Journal of Algebra 339 (2011) 1–26

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Generators and relations for partition monoids and algebras

James East [1]

School of Computing and Mathematics, University of Western Sydney, Locked Bag 1797, Penrith NSW 2751, Australia

ARTICLE INFO

Article history:
Received 1 May 2008
Available online 25 May 2011
Communicated by Michel Broué

MSC:
20M05
20M20

Keywords:
Partition algebras
Transformation semigroups
Presentations

ABSTRACT

We investigate the manner in which the partition monoid $\mathscr{P}_n$ and algebra $\mathscr{P}_n^\xi$ may be presented by generators and relations. Making use of structural properties of $\mathscr{P}_n$, as well as presentations for several key submonoids, we obtain a number of presentations for $\mathscr{P}_n$, including that given (without a complete proof) by Halverson and Ram in 2005. We then conclude by showing how each of these presentations gives rise to an algebra presentation for $\mathscr{P}_n^\xi$.

1. Introduction

Since the introduction of the Brauer algebra [2] in 1937, diagram algebras (i.e. algebras with a basis consisting of diagrams) have received considerable attention. Other diagram algebras include the Temperly–Lieb algebra [17] and the Jones algebra [22]. The motivation for studying such algebras is considerable, and they arise in a wide variety of mathematical disciplines such as statistical mechanics, knot theory, and the study of algebraic groups. The three algebras mentioned above, as well as many others, are all subalgebras of the so-called *partition algebra* [26], whose basis consists of all set-partitions of a $2n$-element set; these partitions are represented diagrammatically as (equivalence classes of) graphs on $2n$ vertices. The partition algebra arises naturally when studying Schur–Weyl duality in the representation theory of the symmetric group, and a thorough exposition may be found in the survey-style article of Halverson and Ram [19] as well as an extensive list of references; see also the more recent studies [10,12,15]. A number of prominent semigroup examples may also be thought of as diagram algebras, including the (full) transformation semigroups (see for example [20] or [21]), and the symmetric and dual symmetric inverse semigroups (see [24] and [16] respectively);

E-mail address: J.East@uws.edu.au.

[1] This work was completed while the author was a postdoctoral fellow at La Trobe University.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2011.04.008

see also [14,23] for some other examples. In fact, Wilcox [29] realized the partition algebra as a twisted semigroup algebra of the *partition monoid* $\mathscr{P}_n$ also considered in [19]; this point of view allows much information, such as cellularity [18] of the algebra, to be deduced from various aspects of the structure of the monoid. The main results of [19] include a characterization of the semisimplicity of the partition algebras, the construction of Specht modules and Murphy elements, and a presentation for $\mathscr{P}_n$ in terms of generators and relations, although the proof given for this presentation is far from being complete, with several key steps being illustrated by examples. The importance of having such a presentation lies in the fact that representations (homomorphisms into other algebras) may be constructed from presentations by choosing the images of the generators, and checking that the relations are preserved. (The complex nature of the partition algebra's multiplication makes this method of constructing representations highly desirable.) Since the well-definedness of such a representation relies on the soundness of the presentation used, it is important to not only have a presentation, but also a complete proof, and that is the purpose of the current work. The approach we use here is very different from that of [19], and bears more similarity to the techniques introduced in [6,8]. We first analyze aspects of the structure of $\mathscr{P}_n$, discussing concepts related to ideas from the theory of transformation semigroups and, in particular, we show that there is a natural factorization $\mathscr{P}_n = \mathscr{L}_n \mathscr{I}_n \mathscr{R}_n$, where $\mathscr{I}_n$ is (isomorphic to) the symmetric inverse semigroup, $\mathscr{L}_n$ is a submonoid of (an isomorphic copy of) the full transformation semigroup, and $\mathscr{R}_n$ is an anti-isomorphic copy of $\mathscr{L}_n$. We then use this factorization, as well as known presentations for the three submonoids, to deduce a presentation for $\mathscr{P}_n$. This presentation contains n^2 generators (the Halverson–Ram presentation uses $3n-2$), so we then set out to simplify the presentation. Our first simplified presentation involves $n+1$ generators and includes both Popova's presentation [28] for the symmetric inverse semigroup and FitzGerald's presentation [13] for the largest factorizable inverse submonoid of the dual symmetric inverse semigroup. From this, we are able to deduce (and therefore prove correct) the Halverson–Ram presentation, and we also give a presentation on four generators. We conclude by showing how each of these presentations gives rise to a presentation for the partition algebra, by developing a general theory of presentations for twisted semigroup algebras.

We now pause to establish the language and notation we will be using for monoids and presentations.

An equivalence $\sim$ on a monoid M is a *congruence* if $ac \sim bd$ whenever $a, b, c, d \in M$ with $a \sim b$ and $c \sim d$. The quotient $M/{\sim}$ is the set of all $\sim$-classes of M, which is itself a monoid under the induced operation. The *kernel* of a monoid homomorphism $\varphi : M \to N$ is the congruence $\ker(\varphi) = \{(s,t) \in M \times M \mid s\varphi = t\varphi\}$, and the fundamental homomorphism theorem (for monoids) states that $M/\ker(\varphi) \cong \mathrm{im}(\varphi)$.

Let X be an alphabet (a set whose elements are called letters), and denote by X^* the free monoid on X, which consists of all words over X (including the empty word which will be denoted by 1), under the operation of concatenation. Let $R \subseteq X^* \times X^*$ and write $R^\sharp$ for the congruence on X^* generated by R. We say that a monoid M has (monoid) presentation $\langle X \mid R \rangle$ if $M \cong X^*/R^\sharp$ or, equivalently, if there is an epimorphism $X^* \to M$ with kernel $R^\sharp$. If φ is such an epimorphism, then we say that M has presentation $\langle X \mid R \rangle$ via φ. Elements of X are also called generators, and an element $(w_1, w_2) \in R$ is called a relation and will generally be displayed as an equation: $w_1 = w_2$. All presentations we consider will be monoid presentations, until in Section 6 where we will also consider algebra presentations.

It should also be noted that all numbers considered in this article are assumed to be integers, so a statement such as "let $1 \leqslant i \leqslant 5$" should be read as "let i be an integer for which $1 \leqslant i \leqslant 5$".

2. The partition monoid

Let n be a positive integer which we fix throughout this article, and write $\mathbf{n}$ for the finite set $\{1, \ldots, n\}$. We also write $\mathbf{n}' = \{1', \ldots, n'\}$ for a set in one–one correspondence with $\mathbf{n}$. A *partition on* $\mathbf{n} \cup \mathbf{n}'$ (or simply a *partition*) is a collection $\alpha = \{A_1, \ldots, A_k\}$ of pairwise-disjoint non-empty sets whose union is $\mathbf{n} \cup \mathbf{n}'$. The sets $A_1, \ldots, A_k$ are called the *blocks* of α. The set $\mathscr{P}_n$ of all partitions on $\mathbf{n} \cup \mathbf{n}'$ forms a monoid, the so-called *partition monoid*, under an associative operation we will describe shortly. A partition $\alpha \in \mathscr{P}_n$ may be represented by a graph on the vertex set $\mathbf{n} \cup \mathbf{n}'$ as follows. We

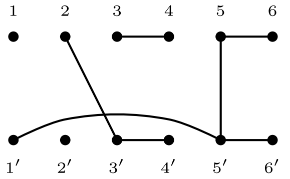


Fig. 1. A graphical representation of a partition from $\mathscr{P}_6$.

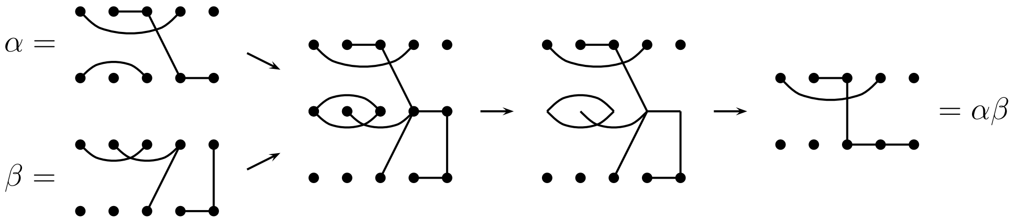


Fig. 2. Calculating the product of two partitions $\alpha, \beta \in \mathscr{P}_5$.

arrange vertices $1, \ldots, n$ in a row (increasing from left to right) and vertices $1', \ldots, n'$ in a parallel row directly below. We then add edges in such a way that two vertices are connected by a path if and only if they belong to the same block of α. For example, the partition

$$\big\{\{1\}, \{2, 3', 4'\}, \{3, 4\}, \{5, 6, 1', 5', 6'\}, \{2'\}\big\} \in \mathscr{P}_6$$

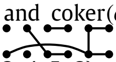
is represented by the graph pictured in Fig. 1. Of course, such a graphical representation is not unique, but we will identify two graphs on the vertex set $\mathbf{n} \cup \mathbf{n}'$ if they have the same connected components. In the same way, we will not distinguish between a partition and a graph which represents it. In order to describe the product alluded to above, let $\alpha, \beta \in \mathscr{P}_n$. We first stack (graphs representing) α and β so that vertices $1', \ldots, n'$ of α are identified with vertices $1, \ldots, n$ of β. The connected components of this graph are then constructed, and we finally delete the middle row of vertices as well as any edges which are not connected to an upper or lower vertex; the resulting graph is the product $\alpha\beta$. An example is given in Fig. 2.

We now introduce some notation and terminology which we will use when studying partitions. These definitions are inspired by analogous concepts in the theory of transformation semigroups. With this in mind, let $\alpha \in \mathscr{P}_n$. For $i \in \mathbf{n} \cup \mathbf{n}'$, let $[i]_\alpha$ denote the block of α containing i. We define the *domain* and *codomain* of α to be the sets

$$\operatorname{dom}(\alpha) = \big\{i \in \mathbf{n} \;\big|\; [i]_\alpha \cap \mathbf{n}' \neq \emptyset\big\},$$
$$\operatorname{codom}(\alpha) = \big\{i \in \mathbf{n} \;\big|\; [i']_\alpha \cap \mathbf{n} \neq \emptyset\big\}.$$

We also define the *kernel* and *cokernel* of α to be the equivalences

$$\ker(\alpha) = \big\{(i, j) \in \mathbf{n} \times \mathbf{n} \;\big|\; [i]_\alpha = [j]_\alpha\big\},$$
$$\operatorname{coker}(\alpha) = \big\{(i, j) \in \mathbf{n} \times \mathbf{n} \;\big|\; [i']_\alpha = [j']_\alpha\big\}.$$

The equivalence classes of $\mathbf{n}$ with respect to $\ker(\alpha)$ and $\operatorname{coker}(\alpha)$ are called the *kernel-classes* and *cokernel-classes* of α. To illustrate these ideas, let $\alpha =$  $\in \mathscr{P}_6$ denote the partition pictured in Fig. 1. Then $\operatorname{dom}(\alpha) = \{2, 5, 6\}$ and $\operatorname{codom}(\alpha) = \{1, 3, 4, 5, 6\}$, and we see that α has kernel-classes $\{1\}$, $\{2\}$, $\{3, 4\}$, $\{5, 6\}$ and cokernel-classes $\{1, 5, 6\}$, $\{2\}$, $\{3, 4\}$.

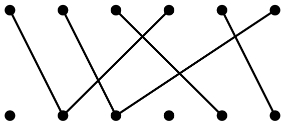


Fig. 3. An element of $\mathscr{T}_6$ corresponding to the transformation $\binom{1\,2\,3\,4\,5\,6}{2\,3\,5\,2\,6\,3} \in \mathcal{T}_6$.

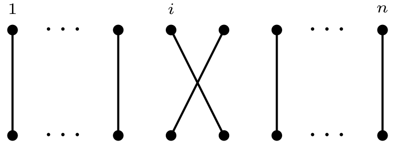


Fig. 4. The simple transposition $\bar{s}_i \in \mathscr{S}_n$.

It is immediate from the definitions that

$$\operatorname{dom}(\alpha\beta) \subseteq \operatorname{dom}(\alpha), \qquad \ker(\alpha) \subseteq \ker(\alpha\beta),$$
$$\operatorname{codom}(\alpha\beta) \subseteq \operatorname{codom}(\beta), \qquad \operatorname{coker}(\beta) \subseteq \operatorname{coker}(\alpha\beta)$$

for all $\alpha, \beta \in \mathscr{P}_n$.

3. Important subsemigroups

The partition monoid contains a number of well-known and well-studied subsemigroups which we will discuss in this section. These include the (full) transformation semigroup and the symmetric and dual symmetric inverse semigroups. We will treat these semigroups in Sections 3.1–3.3 and, in Section 3.4, we show how the partition monoid is "built up" out of them.

3.1. The (full) transformation semigroup

The *(full) transformation semigroup on* $\mathbf{n}$ is the (regular) semigroup $\mathcal{T}_n$ of all transformations on $\mathbf{n}$ (i.e. all functions from $\mathbf{n}$ to itself) under the operation of composition. The group of units of $\mathcal{T}_n$ is the symmetric group $\mathcal{S}_n$, which consists of all permutations on $\mathbf{n}$. Now put

$$\mathscr{T}_n = \{\alpha \in \mathscr{P}_n \mid \text{each block of } \alpha \text{ contains exactly one element of } \mathbf{n}'\}.$$

By considering the graphical representation of the elements of $\mathscr{T}_n$, we see that there is a natural one–one correspondence $\varphi : \mathcal{T}_n \to \mathscr{T}_n$; see Fig. 3 for an example. It is immediate from the rules for multiplication in $\mathcal{T}_n$ and $\mathscr{P}_n$ that $\varphi : \mathcal{T}_n \to \mathscr{T}_n$ is in fact an isomorphism, and so we may identify the submonoid $\mathscr{T}_n \subseteq \mathscr{P}_n$ with the transformation semigroup $\mathcal{T}_n$. In this way, given $\alpha \in \mathscr{T}_n$ and $i \in \mathbf{n}$, we will often write $i\alpha$ for the image of i under (the transformation associated to) α.

