ and $\varepsilon_i > 0$ for any $i \in I$. This means that, given any arbitrary $x \in D$, there exists $j \in I$ for which

$$\rho(x, y_j) < -\varepsilon_j \leqslant -\min_{i\in I} \varepsilon_i =: -\epsilon. \tag{3.13}$$

We define the functions $f_i(x) := -\rho(x, y_i) - \epsilon$, for any $i \in I$, and the sets $C_1, C_2 \subset \mathbb{R}^m$ by

$$C_1 := \big\{(\eta_1, \ldots, \eta_m)\colon\ \exists x \in D \text{ such that } f_i(x) \leqslant \eta_i,\ \forall i \in I\big\}$$

and

$$C_2 := \big\{(\eta_1, \ldots, \eta_m)\colon\ \eta_i \leqslant 0,\ \forall i \in I\big\}.$$

Thanks to the convexity of the set D and the functions $-\rho(\cdot, y_i)$ for any $i \in I$, C_1 is deduced to be convex. On the other hand C_2 is also convex and moreover $C_1 \cap C_2 = \emptyset$ by inequality ((3.13)). Therefore we can apply Hahn–Banach Theorem in $\mathbb{R}^m$ to get the existence of a hyperplane $H := \{(\eta_1, \ldots, \eta_m)\colon\ \sum_{i\in I} \lambda_i \eta_i = \alpha\}$ which separates C_1 from C_2. In particular, we may assume that $\sum_{i\in I} \lambda_i = 1$ and that for any $(\zeta_1, \ldots, \zeta_m) \in C_1$ and any $(\xi_1, \ldots, \xi_m) \in C_2$

$$\sum_{i\in I}\lambda_i\xi_i \leqslant \alpha \leqslant \sum_{i\in I}\lambda_i\zeta_i \tag{3.14}$$

holds. From the first inequality in (3.14) it is easily derived that $\alpha \geqslant 0$ and that for any $i \in I$, $\lambda_i \geqslant 0$. Moreover, since $(f_1(x), \dots, f_m(x))$ belongs to C_1, we deduce that

$$\sum_{i\in I}\lambda_i f_i(x) \geqslant 0.$$

We have proved that there exist $\lambda_1, \dots, \lambda_m \geqslant 0$ such that $\sum_{i=1}^m \lambda_i = 1$ and

$$\sum_{i=1}^m \lambda_i \rho(x, y_i) \leqslant -\epsilon.$$

By the convexity of $\rho(x,\cdot)$ we get the existence of a point $\hat{y} \in \mathrm{co}(\{y_1, \dots, y_m\}) \subseteq K$ such that

$$\rho(x, \hat{y}) \leqslant -\epsilon < 0.$$

Since this is true for any arbitrary $x \in D$, it contradicts the assumption (3.12). □

Given a set-valued mapping $T : K \to 2^K$ where K is a compact convex set, define the bifunction $F : K \times K \to \mathbb{R}$ by

$$F(x, y) = \max\bigl\{-\bigl\langle \exp_x^{-1} z, \exp_x^{-1} y\bigr\rangle\colon\ z \in T(x)\bigr\}, \quad \forall x, y \in K. \tag{3.15}$$

Assume that $T(x)$ is compact and convex for any $x \in K$. Then the set of equilibrium points of F is the fixed point set of T; that is,

$$\mathrm{EP}(F) = \mathrm{Fix}(T) := \bigl\{x \in K\colon\ x \in T(x)\bigr\}.$$

In fact, it is readily proved that any fixed point of T is an equilibrium point. Conversely, if $x \in K$ is an equilibrium point, then $F(x, y) \geqslant 0$ for any $y \in K$. Thus

$$\max\bigl\{-\bigl\langle \exp_x^{-1} z, \exp_x^{-1} y\bigr\rangle\colon\ z \in T(x)\bigr\} \geqslant 0,$$

for any $y \in K$. Since $T(x)$ is compact, it follows from Lemma 3.8 that there exists $z \in T(x)$ such that

$$-\bigl\langle \exp_x^{-1} z, \exp_x^{-1} y\bigr\rangle \geqslant 0, \quad \forall y \in K.$$

Therefore $d(x, z) \leqslant 0$ and so $x = z \in T(x)$.

