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Generalised Chern–Simons actions for 3d gravity and κ-Poincaré symmetry

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Abstract

We consider Chern–Simons theories for the Poincaré, de Sitter and anti-de Sitter groups in three dimensions which generalise the Chern–Simons formulation of 3d gravity. We determine conditions under which κ-Poincaré symmetry and its de Sitter and anti-de Sitter analogues can be associated to these theories as quantised symmetries. Assuming the usual form of those symmetries, with a timelike vector as deformation parameter, we find that such an association is possible only in the de Sitter case, and that the associated Chern–Simons action is not the gravitational one. Although the resulting theory and 3d gravity have the same equations of motion for the gauge field, they are not equivalent, even classically, since they differ in their symplectic structure and the coupling to matter. We deduce that κ-Poincaré symmetry is not associated to either classical or quantum gravity in three dimensions. Starting from the (non-gravitational) Chern–Simons action we explain how to construct a multi-particle model which is invariant under the classical analogue of κ-de Sitter symmetry, and carry out the first steps in that construction.

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1. Introduction

1.1. Motivation

The possibility of deforming Poincaré symmetry with a dimensionful parameter (traditionally called κ) has attracted much interest since its discovery seventeen years ago [1,2]. The deformed symmetry, often called κ-Poincaré algebra, is a Hopf algebra, whose mathematical structure is now well understood [3]. However, both the role of κ-Poincaré symmetry as a fundamental symmetry in physics and its phenomenological status remain unclear. It has been argued that κ-Poincaré symmetry arises in a low-energy limit of quantum gravity in four dimensions [4]. However, since a quantum theory of gravity in four dimensions could not be constructed so far, these arguments are largely heuristic. At the same time, the derivation of experimentally testable consequences from κ-Poincaré symmetry has been hampered by the lack of κ-Poincaré-invariant theories with non-trivial interactions.

In this paper we address both of the above issues in three dimensions, where they turn out to be closely linked. We consider a family of Chern–Simons theories which includes, for a particular choice of parameters, the Chern–Simons formulation of three-dimensional gravity, with general values of the cosmological constant [5,6]. We explain how, for a different choice of parameters, the theory can be used to construct a model of interacting particles which is invariant under the de Sitter version of κ-Poincaré symmetry. More generally, we consider the de Sitter and anti-de Sitter versions of κ-Poincaré symmetry in three dimensions [7,8], and clarify which of these is associated with a Chern–Simons model. Our analysis shows clearly that, for the usual timelike deformations, neither κ-Poincaré symmetry nor its de Sitter and anti-de Sitter versions are compatible with 3d gravity. This conclusion is in disagreement with claims in the literature, for example in the paper [9], that κ-Poincaré symmetry does arise in 3d quantum gravity. We discuss this claim and our reasons for disagreeing with it at the end of our conclusion.

1.2. Background: Poisson–Lie groups, r-matrices and Hopf algebras

Before we can explain our approach in more detail we need to review some basic aspects of Hopf algebras and their classical analogues, Poisson–Lie groups. Both provide generalisations of the usual implementation of a symmetry via a group. The following summary is geared towards spacetime symmetries in relativistic physics. For a more general discussion and further details we refer the reader to [10–12].

In usual special-relativistic physics, the generators of the (undeformed) Poincaré–Lie algebra arise in two, physically quite distinct ways. In the first instance they play the role of infinitesimal symmetry generators of Minkowski space. As such they exponentiate to elements of the symmetry group of Minkowski space i.e. the Poincaré group itself. However, we also come across the Poincaré Lie algebra when we study the phase space of a free relativistic particle. Here, the Poisson brackets of the components of momentum and generalised angular momentum[1] reproduce the Lie algebra of the Poincaré group. This is a familiar fact, which follows from Noether's theorem. However, in the current context it is important to keep in mind the different guises in which the Poincaré generators appear—as generators of a symmetry group and as coordinate

[1] We use "generalised angular momentum" to mean both the angular momentum components and the quantities which are conserved due to boost invariance.

functions on a particle phase space—and to understand how they are related. It turns out that, mathematically, the phase space of a free relativistic particle is a co-adjoint orbit in the dual of the Poincaré Lie algebra. This makes it natural to think of the Lie algebra generators as (linear) functions on phase, and shows that the Lie algebra of the symmetry and the phase space are dual to each other.

When considering classical systems corresponding to quantum systems with quantum group symmetry, the two appearances of the symmetry described above become structurally richer while retaining their duality. The symmetry group (generalising the Poincaré group above) gets equipped with a Poisson structure, and the phase space (generalising the particle phase space) becomes embedded in a group. Both the symmetry group and the ambient group for the phase space become what is known as a Poisson–Lie group—a space which is both a Lie group and a Poisson manifold in such a way that these two structures are compatible. Furthermore, the "symmetry" and "phase space" Poisson–Lie groups turn out to be dual to each other, in a mathematically precise way.

Poisson–Lie groups have associated infinitesimal structures, called Lie bi-algebras. They encode infinitesimal versions of the Lie group structure via the Lie bracket and of the Poisson structure via an additional structure, called co-commutator. If two Poisson–Lie groups are in duality, so are their Lie bi-algebras: the commutator of one determines the co-commutator of the other and vice-versa. Thus, in the terminology of the previous paragraph, the commutator of the "symmetry" Poisson–Lie group agrees with the Poisson brackets of the "phase space" Poisson–Lie group near the identity.

In most applications, the co-commutator of a Lie bi-algebra is given in terms of a special element of the tensor product of two copies of the Lie algebra, called the classical r-matrix. When this is the case, the Lie bi-algebra is called quasitriangular and the Poisson brackets of the associated Poisson–Lie group can be expressed in terms of the r-matrix. The resulting Poisson bracket is called the Sklyanin bracket, and what was called "symmetry" Poisson–Lie group is called Sklyanin Poisson–Lie group in this case. By duality, the r-matrix also fixes the commutator of the dual Lie algebra, and hence the Lie group structure of the dual or "phase space" Poisson–Lie group. Thus, the knowledge of the original Lie brackets together with the r-matrix is, in principle, sufficient to compute both the Sklyanin Poisson–Lie structure and its dual.

When the r-matrix satisfies an additional non-degeneracy condition, one can use it to define a diffeomorphism between the original ("symmetry") Poisson–Lie group and its dual. Using this diffeomorphism, one can pull back the dual Poisson structure to the original Poisson–Lie group, thus defining a second Poisson structure on it. This Poisson structure is the "phase space" Poisson bracket, but written in terms of the original Lie group. We shall refer to it as the dual Poisson structure; it can again be expressed in terms of the original r-matrix.

The quantisation of classical systems with Poisson–Lie symmetry typically leads to quantum systems whose symmetries are implemented by Hopf algebras [10]. In a precise sense, one can regard the Hopf algebra as the quantisation of the Poisson–Lie group in that case. Moreover, quantisation of mutually dual Poisson–Lie groups leads to mutually dual Hopf algebras [10]. Thus, for every Poisson–Lie structure on a given group one expects there to be two, mutually dual associated Hopf algebras.

In the case of the Poincaré group in four dimensions, two dual Hopf algebra deformations were, historically, constructed directly, and not via quantisation of a Poisson–Lie structure. The first version that was discovered [1] is now known as the κ-Poincaré algebra; the other, discovered shortly afterwards [13–15], is often referred to as the κ-Poincaré group. The duality between the κ-Poincaré group and κ-Poincaré algebra is explicitly exhibited in [16]. Both are sometimes referred to as κ-Poincaré symmetries, and we followed that practice in our title and abstract. In this paper we will also consider de Sitter and anti-de Sitter versions of the κ-Poincaré algebra and group, and sometimes call them the κ-(anti-) de Sitter algebra and group. For detailed definitions and properties of these Hopf algebras in three dimensions we refer the reader to [7,8].

Even though the κ-Poincaré algebra and the κ-Poincaré group were not constructed via quantisation, we can associate them to the classical symmetries discussed above by taking the classical limit. One then finds that the classical limit of the κ-Poincaré algebra is the "phase space" Poisson–Lie group described above, and that of the κ-Poincaré group is the "symmetry" Poisson–Lie group. In practice, the calculation of the classical limit involves computing the first order deformation in the co-product of κ-Poincaré algebra, and extracting a classical r-matrix. This is explained in detail in [8] for the κ-Poincaré algebra and its de Sitter and anti-de Sitter versions in three dimensions.

1.3. Chern–Simons theory and Poisson–Lie symmetries: The Fock–Rosly construction

In general, it is difficult to construct physically motivated and mathematically non-trivial phase spaces with (non-trivial) Poisson–Lie symmetries. In this paper we will use a method developed by Fock and Rosly for such a construction. Fock and Rosly showed in [17] that the phase space and Poisson structure of Chern–Simons theory with gauge group H on manifolds of topology $\mathbb{R} \times S$, where S is a closed, oriented two-surface, possibly with handles and punctures, can be described in terms of an auxiliary Poisson structure. If S has n punctures and g handles, the auxiliary Poisson structure is defined on the space H^{n+2g}; the physical phase space of the associated Chern–Simons theory with its canonical Poisson structure induced by the Chern–Simons action is obtained after implementing a constraint and dividing by the conjugation action of H.

Fock and Rosly's auxiliary Poisson structure is defined in terms of a classical r-matrix, which is required to be compatible with the Chern–Simons action in the following sense. Recall that a Chern–Simons action for the gauge group H requires for its definition an Ad-invariant, non-degenerate and symmetric bilinear form on the Lie algebra of H. In this paper we call a classical r-matrix compatible with the Chern–Simons action if it lies in the Lie algebra of the group H tensored with itself, satisfies the classical Yang–Baxter equation and has a symmetric part which equals the Casimir associated to the symmetric form used in the Chern–Simons action. The r-matrix used in the Fock–Rosly construction can be used to define Lie bi-algebras and Poisson–Lie groups, as explained above. The compatibility requirement ensures in particular that the symmetric part of the r-matrix is non-degenerate, so that the (local) diffeomorphism between the Sklyanin Poisson–Lie group H and its dual exists in this case.

The Fock–Rosly Poisson structure is such that the Sklyanin Poisson–Lie group H is a symmetry i.e. acts via Poisson-isomorphisms on the auxiliary phase space. Moreover, it was demonstrated by Alekseev and Malkin [18] that the contribution of different handles and punctures to this Poisson structure can be decoupled and related to standard Poisson structures: each puncture corresponds to a copy of the dual Poisson structure on H described above, while each handle is related to a copy of the so-called Heisenberg double Poisson structure [10,19]. Thus, the Fock–Rosly Poisson-structure, and hence the Poisson structure on the physical phase space of the associated Chern–Simons theory, is completely determined if the dual and Heisenberg double Poisson structures are given.

In physical applications, the surface S is usually interpreted as "space", and the punctures on it as particles. Thus, if the number of punctures is $n > 1$, the Fock–Rosly construction leads to a Poisson algebra that can serve as model for n interacting particles. The detailed physical interpretation of the phase space coordinates can be quite involved; for details in the context of the Chern–Simons formulation of 3d gravity we refer the reader to [20–22].

1.4. Overview of the paper

After this brief review of Poisson–Lie theory, we can summarise the paper in more precise and technical terms. We begin, in Section 2, with a review of the Lie algebras which arise as infinitesimal local symmetries in 3d gravity with or without a cosmological constant. We give a detailed discussion of the space of Ad-invariant, symmetric bilinear forms on these Lie algebras and derive a simple non-degeneracy criterion for such forms. We use a description of the Lie algebras as the Lorentz (or, in the Euclidean case, rotation) Lie algebra over a ring whose multiplication law depends on the cosmological constant. This formalism, invented in [23], turns out to be the most efficient way of treating all signs of the cosmological constants and both the Lorentzian and Euclidean signature in a unified way.