Under the identification of $\mathcal{T}_n$ with $\mathscr{T}_n$, we see that the symmetric group $\mathcal{S}_n \subseteq \mathcal{T}_n$ is isomorphic to the subgroup

$$\mathscr{S}_n = \{\alpha \in \mathscr{P}_n \mid \text{every block of } \alpha \text{ contains one element of } \mathbf{n} \text{ and one of } \mathbf{n}'\},$$

which is easily seen to be the group of units of $\mathscr{P}_n$. In 1897, Moore [27] gave a presentation for the symmetric group $\mathscr{S}_n$ which we now describe. First, for $1 \leqslant i \leqslant n-1$, we define $\bar{s}_i \in \mathscr{S}_n$ to be the simple transposition which interchanges i and $i+1$; see Fig. 4. (The reason for our use of this over-line notation will become clear shortly.)

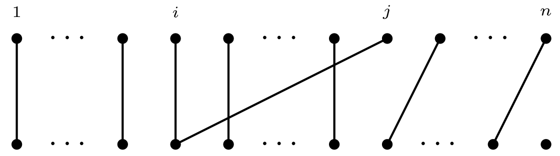


Fig. 5. The map $\bar{\lambda}_{ij} \in \mathscr{L}_n$.

Define an alphabet $S = \{s_1, \ldots, s_{n-1}\}$, and consider the relations

$$s_i^2 = 1 \qquad \text{for all } i, \tag{S1}$$

$$s_i s_j = s_j s_i \qquad \text{if } |i - j| > 1, \tag{S2}$$

$$s_i s_j s_i = s_j s_i s_j \qquad \text{if } |i - j| = 1. \tag{S3}$$

Theorem 1. *(See Moore [27].) The symmetric group $\mathscr{S}_n$ has (monoid) presentation $\langle S \mid \text{(S1)–(S3)} \rangle$ via*

$$\phi_S : S^* \to \mathscr{S}_n : s_i \mapsto \bar{s}_i.$$

We will write $\sim_S$ for the congruence on S^* generated by relations (S1)–(S3) and, for $w = s_{i_1} \cdots s_{i_k} \in S^*$, we will write w^{-1} for the word $s_{i_k} \cdots s_{i_1}$ obtained by reversing w, noting that $ww^{-1} \sim_S w^{-1}w \sim_S 1$ for all $w \in S^*$.

A presentation for $\mathscr{T}_n$, which extends Moore's presentation for $\mathscr{S}_n$, may be found in [1], although we will not require it here. Rather, we will make use of a presentation for a submonoid $\mathscr{L}_n \subseteq \mathscr{T}_n$ which we now describe. Let $\alpha \in \mathscr{T}_n$, and suppose the kernel-classes of α are $K_1, \ldots, K_r$. We write $\mathrm{Mins}(\alpha)$ for the set $\{\min(K_1), \ldots, \min(K_r)\}$, and we say that α is *block-order-preserving* if $i\alpha < j\alpha$ whenever $i, j \in \mathrm{Mins}(\alpha)$ and $i < j$. Put

$$\mathscr{L}_n = \{\alpha \in \mathscr{T}_n \mid \alpha \text{ is block-order-preserving and } \mathrm{codom}(\alpha) = \mathbf{k} \text{ for some } k \in \mathbf{n}\}.$$

This set was shown to be a submonoid of $\mathscr{T}_n$ in [8], where it was studied in relation to the semigroup $\mathcal{T}_n \setminus \mathcal{S}_n$ of all non-invertible transformations on $\mathbf{n}$. (The monoid $\mathscr{L}_n$ was denoted by $\mathcal{A}_n$ in [8], and also played a crucial role in the study of other semigroups in [9,11].) For $1 \leqslant i < j \leqslant n$, let $\bar{\lambda}_{ij} \in \mathscr{L}_n$ denote the map defined by

$$x\bar{\lambda}_{ij} = \begin{cases} x & \text{if } 1 \leqslant x < j, \\ i & \text{if } x = j, \\ x - 1 & \text{if } j < x \leqslant n. \end{cases}$$

This map is also pictured in Fig. 5. Define an alphabet $L = \{\lambda_{ij} \mid 1 \leqslant i < j \leqslant n\}$, and consider the relations

$$\lambda_{kl}\lambda_{in} = \lambda_{kl} \qquad \text{for all } i, k, l, \tag{L1}$$

$$\lambda_{jk}\lambda_{ij} = \lambda_{ik}\lambda_{ij} = \lambda_{ij}\lambda_{i,k-1} \qquad \text{if } i < j < k, \tag{L2}$$

$$\lambda_{kl}\lambda_{ij} = \begin{cases} \lambda_{ij}\lambda_{k-1,l-1} & \text{if } i < j < k < l, \\ \lambda_{ij}\lambda_{k,l-1} & \text{if } i < k < j < l, \\ \lambda_{i,j+1}\lambda_{kl} & \text{if } i < k < l \leqslant j \leqslant n-1. \end{cases} \tag{L3)–(L5}$$

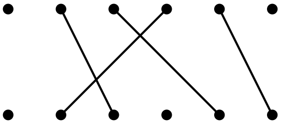


Fig. 6. An element of $\mathscr{I}_6$ corresponding to the partial permutation $\binom{1\ 2\ 3\ 4\ 5\ 6}{-\ 3\ 5\ 2\ 6\ -} \in \mathcal{I}_6$.

Theorem 2. *(See East [8].) The monoid $\mathscr{L}_n$ has presentation $\langle L \mid (\mathrm{L1})\text{–}(\mathrm{L5})\rangle$ via*

$$\phi_L : L^* \to \mathscr{L}_n : \lambda_{ij} \mapsto \bar{\lambda}_{ij}.$$

We denote by $\sim_L$ the congruence on L^* generated by relations (L1)–(L5) and, for $w \in L^*$, we write $\overline{w}$ for the transformation $w\phi_L \in \mathscr{L}_n$. We will now describe a set of normal forms for words over L which will be used in Section 4. In order to do this, we first define an order $\prec$ on the alphabet L by

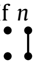
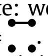
$$\lambda_{ij} \prec \lambda_{rs} \quad \text{if and only if} \quad i < r \quad \text{or} \quad i = r \text{ and } j > s.$$

We say that a word $\lambda_{i_1 j_1} \cdots \lambda_{i_k j_k} \in L^*$ is *ascending* if $\lambda_{i_1 j_1} \prec \cdots \prec \lambda_{i_k j_k}$. An ascending word $\lambda_{i_1 j_1} \cdots \lambda_{i_k j_k}$ is said to be *normal* if $j_r \leqslant n - r + 1$ for all $r \in \mathbf{k}$. Let $N_L \subseteq L^*$ denote the set of all normal words over L. For $0 \leqslant k \leqslant n-1$, write $N_L^{(k)}$ for the set of all normal words of length k. The following facts were proved in [8], as key steps towards the proof of Theorem 2.

Lemma 3.

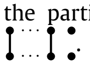
(i) *Every word over L is $\sim_L$-equivalent to a unique normal word.*
(ii) *If $w \in N_L^{(k)}$, then* $\mathrm{codom}(\overline{w}) = \{1, \ldots, n-k\}$.

3.2. The symmetric inverse semigroup

The *partial transformation semigroup on* $\mathbf{n}$ is the (regular) semigroup $\mathcal{PT}_n$ of all partial transformations on $\mathbf{n}$ (that is, all partially defined functions from $\mathbf{n}$ to itself) under the operation of "compose wherever possible". As was the case in the previous section, there is a natural injective map $\varphi' : \mathcal{PT}_n \to \mathscr{P}_n$. If $n \geqslant 2$, this map is not an embedding (i.e. not a homomorphism), as the elements $\alpha =$  and $\beta =$  of $\mathcal{PT}_2$ demonstrate: we see that the rule for multiplication in $\mathcal{PT}_2$ gives $\alpha\beta =$ , while in $\mathscr{P}_2$, we have $(\alpha\varphi')(\beta\varphi') =$ . However, it is easy to check that we *do* obtain an embedding when we restrict φ' to the *symmetric inverse semigroup* $\mathcal{I}_n \subseteq \mathcal{PT}_n$ which is the (factorizable inverse) semigroup of all *injective* partial transformations on $\mathbf{n}$. Thus we may identify $\mathcal{I}_n$ with its image

$$\mathscr{I}_n = \{\alpha \in \mathscr{P}_n \mid \text{every non-trivial block of } \alpha \text{ contains one element of } \mathbf{n} \text{ and one of } \mathbf{n}'\}$$

under φ'. An element of $\mathscr{I}_6$ is pictured in Fig. 6.

We now describe Popova's presentation [28] for $\mathscr{I}_n$ which also extends Moore's presentation for $\mathscr{S}_n$. For $A \subseteq \mathbf{n}$, denote by $\mathrm{id}_A \in \mathscr{I}_n$ the partial permutation obtained by restricting the identity map to A, and put $\bar{\varepsilon} = \mathrm{id}_{\{1,\ldots,n-1\}} =$ . Define an alphabet $I = S \cup \{\varepsilon\}$, and consider the relations

$$s_i^2 = 1 \qquad \text{for all } i, \tag{I1}$$

$$s_i s_j = s_j s_i \qquad \text{if } |i-j| > 1, \tag{I2}$$

$$s_i s_j s_i = s_j s_i s_j \qquad \text{if } |i-j| = 1, \tag{I3}$$

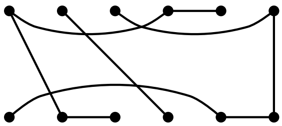


Fig. 7. An element of $\mathscr{J}_6$ corresponding to the block bijection $\left(\begin{smallmatrix} 1,4,5 \\ 2,3 \end{smallmatrix} \,\middle|\, \begin{smallmatrix} 2 \\ 4 \end{smallmatrix} \,\middle|\, \begin{smallmatrix} 3,6 \\ 1,5,6 \end{smallmatrix}\right) \in \mathcal{J}_6$.

$$\varepsilon^2 = \varepsilon, \tag{I4}$$

$$\varepsilon s_i = s_i \varepsilon \qquad \text{if } i \leqslant n-2, \tag{I5}$$

$$\varepsilon s_{n-1} \varepsilon s_{n-1} = s_{n-1} \varepsilon s_{n-1} \varepsilon = \varepsilon s_{n-1} \varepsilon. \tag{I6}$$

Theorem 4. *(See Popova [28].) The symmetric inverse semigroup $\mathscr{I}_n$ has presentation $\langle I \mid (\mathrm{I1})\text{–}(\mathrm{I6}) \rangle$ via*

$$\phi_I : I^* \to \mathscr{I}_n : \begin{cases} s_i \mapsto \bar{s}_i, \\ \varepsilon \mapsto \bar{\varepsilon}. \end{cases}$$

We write $\sim_I$ for the congruence on I^* generated by relations (I1)–(I6) and again, for $w \in I^*$, we denote by $\overline{w}$ the partial permutation $w\phi_I \in \mathscr{I}_n$. We define words

$$\varepsilon_i = (s_i \cdots s_{n-1}) \varepsilon (s_{n-1} \cdots s_i) \quad \text{for each } 1 \leqslant i \leqslant n,$$

$$e_i = \varepsilon_n \cdots \varepsilon_i \quad \text{for each } 1 \leqslant i \leqslant n+1.$$

(Note that $\varepsilon_n = e_n = \varepsilon$, and that $e_{n+1} = 1$.) It is easy to check, diagrammatically, that $\bar{\varepsilon}_i = \mathrm{id}_{\mathbf{n} \setminus \{i\}}$ and $\bar{e}_i = \mathrm{id}_{\{1,\ldots,i-1\}}$. Together with Theorem 4, this proves the following.