By means of this equivalence, a counterpart of Kakutani Theorem (see, for instance, [46]) in this setting is obtained. To this end, the concepts of upper semicontinuous and upper Kuratowski semicontinuous set-valued mapping, first defined for vector fields on Hadamard manifolds in [29], are necessary.

Definition 3.9. Given $T : M \to 2^M$ and $x_0 \in M$, the mapping T is said to be

- *upper semicontinuous*, USC, at x_0 if for any open set $V \subseteq M$ satisfying $T(x_0) \subseteq V$, there exists an open neighborhood $U(x_0)$ of x_0 such that $T(x) \subseteq V$ for any $x \in U(x_0)$;
- *upper Kuratowski semicontinuous*, UKSC, at x_0 if for any sequences $\{x_k\}, \{u_k\} \subset M$ with each $u_k \in T(x_k)$, the relations $\lim_{k\to\infty} x_k = x_0$ and $\lim_{k\to\infty} u_k = u_0$ imply $u_0 \in T(x_0)$.

Theorem 3.10. *Let $K \subseteq M$ be a compact convex set and $T : K \to 2^K$ an UKSC mapping. Assume that for any $x \in K$, $T(x)$ is closed and convex. Then there exists a fixed point of T.*

Proof. Given the bifunction $F : K \times K \to \mathbb{R}$ defined by (3.15), since its equilibrium points coincide with the fixed points of T, if we prove that F satisfies the hypotheses in Corollary 3.3 then we obtain the existence of a fixed point of T. The fact that, for any $x \in K$, $F(x, x) = 0$ is evident. For any $x \in K$, by Proposition 2.9 the function $F(x,\cdot)$ is convex. Hence the set $\{y \in K\colon\ F(x, y) < 0\}$ is convex and hypothesis (ii) holds. It remains to prove that for any fixed $y \in K$ the function $x \mapsto F(x, y)$ is upper semicontinuous. To this end, given $y \in K$, we define the set-valued mapping $\widehat{T} : K \to 2^{\mathbb{R}}$ as

$$\widehat{T}(x) = \bigl\{\bigl\langle \exp_x^{-1} z, \exp_x^{-1} y\bigr\rangle\colon\ z \in T(x)\bigr\}, \tag{3.16}$$

and the function $f : K \to \mathbb{R}$ as

$$f(x) = \min \widehat{T}(x). \tag{3.17}$$

Then the bifunction F, for any $x \in K$, can be rewritten as

$$F(x, y) = -f(x). \tag{3.18}$$

Since T is UKSC, so is $\widehat{T}$. Indeed, given $x_n \to x_0 \in K$ and $u_n \to u_0 \in \mathbb{R}$ such that $u_n \in \widehat{T}(x_n)$ for any $n \geqslant 0$, we have that $u_n = \langle \exp_{x_n}^{-1} z_n, \exp_{x_n}^{-1} y \rangle$ for some $z_n \in T(x_n)$. Letting $n \to \infty$, we obtain that $u_0 = \langle \exp_{x_0}^{-1} z_0, \exp_{x_0}^{-1} y \rangle$ where $z_n \to z_0 \in K$. On the other hand, T is UKSC which implies that $z_0 \in T(x_0)$. This means that $u_0 \in T(x_0)$, so $\widehat{T}$ is UKSC.

Let us see now that f is lower semicontinuous; in other words, if $x_n \to x_0$, then

$$\liminf_{n\to\infty} f(x_n) \geqslant f(x_0). \tag{3.19}$$

Suppose that $u_n \in \widehat{T}(x_n)$ and $u_n \to u_0$. By the upper Kuratowski semicontinuity of $\widehat{T}$ we know that $u_0 \in \widehat{T}(x_0)$. Then $u_0 \geqslant f(x_0)$ by definition. Therefore inequality (3.19) holds. $\square$

Remark 3.11. Clearly, the upper semicontinuity implies the upper Kuratowski semicontinuity, so the previous result remains true assuming that T is USC instead.

3.2.3. Nash equilibrium for non-cooperative games

The classical existence result of a Nash equilibria in an n-person non-cooperative game, [33], was proved under the assumptions that the strategy sets are compact and convex on a Hausdorff topological vector spaces and the payoff functions are continuous and convex. The numerous applications in many areas of economics have led researchers from different fields to investigate the possibility of weakening the convexity condition on either the strategy sets or the payoff functions; see, for instance, [35,42]. Inspired by Udriste's [43] geometrical approach for minimization problems, our aim here is to prove the existence of Nash equilibrium points in the setting of Hadamard manifolds to provide an approach for problems with either nonconvex strategy sets or nonconvex payoff functions which can be transformed to convex ones by introducing an adequate Riemannian metric on the strategy sets (following the ideas in [38] and [11]).