In Section 3 we discuss Chern–Simons theory with the local isometry groups of 3d gravity as gauge groups, using the most general Ad-invariant, non-degenerate symmetric bilinear form of Section 2. The Chern–Simons actions in this section were first considered in [6] and are generalisations of the Chern–Simons formulation of 3d gravity. We describe the coupling of the Chern–Simons field to point particles, and show that the equations of motion in the absence of particles are independent of the symmetric form, but that the Poisson structure of phase space and the coupling to point particles depend on it. We argue that, as a result, Chern–Simons theories for different choices of the non-degenerate symmetric bilinear form are physically inequivalent. The observation about the variation of the Poisson structure with the symmetric form was made, in different notation, in [24], where the similarity with the variation of the Immirzi parameter in four-dimensional gravity was emphasised.

Section 4 deals with classical r-matrices, and contains the main result of our paper. We consider r-matrices obtained from the κ-Poincaré algebra and its de Sitter and anti-de Sitter analogues [8] via the classical limit explained at the end of Section 1.2; the r-matrices obtained in this way are all anti-symmetric. We then check if they can be made compatible with the Chern–Simons action of Section 3, in the sense defined in Section 1.3. This amounts to checking if the anti-symmetric r-matrix obtained via the classical limit can be combined with the (symmetric) Casimir corresponding to the Ad-invariant, non-degenerate symmetric bilinear form used in the Chern–Simons action to give a solution of the classical Yang–Baxter equation. We show that this is the case provided a vector appearing in the anti-symmetric part of the r-matrix satisfies a certain condition, which we discuss. One finding, worth stressing at this stage, is that the classical r-matrix obtained from the κ-Poincaré algebra is only compatible with the Chern–Simons action of 3d gravity if the vector appearing in the r-matrix is taken to be spacelike. Since this vector is timelike in the usual form of the κ-Poincaré algebra, we conclude that the usual κ-Poincaré symmetry is not associated with 3d gravity.

In Sections 5 and 6 we compute Lie bi-algebra and Poisson–Lie structures associated to the r-matrices of Section 4. We compute and discuss the Sklyanin and dual Poisson brackets, focussing on the de Sitter case and the situation where the special vector appearing in the r-matrix is timelike. We obtain general formulas, valid for any value of the cosmological constant, and discuss both their linearisation near the identity, and their limit as $\Lambda \to 0$.

Section 7 contains our conclusions and outlook. We explain how our calculations enable one to construct multi-particle models which are invariant under the Poisson–Lie group which arises as the classical limit of the κ-de Sitter group. We end with comments on the relation between κ-Poincaré symmetry and gravity in three dimensions.

2. Local isometry groups and their Lie algebras

In discussions of three-dimensional spacetimes we adopt the notational conventions of [25], which we briefly review. We set the speed of light to 1, and write $\eta^E = \mathrm{diag}(1,1,1)$ for the three-dimensional Euclidean metric and $\eta^L = \mathrm{diag}(1,-1,-1)$ for the three-dimensional Minkowski metric; we omit the superscript in formulas valid for both signatures. In particular, we use the abbreviations

$$\boldsymbol{p}\cdot\boldsymbol{q} = \eta_{ab}p^a q^b, \quad \text{with } \boldsymbol{p} = \left(p^0, p^1, p^2\right), \quad \boldsymbol{q} = \left(q^0, q^1, q^2\right) \in \mathbb{R}^3, \tag{2.1}$$

as well as $\boldsymbol{pq}$ for $\boldsymbol{p}\cdot\boldsymbol{q}$ and $\boldsymbol{p}^2$ for $\boldsymbol{p}\cdot\boldsymbol{p}$.

We denote by J_a, $a = 0,1,2$, the generators of both the three-dimensional rotation algebra $\mathfrak{su}(2)$ and the three-dimensional Lorentz algebra $\mathfrak{su}(1,1)$, and use the letter $\mathfrak{h}$ for either of these Lie algebras. The Lie brackets are

$$[J_a, J_b] = \epsilon_{abc}J^c, \tag{2.2}$$

where ϵ denotes the fully antisymmetric tensor in three dimensions with the convention $\epsilon_{012} = \epsilon^{012} = 1$ (for both signatures), and the indices are raised with η^E in the Euclidean case, and with η^L in the Lorentzian case. In the Lorentzian case J_0 is the generator of the spatial rotations and J_1, J_2 are the generators of the boosts. As explained in Appendix A, the matrix of the Killing form on $\mathfrak{h}$ turns out to be $-2\eta_{ab}$ in this basis; dividing the Killing form by the factor -2 we obtain the invariant, non-degenerate bilinear form η satisfying

$$\eta(J_a, J_b) = \eta_{ab}. \tag{2.3}$$

In 3d gravity, solutions of the Einstein solution are locally isometric to certain "model spacetimes" which are completely determined by the signature of spacetime and the cosmological constant. The isometry groups of these model spacetimes are hence local isometries of 3d gravity, a situation which differs considerably from the four-dimensional case where one has only local Lorentz symmetry. We list the groups we want to consider in Table 1. If we do not need to specify the group, we use H to stand for any of the groups in Table 1.

The local isometry groups have Lie algebras which can be expressed in a unified fashion, with the cosmological constant Λ_c playing the role of a deformation parameter [6]. Defining

$$\Lambda = \begin{cases} \Lambda_c & \text{for Euclidean signature,} \\ -\Lambda_c & \text{for Lorentzian signature} \end{cases} \tag{2.4}$$

Table 1
Local isometry groups in 3d gravity

Λ_c	Euclidean signature	Lorentzian signature
$=0$	$SU(2) \ltimes \mathbb{R}^3$	$SU(1,1) \ltimes \mathbb{R}^3$
>0	$SU(2) \times SU(2)$	$SL(2,\mathbb{C})$
<0	$SL(2,\mathbb{C})$	$SU(1,1) \times SU(1,1)$

these Lie algebras, in the following denoted by $\mathfrak{h}_\Lambda$, are the six-dimensional Lie algebras with generators J_a, P_a, $a = 0, 1, 2$, and Lie brackets[2]

$$[J_a, J_b] = \epsilon_{abc} J^c, \qquad [J_a, P_b] = \epsilon_{abc} P^c, \qquad [P_a, P_b] = \Lambda \epsilon_{abc} J^c. \tag{2.5}$$

For $\Lambda = 0$, the bracket of the generators P_a vanishes, and the Lie algebra $\mathfrak{h}_\Lambda$ is the three-dimensional Euclidean and Poincaré algebra. For $\Lambda < 0$, the brackets (2.5) are those of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ for both Euclidean and Lorentzian signature. For $\Lambda > 0$ the brackets are those of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ in the Euclidean and of $\mathfrak{su}(1, 1) \oplus \mathfrak{su}(1, 1)$ in the Lorentzian case.

For a unified description of the local isometry groups and their Lie algebras it is convenient to use a trick which was discovered in [23] and used extensively in [25]. The idea is to introduce a formal parameter θ which satisfies $\theta^2 = \Lambda$ and to identify the generators P_a in (2.5) with θJ_a. It is easy to check that the brackets (2.5) then follow from (2.2) by extending (2.2) linearly over θ.

As explained in [23] this construction amounts to considering the commutative ring R_Λ consisting of elements of the form $a + \theta b$, $a, b \in \mathbb{R}$ and viewing the Lie algebra $\mathfrak{h}_\Lambda$ as the realification of the Lie algebra $\mathfrak{h}$ tensored with R_Λ. We refer to [23] for a formal definition and details, but recall the notation

$$\mathrm{Re}_\theta(a + \theta b) = a, \qquad \mathrm{Im}_\theta(a + \theta b) = b, \quad \forall a, b \in \mathbb{R}, \tag{2.6}$$

and the conjugation

$$\overline{(a + \theta b)} = a - \theta b. \tag{2.7}$$

The ring R_Λ is a field in the case $\Lambda < 0$ (the complex numbers) but has zero divisors when $\Lambda \geqslant 0$, as we shall see further below.

To illustrate the use of the parameter θ, consider the Ad-invariant symmetric bilinear forms on $\mathfrak{h}_\Lambda$. As pointed out in [6] there is a two-dimensional vector space of such forms, with a basis is given by

$$t(J_a, J_b) = 0, \qquad t(P_a, P_b) = 0, \qquad t(J_a, P_b) = \eta_{ab}, \tag{2.8}$$

$$s(J_a, J_b) = \eta_{ab}, \qquad s(J_a, P_b) = 0, \qquad s(P_a, P_b) = \Lambda \eta_{ab}. \tag{2.9}$$

It was shown in [23] that these forms can be obtained as the real and imaginary part of the invariant, symmetric bilinear form η (2.3) on $\mathfrak{h} \otimes R_\Lambda$: for $X, Y \in \mathfrak{h}_\Lambda$ we have

$$s(X, Y) = \mathrm{Re}_\theta\big(\eta(X, Y)\big), \qquad t(X, Y) = \mathrm{Im}_\theta\big(\eta(X, Y)\big), \tag{2.10}$$

where P_a should be interpreted as θJ_a when it appears on the right-hand side. More generally, we can consider linear combinations of the forms s and t, which we write as

$$(\cdot,\cdot)_\tau = \alpha t(\cdot,\cdot) + \beta s(\cdot,\cdot) = \mathrm{Im}_\theta\big(\tau \eta(\cdot,\cdot)\big), \tag{2.11}$$

with $\tau = \alpha + \theta\beta$. Explicitly

$$(J_a, J_b)_\tau = \beta \eta_{ab}, \qquad (J_a, P_b)_\tau = \alpha \eta_{ab}, \qquad (P_a, P_b)_\tau = \Lambda \beta \eta_{ab}. \tag{2.12}$$

For the construction of Chern–Simons actions in the next section we require an Ad-invariant, symmetric bilinear form on $\mathfrak{h}_\Lambda$ which is non-degenerate. The following lemma gives a simple criterion for the non-degeneracy of $(\cdot,\cdot)_\tau$.

[2] Our parameter Λ is called λ in [6].

Lemma 2.1. *The* Ad-*invariant, symmetric bilinear form* $(\cdot,\cdot)_\tau$ *is non-degenerate iff*

$$\tau\bar{\tau} = \alpha^2 - \Lambda\beta^2 \neq 0. \tag{2.13}$$

Note that the condition (2.13) is always satisfied (for non-zero τ) if $\Lambda < 0$ but that it is non-trivial in the other cases.