Lemma 5. *Suppose that $w \in I^*$ and that $1 \leqslant i, j \leqslant n+1$. Then*

$$\mathrm{dom}(\overline{e_i w e_j}) \subseteq \{1, \ldots, i-1\} \quad \textit{and} \quad \mathrm{codom}(\overline{e_i w e_j}) \subseteq \{1, \ldots, j-1\}.$$

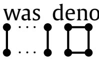
3.3. *The dual symmetric inverse semigroup*

The *dual symmetric inverse semigroup on* $\mathbf{n}$ is the (inverse) semigroup $\mathcal{J}_n$ of all *block bijections* on $\mathbf{n}$. Dual symmetric inverse semigroups were introduced in [16] as categorical duals to the symmetric inverse semigroups; there the notation $\mathcal{I}_n^*$ was used, but here we use the notation $\mathcal{J}_n$ in order to avoid confusion with free monoids. Out of the many descriptions of $\mathcal{J}_n$ from [16], we prefer to think of a block bijection as a bijective map between quotients of $\mathbf{n}$. In this way, we see that there is an injective map $\varphi'' : \mathcal{J}_n \to \mathscr{P}_n$ with image

$$\mathscr{J}_n = \{\alpha \in \mathscr{P}_n \mid \mathrm{dom}(\alpha) = \mathrm{codom}(\alpha) = \mathbf{n}\}.$$

An element of $\mathscr{J}_6$ is pictured in Fig. 7. It was shown in [25] that φ'' is in fact a homomorphism, and so we may identify $\mathcal{J}_n$ with the submonoid $\mathscr{J}_n \subseteq \mathscr{P}_n$. A presentation for $\mathscr{J}_n$, again extending Moore's presentation for $\mathscr{S}_n$, was discovered in [4], although we will not require it here. Rather, we make use of a presentation [13] for the set

$$\mathscr{F}_n = \{\alpha \in \mathscr{J}_n \mid \text{each block } A \text{ of } \alpha \text{ satisfies } |A \cap \mathbf{n}| = |A \cap \mathbf{n}'|\}$$

which was shown in [16] to be the largest factorizable inverse submonoid of $\mathscr{J}_n$. (In [16], this set was denoted $\mathscr{F}_n^*$, and its elements were called *uniform*.) With this presentation in mind, put $\bar{t} =$  $\in \mathscr{F}_n$. Define an alphabet $F = S \cup \{t\}$, and consider the relations

$$s_i^2 = 1 \qquad \text{for all } i, \tag{F1}$$

$$s_i s_j = s_j s_i \qquad \text{if } |i-j| > 1, \tag{F2}$$

$$s_i s_j s_i = s_j s_i s_j \qquad \text{if } |i-j| = 1, \tag{F3}$$

$$t^2 = t = s_{n-1} t = t s_{n-1}, \tag{F4}$$

$$t s_i = s_i t \qquad \text{if } i \leqslant n-3, \tag{F5}$$

$$t s_{n-2} t s_{n-2} = s_{n-2} t s_{n-2} t, \tag{F6}$$

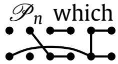
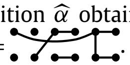
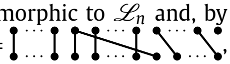
$$t(s_{n-2}s_{n-1}s_{n-3}s_{n-2})t(s_{n-2}s_{n-3}s_{n-1}s_{n-2}) = (s_{n-2}s_{n-1}s_{n-3}s_{n-2})t(s_{n-2}s_{n-3}s_{n-1}s_{n-2})t. \tag{F7}$$

Theorem 6. *(See FitzGerald [13].) The monoid $\mathscr{F}_n$ of all uniform block bijections on* **n** *has presentation $\langle F \mid \text{(F1)–(F7)} \rangle$ via*

$$\phi_F : F^* \to \mathscr{F}_n : \begin{cases} s_i \mapsto \bar{s}_i, \\ t \mapsto \bar{t}. \end{cases}$$

This presentation will not be used until Section 5.1, where the presentation for $\mathscr{P}_n$ obtained in Section 4 is simplified.

3.4. A structural result

There is a natural anti-involution $\widehat{\ } : \mathscr{P}_n \to \mathscr{P}_n$ which maps $\alpha \in \mathscr{P}_n$ to the partition $\widehat{\alpha}$ obtained by reflecting α vertically. For example, if $\alpha =$  $\in \mathscr{P}_6$, then we have $\widehat{\alpha} =$ . Let $\mathscr{R}_n = \widehat{\mathscr{L}_n}$ be the image of the monoid $\mathscr{L}_n$ under the $\widehat{\ }$ map. So $\mathscr{R}_n$ is anti-isomorphic to $\mathscr{L}_n$ and, by the results of Section 3.1, we see that $\mathscr{R}_n$ is generated by the partitions $\overline{\rho}_{ij} =$ , $1 \leqslant i < j \leqslant n$, and that a presentation for $\mathscr{R}_n$ may be obtained from that of $\mathscr{L}_n$ in Theorem 2 by reversing all words in the relations. Specifically, define an alphabet $R = \{\rho_{ij} \mid 1 \leqslant i < j \leqslant n\}$, and consider the relations

$$\rho_{in}\rho_{kl} = \rho_{kl} \qquad \text{for all } i,k,l, \tag{R1}$$

$$\rho_{ij}\rho_{jk} = \rho_{ij}\rho_{ik} = \rho_{i,k-1}\rho_{ij} \qquad \text{if } i<j<k, \tag{R2}$$

$$\rho_{ij}\rho_{kl} = \begin{cases} \rho_{k-1,l-1}\rho_{ij} & \text{if } i<j<k<l, \\ \rho_{k,l-1}\rho_{ij} & \text{if } i<k<j<l, \\ \rho_{kl}\rho_{i,j+1} & \text{if } i<k<l \leqslant j \leqslant n-1. \end{cases} \tag{R3)–(R5}$$

Theorem 7. *The monoid $\mathscr{R}_n$ has presentation $\langle R \mid \text{(R1)–(R5)} \rangle$ via*

$$\phi_R : R^* \to \mathscr{R}_n : \rho_{ij} \mapsto \overline{\rho}_{ij}.$$

We write $\sim_R$ for the congruence on R^* generated by (R1)–(R5) and, for $w \in R^*$, we write $\overline{w}$ for the partition $w\phi_R \in \mathscr{R}_n$. The duality between relations (L1)–(L5) and (R1)–(R5) allows us to translate information concerning $\mathscr{L}_n$ into information concerning $\mathscr{R}_n$ (and vice versa). More formally, we define an anti-isomorphism

$$\widehat{\ } : L^* \to R^* : \lambda_{ij} \mapsto \rho_{ij}.$$

Then for all $w_1, w_2 \in L^*$, we have $w_1 \sim_L w_2$ if and only if $\widehat{w}_1 \sim_R \widehat{w}_2$. Recall that N_L denotes the set of all normal words over L. We now define normal words over R to be elements of the set $N_R = \widehat{N_L} = \{\widehat{w} \mid w \in N_L\}$, and we also write $N_R^{(k)}$ for the set of all normal words (over R) of length k.

Lemma 8.

(i) *Every word over R is $\sim_R$-equivalent to a unique normal word.*
(ii) *If $w \in N_R^{(k)}$, then* $\mathrm{dom}(\overline{w}) = \{1, \dots, n-k\}$.

Our next result shows that there is a natural factorization $\mathscr{P}_n = \mathscr{L}_n \mathscr{I}_n \mathscr{R}_n$, and provides the basis for our derivation of a presentation for $\mathscr{P}_n$ in the next section.

Proposition 9. *Let $\alpha \in \mathscr{P}_n$. Then $\alpha = \beta\gamma\delta$ for unique $\beta \in \mathscr{L}_n$, $\gamma \in \mathscr{I}_n$, and $\delta \in \mathscr{R}_n$, with* $\mathrm{dom}(\gamma) \subseteq \mathrm{codom}(\beta)$ *and* $\mathrm{codom}(\gamma) \subseteq \mathrm{dom}(\delta)$.

Proof. Choose the unique partitions $\beta \in \mathscr{L}_n$ and $\delta \in \mathscr{R}_n$ which satisfy $\ker(\beta) = \ker(\alpha)$ and $\mathrm{coker}(\delta) = \mathrm{coker}(\alpha)$. Suppose that α has kernel-classes $K_1, \dots, K_a$ and cokernel-classes $C_1, \dots, C_b$, where $\min(K_1) < \cdots < \min(K_a)$ and $\min(C_1) < \cdots < \min(C_b)$. Define $\gamma \in \mathscr{I}_n$ by

$$i\gamma = \begin{cases} j & \text{if } i \in \mathbf{a} \text{ and } K_i \cup C_j \text{ is a block of } \alpha, \\ - & \text{otherwise.} \end{cases}$$

Then it is easy to check that $\alpha = \beta\gamma\delta$. By construction, we have $\mathrm{dom}(\gamma) \subseteq \mathbf{a} = \mathrm{codom}(\beta)$ and $\mathrm{codom}(\gamma) \subseteq \mathbf{b} = \mathrm{dom}(\delta)$. This proves the "existence" part of the proposition.

To prove the "uniqueness" part, suppose that $\alpha = \beta'\gamma'\delta'$ where β', γ', δ' satisfy the requirements of the proposition. The conditions $\mathrm{dom}(\gamma') \subseteq \mathrm{codom}(\beta')$ and $\mathrm{codom}(\gamma') \subseteq \mathrm{dom}(\delta')$ imply that $\ker(\beta') = \ker(\alpha) = \ker(\beta)$ and $\mathrm{coker}(\delta') = \mathrm{coker}(\alpha) = \mathrm{coker}(\delta)$, so that $\beta' = \beta$ and $\delta' = \delta$. The same conditions imply that $K_i \cup C_j$ is a block of $\alpha = \beta\gamma'\delta$ if and only if $i \in \mathrm{dom}(\gamma')$ and $j = i\gamma'$, and it follows that $\gamma' = \gamma$. This completes the proof. □

4. A presentation for the partition monoid

We now use the results of the previous section to derive a presentation for the partition monoid $\mathscr{P}_n$. By Proposition 9 we may define an epimorphism

$$\phi : (L \cup I \cup R)^* \to \mathscr{P}_n : \begin{cases} \lambda_{ij} \mapsto \bar{\lambda}_{ij}, \\ s_r \mapsto \bar{s}_r, \\ \varepsilon \mapsto \bar{\varepsilon}, \\ \rho_{ij} \mapsto \bar{\rho}_{ij}. \end{cases}$$

Consider now the relations

$$\rho_{kl}\lambda_{ij} = \begin{cases} \varepsilon\lambda_{i-1,j-1}\rho_{kl} & \text{if } l < i, \\ \varepsilon\lambda_{k,j-1}\rho_{kl} & \text{if } l = i, \\ \varepsilon\lambda_{i,j-1}\rho_{kl} & \text{if } i < l < j, \\ \varepsilon\lambda_{ki}\rho_{ki} & \text{if } k < i < j = l, \\ \varepsilon & \text{if } i = k < j = l, \\ \varepsilon\lambda_{ik}\rho_{ik} & \text{if } i < k < l = j, \\ \varepsilon\lambda_{ij}\rho_{k,l-1} & \text{if } k < j < l, \\ \varepsilon\lambda_{ij}\rho_{i,l-1} & \text{if } j = k, \\ \varepsilon\lambda_{ij}\rho_{k-1,l-1} & \text{if } j < k, \end{cases} \qquad \text{(RL1)–(RL9)}$$

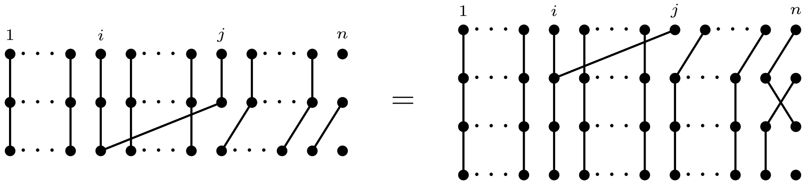


Fig. 8. Relation (EL3): $\varepsilon\lambda_{ij} = \lambda_{ij}s_{n-1}\varepsilon$ if $1 \leqslant i < j \leqslant n-1$.

$$s_r\lambda_{ij} = \begin{cases} \lambda_{ij}s_{r-1} & \text{if } r > j, \\ \lambda_{i,j+1} & \text{if } r = j, \\ \lambda_{i,j-1} & \text{if } r = j-1 > i, \\ \lambda_{ij} & \text{if } r = j-1 = i, \\ \lambda_{ij}s_r & \text{if } i < r < j-1, \\ \lambda_{i+1,j}s_r & \text{if } i = r < j-1, \\ \lambda_{i-1,j}s_r & \text{if } r = i-1, \\ \lambda_{ij}s_r & \text{if } r < i-1, \end{cases} \tag{SL1)–(SL8}$$

$$\rho_{ij}s_r = \begin{cases} s_{r-1}\rho_{ij} & \text{if } r > j, \\ \rho_{i,j+1} & \text{if } r = j, \\ \rho_{i,j-1} & \text{if } r = j-1 > i, \\ \rho_{ij} & \text{if } r = j-1 = i, \\ s_r\rho_{ij} & \text{if } i < r < j-1, \\ s_r\rho_{i+1,j} & \text{if } i = r < j-1, \\ s_r\rho_{i-1,j} & \text{if } r = i-1, \\ s_r\rho_{ij} & \text{if } r < i-1, \end{cases} \tag{RS1)–(RS8}$$

$$\lambda_{ij}\varepsilon = \lambda_{ij} \quad \text{for all } i, j, \tag{EL1}$$

$$\varepsilon\lambda_{in} = \varepsilon \quad \text{for all } i, \tag{EL2}$$

$$\varepsilon\lambda_{ij} = \lambda_{ij}s_{n-1}\varepsilon \quad \text{if } j < n, \tag{EL3}$$

$$\varepsilon\rho_{ij} = \rho_{ij} \quad \text{for all } i, j, \tag{RE1}$$

$$\rho_{in}\varepsilon = \varepsilon \quad \text{for all } i, \tag{RE2}$$

$$\rho_{ij}\varepsilon = \rho_{ij}s_{n-1}\varepsilon \quad \text{if } j < n, \tag{RE3}$$

and denote by $\sim$ the congruence on $(L \cup I \cup R)^*$ generated by the relations (L1)–(L5), (R1)–(R5), (I1)–(I6), (RL1)–(RL9), (SL1)–(SL8), (RS1)–(RS8), (EL1)–(EL3), (RE1)–(RE3). We aim to show that $\sim = \ker\phi$, so that $\mathscr{P}_n$ has presentation

$$\left\langle L \cup I \cup R \;\middle|\; \begin{array}{c} \text{(L1)–(L5), (R1)–(R5), (I1)–(I6), (RL1)–(RL9),} \\ \text{(SL1)–(SL8), (RS1)–(RS8), (EL1)–(EL3), (RE1)–(RE3)} \end{array} \right\rangle$$

via ϕ. We will combine our previous uses of the over-line notation, writing $\overline{w} = w\phi \in \mathscr{P}_n$ for any word $w \in (L \cup I \cup R)^*$.