Let $I = \{1, 2, \ldots, m\}$ be a finite index set which denotes the set of players. For any $i \in I$, consider M_i a Hadamard manifold where the strategy set $K_i \subseteq M_i$ of the i-th player will be given. Let $K := K_1 \times K_2 \times \cdots \times K_m$ belonging to the Hadamard manifold $M = M_1 \times \cdots \times M_m$ (cf. [8]). Suppose that for every $i \in I$ there exists a payoff function $f_i : K \to \mathbb{R}$ representing the loss of each player, depending on the strategies of all the player. The Nash equilibrium problem associated to $\{K_i\}_{i\in I}$ and $\{f_i\}_{i\in I}$ consists of finding $x = (x_i)_{i\in I} \in K$ such that for all $i \in I$,

$$f_i(x) \leqslant f_i(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_m), \tag{3.20}$$

for all $y_i \in K_i$. In other words, no player can reduce his loss by varying his strategy alone. The point x is called a Nash equilibrium point. This problem can be formulated as an equilibrium problem by defining the bifunction $F : K \times K \to \mathbb{R}$ as

$$F(x, y) = \sum_{i\in I} \bigl(f_i(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_m) - f_i(x)\bigr) \tag{3.21}$$

for any $(x, y) \in K \times K$. In fact, $x \in K$ is a Nash equilibrium point if and only if it is an equilibrium point of F. The direct implication is clear. Conversely, if $x \in K$ satisfying that $F(x, y) \geqslant 0$ for any $y \in K$, then for any $i \in I$, by choosing $y \in K$ such that $y_j = x_j$ for any $j \neq i$, we obtain

$$F(x, y) = f_i(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_m) - f_i(x) \geqslant 0.$$

Then, under suitable conditions, as a consequence of the results for equilibrium problems stated in Section 3.1 we obtain the following theorem regarding the existence of Nash equilibrium points.

Theorem 3.12. *For any $i \in I$, let $K_i \subseteq M_i$ be a compact convex set and $f_i : K \to \mathbb{R}$ a continuous function such that it is convex in the i-th variable. Then there exists a Nash equilibrium point.*

Proof. Note that $K := K_1 \times K_2 \times \cdots \times K_m$ is a compact convex set of the Hadamard manifold $M = M_1 \times \cdots \times M_m$. Since the Nash equilibrium problem is equivalent to the equilibrium problem for the bifunction F defined in (3.21), it is enough to prove that F satisfies the conditions in Corollary 3.3. Obviously for any $x \in K$, $F(x, x) = 0$. Given $x \in K$, the set $\{y \in K: F(x, y) < 0\}$ is convex because so are the functions f_i in the i-th variable, for any $i \in I$. Finally, since every f_i is continuous, we can ensure that for any $y \in K$, the function $x \in K \mapsto F(x, y)$ is continuous and then the result of Corollary 3.3 holds. $\square$

Remark 3.13. It is worth mentioning that the previous theorem can be deduced from a more general one concerning the existence of Nash equilibrium points in complete finite-dimensional Riemannian manifolds which was published in [27] by Kristaly.

Example 3.14. Let $M_1 = \mathbb{R}$ the Euclidean space, and $M_2 = \mathbb{H}^1$ the hyperbolic space defined in Example 3.4. We define the strategy sets as $K_1 = [0,1]$ and $K_2 = \{(y,z) \in \mathbb{R}^2\colon\ y^2 - z^2 = -1,\ -1 \leqslant y \leqslant 1,\ z \geqslant 0\}$. It is straightforward to see that K_i is compact and convex in M_i for any $i = 1,2$. Now let $f_i : K = K_1 \times K_2 \to \mathbb{R}$, for $i = 1,2$, be defined as

$$f_1(x,y,z) = 4x - y, \tag{3.22}$$

$$f_2(x,y,z) = (1-x)\big(y^2 + z^2\big), \tag{3.23}$$

for any $(x,y,z) \in M = M_1 \times M_2$. Those are continuous functions in M. On the other hand, given $x \in \mathbb{R}$ fixed, under the Riemannian metric endowed in M_2, by using the expressions of the geodesics (see Example 3.4) the functions f_1 and f_2 are convex in the first and second variable, respectively. Therefore, Theorem 3.12 ensures the existence of a Nash equilibrium point.