Proof. Recall that a non-degenerate bilinear form on a vector space establishes an isomorphism between the vector space and its dual. We will need this map later, so we show that $(\cdot,\cdot)_\tau$ is non-degenerate if (2.13) holds by explicitly giving the map $\phi_\tau : \mathfrak{h}^*_\Lambda \mapsto \mathfrak{h}_\Lambda$ which satisfies

$$\xi(X) = \big(\phi_\tau(\xi), X\big)_\tau, \quad \forall X \in \mathfrak{h}_\Lambda \tag{2.14}$$

and by showing that it is bijective. Consider the basis

$$B = \{J_0, J_1, J_2, P_0, P_1, P_2\} \tag{2.15}$$

of $\mathfrak{h}_\Lambda$ and the dual basis of $\mathfrak{h}^*_\Lambda$:

$$B^* = \{J_0^*, J_1^*, J_2^*, P_0^*, P_1^*, P_2^*\}. \tag{2.16}$$

It is easy to check that (2.14) is satisfied if we set

$$\begin{aligned}
\phi_\tau(J_a^*) &= \frac{\theta}{\tau} J_a = \frac{1}{\alpha^2 - \Lambda\beta^2}(\alpha P_a - \Lambda\beta J_a),\\
\phi_\tau(P_a^*) &= \frac{1}{\tau} J_a = \frac{1}{\alpha^2 - \Lambda\beta^2}(\alpha J_a - \beta P_a),
\end{aligned} \tag{2.17}$$

showing that ϕ_τ is well-defined if and only if $\alpha^2 - \Lambda\beta^2 = \tau\bar{\tau} \neq 0$. The inverse is given by

$$\phi_\tau^{-1}(J_a) = (\alpha P_a^* + \beta J_a^*), \qquad \phi_\tau^{-1}(P_a) = (\alpha J_a^* + \beta\Lambda P_a^*). \tag{2.18}$$

Thus ϕ_τ exists and is invertible iff $\tau\bar{\tau} \neq 0$, as claimed. □

3. Chern–Simons action, Poisson structure and the coupling to particles

A Chern–Simons theory on a three-dimensional manifold depends for its definition on a choice of gauge group and an Ad-invariant, non-degenerate symmetric bilinear form on the Lie algebra of that gauge group. Remarkably, as was shown in [5,6], one obtains the Einstein–Hilbert action for three-dimensional gravity for any signature and value of the cosmological constant from the Chern–Simons action by picking the appropriate local isometry group from Table 1 as gauge group and using the non-degenerate form $t(\cdot,\cdot)$ (2.8). In this section we consider what happens if, instead of $t(\cdot,\cdot)$, we use the more general non-degenerate form $(\cdot,\cdot)_\tau$ to define a Chern–Simons theory. We couple the gauge field to matter in the form of point particles, and study the resulting phase space and its Poisson structure. We assume basic facts about Chern–Simons theory, and refer the reader to [26] for a more detailed treatment in a related context. The generalised Chern–Simons action that we study here was already considered by Witten in [6], and some of the results regarding Poisson brackets were derived in different notation in [24]. However, the coupling to point particles does not appear to have been considered elsewhere. In the context of this paper it is important to understand to what extent the variation of the form $(\cdot,\cdot)_\tau$ leads to physically inequivalent theories. We comment on this issue at the end of this section.

We work on a three-manifold M ("spacetime") of topology $\mathbb{R} \times S$, where S is an oriented two-dimensional manifold of genus g with possible punctures. The punctures are needed in order to introduce matter in the form of point particles into the model. We also need a coordinate x^0 on $\mathbb{R}$, and write $x = (x^1, x^2)$ for local coordinates on S. To keep our formulas simple we consider only one puncture with coordinate x_* on S; the generalisation to several punctures is straightforward [26]. The gauge field of the Chern–Simons theory is locally a one-form on spacetime with values in the Lie algebra $\mathfrak{h}_\Lambda$ of one of the isometry groups in Table 1. In terms of the generators J_a and P_a we have the expansion

$$A = \omega_a J^a + e_a P^a, \tag{3.1}$$

where $\omega = \omega^a J_a$ is geometrically interpreted as the spin connection on the frame bundle and the set of one-forms $\{e_0, e_1, e_2\}$ as a dreibein (provided it is invertible). The curvature of A is

$$F = dA + \frac{1}{2}[A \wedge A] = R + C + T, \tag{3.2}$$

and contains the Riemann curvature

$$R = d\omega + \frac{1}{2}[\omega \wedge \omega], \tag{3.3}$$

the cosmological term

$$C = \frac{\Lambda}{2}\epsilon^{abc} e_a \wedge e_b J_c, \tag{3.4}$$

and the torsion

$$T = \left(de^c + \epsilon^{abc}\omega_a \wedge e_b\right) P_c. \tag{3.5}$$

Using the product structure $M = \mathbb{R} \times S$ we decompose the gauge field as

$$A = A_0\, dx^0 + A_S, \tag{3.6}$$

where A_S is an x^0-dependent and Lie algebra valued one-form on S and A_0 is a Lie algebra valued function on $\mathbb{R} \times S$. We use the usual notation d for the exterior derivative on $\mathbb{R} \times S$ and write d_S for the exterior derivative on S. With this notation, the field strength two-form can be decomposed as

$$F = dA + A \wedge A = dx^0 \wedge \left(\partial_0 A_S - d_S A_0 + [A_0, A_S]\right) + F_S, \tag{3.7}$$

where F_S is the curvature two-form on S:

$$F_S = d_S A_S + A_S \wedge A_S. \tag{3.8}$$

The Chern–Simons action for the gauge field A is

$$I_\tau(A) = \int_M (A \wedge dA)_\tau + \frac{1}{3}\left(A \wedge [A, A]\right)_\tau. \tag{3.9}$$

In order to read off the constraints and the symplectic structure defined by this action, it is useful to perform a $(2+1)$-decomposition of the action:

$$I_\tau[A_S, A_0] = \int_{\mathbb{R}} dx^0 \int_S (\partial_0 A_S \wedge A_S)_\tau + (A_0\,, F_S)_\tau. \tag{3.10}$$

Variation with respect to A_0, which acts as a Lagrange multiplier, gives the constraint

$$F_S(x) = 0, \tag{3.11}$$

and variation with respect to A_S gives the evolution equation

$$\partial_0 A_S = d_S A_0 + [A_S, A_0]. \tag{3.12}$$

Together, these two equations are equivalent to the statement that the field strength F is zero. Geometrically this means that the torsion T vanishes, and that the curvature is constant:

$$T = 0, \qquad R + C = 0. \tag{3.13}$$

Note, in particular, that both the constraint and the evolution equation are independent of the choice of $(\cdot,\cdot)_\tau$. One might think that this means that theories corresponding to different choices of $(\cdot,\cdot)_\tau$ are physically equivalent. However, this is not case, as we shall explain at the end of this section.

To see how the theory is affected by changing τ, and to gain a better understanding of the physical interpretation of the action (3.9) we use the decomposition (3.1). After integration by parts and dropping a boundary term the action (3.9) can be written as

$$\begin{aligned} I_\tau(A) = \alpha \int_M \left(2e^a \wedge R_a + \frac{\Lambda}{3}\epsilon_{abc} e^a \wedge e^b \wedge e^c\right) \\ + \beta \int_M \left(\omega^a \wedge d\omega_a + \frac{1}{3}\epsilon_{abc}\omega^a \wedge \omega^b \wedge \omega^c + \Lambda e^a \wedge T_a\right), \end{aligned} \tag{3.14}$$

where T_a are the components of the torsion (3.5). Note that first line contains the usual action for 3d gravity with cosmological constant Λ. The first term in the second line is simply the Chern–Simons action for the spin connection ω, with ω treated as an independent variable. The formula (3.14) also shows clearly that in the non-gravitational case $\alpha = 0$ the action becomes independent of the dreibein e_a and hence degenerate in the limit $\Lambda \to 0$.

In order to study the symplectic structure associated with this action, we write out the terms in (3.10):

$$\begin{aligned} I_\tau[A_S, A_0] = \alpha \int_{\mathbb{R}} dx^0 \int_S \eta_{ab}\big(\partial_0 e^a_S \wedge \omega^b_S + \partial_0 \omega^a_S \wedge e^b_S\big) \\ + \beta \int_{\mathbb{R}} dx^0 \int_S \eta_{ab}\big(\partial_0 \omega^a_S \wedge \omega^b_S + \Lambda \partial_0 e^a_S \wedge e^b_S\big) \\ + \text{constraint}. \end{aligned} \tag{3.15}$$

This expression is instructive in a number of ways. First of all, it allows us to determine the physical dimensions of the coupling constants α and β. The dimension of the dreibein e_a is length and the spin connection ω is dimensionless; thus, working in units where the speed of light is 1, the constant α has to have the dimension of mass in order for the action to have the correct dimension. In the usual gravitational interpretation of the action (3.14) with $\beta = 0$, α is identified with the inverse of the gravitational constant G in three dimensions. Recalling that the cosmological constant has the dimension of inverse length squared, we see that the terms in the second row of (3.15) have the correct dimension if β has the dimension of an action i.e. mass times length.

The second important lesson we learn from (3.15) is that the symplectic structure of the theory depends on τ. Specifically, we can read off from (3.15) which fields are canonically conjugate to each other for given τ. We refer the reader to [24] where the general formulas for conjugate variables are given (in different notation) and the similarity between the parameters α and β and the Immirzi parameter in four-dimensional gravity is pointed out. For our purposes it is worth emphasising two cases. For the gravitational form t (i.e. $\tau = \alpha$) the spatial components of the dreibein e^a are conjugate to the spatial components of the spin connection. With $e^a_S = e^a_i dx^i$, and $\omega^a_S = \omega^a_i dx^i$ the only non-vanishing brackets are

$$\left\{e^a_i(x), \omega^b_j(y)\right\} = \frac{1}{2\alpha}\eta^{ab}\epsilon_{ij}\delta^{(2)}(x-y). \tag{3.16}$$

In the extreme non-gravitational case $\tau = \theta\beta$, the dreibein and the spin connection are self-conjugate:

$$\begin{aligned}\left\{e^a_i(x), e^b_j(y)\right\} &= \frac{1}{2\Lambda\beta}\eta^{ab}\epsilon_{ij}\delta^{(2)}(x-y),\\ \left\{\omega^a_i(x), \omega^b_j(y)\right\} &= \frac{1}{2\beta}\eta^{ab}\epsilon_{ij}\delta^{(2)}(x-y).\end{aligned} \tag{3.17}$$

The way geometry is coupled to matter in Einstein gravity can be emulated in the gauge theory formulation by the following procedure: each puncture is decorated with the action of a free (relativistic) particle moving in the model spacetime, and this action is minimally coupled to the gauge field. The action of a free particle in the model spacetime has, in turn, a simple description in terms of co-adjoint orbits of the local isometry group H. In the current context it is worth stressing that no invariant, non-degenerate symmetric form is needed on the Lie algebra $\mathfrak{h}_\Lambda$ of H in order to define the free particle action. However, in order to derive the equations of motion of the combined gauge theory–particle system, we do require such a form. The form thus describes the coupling of the particle and the bulk degrees of freedom. We therefore formulate the particle action in terms of the form $(\cdot,\cdot)_\tau$ from the outset.