Lemma 10. *We have the inclusion* $\sim \subseteq \ker\phi$.

Proof. We need to check (diagrammatically) that each relation holds as an equation in $\mathscr{P}_n$ when the generators are replaced by their image under ϕ. This is straightforward (though laborious) and we include only one sample calculation; see Fig. 8 for a proof of relation (EL3). Geometrically, the reader may like to think of this relation as illustrating the two ways to use $\bar{\varepsilon}$ to "snip" the last "string" of $\bar{\lambda}_{ij}$. □

Our next goal is to show (see Corollary 13 below) that $(L \cup I \cup R)^* \sim L^* I^* R^*$, by which we mean that every word w over $L \cup I \cup R$ is $\sim$-equivalent to a word of the form $w_1 w_2 w_3$, where $w_1 \in L^*$, $w_2 \in I^*$, and $w_3 \in R^*$; this gives a "word version" of the factorization $\mathscr{P}_n = \mathscr{L}_n \mathscr{I}_n \mathscr{R}_n$. We then improve this result (in Proposition 16) to incorporate properties of $\mathrm{dom}(\overline{w}_2)$ and $\mathrm{codom}(\overline{w}_2)$ analogous to Proposition 9.

Lemma 11. *If $w \in (L \cup I \cup R)^*$, then $w \sim w_1 w_2$ for some $w_1 \in (L \cup I)^*$ and $w_2 \in R^*$.*

Proof. The proof is by induction on the number k of generators from R appearing in w. If $k = 0$, then $w \in (L \cup I)^*$ and we are done already (with $w_1 = w$ and $w_2 = 1$). Otherwise, we may use relations (RL1)–(RL9), (RE2)–(RE3), and (RS1)–(RS8) to move the right-most generator from R to the right (or make it disappear), and we are done after applying an induction hypothesis. □

Lemma 12. *If $w \in (L \cup I)^*$, then $w \sim w_1 w_2$ for some $w_1 \in L^*$ and $w_2 \in I^*$.*

Proof. This lemma is proved in a similar way to the last, this time using (SL1)–(SL8) and (EL2)–(EL3) and an induction on the number of letters from L appearing in w. □

Corollary 13. *If $w \in (L \cup I \cup R)^*$, then $w \sim w_1 w_2 w_3$ for some $w_1 \in L^*$, $w_2 \in I^*$, and $w_3 \in R^*$.*

Proof. This follows immediately from the previous two lemmas. □

The next two lemmas concern the words ε_i and e_i defined before Lemma 5.

Lemma 14. *If $1 \leqslant i < j \leqslant k < n$, then*

(i) $\lambda_{ij}\varepsilon_k \sim \varepsilon_{k+1}\lambda_{ij}$; *and*
(ii) $\varepsilon_k \rho_{ij} \sim \rho_{ij}\varepsilon_{k+1}$.

Proof. We have

$$\begin{aligned}
\varepsilon_{k+1}\lambda_{ij} &= (s_{k+1}\cdots s_{n-1})\varepsilon(s_{n-1}\cdots s_{k+1})\lambda_{ij} \\
&\sim (s_{k+1}\cdots s_{n-1})\varepsilon\lambda_{ij}(s_{n-2}\cdots s_k) && \text{by (SL1)} \\
&\sim (s_{k+1}\cdots s_{n-1})\lambda_{ij}s_{n-1}\varepsilon(s_{n-2}\cdots s_k) && \text{by (EL3)} \\
&\sim (s_{k+1}\cdots s_{n-1})\lambda_{ij}\varepsilon s_{n-1}\varepsilon(s_{n-2}\cdots s_k) && \text{by (EL1)} \\
&\sim (s_{k+1}\cdots s_{n-1})\lambda_{ij}\varepsilon s_{n-1}\varepsilon s_{n-1}(s_{n-2}\cdots s_k) && \text{by (I6)} \\
&\sim (s_{k+1}\cdots s_{n-1})\lambda_{ij}s_{n-1}\varepsilon s_{n-1}(s_{n-2}\cdots s_k) && \text{by (EL1)} \\
&\sim \lambda_{ij}(s_k\cdots s_{n-2})s_{n-1}\varepsilon s_{n-1}(s_{n-2}\cdots s_k) && \text{by (SL1)} \\
&= \lambda_{ij}\varepsilon_k,
\end{aligned}$$

so that (i) holds. Part (ii) is proved in an almost identical fashion. □

Lemma 15. *If* $0 \leqslant k \leqslant n-1$, *then*

(i) $w \sim we_{n-k+1}$ *for all* $w \in N_L^{(k)}$; *and*
(ii) $w \sim e_{n-k+1}w$ *for all* $w \in N_R^{(k)}$.

Proof. Again, we only prove (i), the proof of (ii) being almost identical. If $k=0$ then $e_{n-k+1}=1$ and there is nothing to show, so suppose $k \geqslant 1$. Then $w = w'\lambda_{ij}$ for some $w' \in N_L^{(k-1)}$ and some $1 \leqslant i < j \leqslant n-k+1$. But then

$$\begin{aligned} we_{n-k+1} &= w'\lambda_{ij}\varepsilon_n\varepsilon_{n-1}\cdots\varepsilon_{n-k+1} \\ &\sim w'\lambda_{ij}\varepsilon_{n-1}\cdots\varepsilon_{n-k+1} && \text{by (EL1), noting that } \varepsilon_n=\varepsilon \\ &\sim w'\varepsilon_n\cdots\varepsilon_{n-k}\lambda_{ij} && \text{by Lemma 14(i)} \\ &= w'e_{n-(k-1)+1}\lambda_{ij} \\ &\sim w'\lambda_{ij} && \text{by an induction hypothesis} \\ &= w, \end{aligned}$$

and we are done. □

Proposition 16. *If* $w \in (L \cup I \cup R)^*$, *then* $w \sim w_1w_2w_3$ *for some* $w_1 \in L^*$, $w_2 \in I^*$, *and* $w_3 \in R^*$ *with* $\text{dom}(\overline{w}_2) \subseteq \text{codom}(\overline{w}_1)$ *and* $\text{codom}(\overline{w}_2) \subseteq \text{dom}(\overline{w}_3)$.

Proof. By Corollary 13 we have $w \sim w_1'w_2'w_3'$ for some $w_1' \in L^*$, $w_2' \in I^*$, and $w_3' \in R^*$. Now $w_1' \sim w_1$ and $w_3' \sim w_3$ for some $w_1 \in N_L$ and $w_3 \in N_R$ by Lemmas 3(i) and 8(i). Write k and l for the lengths of w_1 and w_3 (respectively), and put $w_2 = e_{n-k+1}w_2'e_{n-l+1} \in I^*$. Then by Lemma 15 we have

$$w \sim w_1w_2'w_3 \sim w_1e_{n-k+1}w_2'e_{n-l+1}w_3 = w_1w_2w_3.$$

Now by Lemmas 3(ii), 5, and 8(ii), we have

$$\text{dom}(\overline{w}_2) \subseteq \{1,\dots,n-k\} = \text{codom}(\overline{w}_1) \quad \text{and} \quad \text{codom}(\overline{w}_2) \subseteq \{1,\dots,n-l\} = \text{dom}(\overline{w}_3),$$

and the proof is complete. □

We are now ready to prove the main result of this section.

Theorem 17. *The partition monoid* $\mathscr{P}_n$ *has presentation*

$$\left\langle L \cup I \cup R \;\middle|\; \begin{array}{c} \text{(L1)–(L5), (R1)–(R5), (I1)–(I6), (RL1)–(RL9),} \\ \text{(SL1)–(SL8), (RS1)–(RS8), (EL1)–(EL3), (RE1)–(RE3)} \end{array} \right\rangle$$

via ϕ.

Proof. It remains only to show that $\ker\phi \subseteq \sim$, so suppose $(w,w') \in \ker\phi$. By Proposition 16 we have $w \sim w_1w_2w_3$ and $w' \sim w_1'w_2'w_3'$ for appropriate $w_1, w_1' \in L^*$, $w_2, w_2' \in I^*$, and $w_3, w_3' \in R^*$. But then $\overline{w}_1\overline{w}_2\overline{w}_3 = \overline{w} = \overline{w}' = \overline{w}_1'\overline{w}_2'\overline{w}_3'$ and it follows, by Proposition 9, that $\overline{w}_1 = \overline{w}_1'$, $\overline{w}_2 = \overline{w}_2'$, and $\overline{w}_3 = \overline{w}_3'$. Theorems 2, 4, and 7 then imply that $w_1 \sim w_1'$, $w_2 \sim w_2'$, and $w_3 \sim w_3'$, so that $w \sim w'$. This completes the proof of the theorem. □

5. Refining the presentation

The presentation for $\mathscr{P}_n$ given in Theorem 17 contains $|L|+|I|+|R| = \binom{n}{2}+n+\binom{n}{2} = n^2$ generators, and a great many relations (the number of relations is quartic in n). With this in mind, it is our goal in this section to use Tietze transformations to reduce the size of the generating set, and find simpler presentations. In Section 5.1 we derive the first such presentation which uses $n+1$ generators, and simultaneously extends Popova's presentation [28] for the symmetric inverse semigroup $\mathscr{I}_n \subseteq \mathscr{P}_n$ and FitzGerald's presentation [13] for the monoid $\mathscr{F}_n \subseteq \mathscr{P}_n$ of all uniform block bijections on $\mathbf{n}$. From this presentation we then deduce the Halverson–Ram presentation [19] in Section 5.2. Working again from the presentation from Section 5.1, we obtain a presentation with just four generators in Section 5.3.

5.1. A simpler presentation

We begin with the presentation

$$\left\langle L \cup I \cup R \;\middle|\; \begin{array}{c} \text{(L1)–(L5), (R1)–(R5), (I1)–(I6), (RL1)–(RL9),} \\ \text{(SL1)–(SL8), (RS1)–(RS8), (EL1)–(EL3), (RE1)–(RE3)} \end{array} \right\rangle$$

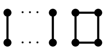
from Theorem 17 and we add a new generator t, along with the relation

$$t = \lambda_{n-1,n}\rho_{n-1,n} \tag{T}$$

which defines it in terms of the existing generators. Write $P = S \cup \{\varepsilon, t\}$, and let

$$\phi' : (L \cup P \cup R)^* \to \mathscr{P}_n$$

be the epimorphism which extends $\phi : (L \cup I \cup R)^* \to \mathscr{P}_n$ by further defining

$$t\phi' = \bar{t} =$$  .

Corollary 18. *The partition monoid $\mathscr{P}_n$ has presentation*

$$\left\langle L \cup P \cup R \;\middle|\; \begin{array}{c} \text{(T), (L1)–(L5), (R1)–(R5), (I1)–(I6), (RL1)–(RL9),} \\ \text{(SL1)–(SL8), (RS1)–(RS8), (EL1)–(EL3), (RE1)–(RE3)} \end{array} \right\rangle$$

via ϕ'.

Write $\sim' = \ker \phi'$ which is the congruence on $(L \cup P \cup R)^*$ generated by the relations in the above presentation. Our first goal is to remove the generators from $L \cup R$. With this in mind we define words

$$\begin{array}{ll} c_i = s_i \cdots s_{n-1} & \text{for } i \in \mathbf{n}, \\ \varepsilon_i = c_i \varepsilon c_i^{-1} & \text{for } i \in \mathbf{n}, \\ d_{ij} = (s_j \cdots s_{n-1})(s_i \cdots s_{n-2}) & \text{for } 1 \leqslant i < j \leqslant n, \\ t_{ij} = d_{ij} t d_{ij}^{-1} & \text{for } 1 \leqslant i < j \leqslant n, \\ L_{ij} = t_{ij}\varepsilon_j c_j & \text{for } 1 \leqslant i < j \leqslant n, \\ R_{ij} = c_j^{-1}\varepsilon_j t_{ij} & \text{for } 1 \leqslant i < j \leqslant n. \end{array}$$

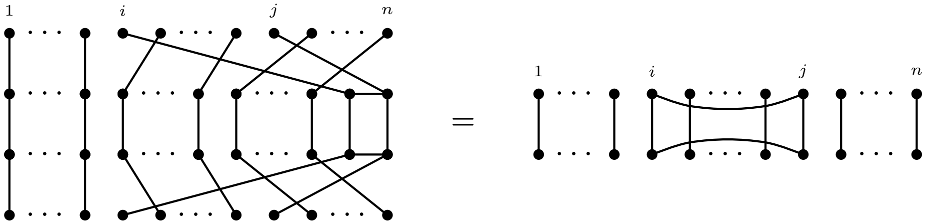


Fig. 9. The partition $t_{ij}\phi' \in \mathscr{P}_n$.