4. Approximation of equilibrium points

The approach that we follow to approximate a solution of the equilibrium problem (1.1) involves the resolvent of the bifunction F, which is a firmly nonexpansive mapping whose fixed point set coincides with the equilibrium point set of F. As a first step we introduce the concept of firmly nonexpansive mappings in this framework along with an important characterization. Then we prove the convergence of Picard iteration for firmly nonexpansive mappings.

4.1. Picard iteration for firmly nonexpansive mappings

The concept of firmly nonexpansive mappings was first introduced in the realm of Banach spaces [6] and the Hilbert ball with the hyperbolic metric [20], so-called firmly nonexpansive mapping of the first kind in the latter case. In [31], this notion was extended to the setting of Hadamard manifolds.

Definition 4.1. Given a mapping $T : K \to K$ defined on $K \subseteq M$, we say that T is *firmly nonexpansive* if for any $x, y \in K$, the function $\Phi : [0,1] \to [0,\infty]$ defined by

$$\Phi(t) = \mathrm{d}\big(\gamma_1(t), \gamma_2(t)\big) \tag{4.1}$$

is nonincreasing, where γ_1 and γ_2 denote the geodesics joining x to $T(x)$ and y to $T(y)$, respectively.

From the definition we deduce that any firmly nonexpansive mapping is nonexpansive, that is, for all $x, y \in K$

$$\mathrm{d}\big(T(x), T(y)\big) \leqslant \mathrm{d}(x,y). \tag{4.2}$$

The following result was given in [31].

Proposition 4.2. *(See [31].) A mapping $T : K \to K$ is firmly nonexpansive if and only if for any $x, y \in K$*

$$\big\langle \exp^{-1}_{T(x)} T(y), \exp^{-1}_{T(x)} x \big\rangle + \big\langle \exp^{-1}_{T(y)} T(x), \exp^{-1}_{T(y)} y \big\rangle \leqslant 0. \tag{4.3}$$

As in Banach spaces and the Hilbert ball [40], the class of firmly nonexpansive mappings is characterized by the good asymptotic behavior of the sequence of Picard iterates $\{T^n(x)\}$. In order to prove the convergence of this sequence, the following definition and results are necessary.

Definition 4.3. Let X be a complete metric space and $C \subseteq X$ be a nonempty set. A sequence $\{x_n\} \subset X$ is called *Fejér monotone* with respect to C if

$$\mathrm{d}(x_{n+1}, y) \leqslant \mathrm{d}(x_n, y)$$

for all $y \in C$ and $n \geqslant 0$.

Lemma 4.4. *(See [5,17].) Let X be a complete metric space. If $\{x_n\} \subset X$ is Fejér monotone with respect to a nonempty set $C \subseteq X$, then $\{x_n\}$ is bounded. Moreover, if a cluster point x of $\{x_n\}$ belongs to C, then $\{x_n\}$ converges to x.*

Theorem 4.5. *Let $T : K \to K$ be a firmly nonexpansive mapping such that its fixed point set* $\mathrm{Fix}(T) \neq \emptyset$. *Then for each $x \in K$, the sequence of iterates $\{T^n(x)\}$ converges to a fixed point of T.*

Proof. Let $x_n = T^n(x)$ for any $n \geqslant 0$. Note that, since K is closed and convex, it is a complete metric space. Thus, by Lemma 4.4, it suffices to prove that $\{x_n\}$ is Fejér monotone with respect to $\mathrm{Fix}(T)$ and that a cluster point of $\{x_n\}$ belongs to $\mathrm{Fix}(T)$. To this end, let $n \geqslant 0$ and $y \in \mathrm{Fix}(T)$ be fixed. Since T is nonexpansive,

$$\mathrm{d}(x_{n+1}, y) = \mathrm{d}\big(T(x_n), T(y)\big) \leqslant \mathrm{d}(x_n, y).$$