For our discussion we consider one particle, with mass m and spin s. We encode both in one element $\hat{\xi}$ of the dual Lie algebra $\mathfrak{h}^*_\Lambda$ via

$$\hat{\xi} = mP^*_0 + sJ^*_0. \tag{3.18}$$

The momentum p_a and generalised angular momentum j_a of the particle in a general state of motion is obtained via the co-adjoint action of an element $h \in H$ on $\hat{\xi}$:

$$\xi = \mathrm{Ad}^*(h)(\hat{\xi}) = p^a P^*_a + j^a J^*_a; \tag{3.19}$$

their Poisson brackets reproduce the Lie brackets (2.5):

$$\{j_a, j_b\} = \epsilon_{abc}j^c, \qquad \{j_a, p_b\} = \epsilon_{abc}p^c, \qquad \{p_a, p_b\} = \Lambda\epsilon_{abc}j^c. \tag{3.20}$$

The Lie algebra elements $\phi_\tau(\hat{\xi})$ and $\phi_\tau(\xi)$ associated to $\hat{\xi}$ and ξ via (2.14) are related by the adjoint action:

$$\phi_\tau(\xi) = h\phi_\tau(\hat{\xi})h^{-1}. \tag{3.21}$$

The combined field and particle action now takes the following form:

$$I_\tau[A_S, A_0, h] = \int_{\mathbb{R}} dx^0 \int_S (\partial_0 A_S \wedge A_S)_\tau - \int_{\mathbb{R}} dx^0 \big(\phi_\tau(\hat{\xi}), h^{-1}\partial_0 h\big)_\tau$$
$$+ \int_{\mathbb{R}} dx^0 \int_S \big(A_0, F_S - \phi_\tau(\xi)\delta^{(2)}(x - x_*)\, dx^1 \wedge dx^2\big)_\tau. \tag{3.22}$$

Varying with respect to the Lagrange multiplier A_0 we obtain the constraint

$$F_S(x) = \phi_\tau(\xi)\delta^{(2)}(x - x_*)\, dx^1 \wedge dx^2. \tag{3.23}$$

Variation with respect to A_S gives again the evolution equation (3.12). Together with (3.23) this means that the curvature F vanishes except at the "worldline" of the puncture, where it is given by the constraint (3.23). In order to interpret the constraint geometrically and physically we use the decomposition (3.2) of the curvature and the map (2.17):

$$R_S^a + C_S^a = \frac{1}{\tau\bar{\tau}}\big(\alpha p^a - \Lambda\beta j^a\big)\delta^{(2)}(x - x_*)\, dx^1 \wedge dx^2,$$
$$T_S^a = \frac{1}{\tau\bar{\tau}}\big(\alpha j^a - \beta p^a\big)\delta^{(2)}(x - x_*)\, dx^1 \wedge dx^2. \tag{3.24}$$

The extreme cases are again most easily interpreted: in the gravitational case $\tau = \alpha$, the momentum is a source of curvature and the generalised angular momentum is a source of torsion. In the non-gravitational case $\tau = \theta\beta$, the momentum is a source of torsion and the generalised angular momentum a source of curvature. In the generic case, both momentum and generalised angular momentum are sources of both torsion and curvature.

To end this section we discuss how the physics of the Chern–Simons theory with action (3.22) depends on the parameter τ. We begin with an observation regarding the equations of motion in the absence of matter. As we saw (and was previously stressed by Witten in [6]) the classical equation of motion for the gauge field, combining our constraint (3.11) and evolution equation (3.12), is the flatness condition $F = 0$ *regardless* of which non-degenerate form $(\cdot,\cdot)_\tau$ is used in the action. However, we shall now argue that the equation of motion does not capture all of the physics, even classically. The constraint (3.11) states that, for any value of x^0, the restriction of the connection A to the surface S is flat, while the evolution equation (3.12) simply states that the x^0-evolution is via gauge transformations. Together, they imply that the physical phase space, as a manifold, is the space of flat H-connections on S. Thus, it is indeed true that the phase space manifold does not depend on the non-degenerate form $(\cdot,\cdot)_\tau$.

However, any physical interpretation of the phase space and of functions on it (classical observables) also depends on the symplectic structure of the phase space. This is inherited from the Poisson brackets of the gauge field which, as we explained in the discussion preceding (3.17), *does* depend on $(\cdot,\cdot)_\tau$. In the current paper we can only illustrate how the symplectic structure enters the interpretation of the space of flat H connections in terms of spacetime geometries, and refer to [23,27,28] for details. As explained there in the context of the usual gravitational action ($\beta = 0$) for general values of the constant Λ, one can associate to every closed geodesic on the spatial surface two canonical phase space functions computed from the holonomy along the geodesic. These functions generalise the masses and spins associated to point particles and generate geometrical transformations via Poisson brackets on the phase space. For $\beta = 0$ the "mass variable" generates grafting (cutting a spatial surface along the geodesic and inserting a cylinder) and the "spin variable" generates earthquakes (cutting a spatial surface along the geodesic and rotating the edges of the cut against each other). These transformations can be viewed, respectively, as a translation and a rotation associated to a geodesic which supports the interpretation of their Hamiltonians as, respectively, a momentum or mass and angular momentum or spin variable associated to the geodesic. If one studied the Chern–Simons theory with the same gauge group but with $\alpha = 0$ and $\beta \neq 0$, the physical role of "mass" and "spin" would be reversed [27], so the same phase space function now has a different physical interpretation.

The dependence of the physics on $(\cdot,\cdot)_\tau$ becomes even clearer when one considers a manifold with boundary or when one couples the gauge field to matter. In the Chern–Simons formulation of 3d gravity, a boundary at "spatial infinity" can be modelled as a non-standard puncture [22]. There are phase space functions associated to that puncture which are interpreted as the "total mass of the universe" and "total spin of the universe". Via the Poisson brackets on the phase space induced by $(\cdot,\cdot)_\tau$, with $\beta = 0$, these phase space functions generate, respectively, time evolution and rotation of the universe relative to a chosen centre-of-mass frame. If, instead, one sets $\alpha = 0$, keeping $\beta \neq 0$, the roles of those functions are reversed, with the total spin now generating time translations, and the total mass rotations. This effect is the analogue of what we observed above for the coupling of the gauge field to point particles. In the theory with $\alpha \neq 0$, $\beta = 0$ the particle's momentum is a source of curvature and its generalised angular momentum is a source of torsion, as expected in any formulation of gravity. However, in the Chern–Simons theory with $\alpha = 0$, $\beta \neq 0$ the roles of momentum and generalised angular momentum are reversed, leading to a coupling between matter and geometry which is quite different from what happens in gravity.

All these considerations show that Chern–Simon theories with different values of τ are physically inequivalent. Even though the phase space is independent of τ *as a manifold*, the symplectic structure and hence the physical interpretation depends on τ. Generally speaking, we can summarise the above discussion by saying that exchanging the gravitational case $\alpha \neq 0$, $\beta = 0$ with the non-gravitational case $\beta \neq 0$, $\alpha = 0$ amounts to exchanging the roles of momentum and generalised angular momentum. Note, finally, that one can also detect the difference between these two cases by taking the limit $\Lambda \to 0$: in the gravitational case this limit exists, in the non-gravitational case it does not.

4. Classical r-matrices and their compatibility with Chern–Simons gauge theory

As explained in the introduction, classical r-matrices provide the link between Chern–Simons theory on the one hand and Hopf algebras on the other. An r-matrix is associated to a given Chern–Simons theory via the Fock–Rosly construction, which we reviewed in Section 1.3. Recall that, in that review, we called an r-matrix compatible with a Chern–Simons action if it satisfies the classical Yang–Baxter equation and its symmetric part is equal to the Casimir associated to the Ad-invariant, non-degenerate symmetric bilinear form used in the Chern–Simons action. The connection between r-matrices and Hopf algebras is established by taking a classical limit, as explained in Section 1.2. In this section we will establish a condition for the r-matrix obtained via the classical limit of the κ-Poincaré algebra and its de Sitter and anti-de Sitter version to be compatible with Chern–Simons action (3.14).

We begin by fixing some terminology and notation. The classical Yang–Baxter equation for a Lie algebra $\mathfrak{g}$ is an equation for an element $r = r_{mn} X^m \otimes X^n \in \mathfrak{g} \otimes \mathfrak{g}$. With the standard notation $r_{12} = r_{mn} X^m \otimes X^n \otimes 1$, $r_{13} = r_{mn} X^m \otimes 1 \otimes X^n$, $r_{23} = r_{mn} 1 \otimes X^m \otimes X^n$ the equation reads [10]

$$[[r,r]] := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0. \tag{4.1}$$

If the right-hand side is not zero but an invariant element of $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ we say that r satisfies the modified classical Yang–Baxter equation.

Now consider the anti-symmetric elements $r_A \in \mathfrak{h}_\Lambda \otimes \mathfrak{h}_\Lambda$ which are obtained in [8] from the κ-Poincaré algebra and its de Sitter and anti-de Sitter analogues in three dimensions by taking the classical limit, or, more precisely, by looking at the first-order terms in the co-product of those Hopf algebras. The formulas given in [8] contain implicitly the unit timelike vector $(1,0,0)$; we generalise them by considering instead an arbitrary vector $\boldsymbol{m}$. In terms of the element $M = m_a J^a \in \mathfrak{h}$ our generalisation of the r-matrices in [8] can be written as

$$r_A = J_a \otimes \left[M, P^a\right] + P_a \otimes \left[M, J^a\right]. \tag{4.2}$$

We have the following

Lemma 4.1. *For any $M = m_a J^a \in \mathfrak{h}$, the antisymmetric element r_A in (4.2) satisfies*

$$[[r_A, r_A]] = \boldsymbol{m}^2 \epsilon_{abc}\big(\Lambda J^a \otimes J^b \otimes J^c + J^a \otimes P^b \otimes P^c + P^a \otimes J^b \otimes P^c + P^a \otimes P^b \otimes J^c\big). \tag{4.3}$$

Proof. This can be shown by a lengthy but direct computation. Alternatively, it can be deduced from Eq. (A.7) derived in Appendix A by multiplying both sides with

$$\Lambda 1 \otimes 1 \otimes 1 + 1 \otimes \theta \otimes \theta + \theta \otimes 1 \otimes \theta + \theta \otimes \theta \otimes 1. \qquad \square$$

Note that the right-hand side of (4.3) is an invariant element of $\mathfrak{h}_\Lambda \otimes \mathfrak{h}_\Lambda \otimes \mathfrak{h}_\Lambda$; hence r_A satisfies the modified classical Yang–Baxter equation. This is expected since one can show quite generally that the first-order terms from which r_A was obtained define a co-commutator on $\mathfrak{h}_\Lambda$ which gives it the structure of a Lie bi-algebra. The modified classical Yang–Baxter equation is precisely the condition for r_A to give rise to a Lie bi-algebra structure on $\mathfrak{h}_\Lambda$.