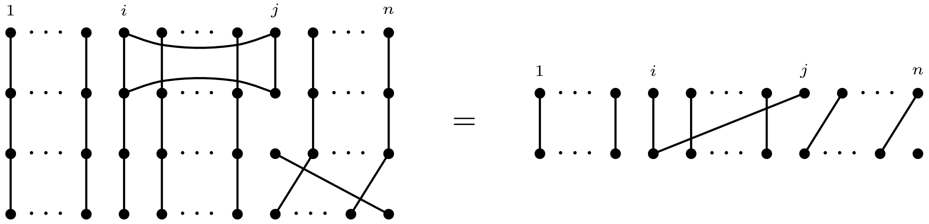


Fig. 10. A verification of the equality $L_{ij}\phi' = \bar{\lambda}_{ij}$.

Note that the words ε_i defined here are the same as those introduced before Lemma 5. The partition $t_{ij}\phi'$ is pictured in Fig. 9. It is now easy to check, diagrammatically, that $L_{ij}\phi' = \bar{\lambda}_{ij}$ and $R_{ij}\phi' = \overline{\rho}_{ij}$ for all i, j. A proof of the first of these equalities is given in Fig. 10; the second may be proved by turning Fig. 10 upside down. It then follows from Corollary 18 that $\lambda_{ij} \sim' L_{ij}$ and $\rho_{ij} \sim' R_{ij}$ for all i, j. As a result, we may remove all generators λ_{ij} and ρ_{ij}, replacing their every occurrence in the relations by the words L_{ij} and R_{ij} (respectively). We denote the relations modified in this way by (L1)′–(L5)′, (RL1)′–(RL9)′, etc. (At this point we remark that the words L_{ij} and R_{ij} are not the simplest words over P which are $\sim'$-equivalent to λ_{ij} and ρ_{ij}, but they are expressed in a form specifically designed to make subsequent calculations easier to carry out.)

Corollary 19. *The partition monoid $\mathscr{P}_n$ has presentation*

$$\left\langle P \;\middle|\; \begin{array}{c} \text{(T)}', \text{(L1)}'\text{–(L5)}', \text{(R1)}'\text{–(R5)}', \text{(I1)–(I6)}, \text{(RL1)}'\text{–(RL9)}', \\ \text{(SL1)}'\text{–(SL8)}', \text{(RS1)}'\text{–(RS8)}', \text{(EL1)}'\text{–(EL3)}', \text{(RE1)}'\text{–(RE3)}' \end{array} \right\rangle$$

via

$$\Phi : P^* \to \mathscr{P}_n : \begin{cases} s_r \mapsto \bar{s}_r, \\ \varepsilon \mapsto \bar{\varepsilon}, \\ t \mapsto \bar{t}. \end{cases}$$

Write $\sim'' = \ker \Phi$, which is the congruence on P^* generated by the relations in the above presentation. Again, for $w \in P^*$, we will write $\overline{w}$ for the partition $w\Phi \in \mathscr{P}_n$. By Corollary 19, and a simple diagram-aided check, we have the following.

Lemma 20. *The following relations are in $\sim''$:*

$$s_i^2 = 1 \qquad \textit{for all } i, \tag{P1}$$

$$s_i s_j = s_j s_i \qquad \textit{if } |i-j| > 1, \tag{P2}$$

$$s_is_js_i = s_js_is_j \qquad \textit{if } |i-j| = 1, \tag{P3}$$

$$\varepsilon^2 = \varepsilon = \varepsilon t\varepsilon, \tag{P4}$$

$$t^2 = t = t\varepsilon t = ts_{n-1} = s_{n-1}t, \tag{P5}$$

$$\varepsilon s_i = s_i\varepsilon \qquad \textit{if } i \leqslant n-2, \tag{P6}$$

$$ts_i = s_it \qquad \textit{if } i \leqslant n-3, \tag{P7}$$

$$s_{n-1}\varepsilon s_{n-1}\varepsilon = \varepsilon s_{n-1}\varepsilon s_{n-1} = \varepsilon s_{n-1}\varepsilon, \tag{P8}$$

$$ts_{n-2}ts_{n-2} = s_{n-2}ts_{n-2}t, \tag{P9}$$

$$t(s_{n-2}s_{n-1}s_{n-3}s_{n-2})t(s_{n-2}s_{n-3}s_{n-1}s_{n-2}) = (s_{n-2}s_{n-1}s_{n-3}s_{n-2})t(s_{n-2}s_{n-3}s_{n-1}s_{n-2})t, \tag{P10}$$

$$t(s_{n-2}s_{n-1}\varepsilon s_{n-1}s_{n-2}) = (s_{n-2}s_{n-1}\varepsilon s_{n-1}s_{n-2})t. \tag{P11}$$

So we may add relations (P1)–(P11) to the presentation from Corollary 19. We aim to show that all the other relations may be removed. With this in mind, write $\approx$ for the congruence on P^* generated by relations (P1)–(P11). Notice that the alphabet P contains both alphabets I and F, and that the relations (I1)–(I6) and (F1)–(F7) are all contained in (P1)–(P11). As a result, we see by Theorems 4 and 6 that $w_1 \approx w_2$ whenever w_1 and w_2 are both words over I (or F) with $\overline{w}_1 = \overline{w}_2$. In particular, we have the following.

Proposition 21. *For all $i, j \in \mathbf{n}$ and $w \in S^*$ we have*

(i) $\varepsilon_i^2 \approx \varepsilon_i$;
(ii) $\varepsilon_i\varepsilon_j \approx \varepsilon_j\varepsilon_i$; *and*
(iii) $w^{-1}\varepsilon_iw \approx \varepsilon_{i\overline{w}}$ *and* $\varepsilon_iw \approx w\varepsilon_{i\overline{w}}$.

Proposition 22. *For all $1 \leqslant i < j \leqslant n$, $1 \leqslant k < l \leqslant n$, $1 \leqslant a < b < c \leqslant n$, and $w \in S^*$ we have*

(i) $t_{ij}^2 \approx t_{ij}$;
(ii) $t_{ij}t_{kl} \approx t_{kl}t_{ij}$;
(iii) $t_{ab}t_{bc} \approx t_{bc}t_{ac} \approx t_{ac}t_{ab}$; *and*
(iv) $w^{-1}t_{ij}w \approx t_{i\overline{w},j\overline{w}}$ *and* $t_{ij}w \approx wt_{i\overline{w},j\overline{w}}$.

In the previous proposition, and for the remainder of the section, it is convenient to write $t_{ji} = t_{ij}$ for all $1 \leqslant i < j \leqslant n$. The next lemma will be used frequently.

Lemma 23. *If $1 \leqslant i < j \leqslant n$ and $k \in \mathbf{n} \setminus \{i, j\}$, then $t_{ij}\varepsilon_k \approx \varepsilon_kt_{ij}$.*

Proof. Choose any $w \in S^*$ such that $(n-2, n-1, n)\overline{w} = (k, i, j)$. Then

$$\begin{aligned}
t_{ij}\varepsilon_k &= t_{(n-1)\overline{w},n\overline{w}}\varepsilon_{(n-2)\overline{w}} \\
&\approx w^{-1}t_{n-1,n}ww^{-1}\varepsilon_{n-2}w && \text{by Propositions 21 and 22} \\
&\approx w^{-1}t_{n-1,n}\varepsilon_{n-2}w && \text{by (P1)} \\
&\approx w^{-1}\varepsilon_{n-2}t_{n-1,n}w && \text{by (P11)} \\
&\approx w^{-1}\varepsilon_{n-2}ww^{-1}t_{n-1,n}w && \text{by (P1)}
\end{aligned}$$

$$\approx \varepsilon_{(n-2)\overline{w}} t_{(n-1)\overline{w},n\overline{w}} \qquad \text{by Propositions 21 and 22}$$
$$= \varepsilon_k t_{ij},$$

completing the proof. □

The next lemma substantially reduces the amount of calculations required.

Lemma 24. *Define an anti-isomorphism*

$$\widehat{\ }: P^* \to P^* : \begin{cases} s_r \mapsto s_r, \\ \varepsilon \mapsto \varepsilon, \\ t \mapsto t. \end{cases}$$

Then for all $w_1, w_2 \in P^*$, we have $w_1 \approx w_2$ if and only if $\widehat{w}_1 \approx \widehat{w}_2$.

Proof. This follows from the fact that the relations (P1)–(P11) are symmetric. □

We now show, one-by-one, that relations (T)′, (L1)′–(L5)′, (R1)′–(R5)′, (I1)–(I6), (RL1)′–(RL9)′, (SL1)′–(SL8)′, (RS1)′–(RS8)′, (EL1)′–(EL3)′, (RE1)′–(RE3)′ may be removed. In fact, relations (I1)–(I6) are contained in (P1)–(P11), so we remove them immediately.

Lemma 25. *Relation* (T)′ *is in* $\approx$.

Proof. By (P4) and (P5) we have $L_{n-1,n} R_{n-1,n} = (t\varepsilon)(\varepsilon t) \approx t\varepsilon t \approx t$. □

Lemma 26. *Relations* (EL1)′–(EL3)′ *are in* $\approx$.

Proof. For (EL1)′ suppose $1 \leqslant i < j \leqslant n$. Then we have

$$L_{ij}\varepsilon = t_{ij}\varepsilon_j c_j \varepsilon_n \approx t_{ij}\varepsilon_j \varepsilon_{n\bar{c}_j^{-1}} c_j = t_{ij}\varepsilon_j\varepsilon_j c_j \approx t_{ij}\varepsilon_j c_j = L_{ij},$$

by Proposition 21. For (EL2)′, if $1 \leqslant i < n$, then

$$\varepsilon L_{in} = \varepsilon_n t_{in}\varepsilon_n = \varepsilon_n d_{in} t d_{in}^{-1}\varepsilon_n \approx d_{in}\varepsilon_{n\bar{d}_{in}} t \varepsilon_{n\bar{d}_{in}} d_{in}^{-1} = d_{in}\varepsilon_n t\varepsilon_n d_{in}^{-1} \approx d_{in}\varepsilon_n d_{in}^{-1} \approx \varepsilon_{n\bar{d}_{in}^{-1}} = \varepsilon,$$

by Proposition 22 and (P4). Finally, for (EL3)′, if $1 \leqslant i < j \leqslant n-1$, then

$$\begin{aligned} \varepsilon L_{ij} &= \varepsilon_n t_{ij}\varepsilon_j c_j \approx t_{ij}\varepsilon_n\varepsilon_j c_j \approx t_{ij}\varepsilon_j\varepsilon_n c_j \approx t_{ij}\varepsilon_j c_j \varepsilon_{n\bar{c}_j} = L_{ij}\varepsilon_{n-1} \\ &= L_{ij} s_{n-1}\varepsilon s_{n-1} \approx L_{ij}\varepsilon s_{n-1}\varepsilon s_{n-1} \approx L_{ij}\varepsilon s_{n-1}\varepsilon \approx L_{ij} s_{n-1}\varepsilon, \end{aligned}$$

where we have used Lemma 23, Proposition 21, (P1), (P8), and (EL1)′. □

By Lemma 24 we see that relations (RE1)′–(RE3)′ are also in $\approx$.

Lemma 27. *Relations* (SL1)′–(SL8)′ *are in* $\approx$.

Proof. All of these follow quickly from Propositions 21 and 22, and the easily checked fact that

$$s_r c_j \approx \begin{cases} c_j s_r & \text{if } r < j-1, \\ c_{j-1} & \text{if } r = j-1, \\ c_{j+1} & \text{if } r = j, \\ c_j s_{r-1} & \text{if } r > j. \end{cases}$$

For example, if $1 \leqslant i < j < r \leqslant n-1$, then

$$s_r L_{ij} = s_r t_{ij} \varepsilon_j c_j \approx t_{ij} s_r \varepsilon_j c_j \approx t_{ij} \varepsilon_j s_r c_j \approx t_{ij} \varepsilon_j c_j s_{r-1} = L_{ij} s_{r-1},$$

establishing (SL1)$'$. For (SL2)$'$, if $1 \leqslant i < j \leqslant n-1$, then we have

$$s_j L_{ij} = s_j t_{ij} \varepsilon_j c_j \approx t_{i,j+1} s_j \varepsilon_j c_j \approx t_{i,j+1} \varepsilon_{j+1} s_j c_j \approx t_{i,j+1} \varepsilon_{j+1} c_{j+1} = L_{i,j+1}.$$

The other relations may be checked analogously. □

Again, Lemma 24 implies that relations (RS1)$'$–(RS8)$'$ are also in $\approx$. We now pause to prove two intermediate lemmas.

Lemma 28. *If* $1 \leqslant i < j \leqslant n$ *and* $1 \leqslant p < q \leqslant n-1$, *then* $L_{ij} L_{pq} s_{n-1} \approx L_{ij} L_{pq}$.

Proof. By (EL1)$'$ and (EL3)$'$ we have $L_{ij} L_{pq} \approx L_{ij} \varepsilon L_{pq} \approx L_{ij} L_{pq} s_{n-1} \varepsilon \approx L_{ij} L_{pq} \varepsilon s_{n-1} \varepsilon$, and we are done, since $\varepsilon s_{n-1} \varepsilon \approx \varepsilon s_{n-1} \varepsilon s_{n-1}$ by (P8). □

Lemma 29. *If* $1 \leqslant i < j \leqslant n$, *then* $\varepsilon_j t_{ij} \varepsilon_j \approx \varepsilon_j$ *and* $t_{ij} \varepsilon_j t_{ij} \approx t_{ij}$.