Hence $\{x_n\}$ is Fejér monotone with respect to $\mathrm{Fix}(T)$. Now let x be a cluster point of $\{x_n\}$. Then there exists a subsequence $\{n_k\}$ of $\{n\}$ such that $x_{n_k} \to x$. On the other hand, one has that

$$\begin{aligned}\mathrm{d}\big(x, T(x)\big) &\leqslant \mathrm{d}(x, x_{n_k}) + \mathrm{d}\big(x_{n_k}, T(x_{n_k})\big) + \mathrm{d}\big(T(x_{n_k}), T(x)\big)\\ &\leqslant 2\,\mathrm{d}(x_{n_k}, x) + \mathrm{d}\big(x_{n_k}, T(x_{n_k})\big).\end{aligned}$$

Then we just need to prove that

$$\lim_{n\to\infty} \mathrm{d}\big(x_n, T(x_n)\big) = 0, \tag{4.4}$$

because if so, taking limit, we obtain that $\mathrm{d}(x, Tx) = 0$, which means that $x \in \mathrm{Fix}(T)$. Let $y \in \mathrm{Fix}(T)$. Since $\{x_n\}$ is Fejér monotone to $\mathrm{Fix}(T)$, there exists the limit $\lim_{n\to\infty} \mathrm{d}(x_n, y) = \lim_{n\to\infty} \mathrm{d}(T(x_n), y) = d$. Given $n \geqslant 0$ fixed, let $\gamma_n : [0,1] \to M$ the geodesic joining x_n to $T(x_n)$. Then $\gamma_n(1/2) = m_n$ verifies

$$\mathrm{d}(m_n, x_n) = \mathrm{d}\big(m_n, T(x_n)\big) = \mathrm{d}\big(x_n, T(x_n)\big)/2.$$

Since T is firmly nonexpansive,

$$\mathrm{d}\big(T(x_n), y\big) \leqslant \mathrm{d}(m_n, y) \leqslant \mathrm{d}(x_n, y).$$

Then $\lim_{n\to\infty} \mathrm{d}(m_n, y) = d$. By inequality (2.6) of Lemma 2.5, we obtain

$$\frac{1}{4} d^2\big(x_n, T(x_n)\big) \leqslant \frac{1}{2} d^2(x_n, y) + \frac{1}{2} d^2\big(T(x_n), y\big) - d^2(m_n, y).$$

Taking limit as $n \to \infty$ we have that (4.4) holds. $\square$

4.2. Resolvents of bifunctions

The definition of the resolvent of a bifunction in the setting of a Hilbert space H appears implicitly in [4] and was first given in [10]. In order to distinguish the resolvent of vector fields and the resolvent of bifunctions we denote the latter with an upper index, J^F. Given a bifunction $F : K \times K \to \mathbb{R}$, where $K \subseteq H$ is nonempty closed and convex, the resolvent of F is the set-valued operator $J^F : H \to 2^K$ such that for any $x \in H$,

$$J^F(x) = \big\{z \in K \;\big|\; (\forall y \in K)\ F(z, y) + \langle z - x, y - z\rangle \geqslant 0\big\}.$$

Under some conditions on the bifunction F, J^F can be proved to be well defined, single-valued and firmly nonexpansive, and its fixed point set turns out to be the equilibrium point set of F; see [10].

The following definition extends the previous one to the setting of a Hadamard manifold M.

Definition 4.6. Let $F : K \times K \to \mathbb{R}$. For any $\lambda > 0$, the *resolvent* of F is the set-valued operator $J^F_\lambda : M \to 2^K$ defined by

$$J^F_\lambda(x) = \big\{z \in K \;\big|\; \lambda F(z, y) - \big\langle \exp_z^{-1} x, \exp_z^{-1} y\big\rangle \geqslant 0,\ \forall y \in K\big\}, \quad \forall x \in M.$$

Theorem 4.7. *Let $F : K \times K \to \mathbb{R}$ be a bifunction satisfying the following conditions*:

(1) *F is monotone, that is, for any $(x, y) \in K \times K$,*

$$F(x, y) + F(y, x) \leqslant 0;$$

(2) *for each $\lambda > 0$, J^F_λ is properly defined, that is, the domain $\mathcal{D}(J^F_\lambda) \neq \emptyset$.*

Then for any $\lambda > 0$,

(i) *the resolvent J^F_λ is single-valued*;
(ii) *the resolvent J^F_λ is firmly nonexpansive*;
(iii) *the fixed point set of J^F_λ is the equilibrium point set of F,*

$$\mathrm{Fix}\big(J^F_\lambda\big) = \mathrm{EP}(F);$$

(iv) *if $\mathcal{D}(J^F_\lambda)$ is closed and convex, the equilibrium point set* $\mathrm{EP}(F)$ *is closed and convex.*

Proof.