Next we turn to the quadratic Casimir element associated to the metric $(\cdot,\cdot)_\tau$ used in defining the Chern–Simons action. The general form of that Casimir element is

$$K_\tau = \frac{\alpha}{\tau\bar{\tau}}\big(J_a \otimes P^a + P_a \otimes J^a\big) - \frac{\beta}{\tau\bar{\tau}}\big(\Lambda J_a \otimes J^a + P_a \otimes P^a\big). \tag{4.4}$$

In particular, for $\tau = 1$ we obtain the Casimir associated to the form s

$$K_s = J_a \otimes P^a + P_a \otimes J^a, \tag{4.5}$$

and for $\tau = \theta$ we obtain the Casimir associated to the form t

$$K_t = J_a \otimes J^a + \frac{1}{\Lambda} P_a \otimes P^a. \tag{4.6}$$

In order to check the compatibility of r_A with the Chern–Simons action we need to check if

$$r = K_\tau + r_A \tag{4.7}$$

satisfies the classical Yang–Baxter equation. As explained on p. 54 of [10], it follows from the $\mathfrak{h}_\Lambda$-invariance of K_τ that for any anti-symmetric element $r_A \in \mathfrak{h}_\Lambda \otimes \mathfrak{h}_\Lambda$

$$[[K_\tau + r_A, K_\tau + r_A]] = [[K_\tau, K_\tau]] + [[r_A, r_A]]. \tag{4.8}$$

Thus we need to check if $[[r_A, r_A]] = -[[K_\tau, K_\tau]]$. A straightforward calculation shows

$$\begin{aligned}
[[K_\tau, K_\tau]] = {} & \frac{\alpha^2 + \Lambda\beta^2}{(\tau\bar{\tau})^2}\epsilon_{abc}\big(\Lambda J^a \otimes J^b \otimes J^c + J^a \otimes P^b \otimes P^c + P^a \otimes J^b \otimes P^c \\
& + P^a \otimes P^b \otimes J^c\big) \\
& - \frac{\alpha\beta}{(\tau\bar{\tau})^2}\epsilon_{abc}\big(P^a \otimes P^b \otimes P^c + \Lambda P^a \otimes J^b \otimes J^c + \Lambda J^a \otimes J^b \otimes P^c \\
& + \Lambda J^a \otimes P^b \otimes J^c\big).
\end{aligned} \tag{4.9}$$

Combining (4.9) and (4.3) we arrive at

Theorem 4.2. *The element $r = K_\tau + r_\Lambda \in \mathfrak{h}_\Lambda \otimes \mathfrak{h}_\Lambda$ satisfies the classical Yang–Baxter equation iff*

$$\boldsymbol{m}^2 = -\frac{1}{\tau^2} \tag{4.10}$$

or, writing real and imaginary components explicitly,

$$\alpha\beta = 0 \qquad \textit{and} \qquad \boldsymbol{m}^2 = -\frac{\alpha^2 + \Lambda\beta^2}{(\tau\bar{\tau})^2}. \tag{4.11}$$

Proof. This is a direct consequence of (4.8), (4.9) and (4.3). □

To interpret the condition (4.11) we focus on the Lorentzian case, and recall that Λ is minus the cosmological constant in that case. Clearly either α or β have to vanish. If $\beta = 0$ then $\boldsymbol{m}^2 < 0$ so $\boldsymbol{m}$ has to be spacelike. If $\alpha = 0$ then

$$\boldsymbol{m}^2 = -\frac{1}{\Lambda\beta^2}, \tag{4.12}$$

so $\boldsymbol{m}$ is timelike if $\Lambda < 0$ and spacelike if $\Lambda > 0$.

As mentioned in the introduction, the vector $\boldsymbol{m}$ in appearing in the r-matrix (4.2) is chosen to be timelike in the usual formulation of the κ-Poincaré algebra and its de Sitter and anti-de Sitter analogues. We also concentrate on this case from now on, and therefore set $\alpha = 0$. We introduce a new constant $\kappa > 0$ via

$$\kappa = \sqrt{|\Lambda|}\beta, \tag{4.13}$$

which has the dimension of mass. In the case $\Lambda < 0$ the condition (4.12) on the timelike vector $\boldsymbol{m}$ thus becomes

$$\boldsymbol{m}^2 = \frac{1}{\kappa^2}. \tag{4.14}$$

Note that the condition (4.10), which matches the coefficients in K_τ to the norm squared of the vector $\boldsymbol{m}$ in r_Λ, is a requirement of the Fock–Rosly construction. If we were only interested in classifying Lie bi-algebras we would have the weaker condition that r_Λ satisfies the modified classical Yang–Baxter equation. However, since the right-hand side of (4.3) is an invariant element of $\mathfrak{h}_\Lambda \otimes \mathfrak{h}_\Lambda \otimes \mathfrak{h}_\Lambda$ for any $\boldsymbol{m}$, this would impose no condition on $\boldsymbol{m}$.

5. Lie bi-algebra structures and their physical interpretation

In this section we are going to compute the Lie bi-algebra structure associated to the r-matrix (4.7). We will show that we obtain infinitesimal versions of the κ-Poincaré algebra and κ-Poincaré group, thus allowing us to interpret the generators of $\mathfrak{h}_\Lambda$ and its dual physically. We will see that the parameter κ introduced in (4.13) should indeed be identified with the parameter that gives the κ-Poincaré algebra its name. We will also study the Lie bi-algebra structure in an alternative set of generators for the Lie algebra $\mathfrak{h}_\Lambda$ which will be useful for calculations in the next section.

We focus initially on the case associated with κ-de Sitter symmetry, i.e. $\Lambda < 0$ and $\alpha = 0$ so that $\tau = \theta\beta$; the limit $\Lambda \to 0$, which takes us from the de Sitter to the Poincaré case, is discussed further below. We introduce a timelike vector $\boldsymbol{n}$ which satisfies

$$\boldsymbol{n}^2 = -\Lambda. \tag{5.1}$$

Then we can meet the requirement (4.12) by setting $\boldsymbol{m} = \frac{1}{\Lambda\beta}\boldsymbol{n}$; the classical r-matrix of Theorem 4.2 becomes

$$r = \frac{1}{\Lambda\beta}\big(\Lambda J_a \otimes J^a + P_a \otimes P^a + \epsilon_{abc} n^a \big(J^b \otimes P^c - P^c \otimes J^b\big)\big). \tag{5.2}$$

This r-matrix defines a Lie bi-algebra structure on $\mathfrak{h}_\Lambda$. We refer the reader to [10] for a definition of a Lie bi-algebra; for our purposes the most important aspect of this structure is that it gives rise to a Lie bracket on the dual space of $\mathfrak{h}_\Lambda$. To define this, one first computes co-commutators from the r-matrix via

$$\delta(J_a) = (1 \otimes \mathrm{ad}_{J_a} + \mathrm{ad}_{J_a} \otimes 1) r = \frac{1}{\Lambda\beta}(J_a \wedge \boldsymbol{nP} + P_a \wedge \boldsymbol{nJ}), \tag{5.3}$$

where $X \wedge Y = X \otimes Y - Y \otimes X$. Similarly

$$\delta(P_a) = (1 \otimes \mathrm{ad}_{P_a} + \mathrm{ad}_{P_a} \otimes 1) r = \frac{1}{\Lambda\beta}(P_a \wedge \boldsymbol{nP} + \Lambda J_a \wedge \boldsymbol{nJ}). \tag{5.4}$$

Note that the co-commutators only depend on the antisymmetric part of r since the symmetric part is a Casimir and hence invariant. The Lie brackets $[\cdot,\cdot]_*$ in the dual of Lie algebra $\mathfrak{h}_\Lambda$ are defined via

$$(\xi \otimes \eta)\big(\delta(X)\big) = [\xi, \eta]_*(X), \tag{5.5}$$

where $X \in \mathfrak{h}_\Lambda$, $\xi, \eta \in \mathfrak{h}^*_\Lambda$. For the dual basis (2.16) the Lie brackets are

$$\begin{aligned}
&[P^*_a, P^*_b]_* = \frac{1}{\Lambda\beta}(P^*_a n_b - P^*_b n_a),\\
&[P^*_a, J^*_b]_* = \frac{1}{\Lambda\beta}(J^*_a n_b - J^*_b n_a),\\
&[J^*_a, J^*_b]_* = \frac{1}{\beta}(P^*_a n_b - P^*_b n_a).
\end{aligned} \tag{5.6}$$

Interestingly we can recover these from the bracket

$$[P^*_a, P^*_b]_* = \frac{1}{\Lambda\beta}(P^*_a n_b - P^*_b n_a) \tag{5.7}$$

by setting $J_a^* = \theta P_a^*$. Following [25] we denote the three-dimensional Lie algebra with the bracket (5.7) by $\mathfrak{an}(2)$. Thus, just like the original Lie algebra $\mathfrak{h}_\Lambda$ can be obtained from $\mathfrak{h}$ by tensoring with the ring R_Λ, so the dual $\mathfrak{h}_\Lambda^*$ is obtained by tensoring $\mathfrak{an}(2)$ with R_Λ.

In order to make contact with the usual formulation of the κ-Poincaré and κ-de Sitter algebras and groups we pick $\boldsymbol{n} = (\sqrt{|\Lambda|}, 0, 0)$ and express the co-commutators and dual commutators in terms of the parameter κ (4.13). Then the co-commutators are

$$\begin{aligned}
&\delta(J_a) = -\frac{1}{\kappa}(J_a \wedge P_0 + P_a \wedge J_0),\\
&\delta(P_a) = -\frac{1}{\kappa}(P_a \wedge P_0 + \Lambda J_a \wedge J_0)
\end{aligned} \tag{5.8}$$

and the dual brackets take the form

$$\begin{aligned}
&[P_a^*, P_b^*]_* = -\frac{1}{\kappa}\big(P_a^*\delta_b^0 - P_b^*\delta_a^0\big),\\
&[P_a^*, J_b^*]_* = -\frac{1}{\kappa}\big(J_a^*\delta_b^0 - J_b^*\delta_a^0\big),\\
&[J_a^*, J_b^*]_* = -\frac{\Lambda}{\kappa}\big(P_a^*\delta_b^0 - P_b^*\delta_a^0\big).
\end{aligned} \tag{5.9}$$

These co-commutators and brackets and their relation to the κ-de Sitter algebra and group are discussed in [8]. The letters for the generators used there, and in most of the literature on κ-Poincaré symmetries, differ from ours and are related to them as follows. The rotation generator J_0 is called $-J$, the boost generator J_1 is denoted K_2 and the boost generator J_2 is denoted $-K_1$. The notation for the translation generators P_a is the same as ours, but their duals are denoted by X_a, to emphasise their interpretation as space–time coordinates. In terms of $X_0 = P_0^*$ and $X_j = P_j^*$, with $j = 1, 2$, the first of the brackets in (5.9) gives the familiar space–time non-commutativity

$$[X_i, X_j]_* = 0, \qquad [X_0, X_j]_* = \frac{1}{\kappa}X_j, \quad i, j = 1, 2. \tag{5.10}$$

The notation used in [8] for J_0^* is $\hat{\theta}$ to indicate its interpretation as an angle, and the notation for J_i^* is $\hat{\xi}_i$, $i = 1, 2$, to indicate the interpretation as a boost parameter or rapidity. The interpretation suggested by this notation is thus in agreement with our general remarks in Section 1.2: the generators P_0, P_1, P_2 and J, K_1, K_2 are generators of the symmetry (Sklyanin) Poisson–Lie group, or equivalently "infinitesimal coordinates" (coordinates near the identity) of the phase space (dual) Poisson–Lie group. The dual generators X_0, X_1, X_2 and $\hat{\theta}, \hat{\xi}_1, \xi_2$ are generators of the dual Poisson–Lie groups or, equivalently, "infinitesimal coordinates" on the symmetry Poisson–Lie group, with their names alluding to the second viewpoint. Finally, note that the limit $\Lambda \to 0$, which takes us from the κ-de Sitter algebra and group to the κ-Poincaré algebra and group, has to be accompanied by $\beta \to \infty$ in such a way that κ stays fixed. With the limit taken in that way, all co-commutators and commutators given above have a smooth limit.