Proof. Choose $w \in S^*$ such that $(i, j) = (n-1, n)\overline{w}$. Then by Propositions 21 and 22, and relations (P1) and (P4), we have

$$\varepsilon_j t_{ij} \varepsilon_j \approx \left(w^{-1} \varepsilon w\right)\left(w^{-1} t w\right)\left(w^{-1} \varepsilon w\right) \approx w^{-1} \varepsilon t \varepsilon w \approx w^{-1} \varepsilon w \approx \varepsilon_j,$$

showing that the first relation holds. The second is proved in an almost identical fashion, using (P5) rather than (P4). □

Lemma 30. *Relations* (L1)$'$–(L5)$'$ *are in* $\approx$.

Proof. For (L1)$'$, suppose $1 \leqslant k < l \leqslant n$ and $1 \leqslant i < n$. Then by Proposition 21 and Lemma 29, we have

$$L_{kl} L_{in} = t_{kl} \varepsilon_l c_l t_{in} \varepsilon_n \approx t_{kl} c_l \varepsilon_n t_{in} \varepsilon_n \approx t_{kl} c_l \varepsilon_n \approx t_{kl} \varepsilon_l c_l = L_{kl}.$$

Next, to prove (L2)$'$, suppose $1 \leqslant i < j < k \leqslant n$. Then by Proposition 22 and Lemma 23, we have

$$L_{jk} L_{ij} = t_{jk} \varepsilon_k c_k t_{ij} \varepsilon_j c_j \approx t_{jk} t_{ij} \varepsilon_k c_k \varepsilon_j c_j \approx t_{ik} t_{ij} \varepsilon_k c_k \varepsilon_j c_j \approx t_{ik} \varepsilon_k c_k t_{ij} \varepsilon_j c_j = L_{ik} L_{ij}.$$

To prove the second part of (L2)$'$, observe first that $c_p c_{q-1} \approx c_q c_p s_{n-1}$ if $p < q$ (as may easily be checked by drawing diagrams and applying Theorem 1). Using this observation, as well as Propositions 21 and 22, and Lemmas 23 and 28, we see that

$$L_{ij}L_{i,k-1} = t_{ij}\varepsilon_j c_j t_{i,k-1}\varepsilon_{k-1}c_{k-1} \approx t_{ij}\varepsilon_j t_{ik} c_j \varepsilon_{k-1}c_{k-1} \approx t_{ij}t_{ik}\varepsilon_j\varepsilon_k c_j c_{k-1} \approx t_{ik}t_{ij}\varepsilon_k\varepsilon_j c_k c_j s_{n-1}$$
$$\approx t_{ik}\varepsilon_k t_{ij} c_k \varepsilon_j c_j s_{n-1} \approx t_{ik}\varepsilon_k c_k t_{ij}\varepsilon_j c_j s_{n-1} = L_{ik}L_{ij}s_{n-1} \approx L_{ik}L_{ij}.$$

Relations (L3)′–(L5)′ are each treated in a similar way to the second half of (L2)′. □

By Lemma 24 we conclude that relations (R1)′–(R5)′ are also in $\approx$.

Lemma 31. *Relations* (RL1)′–(RL9)′ *are in* $\approx$.

Proof. Again, the various proofs for these relations bear many similarities, so we just give a number of representative calculations. For (RL1)′, suppose $1 \leqslant k < l < i < j \leqslant n$. We use Propositions 21 and 22, Lemma 23, relations (RE1)′ and (EL3)′, and the easily checked fact that $c_p^{-1}c_q \approx c_{q-1}s_{n-1}c_p^{-1}$ if $p < q$, to calculate

$$R_{kl}L_{ij} = c_l^{-1}\varepsilon_l t_{kl} t_{ij}\varepsilon_j c_j \approx c_l^{-1}\varepsilon_l t_{ij} t_{kl}\varepsilon_j c_j \approx c_l^{-1} t_{ij}\varepsilon_l\varepsilon_j t_{kl} c_j \approx t_{i-1,j-1}c_l^{-1}\varepsilon_j\varepsilon_l c_j t_{kl}$$
$$\approx t_{i-1,j-1}\varepsilon_{j-1}c_l^{-1}c_j\varepsilon_l t_{kl} \approx t_{i-1,j-1}\varepsilon_{j-1}c_{j-1}s_{n-1}c_l^{-1}\varepsilon_l t_{kl}$$
$$= L_{i-1,j-1}s_{n-1}R_{kl} \approx L_{i-1,j-1}s_{n-1}\varepsilon R_{kl} \approx \varepsilon L_{i-1,j-1}R_{kl}.$$

For (RL4)′, suppose $1 \leqslant k < i < j \leqslant n$. Then by Propositions 21 and 22, Lemmas 23 and 29, and (P1), we have

$$R_{kj}L_{ij} = c_j^{-1}\varepsilon_j t_{kj} t_{ij}\varepsilon_j c_j \approx c_j^{-1}\varepsilon_j t_{ki} t_{ij}\varepsilon_j c_j \approx t_{ki}c_j^{-1}\varepsilon_j t_{ij}\varepsilon_j c_j \approx t_{ki}c_j^{-1}\varepsilon_j c_j \approx t_{ki}\varepsilon_n$$
$$\approx \varepsilon_n t_{ki} \approx \varepsilon_n t_{ki}\varepsilon_i t_{ki} \approx \varepsilon_n t_{ki}\varepsilon_i\varepsilon_i t_{ki} \approx \varepsilon_n t_{ki}\varepsilon_i c_i c_i^{-1}\varepsilon_i t_{ki} = \varepsilon L_{ki}R_{ki}.$$

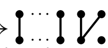
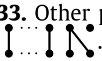
For (RL5)′, suppose $1 \leqslant i < j \leqslant n$. Then by Propositions 21 and 22, and Lemma 29, we have

$$R_{ij}L_{ij} = c_j^{-1}\varepsilon_j t_{ij} t_{ij}\varepsilon_j c_j \approx c_j^{-1}\varepsilon_j t_{ij}\varepsilon_j c_j \approx c_j^{-1}\varepsilon_j c_j \approx \varepsilon_n = \varepsilon.$$

The other relations are proved similarly. □

We have now proved the following.

Theorem 32. *The partition monoid* $\mathscr{P}_n$ *has presentation* $\langle P \mid \text{(P1)–(P11)}\rangle$ *via* Φ.

Remark 33. Other presentations may be found in terms of generators $s_1, \dots, s_{n-1}$, and $\lambda \mapsto$  and $\rho \mapsto$ . These may be found by either (i) starting from Theorem 17 and putting $\lambda = \lambda_{n-1,n}$ and $\rho = \rho_{n-1,n}$, or (ii) starting from Theorem 32 and making the substitution $\lambda = t\varepsilon$ and $\rho = \varepsilon t$. For reasons of space we have not included the relations here, but the author may be contacted for details.

5.2. The Halverson–Ram presentation

We now show how the presentation from [19] may be deduced from Theorem 32. With this in mind, we define partitions $\bar{p}_i$ $(i = 1, \dots, n)$ and $\bar{q}_j$ $(j = 1, \dots, n-1)$ in Fig. 11. Beginning with the presentation $\langle P \mid \text{(P1)–(P11)}\rangle$ from Theorem 32, we first rename $\varepsilon = p_n$ and $t = q_{n-1}$, and we add generators $p_1, \dots, p_{n-1}$ and $q_1, \dots, q_{n-2}$, along with the relations

$$p_i = c_i p_n c_i^{-1} \quad \text{for all } i, \tag{P}$$
$$q_j = d_{j,j+1}q_{n-1}d_{j,j+1}^{-1} \quad \text{for all } j, \tag{Q}$$

Fig. 11. The partitions $\overline{p}_i$ (left) and $\bar{q}_j$ (right) in $\mathscr{P}_n$.

which define them in terms of the existing generators. (The words $c_i, d_{j,j+1} \in S^*$ were defined after Corollary 18.) Put $Q = S \cup \{p_1, \ldots, p_n\} \cup \{q_1, \ldots, q_{n-1}\}$.

Corollary 34. *The partition monoid $\mathscr{P}_n$ has presentation $\langle Q \mid (\mathrm{P}), (\mathrm{Q}), (\mathrm{P1})\text{–}(\mathrm{P11})\rangle$ via*

$$\Psi : Q^* \to \mathscr{P}_n : \begin{cases} s_r \mapsto \bar{s}_r, \\ p_i \mapsto \overline{p}_i, \\ q_j \mapsto \bar{q}_j. \end{cases}$$

Put $\approx' = \ker \Psi$, which is the congruence on Q^* generated by (P), (Q), and (P1)–(P11).

Lemma 35. *The following relations are in $\approx'$:*

$$s_i^2 = 1 \quad \text{for all } i, \tag{Q1}$$

$$s_i s_j = s_j s_i \quad \text{if } |i-j| > 1, \tag{Q2}$$

$$s_i s_j s_i = s_j s_i s_j \quad \text{if } |i-j| = 1, \tag{Q3}$$

$$p_i^2 = p_i \quad \text{for all } i, \tag{Q4}$$

$$p_i p_j = p_j p_i \quad \text{for all } i, j, \tag{Q5}$$

$$s_i p_j = p_j s_i \quad \text{if } j \neq i, i+1, \tag{Q6}$$

$$s_i p_i = p_{i+1} s_i \quad \text{for all } i, \tag{Q7}$$

$$p_i p_{i+1} s_i = p_i p_{i+1} \quad \text{for all } i, \tag{Q8}$$

$$q_i^2 = q_i \quad \text{for all } i, \tag{Q9}$$

$$q_i q_j = q_j q_i \quad \text{for all } i, j, \tag{Q10}$$

$$s_i q_j = q_j s_i \quad \text{if } |i-j| > 1, \tag{Q11}$$

$$s_i s_j q_i = q_j s_i s_j \quad \text{if } |i-j| = 1, \tag{Q12}$$

$$q_i s_i = s_i q_i = q_i \quad \text{for all } i, \tag{Q13}$$

$$q_i p_j = p_j q_i \quad \text{if } j \neq i, i+1, \tag{Q14}$$

$$q_i p_j q_i = q_i \quad \text{if } j = i, i+1, \tag{Q15}$$

$$p_j q_i p_j = p_j \quad \text{if } j = i, i+1. \tag{Q16}$$

Proof. This follows from Corollary 34 and a simple diagrammatic check. □

Theorem 36. *(See Halverson and Ram [19].) The partition monoid $\mathscr{P}_n$ has presentation $\langle Q \mid (\mathrm{Q1})\text{–}(\mathrm{Q16})\rangle$ via Ψ.*

Proof. By Lemma 35 we may add (Q1)–(Q16) to the presentation from Corollary 34. We now show that the remaining relations may be removed. For the duration of this proof, let $\approx$ denote the congruence on Q^* generated by relations (Q1)–(Q16). We consider the relations (P), (Q), and (P1)–(P11) one-by-one, showing that they all hold in $\approx$. We first prove (P) by (backwards) induction on i. Now if $i=n$, then there is nothing to show while if $i<n$, then

$$p_i \approx s_i s_i p_i \approx s_i p_{i+1} s_i \approx s_i c_{i+1} p_n c_{i+1}^{-1} s_i = c_i p_n c_i^{-1}$$

by (Q1), (Q7), and an induction hypothesis. We also use induction for (Q). Again, the $j=n-1$ case is trivial, so suppose $j<n-1$. Then

$$q_j \approx s_{j+1} s_j s_j s_{j+1} q_j \approx s_{j+1} s_j q_{j+1} s_j s_{j+1} \approx s_{j+1} s_j d_{j+1,j+2} q_{n-1} d_{j+1,j+2}^{-1} s_j s_{j+1}$$

by (Q1), (Q12), and an induction hypothesis, and we are done since $s_{j+1} s_j d_{j+1,j+2} \approx d_{j,j+1}$ by (Q2). Next note that (P1)–(P3) are identical to (Q1)–(Q3); (P4) is part of (Q4) and (Q16); (P5) is part of (Q9), (Q13), and (Q15); (P6) is part of (Q6); and (P7) is part of (Q11). Next, observe that

$$p_n s_{n-1} p_n \approx p_n s_{n-1} p_n s_{n-1} s_{n-1} \approx p_n s_{n-1} s_{n-1} p_{n-1} s_{n-1} \approx p_n p_{n-1} s_{n-1} \approx p_{n-1} p_n s_{n-1} \approx p_{n-1} p_n$$

by (Q1), (Q5), (Q7), and (Q8). Relation (P8) then follows, since

$$(p_{n-1} p_n) s_{n-1} \approx (p_{n-1} p_n) \quad \text{and} \quad s_{n-1}(p_{n-1} p_n) \approx p_n s_{n-1} p_n \approx p_{n-1} p_n,$$

by (Q7), (Q8), and the observation. Next, we have

$$\begin{aligned} q_{n-1} s_{n-2} q_{n-1} s_{n-2} &\approx q_{n-1} s_{n-1} s_{n-2} q_{n-1} s_{n-2} \approx q_{n-1} q_{n-2} s_{n-1} s_{n-2} s_{n-2} \\ &\approx q_{n-2} q_{n-1} s_{n-1} \approx q_{n-2} q_{n-1} \end{aligned}$$

by (Q1), (Q10), (Q12), and (Q13). A similar calculation, using the same relations, shows that

$$s_{n-2} q_{n-1} s_{n-2} q_{n-1} \approx q_{n-2} q_{n-1},$$

and relation (P9) follows. For (P10), note that

$$(s_{n-2} s_{n-1} s_{n-3} s_{n-2}) q_{n-1} (s_{n-2} s_{n-3} s_{n-1} s_{n-2}) \approx q_{n-3}$$

by (Q), so (P10) follows from (Q10). Similarly, $s_{n-2} s_{n-1} p_n s_{n-1} s_{n-2} \approx p_{n-2}$ by (P), so (P11) follows from (Q14). This completes the proof. $\square$

Remark 37. In [19], the generators q_i were denoted by $p_{i+\frac{1}{2}}$, allowing the relations (Q1)–(Q16) to be given in a more concise form. Actually, the presentation as stated in [19] also contains the redundant relation

$$s_i p_i p_{i+1} = p_i p_{i+1} \quad \text{for all } i,$$

which follows from (Q1), (Q5), (Q7) and (Q8). It is interesting to note however, that the addition of this relation allows relation (Q5) to be removed. (Our list (Q1)–(Q16) still contains redundant relations. For example, we need only the $i=1$ cases of (Q4), (Q8), (Q9), and (Q13); the $(i,j)=(1,2)$ case of (Q5); the $(i,j)=(1,2)$ and $(1,3)$ cases of (Q10); etc.)