(i) We fix $x \in \mathcal{D}(J_\lambda^F)$ and assume that there exist $z_1, z_2 \in J_\lambda^F(x)$. By definition this means that

$$\lambda F(z_1, z_2) - \langle \exp_{z_1}^{-1} x, \exp_{z_1}^{-1} z_2 \rangle \geqslant 0, \tag{4.5}$$
$$\lambda F(z_2, z_1) - \langle \exp_{z_2}^{-1} x, \exp_{z_2}^{-1} z_1 \rangle \geqslant 0. \tag{4.6}$$

By summing inequalities (4.5) and (4.6), by condition (1) and applying inequality (2.7), we get

$$d^2(z_1, z_2) \leqslant \langle \exp_{z_1}^{-1} x, \exp_{z_1}^{-1} z_2 \rangle + \langle \exp_{z_2}^{-1} x, \exp_{z_2}^{-1} z_1 \rangle \leqslant 0.$$

Therefore $z_1 = z_2$.

(ii) To prove that J_λ^F is firmly nonexpansive we consider $x_1, x_2 \in \mathcal{D}(J_\lambda^F)$. By definition of resolvent we get

$$\lambda F\big(J_\lambda^F x_1, J_\lambda^F x_2\big) - \langle \exp_{J_\lambda^F x_1}^{-1} x_1, \exp_{J_\lambda^F x_1}^{-1} J_\lambda^F x_2 \rangle \geqslant 0, \tag{4.7}$$
$$\lambda F\big(J_\lambda^F x_2, J_\lambda^F x_1\big) - \langle \exp_{J_\lambda^F x_2}^{-1} x_2, \exp_{J_\lambda^F x_2}^{-1} J_\lambda^F x_1 \rangle \geqslant 0. \tag{4.8}$$

If we sum inequalities (4.7) and (4.8), it follows that

$$\langle \exp_{J_\lambda^F x_1}^{-1} x_1, \exp_{J_\lambda^F x_1}^{-1} J_\lambda^F x_2 \rangle + \langle \exp_{J_\lambda^F x_2}^{-1} x_2, \exp_{J_\lambda^F x_2}^{-1} J_\lambda^F x_1 \rangle \leqslant 0,$$

for any $x_1, x_2 \in \mathcal{D}(J_\lambda^F)$, which is equivalent to say that J_λ^F is firmly nonexpansive as we proved in Proposition 4.2.

(iii) Given $x \in \mathcal{D}(J_\lambda^F)$,

$$x = J_\lambda^F x \quad \Leftrightarrow \quad F(x, y) - \langle \exp_x^{-1} x, \exp_x^{-1} y \rangle \geqslant 0 \quad (\forall y \in K) \quad \Leftrightarrow \quad F(x, y) \geqslant 0 \quad (\forall y \in K).$$

So $\mathrm{Fix}(J_\lambda^F) = \mathrm{EP}(F)$.

(iv) This item follows from the same argument as in the proof of Corollary 3 in [31]. □

Remark 4.8. The resolvent could be defined for a set-valued bifunction $F : K \times K \to 2^{\mathbb{R}}$ as the set-valued function $J_\lambda^F : M \to 2^K$ such that

$$J_\lambda^F(x) = \big\{ z \in K \;\big|\; \lambda u - \langle \exp_z^{-1} x, \exp_z^{-1} y \rangle \geqslant 0,\ \forall y \in K,\ \forall u \in F(z, y) \big\},$$

for any $\lambda > 0$ and any $x \in M$. Then, assuming that F monotone means that $u + v \leqslant 0$ for any $u \in F(x, y)$, $v \in F(y, x)$ and $x, y \in K$, the previous theorem would remain true except for (iii) which needs F to be single-valued.

The next theorem gives sufficient conditions for the resolvent to have full domain.

Theorem 4.9. *Let $F : K \times K \to \mathbb{R}$ be a bifunction satisfying hypotheses* (i), (iii) *and* (iv) *in Theorem* 3.2. *Additionally, assume that*

(a) *F is monotone*;
(b) *for any fixed $x \in K$, the map $y \mapsto F(x, y)$ is convex.*

Then $\mathcal{D}(J_\lambda^F) = M$.