For the calculations in the next section it is convenient to replace the generators P_a by

$$S_a = P_a + \epsilon_{abc} n^b J^c, \qquad \boldsymbol{n}^2 = -\Lambda. \tag{5.11}$$

The Lie brackets (2.5) on $\mathfrak{h}_\Lambda$ now take the form

$$[J_a, J_b] = \epsilon_{abc} J^c, \qquad [J_a, S_b] = \epsilon_{abc} S^c + n_b J_a - \eta_{ab} \boldsymbol{n}\boldsymbol{J},$$

$$[S_a, S_b] = n_a S_b - n_b S_a. \tag{5.12}$$

The two Casimirs are

$$K_t = S_a \otimes J^a + J_a \otimes S^a,$$
$$K_s = \frac{1}{\Lambda}\big(S_a \otimes S^a - (\boldsymbol{n}\boldsymbol{J}) \otimes (\boldsymbol{n}\boldsymbol{J}) + n_a \epsilon^{abc}(S_b \otimes J_c + J_c \otimes S_b)\big) \tag{5.13}$$

and the r-matrix (5.2) reads

$$r = \frac{1}{\Lambda\beta}\big(S_a \otimes S^a - (\boldsymbol{n}\boldsymbol{J}) \otimes (\boldsymbol{n}\boldsymbol{J}) + 2n_a \epsilon^{abc} S_b \otimes J_c\big). \tag{5.14}$$

One advantage of the generators S_a is that they generate a subalgebra of $\mathfrak{h}_\Lambda$ which is isomorphic to the Lie algebra $\mathfrak{an}(2)$ encountered earlier (5.7). Explicitly, in the case $\Lambda < 0$ and with the choice $\boldsymbol{n} = (\sqrt{|\Lambda|}, 0, 0)$ one finds

$$[S_0, S_i] = \sqrt{|\Lambda|} S_i, \qquad [S_i, S_j] = 0, \quad i, j = 1, 2. \tag{5.15}$$

It is worth commenting on the different role of the $\mathfrak{an}(2)$ bracket here and in (5.10). The brackets (5.15) are part of the Lie algebra of $\mathfrak{h}_\Lambda$ and show that the non-commutativity of infinitesimal translations in the corresponding model spacetime is controlled by the inverse length scale $\sqrt{|\Lambda|}$. The brackets (5.10), by contrast, are part of the dual Lie algebra $\mathfrak{h}^*_\Lambda$ and show that the non-commutativity of infinitesimal translation in momentum space is controlled by the inverse mass scale $1/\kappa$.

If we use the basis

$$\tilde{B} = \{J_0, J_1, J_2, S_0, S_1, S_2\} \tag{5.16}$$

instead of (2.15) we have a new dual basis

$$\tilde{B}^* = \{\tilde{J}^*_0, \tilde{J}^*_1, \tilde{J}^*_2, S^*_0, S^*_1, S^*_2\}, \tag{5.17}$$

which is related to the dual basis (2.16) via

$$S^*_a = P^*_a, \qquad \tilde{J}^*_a = J^*_a + \epsilon_{abc} n^b (P^*)^c. \tag{5.18}$$

The dual brackets now take the form

$$[S^*_a, S^*_b]_* = \frac{1}{\Lambda\beta}(S^*_a n_b - S^*_b n_a),$$
$$[\tilde{J}^*_a, S^*_b]_* = \frac{1}{\Lambda\beta}\big(\tilde{J}^*_a n_b - \tilde{J}^*_b n_a + n_a(\boldsymbol{n} \times \boldsymbol{S}^*)_b\big),$$
$$[\tilde{J}^*_a, \tilde{J}^*_b]_* = \frac{1}{\Lambda\beta}\big((\boldsymbol{n} \times \tilde{\boldsymbol{J}}^*)^*_a n_b - (\boldsymbol{n} \times \tilde{\boldsymbol{J}}^*)^*_b n_a\big) = -\frac{1}{\Lambda\beta}\epsilon_{abc}\big(\Lambda \tilde{J}^c + (\boldsymbol{n}\tilde{\boldsymbol{J}}^*) n^c\big). \tag{5.19}$$

In the next section we require the Lie brackets of $\mathfrak{h}_\Lambda$ in yet a different basis, namely the basis obtained by applying the isomorphism $\phi_\tau : \mathfrak{h}^*_\Lambda \to \mathfrak{h}_\Lambda$ defined in (2.14) to the basis $\tilde{B}^*$ (5.17) With $\tau = \theta\beta$, the map ϕ_τ is simple in the dual basis (2.16)

$$\phi_\tau(J^*_a) = \frac{1}{\beta} J_a, \qquad \phi_\tau(P^*_a) = \frac{1}{\Lambda\beta} P_a. \tag{5.20}$$

Using the bijection (5.18) one finds

$$\phi_\tau(\tilde{J}^*_a) = \frac{1}{\Lambda\beta}\big(-n_a \boldsymbol{n}\boldsymbol{J} + \epsilon_{abc} n^b S^c\big), \qquad \phi_\tau(S^*_a) = \frac{1}{\Lambda\beta}\big(S_a - \epsilon_{abc} n^b J^c\big), \tag{5.21}$$

and in terms of the generators

$$I_a = \phi_\tau(\tilde{J}_a^*), \qquad R_a = \phi_\tau(S_a^*), \tag{5.22}$$

the Lie bracket takes the form

$$\begin{aligned}
[I_a, I_b] &= \frac{1}{\Lambda\beta}\epsilon_{abc}\big(\boldsymbol{n}\boldsymbol{I}n^c + \Lambda I^c\big),\\
[R_a, R_b] &= \frac{1}{\Lambda\beta}\big(\epsilon_{abc}I^c + n_b R_a - n_a R_b\big),\\
[I_a, R_b] &= \frac{1}{\Lambda\beta}\big(\eta_{ab}\boldsymbol{n}\boldsymbol{I} - n_b I_a + n_a\epsilon_{bcd}n^c R^d - \epsilon_{abc}n^c\boldsymbol{n}\boldsymbol{R}\big).
\end{aligned} \tag{5.23}$$

6. Poisson–Lie structures

The r-matrix discussed in Section 4 gives rise to Poisson structures on the local isometry groups H in Table 1: the Sklyanin Poisson structure on H and the dual Poisson structure. As explained in the introductory Section 1.2, H equipped with the Sklyanin bracket is a Poisson–Lie group which generalises the notion of a symmetry group; it is the classical limit of the κ-Poincaré group (or its de Sitter or anti-de Sitter versions). On the other hand, the dual Poisson bracket on H is the pull-back of the Poisson structure on the dual Poisson–Lie group, which generalises the notion of a particle phase space and is the classical limit of the κ-Poincaré algebra (or its de Sitter or anti-de Sitter versions).

We now compute and discuss the Sklyanin and dual Poisson brackets, focussing on the de Sitter case and the situation where the special vector appearing in the r-matrix is timelike. Since the expressions we obtain are complicated we discuss various approximations which give additional insights. In particular we discuss their linearisations and relate them to the Lie bi-algebras of the previous section.

6.1. Parametrisation and coordinates

In order to compute Poisson structures on H we need to pick coordinates on the group manifold. As explained in [25] one obtains a convenient and unified description of the local isometry groups by thinking of them as unit (pseudo) quaternions over the ring R_Λ. To appreciate this point of view, recall the defining relation

$$e_a e_b = -\eta_{ab} + \epsilon_{abc}e^c \tag{6.1}$$

for the unit imaginary (pseudo) quaternions e_0, e_1, e_2. If we now set

$$J_a = \frac{1}{2}e_a, \qquad S_a = \frac{\theta}{2}e_a + \frac{1}{2}\epsilon_{abc}n^b e^c, \qquad \boldsymbol{n}^2 = -\Lambda, \tag{6.2}$$

we obtain the commutation relations (5.12) as a consequence of (6.1) and of $\theta^2 = -\boldsymbol{n}^2 = \Lambda$. Specialising to the Lorentzian case with $\Lambda < 0$ one can cast the realisation (6.2) in more familiar from by using the representation

$$\rho(e_0) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \rho(e_1) = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \qquad \rho(e_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{6.3}$$

Then $\rho(J_a)$ are familiar generators of the Lie algebra $\mathfrak{su}(1,1)$. Since $\Lambda < 0$ we can identify θ with $i\sqrt{|\Lambda|}$ and pick $\boldsymbol{n} = (\sqrt{|\Lambda|}, 0, 0)$. Then

$$\rho(S_0) = \sqrt{|\Lambda|}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \rho(S_1) = \sqrt{|\Lambda|}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \rho(S_2) = \sqrt{|\Lambda|}\begin{pmatrix} 0 & 0 \\ -i & 0 \end{pmatrix}, \tag{6.4}$$

which are familiar generators.

According to Table 1, the local isometry group in the Lorentzian case with $\Lambda < 0$ is $SL(2,\mathbb{C})$. We will use a parametrisation of this group which relies on its factorisation into an element $u \in SU(1,1)$ and an element $s \in AN(2)$. As discussed in [25] such a factorisation is not possible globally; however, the coordinates we obtain cover "half" the group manifold, including a neighbourhood of the identity. The idea is to parametrise $SU(1,1)$ group elements via

$$u = p_3 + p^a J_a, \qquad \frac{\boldsymbol{p}^2}{4} + p_3^2 = 1, \tag{6.5}$$

and $AN(2)$ elements via

$$s = q_3 + q^a S_a, \quad q_3 = \sqrt{1 + (\boldsymbol{q}\boldsymbol{n})^2/4}. \tag{6.6}$$

Here J_a and S_a are as defined in (6.2); one can make contact with familiar matrix representations of $SU(1,1)$ and $AN(2)$ by using the representations (6.3) but this is not essential in what follows. We factorise an element $h \in SL(2,\mathbb{C})$ into an $SU(1,1)$ and an $AN(2)$ element parametrised as above:

$$h = \left(p_3 + p^b J_b\right) \cdot \left(q_3 + q^b S_b\right). \tag{6.7}$$

Then we use the coordinate functions

$$p^a : h \mapsto p^a, \qquad q^a : h \mapsto q^a. \tag{6.8}$$

For the physical interpretation of the coordinates $\{p^0, p^1, p^2\}$ it is important to recall that $SU(1,1)$ is the double cover of the orthochronous Lorentz group $SO^+(2,1)$ in three dimensions. One obtains the $SO^+(2,1)$ matrix $\Lambda^a{}_b$ associated to an $SU(1,1)$ element via the adjoint representation:

$$\left(p_3 + p^a J_a\right) v^a J_a \left(p_3 + p^a J_a\right)^{-1} = \left(p_3 + p^a J_a\right) v^a J_a \left(p_3 - p^a J_a\right) = \Lambda^a{}_b(\boldsymbol{p}) v^b J_a. \tag{6.9}$$

Using the relations (6.1) we calculate the matrix elements $\Lambda^a{}_b(\boldsymbol{p})$ in (6.9) and find

$$\Lambda^a{}_b(\boldsymbol{p}) = \left(1 - \frac{1}{2}\boldsymbol{p}^2\right)\delta^a_b + \frac{1}{2}p^a p_b - p_3 \epsilon^a{}_{bc} p^c, \quad p_3 = \sqrt{1 - \boldsymbol{p}^2/4}. \tag{6.10}$$

For $\Lambda < 0$, $\boldsymbol{n} = (\sqrt{|\Lambda|}, 0, 0)$, we have the following expressions for the matrix elements $\Lambda^a{}_b(\boldsymbol{p})$:

$$\begin{aligned}
&\Lambda^0{}_0(\boldsymbol{p}) = 1 + \frac{1}{2}\left(p_1^2 + p_2^2\right), \qquad \Lambda^1{}_1(\boldsymbol{p}) = 1 - \frac{1}{2}\left(p_0^2 - p_2^2\right),\\
&\Lambda^2{}_2(\boldsymbol{p}) = 1 - \frac{1}{2}\left(p_0^2 - p_1^2\right), \qquad \Lambda^0{}_i(\boldsymbol{p}) = \frac{1}{2}p^0 p_i - p_3 \epsilon^0{}_{ik} p^k,\\
&\Lambda^i{}_0(\boldsymbol{p}) = \frac{1}{2}p_0 p^i - p_3 \epsilon^i{}_{0k} p^k, \qquad \Lambda^i{}_j(\boldsymbol{p}) = \frac{1}{2}p^i p_j - p_3 \epsilon^i{}_{j0} p^0,\\
&\quad i \neq j,\ i, j = 1, 2.
\end{aligned} \tag{6.11}$$

