ELSEVIER

Discrete Mathematics 204 (1999) 27–39

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

A simple bijection for the regions of the Shi arrangement of hyperplanes[1]

Christos A. Athanasiadis[a,*,2], Svante Linusson[b]

[a] *Mathematical Sciences Research Institute, 1000 Centennial Drive, Berkeley, CA 94720, USA*
[b] *Department of Mathematics, Stockholm University, S-106 91 Stockholm, Sweden*

Received 14 February 1997; revised 27 October 1997; accepted 3 August 1998

Abstract

The Shi arrangement $\mathcal{S}_n$ is the arrangement of affine hyperplanes in $\mathbb{R}^n$ of the form $x_i - x_j = 0$ or 1, for $1 \leqslant i < j \leqslant n$. It dissects $\mathbb{R}^n$ into $(n+1)^{n-1}$ regions, as was first proved by Shi. We give a simple bijective proof of this result. Our bijection generalizes easily to any subarrangement of $\mathcal{S}_n$ containing the hyperplanes $x_i - x_j = 0$ and to the extended Shi arrangements. It also implies the fact that the number of regions of $\mathcal{S}_n$ which are relatively bounded is $(n-1)^{n-1}$.

Keywords: Hyperplane arrangement; Region Shi arrangement; Parking functions

1. Introduction

A *hyperplane arrangement* $\mathcal{A}$ is a finite set of affine hyperplanes in $\mathbb{R}^n$. The *regions* of $\mathcal{A}$ are the connected components of the space obtained from $\mathbb{R}^n$ by removing the hyperplanes of $\mathcal{A}$. A classical example is provided by the *braid arrangement* $\mathcal{A}_n$. It consists of the hyperplanes in $\mathbb{R}^n$ of the form $x_i = x_j$ for $1 \leqslant i < j \leqslant n$, i.e. the reflecting hyperplanes of the Coxeter group of type A_{n-1}. Its regions correspond to permutations of the set $[n] := \{1, 2, \ldots, n\}$.

A *deformation* of $\mathcal{A}_n$ [21] is an arrangement each of whose hyperplanes is parallel to one of the hyperplanes of $\mathcal{A}_n$. We will be concerned with a deformation of $\mathcal{A}_n$

* Corresponding author. E-mail: athana@math.upenn.edu.

[1] Both authors were partially supported by the Mathematical Sciences Research Institute, Berkeley, California. The second author was also supported by the Swedish Science Council. Research at MSRI is supported in part by NSF grant DMS-9022140.

[2] Current address: Department of Mathematics, University of Pennsylvania, 209 South 33rd Street, Philadelphia, PA 19104-6395, USA.

PII: S0012-365X(98)00365-3

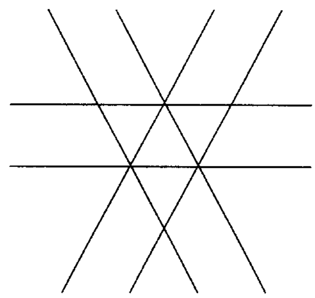

Fig. 1. The Shi arrangement for $n=3$.

which has remarkable combinatorial properties. It is the *Shi arrangement*, denoted by $\mathscr{S}_n$, and consists of the hyperplanes

$$x_i - x_j = 0 \quad \text{for } 1 \leqslant i < j \leqslant n$$

and

$$x_i - x_j = 1 \quad \text{for } 1 \leqslant i < j \leqslant n \tag{1}$$

in $\mathbb{R}^n$. Fig. 1 shows $\mathscr{S}_3$ intersected with the plane $x_1 + x_2 + x_3 = 0$. Shi was the first to consider $\mathscr{S}_n$ in his investigation of the affine Weyl group of type A_{n-1} [18]. He used techniques from combinatorial group theory and gave a constructive proof of the following result.

Theorem 1.1 (Shi [18, Corollary. 7.3.10]). *The number of regions of $\mathscr{S}_n$ is $(n+1)^{n-1}$.*

Shi [19] generalized Theorem 1.1 to a natural analogue of $\mathscr{S}_n$, defined for any irreducible crystallographic root system (see Remark 3 in Section 4). Since Shi's work, the arrangement $\mathscr{S}_n$ has continued to appear in the context of affine Weyl groups [8–10], as an object of independent interest in enumerative combinatorics [3,10,21,23], as a particularly nice example where old [4,24] and new techniques [1,2,15,16] from the theory of hyperplane arrangements apply and more recently, in the context of representations of affine Hecke algebras [17]. Some of this work remains to be done for the various root system analogues. We briefly describe those developments which are relevant here.

Remark. A semi-bijective proof of Theorem 1.1 was given by Headley in his work on affine Weyl groups [9,8]. Assuming Theorem 1.1, Headley went on to compute the *characteristic polynomial* [14, Section 2.3] of $\mathscr{S}_n$ as $\chi(\mathscr{S}_n, q) = q(q-n)^{n-1}$. This statement is stronger than Theorem 1.1 since Zaslavsky's theory [24] expresses the number of regions of a hyperplane arrangement $\mathscr{A}$ in $\mathbb{R}^n$ as $(-1)^n \chi(\mathscr{A}, -1)$. Moreover, Zaslavsky [24] expresses the number of *relatively bounded* regions of $\mathscr{A}$ as $|\chi(\mathscr{A}, 1)|$.