Proof. First of all, note that being F convex in the second variable, it satisfies all the hypotheses in Theorem 3.2. Hence there exists a point $y_0 \in K$ such that

$$-F(y_0, z) \leqslant 0, \quad \forall z \in K. \tag{4.9}$$

Fix $\tilde{x} \in M, \lambda > 0$ and define $G : K \times K \to \mathbb{R}$ by

$$G(z, y) := \lambda F(z, y) - \langle \exp_z^{-1} \tilde{x}, \exp_z^{-1} y \rangle.$$

Let us prove that G satisfies all the assumptions in Theorem 3.2. It easily follows from (i) that $G(z, z) \geqslant 0$ and that for any fixed $y \in K$, the function $z \mapsto G(z, y)$ is upper semicontinuous. To prove that for any fixed $z \in K$, the set $\Gamma := \{y \in K\colon G(z, y) < 0\}$ is convex, observe that the mapping

$$y \mapsto -\langle \exp_z^{-1} \tilde{x}, \exp_z^{-1} y \rangle$$

is convex by Proposition 2.9. Being the sum of two convex functions, $G(z,\cdot)$ itself is convex and hence Γ is convex as well. Finally, fix $y_0 \in K$ such that inequality (4.9) holds and observe that by inequality (2.7) and by the monotonicity of F we have,

$$\begin{aligned} G(z, y_0) &= \lambda F(z, y_0) - \langle \exp_z^{-1} \tilde{x}, \exp_z^{-1} y_0 \rangle \\ &\leqslant -\lambda F(y_0, z) + \langle \exp_{y_0}^{-1} \tilde{x}, \exp_{y_0}^{-1} z \rangle - d^2(y_0, z) \\ &\leqslant \big(\mathrm{d}(\tilde{x}, y_0) - \mathrm{d}(y_0, z) \big) \mathrm{d}(y_0, z). \end{aligned} \tag{4.10}$$

Set L to be the compact set $\{z \in K\colon\ \mathrm{d}(z, y_0) \leqslant \mathrm{d}(\tilde{x}, y_0)\}$. Hence $y_0 \in L$ and for any $z \in K \setminus L$ we have

$$G(z, y_0) < 0$$

by (4.10). That is, assumption (iv) is also satisfied. Hence we can apply Theorem 3.2 to get the existence of $\tilde{z} \in K$ such that

$$G(\tilde{z}, y) \geqslant 0, \quad \forall y \in K.$$

This last inequality implies $J_\lambda^F(\tilde{x}) = \tilde{z}$. $\square$

The previous theorems allow us to approximate a solution to the equilibrium problem associated to a bifunction F, by means of the resolvent and the sequence of iterates $\{(J_\lambda^F)^n x\}$ whose convergence is assured by Theorem 4.5.

Theorem 4.10. *Let $F : K \times K \to \mathbb{R}$ be a monotone bifunction such that $\mathrm{EP}(F) \neq \emptyset$. Let $\lambda > 0$ and assume that the resolvent of F, J_λ^F, is properly defined with $K \subseteq \mathcal{D}(J_\lambda^F)$. Then, for each $x \in \mathcal{D}(J_\lambda^F)$, the sequence defined by*

$$x_{n+1} = \big(J_\lambda^F\big)^n x, \quad \forall n \geqslant 0, \tag{4.11}$$

converges to an equilibrium point of F.

As happens in Hilbert spaces, the resolvent of a bifunction constitutes a generalization of the resolvent of a maximal monotone vector field and the Moreau–Yosida regularization of a convex function. In these cases, as shown in the following examples, we know that the resolvent is properly defined and moreover, $J_\lambda^F(x) \neq \emptyset$ for any $x \in M$, that is $\mathcal{D}(J_\lambda^F) = M$.