The Poisson brackets we are going to compute in this section are defined in terms of vector fields on the group $H = SL(2,\mathbb{C})$. For the computation we need to pick generators X_α, $\alpha = 1,\dots,\dim(\mathfrak{h}_\Lambda)$, of $\mathfrak{h}_\Lambda$ and to compute the right- and left-invariant vector fields X_α^L and X_α^R associated to the generators via

$$X_\alpha^L f(h) = \frac{d}{dt}\bigg|_{t=0} f\big(e^{-tX_\alpha}h\big), \qquad X_\alpha^R f(h) = \frac{d}{dt}\bigg|_{t=0} f\big(he^{tX_\alpha}\big) \quad \forall h\in H,\ f\in \mathcal{C}^\infty(H). \tag{6.12}$$

Expanding the classical r-matrix in terms of these generators $r = r^{\alpha\beta}X_\alpha\otimes X_\beta$, the Sklyanin Poisson–Lie structure is defined via the bi-vector

$$B_S = \frac{1}{2}r^{\alpha\beta}\big(X_\alpha^R\wedge X_\beta^R - X_\alpha^L\wedge X_\beta^L\big). \tag{6.13}$$

The notion of a Poisson bi-vector is discussed in [10], but for our purpose it is sufficient to know that the bracket of two functions $f,g\in\mathcal{C}^\infty(H)$ is obtained by acting with the vector fields on the functions f and g as spelled out in (6.12). The second Poisson structure on H which we want to compute is the dual Poisson structure discussed in the introduction. Its Poisson bi-vector is

$$B_D = \frac{1}{4}r^{\alpha\beta}\big(X_\alpha^R\wedge X_\beta^R + X_\alpha^L\wedge X_\beta^L\big) + \frac{1}{2}r^{\alpha\beta}X_\alpha^R\wedge X_\beta^L. \tag{6.14}$$

For our computations we are going to use the basis $\tilde{B}$ (5.16) of $\mathfrak{h}_\Lambda$. The expressions for the left- and right-invariant vector fields generated by J_a and S_a in terms of the coordinate functions p^a, q^a were first given in [25]. We reproduce them here for the convenience of the reader:

$$\begin{aligned}
&J_a^L p^b = -\eta^{ab}p_3 + \frac{1}{2}\epsilon^{abc}p_c,\\
&J_a^R p^b = \frac{k_-(\boldsymbol{q})}{k_+(\boldsymbol{q})}\left(p_3\delta_a^b + \frac{1}{2}\epsilon_a{}^{bc}p_c\right) + \frac{q_a}{k_+(\boldsymbol{q})}\left(p_3n^b + \frac{1}{2}\epsilon^{bcd}p_cn_d\right),\\
&J_a^L q^b = 0, \qquad J_a^R q^b = \frac{1}{k_+(\boldsymbol{q})}\left(q_3\epsilon_{abc}q_c - \frac{1}{2}q^a\epsilon^{bcd}n_cq_d\right),
\end{aligned} \tag{6.15}$$

$$\begin{aligned}
&S_a^R q^b = k_+(\boldsymbol{q})\,\eta^{ab} - \frac{1}{2}n^aq^b,\\
&S_a^L q^b = -\Lambda^a{}_c(\boldsymbol{p})\left(\left(q_3 - \frac{1}{2}\boldsymbol{qn}\right)\eta^{bc} + \frac{1}{2}n^cq^b\right),\\
&S_a^R p^b = 0, \qquad S_a^L p^b = \frac{p_3}{2}p^b\epsilon_{acd}p^cn^d + p_3^2n_ap^b + \frac{1}{4}\boldsymbol{pn}\,p_ap^b - p_an^b,
\end{aligned} \tag{6.16}$$

where $k_\pm(\boldsymbol{q}) = q_3 \pm \frac{1}{2}\boldsymbol{qn}$ and $\Lambda^a{}_c(\boldsymbol{p})$ is given by (6.10) and (6.11). The evaluation of the brackets (6.13) and (6.14) is straightforward but lengthy and relies on the repeated use of the identity

$$x^b\epsilon^{acd}y_cx_d - x^a\epsilon^{bcd}y_cx_d = \epsilon^{abc}\big(\boldsymbol{x}\boldsymbol{y}x_c - \boldsymbol{x}^2y_c\big). \tag{6.17}$$

We therefore present only its result.

6.2. The Sklyanin bracket and the κ-Poincaré group

The Poisson–Lie group $SL(2,\mathbb{C})$ equipped with the Sklyanin bracket is the classical limit of the κ-de Sitter group. In terms of the coordinates p^a, q^a the Sklyanin bracket for the r-matrix (5.14), (6.13) takes the form

$$\begin{aligned}
\{p^a,p^b\} &= -\frac{1}{\Lambda\beta}p_3\epsilon^{abc}(\boldsymbol{pn}n_c+\Lambda p_c),\\
\{q^a,q^b\} &= \frac{1}{\Lambda\beta}q_3\big(n^bq^a-n^aq^b\big),\\
\{p^a,q^b\} &= \frac{1}{\Lambda\beta}\left(q_3-\frac{1}{2}\boldsymbol{nq}\right)\left(\frac{p_3}{2}p^a\epsilon^{bcd}n_cp_d-n^ap^b+p_3^2p^an^b+\frac{1}{4}\boldsymbol{pn}p^ap^b\right)\\
&\quad+\frac{1}{\Lambda\beta}\bigg(-\frac{1}{2}\boldsymbol{pn}q^bn^a-\frac{\Lambda}{2}p^aq^b\big(1-\boldsymbol{p}_\perp^2/4\big)+p_3n^a\epsilon^{bcd}n_cq_d\\
&\quad-\frac{1}{2}\epsilon^{akl}p_kn_l\epsilon^{bcd}q_cn_d\bigg),
\end{aligned}\tag{6.18}$$

where $\boldsymbol{p}_\perp^2=\frac{1}{\Lambda}(\boldsymbol{p}^2+(\boldsymbol{pn})^2)$.

To understand the structure of these brackets we first consider their linearisations near the identity:

$$\begin{aligned}
\{p^a,p^b\} &= -\frac{1}{\Lambda\beta}\epsilon^{abc}(\boldsymbol{pn}n_c+\Lambda p_c)+\mathcal{O}\big(\boldsymbol{p}^2\big),\\
\{q^a,q^b\} &= \frac{1}{\Lambda\beta}\big(n^bq^a-n^aq^b\big)+\mathcal{O}\big(\boldsymbol{q}^2\big),\\
\{p^a,q^b\} &= \frac{1}{\Lambda\beta}\big(n^bp^a-n^ap^b+n^a\epsilon^{bcd}n_cq_d\big)+\mathcal{O}\big(\boldsymbol{p}^2,\boldsymbol{q}^2,\boldsymbol{pq}\big).
\end{aligned}\tag{6.19}$$

This agrees with expression (5.19) for the dual Lie bracket, as it should according to the general remarks in Section 1.2: the dual Lie bracket is the infinitesimal version of the Sklyanin Poisson bracket.

Next we consider the limit $\Lambda\to 0$. As for the dual Lie brackets in the previous section, we take the limit $\beta\to\infty$ at the same time in such a way that κ, as defined in (4.13), remains constant. Recalling that $\Lambda<0$ and taking $\boldsymbol{n}=(\sqrt{|\Lambda|},0,0)$, the brackets (6.18) become, in this limit,

$$\begin{aligned}
\{p^a,p^b\} &= 0,\\
\{q^a,q^b\} &= -\frac{1}{\kappa}\big(\delta_0^bq^a-\delta_0^aq^b\big),\\
\{p^a,q^b\} &= -\frac{1}{\kappa}\left(\frac{p_3}{2}p^a\epsilon^{b0d}p_d-\delta_0^ap^b+p_3^2\delta_0^bp^a+\frac{p_0}{4}p^ap^b\right).
\end{aligned}\tag{6.20}$$

We thus find that the coordinates q^a have the $\mathfrak{an}(2)$ brackets (5.15), while the bracket of the p^a vanishes and the bracket of q^a and p^a yields a term proportional to p^a. In order to relate these Poisson brackets to the commutators of the κ-Poincaré group in its usual formulation [15,16], we work with the expressions (6.11) for the $SO^+(2,1)$ matrix associated to the coordinates p^a. Using expression (6.20) for the bracket in order $\mathcal{O}(\sqrt{|\Lambda|})$, we can calculate the brackets of the matrix elements $\Lambda^a{}_b$ with the position coordinates q^c explicitly and obtain

$$\{\Lambda^a{}_b,\Lambda^c{}_d\}=0,$$

$$\{q^a, q^b\} = -\frac{1}{\kappa}\left(\delta_0^b q^a - \delta_0^a q^b\right),$$
$$\{\Lambda^a{}_b, q^c\} = -\frac{1}{\kappa}\left(\left(\Lambda^a{}_0 - \delta^a{}_0\right)\Lambda^c{}_b + \eta^{ac}\left(\Lambda^0{}_b - \delta^0{}_b\right)\right). \tag{6.21}$$

As expected, this is the classical limit of the κ-Poincaré group as given in [15,16].

6.3. The dual Poisson structure

Continuing with the de Sitter case ($\Lambda < 0$ and $H = SL(2,\mathbb{C})$) we now compute the dual Poisson structure (6.14) on $H = SL(2,\mathbb{C})$, using the r-matrix (5.14). As explained in the introduction, this is the pull-back of the Poisson structure on the Poisson–Lie group dual to the Sklyanin Poisson–Lie group. Concretely, the pull-back means that the dual Poisson–Lie group is coordinatised in terms of the original group $SL(2,\mathbb{C})$, and its Poisson brackets are given in terms of the coordinates (6.8) on $SL(2,\mathbb{C})$. We compute the dual Poisson structure in this way because this is how it arises in the Fock–Rosly construction: for each puncture, the auxiliary phase space of the Fock–Rosly construction contains a copy of the group $H = SL(2,\mathbb{C})$, equipped with the dual Poisson bracket. Since punctures are interpreted as particles, the dual Poisson bracket thus gives the Poisson structure of the particle phase space—in agreement with our general remarks about the interpretation of the dual Poisson–Lie group as a generalised phase space.

The existence of the pull-back, which is assumed in the general formula (6.14) for the dual bracket, depends on the non-degeneracy of the symmetric part of the r-matrix (5.14). In the Fock–Rosly construction that symmetric part is equal to the Casimir K_τ, which is non-degenerate by assumption. However, in the limit $\Lambda \to 0$, the Casimir K_τ (with $\alpha = 0$) becomes singular. This leads to singularities in the dual Poisson brackets, as we shall see below.