Remark 38. For presentations of $\mathscr{I}_n$ and $\mathscr{F}_n$ involving generating sets $S \cup \{p_1, \dots, p_n\}$ and $S \cup \{q_1, \dots, q_{n-1}\}$ respectively, see [7] and [5].

5.3. A four-generator presentation

The presentation $\langle P \mid (\text{P1})\text{–}(\text{P11}) \rangle$ from Theorem 32 involves one generator when $n = 1$, and $n + 1$ generators when $n \geqslant 2$. If $n \leqslant 2$ this generating set is minimal, as we show below in Proposition 39 where we also show that a minimal generating set for $\mathscr{P}_n$ contains four elements if $n \geqslant 3$. Making use of the fact that the symmetric group may be generated by two elements (namely, a transposition and an n-cycle), we derive a presentation for $\mathscr{P}_n$ with four generators in Theorem 41. For the statement of Proposition 39, recall that the rank of a monoid M, denoted rank(M), is the minimum cardinality of a set which generates M as a monoid.

Proposition 39. *We have*

$$\operatorname{rank}(\mathscr{P}_n) = \begin{cases} 1 & \textit{if } n = 1, \\ 3 & \textit{if } n = 2, \\ 4 & \textit{if } n \geqslant 3. \end{cases}$$

Proof. Now $\mathscr{P}_1 = \{1 =$ , $\bar{\varepsilon} =$ $\} = \langle \bar{\varepsilon} \rangle$, so $\operatorname{rank}(\mathscr{P}_1) = 1$. For the remainder of the proof, suppose $n \geqslant 2$. Now by Theorem 32 we have $\mathscr{P}_n = \langle \bar{s}_1, \dots, \bar{s}_{n-1}, \bar{\varepsilon}, \bar{t} \rangle$. Since the symmetric group $\mathscr{S}_n = \langle \bar{s}_1, \dots, \bar{s}_{n-1} \rangle$ is in fact generated by the transposition $\bar{s}_{n-1}$ and the n-cycle $\bar{s}_{n-1} \cdots \bar{s}_1$, we see that $\mathscr{P}_n$ is generated by the set $\{\bar{s}_{n-1}, \bar{s}_{n-1} \cdots \bar{s}_1, \bar{\varepsilon}, \bar{t}\}$. This set has cardinality 3 if $n = 2$, and 4 if $n \geqslant 3$, and so we have

$$\operatorname{rank}(\mathscr{P}_2) \leqslant 3 \quad \text{and} \quad \operatorname{rank}(\mathscr{P}_n) \leqslant 4 \quad \text{if } n \geqslant 3.$$

To show that these upper bounds are sharp, suppose Ω_n is a generating set for $\mathscr{P}_n$. Now the symmetric group $\mathscr{S}_n$ is contained in $\mathscr{P}_n = \langle \Omega_n \rangle$ and, since the complement $\mathscr{P}_n \setminus \mathscr{S}_n$ is an ideal of $\mathscr{P}_n$, we see that Ω_n must contain a generating set Σ_n for $\mathscr{S}_n$. As mentioned above, we have $|\Sigma_2| \geqslant 1$ and $|\Sigma_n| \geqslant 2$ if $n \geqslant 3$, so it suffices to show that $\Omega_n \setminus \Sigma_n$ contains at least two elements. Now consider an expression $\bar{\varepsilon} = \omega_1 \cdots \omega_k$ where $\omega_1, \dots, \omega_k \in \Omega_n$. Since $\operatorname{dom}(\bar{\varepsilon}) \neq \mathbf{n}$, we see that there exists $i \in \mathbf{k}$ such that $\operatorname{dom}(\omega_i) \neq \mathbf{n}$. Similarly, there exists $j \in \mathbf{k}$ such that $\operatorname{codom}(\omega_j) \neq \mathbf{n}$. If $\omega_i \neq \omega_j$, then we are done, so suppose $\omega_i = \omega_j$. The proof will be complete if we can show that $\Sigma_n \cup \{\omega_i\}$ does not generate all of $\mathscr{P}_n$. But this is obvious, since any element of $\langle \Sigma_n \cup \{\omega_i\} \rangle$ is either a permutation, or has its domain (and codomain) properly contained in $\mathbf{n}$. $\square$

Since $|P|$ is minimal for $n \leqslant 2$, we assume that $n \geqslant 3$ for the remainder of this section, with the goal of finding a presentation for $\mathscr{P}_n$ with four generators. As a first step, we note that a number of small changes may be made to the presentation $\langle P \mid (\text{P1})\text{–}(\text{P11}) \rangle$.

Lemma 40. *In the presence of relations* (P1)–(P8)*, relations* (P9)–(P11) *are equivalent to*

$$t(wtw^{-1}) = (wtw^{-1})t, \tag{P9$'$}$$

$$t(w^2tw^{-2}) = (w^2tw^{-2})t, \tag{P10$'$}$$

$$t(w^2\varepsilon w^{-2}) = (w^2\varepsilon w^{-2})t, \tag{P11$'$}$$

where w denotes the word $s_{n-1} \cdots s_1$.

Proof. Theorem 32 and a diagrammatic check show that relations (P1)–(P11) imply (P9)′–(P11)′. We now prove the converse. For the duration of this proof, let $\approx$ be the congruence on P^* generated by (P1)–(P8) and (P9)′–(P11)′. Then

$$\begin{aligned} ts_{n-2}ts_{n-2} &\approx s_{n-1}ts_{n-1}s_{n-2}(s_{n-3}\cdots s_1)t(s_1\cdots s_{n-3})s_{n-2}s_{n-1}s_{n-1} && \text{by (P1), (P5), and (P7)} \\ &\approx s_{n-1}s_{n-1}s_{n-2}(s_{n-3}\cdots s_1)t(s_1\cdots s_{n-3})s_{n-2}s_{n-1}ts_{n-1} && \text{by (P9)}' \\ &\approx s_{n-2}ts_{n-2}t && \text{by (P1), (P5), and (P7),} \end{aligned}$$

showing that (P9) is in $\approx$. Next put $u = s_{n-2}s_{n-1}s_{n-3}s_{n-2}$. It is easy to check (either diagrammatically, or using the relations) that $w^2 \approx u(s_{n-3}\cdots s_1)^2$. By (P1) and (P7), it then follows that

$$utu^{-1} \approx u(s_{n-3}\cdots s_1)^2 t(s_1\cdots s_{n-3})^2u^{-1} \approx w^2tw^{-2}.$$

But then

$$t\big(utu^{-1}\big) \approx t\big(w^2tw^{-2}\big) \approx \big(w^2tw^{-2}\big)t \approx \big(utu^{-1}\big)t,$$

by (P10)′, so that (P10) is in $\approx$. Relation (P11) is established in similar fashion, after first observing that $w^2 \approx s_{n-2}s_{n-1}(s_{n-2}\cdots s_1)^2$, and $s_{n-2}s_{n-1}\varepsilon s_{n-1}s_{n-2} \approx w^2\varepsilon w^{-2}$. □

Next, note that (P6) is equivalent to the assertion that ε commutes with every word over $\{s_1,\dots,s_{n-2}\}$. Since $\{s_1,\dots,s_{n-2}\}^*\Phi \cong \mathscr{S}_{n-1}$ is generated by the transposition $\bar{s}_{n-2}$ and the $(n-1)$-cycle $\bar{s}_{n-2}\cdots\bar{s}_1$, we see that relation (P6) is equivalent to

$$\varepsilon s_{n-2} = s_{n-2}\varepsilon, \qquad \varepsilon(s_{n-2}\cdots s_1) = (s_{n-2}\cdots s_1)\varepsilon. \tag{P6$'$}$$

Similarly, we see that (P7) is equivalent to

$$ts_{n-3} = s_{n-3}t, \qquad t(s_{n-3}\cdots s_1) = (s_{n-3}\cdots s_1)t. \tag{P7$'$}$$

In fact, using the word $w = s_{n-1}\cdots s_1$ from Lemma 40, we see that relations (P6)′–(P7)′ are equivalent to

$$\varepsilon\big(ws_{n-1}w^{-1}\big) = \big(ws_{n-1}w^{-1}\big)\varepsilon, \qquad \varepsilon(s_{n-1}w) = (s_{n-1}w)\varepsilon, \tag{P6$''$}$$

$$t\big(w^2s_{n-1}w^{-2}\big) = \big(w^2s_{n-1}w^{-2}\big)t, \qquad t\big(ws_{n-1}w^{-1}s_{n-1}w\big) = \big(ws_{n-1}w^{-1}s_{n-1}w\big)t. \tag{P7$''$}$$

So $\mathscr{P}_n$ has presentation

$$\big\langle P \bigm| \text{(P1)–(P5), (P6)}''\text{–(P7)}''\text{, (P8), (P9)}'\text{–(P11)}'\big\rangle$$

via Φ. We now rename $s = s_{n-1}$, and introduce a new generator c, along with the relation

$$c = s_{n-1}\cdots s_1, \tag{C}$$

which defines it in terms of the original generators. By [27] (see also [3]), we may remove the generators $s_1,\dots,s_{n-2}$ and relations (P1)–(P3) and (C), replacing them by

$$c^n = (sc)^{n-1} = s^2 = 1, \tag{U1}$$

$$\left(c^i sc^{n-i}\right)s = s\left(c^i sc^{n-i}\right) \quad \text{for all } 2 \leqslant i \leqslant \frac{n}{2}. \tag{U2}$$

Relations (P4)–(P5), (P6)″–(P7)″, (P8), (P9)′–(P11)′ may then be replaced by

$$\varepsilon^2 = \varepsilon = \varepsilon t \varepsilon, \tag{U3}$$

$$t^2 = t = t\varepsilon t = ts = st, \tag{U4}$$

$$\varepsilon\left(csc^{n-1}\right) = \left(csc^{n-1}\right)\varepsilon, \qquad \varepsilon(sc) = (sc)\varepsilon, \tag{U5}$$

$$t\left(c^2sc^{n-2}\right) = \left(c^2sc^{n-2}\right)t, \qquad t\left(csc^{n-1}sc\right) = \left(csc^{n-1}sc\right)t, \tag{U6}$$

$$s\varepsilon s\varepsilon = \varepsilon s\varepsilon s = \varepsilon s\varepsilon, \tag{U7}$$

$$tctc^{n-1} = ctc^{n-1}t, \tag{U8}$$

$$tc^2tc^{n-2} = c^2tc^{n-2}t, \tag{U9}$$

$$tc^2\varepsilon c^{n-2} = c^2\varepsilon c^{n-2}t. \tag{U10}$$

Rewriting some of these relations, we have the following.

Theorem 41. *The partition monoid* $\mathscr{P}_n$ *has presentation* $\langle U \mid \text{(V1)–(V7)}\rangle$ *via*

$$\Upsilon : U^* \to \mathscr{P}_n : \begin{cases} s \mapsto \bar{s}_{n-1}, \\ c \mapsto \bar{s}_{n-1}\cdots\bar{s}_1, \\ \varepsilon \mapsto \bar{\varepsilon}, \\ t \mapsto \bar{t}, \end{cases}$$

where (V1)–(V7) *denote the relations*

$$c^n = (sc)^{n-1} = s^2 = \left(c^i sc^{n-i}s\right)^2 = 1 \qquad \textit{for all } 2 \leqslant i \leqslant \frac{n}{2}, \tag{V1}$$

$$\varepsilon^2 = \varepsilon = \varepsilon t\varepsilon = sc\varepsilon c^{n-1}s = csc^{n-1}\varepsilon csc^{n-1}, \tag{V2}$$

$$t^2 = t = t\varepsilon t = ts = st = c^2sc^{n-2}tc^2sc^{n-2} = c^{n-1}scsc^{n-1}tcsc^{n-1}sc, \tag{V3}$$

$$s\varepsilon s\varepsilon = \varepsilon s\varepsilon s = \varepsilon s\varepsilon, \tag{V4}$$

$$tctc^{n-1} = ctc^{n-1}t, \tag{V5}$$

$$tc^2tc^{n-2} = c^2tc^{n-2}t, \tag{V6}$$

$$tc^2\varepsilon c^{n-2} = c^2\varepsilon c^{n-2}t. \tag{V7}$$

Remark 42. As in Remark 33, presentations also exist in terms of generators s, c, λ, ρ.