4.3. Resolvents of vector fields

In the single-valued case the resolvent of a maximal monotone vector field can be seen as the resolvent of a bifunction. Let $A : M \to TM$ be a single-valued monotone vector field with full domain $\mathcal{D}(A) = M$; that is, for any $x, y \in M$

$$\langle A(x), \exp_x^{-1} y \rangle \leqslant \langle A(y), -\exp_y^{-1} x \rangle.$$

Assume that A is maximal monotone; that is, for any $x \in M$ and $u \in T_x M$, the condition

$$\langle u, \exp_x^{-1} y \rangle \leqslant \langle A(y), -\exp_y^{-1} x \rangle, \quad \forall y \in M, \tag{4.12}$$

implies that $u = A(x)$; in other words, there exists no other monotone vector field containing A. For any $\lambda > 0$, the resolvent of A, $J_\lambda^A : M \to 2^M$, defined by

$$J_\lambda^A(x) := \{z \in M \mid x = \exp_z \lambda A z\}, \quad \forall x \in M,$$

was proved to be single-valued and firmly nonexpansive with full domain $\mathcal{D}(J_\lambda^A) = M$ (see [31]). Define the bifunction $F : M \times M \to \mathbb{R}$ by

$$F(x, y) = \langle Ax, \exp_x^{-1} y \rangle, \quad \forall x, y \in M. \tag{4.13}$$

The monotonicity of A implies the monotonicity of F. On the other hand, for any $x \in M$ and $\lambda > 0$, the resolvent of F can be written as

$$J_\lambda^F(x) = \big\{ z \in K \bigm| \langle \lambda A(z) - \exp_z^{-1} x, \exp_z^{-1} y \rangle \geqslant 0,\ \forall y \in M \big\}.$$

Thus, if $z = J_\lambda^A(x)$ we have that $\lambda A z = \exp_z^{-1} x$ and then $z \in J_\lambda^F(x)$; that is,

$$J_\lambda^A(x) \subseteq J_\lambda^F(x), \quad \forall x \in M.$$

Therefore the conditions in Theorem 4.7 hold for F defined in (4.13), so the fact that J_λ^F is single-valued implies that

$$J_\lambda^A(x) = J_\lambda^F(x), \quad \forall x \in M.$$

Then the resolvent of A allows us to approximate a solution to the variational inequality problem

$$\langle Ax, \exp_x^{-1} y\rangle \geqslant 0, \quad \forall y \in M, \tag{4.14}$$

whenever it exists, by means of the sequence $\{(J_\lambda^A)^n(x)\}$ for some $x \in M$.

Convergence properties of other iterative methods to solve the variational inequality problem (4.14), or equivalently to find a singularity of a monotone vector field, can be found in the literature (see for instance [11,16,34]).

4.4. Moreau–Yosida regularization of a convex function

Letting $f : M \to \mathbb{R}$ be a convex function, the Moreau–Yosida regularization $f_\lambda : M \to \mathbb{R}$ of f is defined by

$$f_\lambda(x) = \operatorname*{argmin}_{y\in M}\left\{\lambda f(y) + \frac{1}{2}\,\mathrm{d}^2(x, y)\right\}. \tag{4.15}$$

In [17] it was proved that there exists a unique point $y_\lambda = f_\lambda(x)$ for any $x \in M$ and $\lambda \geqslant 0$, which is characterized by

$$\frac{1}{\lambda}\exp_{y_\lambda}^{-1} x \in \partial f(y_\lambda). \tag{4.16}$$

Then the mapping f_λ is well defined and single-valued. On the other hand, the Moreau–Yosida regularization of a convex function is the resolvent of the bifunction $F : M \times M \to \mathbb{R}$ defined by $F(x, y) = f(y) - f(x)$. Indeed, given $x \in M$, let $z = f_\lambda(x)$. This means that $\frac{1}{\lambda}\exp_z^{-1} x \in \partial f(z)$, and by definition of the subdifferential of f, for any $y \in M$,

$$\frac{1}{\lambda}\langle \exp_z^{-1} x, \exp_z^{-1} y\rangle \leqslant f(y) - f(z).$$

Equivalently,

$$\lambda F(z, y) - \langle \exp_z^{-1} x, \exp_z^{-1} y\rangle \geqslant 0,$$

so $z \in J_\lambda^F(x)$. Then F is properly defined. Since F is monotone as well, Theorem 4.7 assures that J_λ^F is single-valued, therefore we get the equality

$$f_\lambda(x) = J_\lambda^F(x), \quad \forall x \in M.$$

Note that a fixed point of f_λ is a solution of the minimization problem

$$\min_{x\in M} f(x). \tag{4.17}$$

Thus the firmly nonexpansivity of f_λ allows us to apply Theorem 4.5 to get the convergence of the sequence $\{(f_\lambda)^n(x)\}$ to a minimizer of f. The resulting sequence is the proximal point algorithm for convex functions, first studied in [17].

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