The dependence of the dual bracket on the symmetric part of r-matrix is manifest in the formula (6.14). This should be contrasted with the Sklyanin bracket, which only depends on the anti-symmetric part. Explicitly, we find the following expression for (6.14) in terms of the coordinates (6.8):

$$\{p^a, p^b\} = \frac{1}{\beta\Lambda} p_3 \epsilon^{abc}(\boldsymbol{pn} n_c + \Lambda p_c),$$
$$\{q^a, q^b\} = \frac{1}{\beta\Lambda}\left(p_3\epsilon^{abc} p_c + q_3 p_3^2\left(q^a n^b - q^b n^a\right)\right)$$
$$+ \frac{1}{\beta\Lambda}\left(\frac{q_3}{4}\boldsymbol{pn}\left(q^a p^b - p^a q^b\right) + \frac{p_3}{4}\boldsymbol{qn}\left(q^b \epsilon^{acd} n_c p_d - q^a \epsilon^{bcd} n_c p_d\right)\right),$$
$$\{p^a, q^b\} = \frac{1}{\beta\Lambda}\left(p_3 n^b \epsilon^{acd} n_c q_d + \frac{p_3}{4}\boldsymbol{qn} p^a \epsilon^{bcd} n_c p_d + \Lambda p_3 \epsilon^{abc} q_c + q_3 \boldsymbol{pn}\eta^{ab}\right.$$
$$\left. - p_3^2 q_3 n^b p^a - \frac{q_3}{4}\boldsymbol{pn} p^a p^b\right) \tag{6.22}$$

with p_3, q_3 given by (6.5) and (6.6).

The functions $\boldsymbol{pq}$ and $p_3 q_3 - \frac{1}{4}\epsilon_{abc} n^a p^b q^c$ are Casimir functions of this Poisson bracket. As explained in [25] these functions have a simple geometrical interpretation in terms of the quaternionic language briefly introduced above. In particular, they are constant on conjugacy classes of the group $H = SL(2,\mathbb{C})$, which, according to the general theory of dressing transformations, are precisely the symplectic leaves of the dual Poisson structure.

The linearisation of the brackets (6.22) near the identity is given by

$$\begin{aligned}
\{p^a, p^b\} &= \frac{1}{\beta\Lambda}\epsilon^{abc}(\boldsymbol{pn}n_c + \Lambda p_c) + \mathcal{O}(\boldsymbol{p}^2),\\
\{q^a, q^b\} &= \frac{1}{\beta\Lambda}(\epsilon^{abc}p_c + q^a n^b - n^a q^b) + \mathcal{O}(\boldsymbol{p}^2, \boldsymbol{pq}, \boldsymbol{q}^2),\\
\{p^a, q^b\} &= \frac{1}{\beta\Lambda}(\boldsymbol{pn}\,\eta^{ab} - n^b p^a + n^a\epsilon^{bcd}n_c q_d - \boldsymbol{qn}\epsilon^{abc}n_c) + \mathcal{O}(\boldsymbol{p}^2, \boldsymbol{pq}, \boldsymbol{q}^2).
\end{aligned} \tag{6.23}$$

Comparing with (5.23) one finds agreement with the Lie bracket on $\mathfrak{h}_\Lambda$ when expressed in terms of the images under the map ϕ_τ of the generators of the dual Lie algebra $\mathfrak{h}^*_\Lambda$. This confirms that the dual Poisson structure reduces to the Lie bracket of $\mathfrak{h}_\Lambda$ near the identity. Note that the existence of the map ϕ_τ depends on the non-degeneracy of the Ad-invariant symmetric bilinear form $(\cdot,\cdot)_\tau$, which is in turn equivalent to the non-degeneracy of the Casimir K_τ. Since the latter is assumed in the expression (6.14) for the dual Poisson structure, it is not surprising that we need the identification between $\mathfrak{h}_\Lambda$ and $\mathfrak{h}^*_\Lambda$ via (2.14) here.

The dual Poisson structure does not have a well-defined limit when $\Lambda \to 0$. Even when we take $\beta \to \infty$ in such a way that κ (4.13) remains constant, the first term in the second line of (6.23) tends to infinity. As explained at the beginning of this subsection, this singularity arises because the pull-back of the Poisson structure on the dual Poisson–Lie group to $SL(2,\mathbb{C})$ becomes singular in the limit $\Lambda \to 0$. Thus, even though the dual Poisson–Lie group does have a smooth limit as $\Lambda \to 0$, the pull-back of its Poisson structure to $SL(2,\mathbb{C})$ becomes ill-defined. We discuss implications of this result in the conclusion.

7. Conclusion and outlook

In this paper we addressed the question if κ-Poincaré symmetry and its de Sitter and anti-de Sitter analogues in three dimensions can be associated to Chern–Simons theories with, respectively, the Poincaré, de Sitter and anti-de Sitter group as gauge group. In practice this meant checking if the classical r-matrices obtained from the κ-Poincaré algebra and its de Sitter and anti-de Sitter analogues are compatible with those Chern–Simons theories via the Fock–Rosly construction. We showed that, if one insists on the vector appearing in the r-matrix being time-like, only the κ-de Sitter algebra can be associated to a Chern–Simons theory in this way. The relevant Chern–Simons action is based on the non-degenerate symmetric form $s(\cdot,\cdot)$, which is not the one used in 3d gravity.

The association between the κ-de Sitter algebra and Chern–Simons gauge theory opens up the possibility of constructing a multi-particle system with κ-de Sitter symmetry. Given the compatibility between the classical r-matrix for the κ-de Sitter algebra with the Chern–Simons action, one can use the Fock–Rosly method to construct the phase space for an arbitrary number of interacting particles coupled to Chern–Simons theory. The Poisson structure on that phase space is invariant under the Sklyanin Poisson–Lie group based on the same r-matrix. In this paper we computed the Sklyanin Poisson bracket and the dual bracket. The Fock–Rosly bracket on the n-particle phase can be expressed in terms of n copies of the dual bracket; doing this explicitly and interpreting the resulting phase space is left for future work.

The limit $\Lambda \to 0$ is subtle, and deserves further comments. On can recover the κ-Poincaré group and algebra, as well as all its associated Lie bi-algebra and Poisson–Lie group structures, from the corresponding de Sitter version by taking the limit $\Lambda \to 0$ while keeping κ fixed.

However, the symmetric form $s(\cdot,\cdot)$ used in the Chern–Simons theory associated to κ-de Sitter symmetry degenerates in this limit, and, as a result, the basic Poisson brackets (3.17) become ill-defined. Not surprisingly, the Fock–Rosly construction of an auxiliary phase space also fails in this situation. We saw this in our discussion of the dual Poisson–Lie structure at the end of Section 6.3: even though the dual Poisson–Lie group does have a smooth limit, the pull-back of its Poisson structure to the group H (which is required in the Fock–Rosly construction) does not. We therefore conclude that one cannot associate a Chern–Simons model to the κ-Poincaré group and algebra in three dimensions by the method used in the present paper.

It is worth stressing that the requirement of compatibility of an r-matrix with a Chern–Simons theory is a much stronger requirement than the modified classical Yang–Baxter equation, which is needed for the construction of Lie bi-algebras and Poisson–Lie groups from the same r-matrix. In particular, the r-matrix (4.2) gives rise to Lie bi-algebras and Poisson–Lie groups for any choice of the vector $\boldsymbol{m}$. The difference between these two requirements is related to the fact that the limit $\Lambda \to 0$ works well for the Lie bi-algebras and Poisson–Lie groups, but is ill-defined for the auxiliary phase space of the Fock–Rosly construction.

If one insists on the vector appearing in the r-matrix being timelike it is impossible to associate either the κ-Poincaré algebra or its de Sitter and anti-de Sitter analogues to the Chern–Simons actions of 3d gravity, which are based on the symmetric form $t(\cdot,\cdot)$. This follows directly from the condition (4.11). The r-matrices which are compatible with the Chern–Simons action of 3d gravity are discussed systematically in [25]. Their form is related to the fact that the Lie algebra $\mathfrak{h}_\Lambda$ with the non-degenerate symmetric form (2.8) has the structure of a classical double. The corresponding Hopf algebras are all quantum doubles, and quite different from the κ-Poincaré, κ-de Sitter and κ-anti-de Sitter algebras, which all have the structure of bicrossproducts [3,11].

Claims in the literature that κ-Poincaré symmetry does arise in 3d gravity tend to be based on the algebra structure alone and ignore the co-algebra. This is the case, for example in [9], which focuses on the algebra structure. However, as algebras both the κ-Poincaré algebra and the quantum double of the Lorentz group (which does arise in 3d gravity [31]) are isomorphic. This is an immediate consequence of the fact that both are isomorphic to the universal enveloping algebra of the Poincaré Lie algebra. For the κ-Poincaré algebra this can be shown explicitly by writing it in a suitable basis [29], and for the quantum double of the Lorentz group this is obvious in the usual formulation [30,31]. To distinguish the κ-Poincaré algebra and the quantum double of the Lorentz group, and to show that either arises in 3d gravity one has to take the full Hopf algebra structure into account, and this was not done in [9]. The relation between bicrossproducts on the one hand and the quantum doubles arising in 3d gravity is discussed further in [32] and [33]. The upshot of the discussion there and in the current paper is that κ-Poincaré symmetry is not directly related to 3d gravity. It is possible to establish a connection using the notion of semi-duality [33], but the physical significance of this remains to be clarified.

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Appendix A. Killing forms and a three-dimensional identity

For any Lie algebra the Killing form is the Ad-invariant symmetric bilinear form defined via

$$k(X,Y) = \mathrm{tr}\big(\mathrm{ad}(X)\mathrm{ad}(Y)\big). \tag{A.1}$$

Its matrix relative to a basis of the Lie algebra can be expressed entirely in terms of the structure constants in that basis. For both the Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{su}(1,1)$ with the generators described in the main text, the matrix of the Killing form turns out to be

$$k(J_a, J_b) = -2\eta_{ab}, \tag{A.2}$$

with $\eta = \mathrm{diag}(1,1,1)$ in the Euclidean case and $\eta = \mathrm{diag}(1,-1,-1)$ in the Lorentzian case.

In the following we make use of two identities which hold for any semisimple Lie algebra, namely the Jacobi identity

$$\big[X,[Y,Z]\big] + \big[Y,[Z,X]\big] + \big[Z,[X,Y]\big] = 0 \tag{A.3}$$

and the invariance of the Killing form

$$k\big([X,Y],Z\big) + k\big(Y,[X,Z]\big) = 0. \tag{A.4}$$

However, we also need a special identity for the double commutator, which holds only for the semi-simple three-dimensional Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{su}(1,1)$:

$$\big[X,[Y,Z]\big] = \eta(X,Z)Y - \eta(X,Y)Z. \tag{A.5}$$

This is equivalent to (6.17) and can be proved by checking it on a basis.

The following lemma can be proved by repeated but straightforward application of the Jacobi identity, the invariance of the Killing form, and the special identity (A.5).

Lemma A.1. *Let X, Y, Z, N be four elements of either $\mathfrak{su}(2)$ or $\mathfrak{su}(1,1)$. Then*

$$\begin{aligned}\eta\big(\big[X,[Y,N]\big],[Z,N]\big) + \eta\big(\big[Y,[N,X]\big],[Z,N]\big) + \eta\big(Z,\big[[X,N],[Y,N]\big]\big)\\ = \eta(N,N)\eta\big([X,Y],Z\big).\end{aligned} \tag{A.6}$$

Eq. (A.6) is equivalent to the identity

$$\big[\big[J_a \otimes \big[N, J^a\big], J_b \otimes \big[N, J^b\big]\big]\big] = \eta(N,N)\epsilon_{abc} J^a \otimes J^b \otimes J^c, \tag{A.7}$$

as can be seen by taking the inner product with $X \otimes Y \otimes Z$.

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