MATHEMATICS

A NOTE ON MODULAR p-GROUPS

BY

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Introduction

All groups in this paper will be finite; notation will be that of HUPPERT's book [1].

In [5] we proved:

Every non-abelian modular p-group with $p \neq 2$, contains a characteristic maximal subgroup.

For $p=2$ a similar statement has been proved by WARD [6], not for modular 2-groups but for quaternion-free 2-groups.

As to the definition of modular resp. quaternion-free:

A group is called modular if its lattice of subgroups is modular; every section of a modular p-group is modular too, see [2].

[A section of a group G is by definition a homomorphic image of a subgroup of G].

A 2-group is called quaternion-free if it has no section isomorphic to the quaternion group Q of eight elements.

It is the purpose of this paper to derive a criterion for a non-abelian p-group to be modular (quaternion-free when $p=2$).

Note that all abelian p-groups are modular and quaternion-free as well.

We denote by $\mathscr{M}_p$ the class of all non-abelian modular p-groups when $p \neq 2$ and if $p=2$, $\mathscr{M}_p = \mathscr{M}_2$ will be the class of all non-abelian quaternion-free 2-groups.

Then we prove the

Theorem: Let the p-group P be non-abelian.

Then every non-abelian section H of P contains a maximal subgroup, characteristic in H, if and only if $P \in \mathscr{M}_p$.

Proof: The "if" part has been done in [5] and [6].

So we prove the "only if" part:

Assume that $P \notin \mathscr{M}_p$; when $p \neq 2$, then P has as section the extra-special p-group P_1 of order p^3 and of exponent p, so

$$P_1 = \langle a, b | a^p = b^p = c^p = 1, [a, b] = c, [a, c] = [b, c] = 1 \rangle.$$

The fact that P_1 is indeed a section of P when $p \neq 2$, has been proved by IWASAWA [3]; see also [4]. And when $P \notin \mathscr{M}_p$ and $p=2$, then Q is a section of P. However, neither P_1 nor Q contain characteristic maximal subgroups, so the proof of the theorem is complete! q.e.d.

Therefore a particular criterion has been found for a non-abelian p-group to be modular, resp. quaternion-free.

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