6. Presentations for the partition algebra

Fix a non-zero complex number $\xi \in \mathbb{C}$. The *partition algebra* $\mathscr{P}_n^\xi$ is the $\mathbb{C}$-algebra with basis $\mathscr{P}_n$, and product $\circ$ induced by the product of $\mathscr{P}_n$ in a way made precise as follows. Let $\alpha, \beta \in \mathscr{P}_n$. When forming the product $\alpha\beta$ in $\mathscr{P}_n$, a key step involved the removal of all connected components in the "interior" of the concatenated diagram of α and β. We write $m(\alpha,\beta)$ for the number of such connected components. The product in $\mathscr{P}_n^\xi$ of the basis elements α and β is defined to be

$$\alpha \circ \beta = \xi^{m(\alpha,\beta)}(\alpha\beta),$$

the scalar multiple of (the monoid product) $\alpha\beta$ by the scalar $\xi^{m(\alpha,\beta)} \in \mathbb{C}$. This operation (extended linearly) gives $\mathscr{P}_n^\xi$ the structure of an associative $\mathbb{C}$-algebra. In this final section, we show how any presentation for $\mathscr{P}_n$ may be translated into an algebra presentation for $\mathscr{P}_n^\xi$. To do this we use the fact, observed by Wilcox [29], that $\mathscr{P}_n^\xi$ is a *twisted semigroup algebra* of $\mathscr{P}_n$. The precise definitions of twisted semigroup algebras are given in the next section.

6.1. Twisted semigroup algebras

Let M be a monoid and F a commutative ring with identity. Without causing confusion, we will write 1 for the identity of both M and F. The group of units of F will be denoted by $G(F)$. A *twisting* from M to F is a map

$$\tau : M \times M \to G(F)$$

which satisfies

$$\tau(s,t)\tau(st,u) = \tau(s,tu)\tau(t,u) \quad \text{for all } s,t,u \in M.$$

Such a map allows us to define an associative F-algebra $F^\tau[M]$, called the *twisted semigroup algebra* of M over F, with basis M and multiplication $\circ$ induced by

$$s \circ t = \tau(s,t)(st) \quad \text{for } s,t \in M.$$

So $F^\tau[M]$ consists of all formal (finite) linear combinations of the elements of M, and the product (in $F^\tau[M]$) of any two basis elements is a scalar multiple of another basis element. Note that if $\tau(s,t) = 1$ for all $s,t \in M$, then $F^\tau[M] = F[M]$ is the (usual) semigroup algebra of M over F.

In this way we see that the partition algebra $\mathscr{P}_n^\xi$ is the twisted semigroup algebra (over $\mathbb{C}$) of the partition monoid $\mathscr{P}_n$ with respect to the twisting $\tau : \mathscr{P}_n \times \mathscr{P}_n \to \mathbb{C}$ defined by $\tau(\alpha,\beta) = \xi^{m(\alpha,\beta)}$.

6.2. Presentations for twisted semigroup algebras

Let X be an alphabet. The *free F-algebra on X* is $F[X^*]$, the semigroup algebra of the free monoid X^*, which is the set of all (finite) polynomials in the (non-commuting) variables X. Suppose now that $R \subseteq F[X^*] \times F[X^*]$, and write $R^\sharp$ for the (algebra) congruence on $F[X^*]$ generated by R. We say that an F-algebra A has algebra presentation $\langle X \mid R\rangle$ if $A \cong F[X^*]/R^\sharp$ or, equivalently, if there exists an F-algebra epimorphism $\varphi : F[X^*] \to A$ with kernel $R^\sharp$. (Note that we still use the semigroup theoretic definition of the kernel of an F-algebra homomorphism as a congruence rather than the subalgebra of elements which map to 0.) In this section we show how an (algebra) presentation for a twisted semigroup algebra maybe obtained from a (monoid) presentation of its corresponding monoid.

With this task in mind, suppose that M is a monoid and $\tau : M \times M \to G(F)$ a twisting. Write $A = F^\tau[M]$ for the twisted semigroup algebra. Suppose further that M has presentation $\langle X \mid R\rangle$ via $\varphi : X^* \to M$ and, for $w \in X^*$, write $\overline{w} = w\varphi \in M$.

Let $w = x_1 \cdots x_k \in X^*$, and put $\widehat{w} = \bar{x}_1 \circ \cdots \circ \bar{x}_k \in A$. For $1 \leqslant i \leqslant k-1$ write $\tau_i(w) = \tau(\bar{x}_i, \bar{x}_{i+1} \cdots \bar{x}_k) \in F$, and put $\tau(w) = \tau_1(w) \cdots \tau_{k-1}(w)$. It is then easy to check that $\widehat{w} = \tau(w)\overline{w}$. Now put

$$\widehat{R} = \{ (\tau(w_2)w_1, \tau(w_1)w_2) \mid (w_1, w_2) \in R \} \subseteq F[X^*] \times F[X^*],$$

and let $\widehat{\varphi} : F[X^*] \to A$ be the F-algebra homomorphism induced by $w\widehat{\varphi} = \widehat{w}$ for all $w \in X^*$. We aim to show that A has presentation $\langle X \mid \widehat{R} \rangle$ via $\widehat{\varphi}$.

For the duration of this current section, we write $\sim = R^\sharp$ and $\approx = \widehat{R}^\sharp$. (There should be no confusion caused by our earlier uses of these symbols.) For simplicity, we will assume that R (and therefore $\widehat{R}$) is symmetric, so a statement such as "assume that $(u, v) \in R$" should be read as "assume that either $(u, v) \in R$ or $(v, u) \in R$".

Lemma 43. *If $u, v \in X^*$ and $u \sim v$, then $u \approx \tau(u)\tau(v)^{-1}v$.*

Proof. Since $u \sim v$, there exist words $w_1, w_2, \dots, w_{k-1}, w_k \in X^*$ such that $u = w_1$, $v = w_k$, and $(w_i, w_{i+1}) \in R$ for each i. We then have $(\tau(w_{i+1})w_i, \tau(w_i)w_{i+1}) \in \widehat{R}$, whence $w_i \approx \tau(w_i)\tau(w_{i+1})^{-1}w_{i+1}$, for all i. Combining these gives $w_1 \approx \tau(w_1)\tau(w_k)^{-1}w_k$, and the proof is complete. □

Theorem 44. *In the notation of this section, the twisted semigroup algebra $A = F^\tau[M]$ has presentation $\langle X \mid \widehat{R} \rangle$ via $\widehat{\varphi}$.*

Proof. If $s \in M$, then $s = \overline{w}$ for some $w \in X^*$, so that $s = (\tau(w)^{-1}w)\widehat{\varphi}$. This shows that $\widehat{\varphi}$ is surjective. To see that $\approx \subseteq \ker\widehat{\varphi}$, suppose $(u, v) \in \widehat{R}$. Then $u = \tau(w_2)w_1$ and $v = \tau(w_1)w_2$ for some $(w_1, w_2) \in R$, so we have

$$u\widehat{\varphi} = \tau(w_2)\tau(w_1)\overline{w}_1 = \tau(w_1)\tau(w_2)\overline{w}_2 = v\widehat{\varphi}.$$

To complete the proof, we must establish the reverse inclusion, so suppose $(u, v) \in \ker\widehat{\varphi}$, and write

$$u = \sum_{i=1}^{k} a_i u_i \quad \text{and} \quad v = \sum_{i=1}^{l} b_i v_i,$$

where $a_i, b_i \in F$ and $u_i, v_i \in X^*$. Then

$$\sum_{i=1}^{k} a_i \tau(u_i)\bar{u}_i = u\widehat{\varphi} = v\widehat{\varphi} = \sum_{i=1}^{l} b_i \tau(v_i)\bar{v}_i.$$

After re-ordering (if necessary) we conclude that $k = l$, and $\bar{u}_i = \bar{v}_i$ and $a_i\tau(u_i) = b_i\tau(v_i)$ for each $i \in \mathbf{k}$. It then follows that $u_i \sim v_i$ for each $i \in \mathbf{k}$. By Lemma 43, we have $u_i \approx \tau(u_i)\tau(v_i)^{-1}v_i$ for all i, and it follows that

$$u = \sum_{i=1}^{k} a_i u_i \approx \sum_{i=1}^{k} a_i \tau(u_i)\tau(v_i)^{-1} v_i = \sum_{i=1}^{k} b_i \tau(v_i)\tau(v_i)^{-1} v_i = \sum_{i=1}^{k} b_i v_i = v,$$

completing the proof. □

6.3. *The partition algebra*

As a result of Theorem 44, we may deduce presentations for $\mathscr{P}_n^\xi$ from each presentation for $\mathscr{P}_n$ derived in Section 5. The presentations $\langle P \mid \text{(P1)–(P11)} \rangle$, $\langle Q \mid \text{(Q1)–(Q16)} \rangle$, $\langle V \mid \text{(V1)–(V7)} \rangle$ yield presentations for $\mathscr{P}_n^\xi$ by simply replacing (P4), (Q4), (V2) by

$$\varepsilon^2 = \xi\varepsilon \quad \text{and} \quad \varepsilon t\varepsilon = \varepsilon, \tag{P4$'$}$$

$$p_i^2 = \xi p_i \quad \text{for all } i, \tag{Q4$'$}$$

$$\varepsilon^2 = \xi\varepsilon \quad \text{and} \quad \varepsilon = \varepsilon t\varepsilon = sc\varepsilon c^{n-1}s = csc^{n-1}\varepsilon csc^{n-1}, \tag{V2$'$}$$

respectively.

Remark 45. These presentations may also be derived by considering an "extended partition monoid" which is generated by $\mathscr{P}_n$ as well as an extra element ξ (central, non-invertible, and of infinite order) which may be thought of as the identity partition together with a "floating loop".

References

[1] A.Ya. Aĭzenštat, Defining relations of finite symmetric semigroups, Mat. Sb. N. S. 45 (87) (1958) 261–280 (in Russian).
[2] R. Brauer, On algebras which are connected with the semisimple continuous groups, Ann. Math. 38 (1937) 854–872.
[3] H.S.M. Coxeter, W.O.J. Moser, Generators and Relations for Discrete Groups, Springer-Verlag, Berlin, 1972.
[4] D. Easdown, J. East, D.G. FitzGerald, A presentation of the dual symmetric inverse semigroup, Internat. J. Algebra Comput. 18 (2) (2008) 357–374.
[5] J. East, The factorizable braid monoid, Proc. Edinb. Math. Soc. 49 (2006) 609–636.
[6] J. East, A presentation of the singular part of the symmetric inverse monoid, Comm. Algebra 34 (2006) 1671–1689.
[7] J. East, Braids and partial permutations, Adv. Math. 213 (2007) 440–461.
[8] J. East, A presentation for the singular part of the full transformation semigroup, Semigroup Forum 81 (2) (2010) 357–379.
[9] J. East, Presentations for singular subsemigroups of the partial transformation semigroups, Internat. J. Algebra Comput. 20 (1) (2010) 1–15.
[10] J. East, Infinite partition monoids, preprint, 2011.
[11] J. East, On the singular part of the partition monoid, Internat. J. Algebra Comput. 21 (1–2) (2011) 147–178.
[12] D.G. FitzGerald, K. Lau, On the partition monoid and some related semigroups, Bull. Aust. Math. Soc. 83 (2) (2011) 273–288.
[13] D.G. FitzGerald, A presentation for the monoid of uniform block permutations, Bull. Aust. Math. Soc. 68 (2003) 317–324.
[14] D.G. FitzGerald, K. Lau, Ideal structure of the Kauffman and related monoids, Comm. Algebra 34 (7) (2006) 2617–2629.
[15] D.G. FitzGerald, K. Lau, On the partition monoid and some related semigroups, Bull. Aust. Math. Soc., in press.
[16] D.G. FitzGerald, J. Leech, Dual symmetric inverse semigroups and representation theory, J. Aust. Math. Soc. 64 (1998) 146–182.
[17] F.M. Goodman, H. Wenzel, The Temperly Lieb algebra at roots of unity, Pacific J. Math. 161 (1993) 307–334.
[18] J.J. Graham, G.I. Lehrer, Cellular algebras, Invent. Math. 123 (1996) 1–34.
[19] T. Halverson, A. Ram, Partition algebras, European J. Combin. 26 (6) (2005) 869–921.
[20] P.M. Higgins, Techniques of Semigroup Theory, Oxford University Press, New York, 1992.
[21] J.M. Howie, An Introduction to Semigroup Theory, Academic Press, New York, 1976.
[22] V. Jones, A quotient of the affine Hecke algebra in the Brauer algebra, Enseign. Math. 40 (1994) 313–344.
[23] G. Kudryavtseva, V. Mazorchuk, On presentations of Brauer-type monoids, Cent. Eur. J. Math. 4 (3) (2006) 413–434 (electronic).
[24] S. Lipscombe, Symmetric Inverse Semigroups, American Mathematical Society, Providence, RI, 1996.
[25] V. Maltcev, On a new approach to the dual symmetric inverse monoid $\mathcal{I}_X^*$, Internat. J. Algebra Comput. 17 (2007) 567–591.
[26] P. Martin, Temperly–Lieb algebras for nonplanar statistical mechanics—the partition algebra construction, J. Knot Theory Ramifications 3 (1994) 51–82.
[27] E.H. Moore, Concerning the abstract groups of order $k!$ and $\frac{1}{2}k!$ holohedrically isomorphic with the symmetric and alternating substitution groups on k letters, Proc. Lond. Math. Soc. 28 (1897) 357–366.
[28] L.M. Popova, Defining relations in some semigroups of partial transformations of a finite set, Uchenye Zap. Leningrad Gos. Ped. Inst. 218 (1961) 191–212 (in Russian).
[29] S. Wilcox, Cellularity of diagram algebras as twisted semigroup algebras, J. Algebra 309 (1) (2007) 10–31.