Therefore, Headley's formula also implies that this number for $\mathscr{S}_n$ is $(n-1)^{n-1}$. Headley [10] generalized his formula to other root systems (see also Remark 4 in Section 4).

A bijective proof of Theorem 1.1 follows from work of Pak and Stanley [21, Section 5]. They established a correspondence between the regions of $\mathscr{S}_n$ and the *parking functions* on $[n]$, which are known to be counted by $(n+1)^{n-1}$. Although this correspondence is easy to define, a lot of effort is needed to prove that it is indeed a bijection (see the proof of [23, Theorem 2.1]). However, it allows one to enumerate the regions according to a certain statistic $d(R)$, the 'distance' of the region R from a fixed base region R_0. Under the bijection, the statistic d corresponds to a natural statistic on the parking functions on $[n]$ which is closely related to the inversion enumerator for trees [13]. We refer the reader to Remark 5 in Section 4 and [23] for more information.

A simple counting proof of Headley's formula was given by an application of the 'finite field method' of [1,2] (see [1, Theorem 3.3; 2, Theorem 6.2.1]). This general method to compute characteristic polynomials of hyperplane arrangements makes use of a combinatorial interpretation of the values of $\chi(\mathscr{A},q)$ at large primes q. Other applications included (but were not limited to) the Shi arrangements for the other infinite families of root systems, the family of arrangements between $\mathscr{A}_n$ and $\mathscr{S}_n$, along with various generalizations, and a simple formula for the face numbers of $\mathscr{S}_n$ of any fixed dimension. This formula yields another interesting generalization of Theorem 1.1, which we describe in Section 4. Other such results, mainly for deformations of $\mathscr{A}_n$, are included in work of Postnikov and Stanley [15, Chapter 1, 16].

The arrangement $\mathscr{S}_n$, as well as the whole family of arrangements between $\mathscr{A}_n$ and $\mathscr{S}_n$, was also studied from the point of view of *freeness* [14, Chapter 4] in [4]. As a special case of the techniques in [4], one can use the classical method of *deletion and restriction* to give simple inductive proofs to Headley's formula and Theorem 1.1, suitably generalized. We give details in Section 4.

The results: We first give a *simple bijective* proof of Theorem 1.1. Our bijection can also be stated in terms of parking functions but is different from that of Pak and Stanley. It has the advantage that it generalizes easily to any arrangement between $\mathscr{A}_n$ and $\mathscr{S}_n$, a problem which provided the main motivation for this work, as follows. Let G be a simple graph on the vertex set $[n]$. We denote by $\mathscr{S}_{n,G}$ the arrangement

$$\begin{aligned} &x_i - x_j = 0 \quad \text{for } 1 \leqslant i < j \leqslant n,\\ &x_i - x_j = 1 \quad \text{for } 1 \leqslant i < j \leqslant n,\ ij \in G \end{aligned} \tag{2}$$

in $\mathbb{R}^n$, first considered in [1, Section 3, 2, Section 6.2] and later in [4]. It specializes to $\mathscr{A}_n$ when G is empty and to $\mathscr{S}_n$ when G is the complete graph. Let $\mathbb{Z}_{n+1}$ denote the abelian group of integers modulo $n+1$ and let H be the cyclic subgroup of $\mathbb{Z}_{n+1}^n$ generated by $(1,1,\ldots,1)$. One can think of $\mathbb{Z}_{n+1}^n$ as the set of all placements of n distinct balls into $n+1$ distinct boxes arranged cyclically. Special cases of the following theorem have appeared in [1,2] (see Theorems 3.2 and 3.3 in Section 3).

Theorem 1.2. *The regions of* $\mathscr{S}_{n,G}$ *are in bijection with the cosets*

$$(a_1, a_2, \ldots, a_n) + H \in \mathbb{Z}_{n+1}^n / H \tag{3}$$

which satisfy the following condition: given i, if j is the smallest integer such that $i<j$ *and* $a_i = a_j$, *then* $ij \in G$.

Furthermore, in a dual formulation, our bijection maps the relatively bounded regions of $\mathscr{S}_n$ to the *prime parking functions* on $[n]$, a concept due to Gessel. The prime parking functions on $[n]$ are known to be counted by $(n-1)^{n-1}$ and hence we derive bijectively the formula for the number of relatively bounded regions of $\mathscr{S}_n$, only derived so far as a corollary to Headley's formula.

Stanley [23] has generalized the correspondence of [21, Section 5] to a bijection between the regions of the *extended Shi arrangement*

$$x_i - x_j = -k+1, -k+2, \ldots, k \quad \text{for } 1 \leqslant i < j \leqslant n \tag{4}$$

and *k-parking functions* on $[n]$, which are counted by $(kn+1)^{n-1}$. Our bijection also generalizes easily in this direction.

This paper is organized as follows: Section 2 contains our bijective proof of Theorem 1.1 and its version for the relatively bounded regions. In Section 3 we prove the more general Theorem 1.2 and derive some special cases, previously obtained with non-bijective methods. We also generalize our bijection to the extended Shi arrangements. In Section 4 we give explicitly the proof of Theorem 1.1 which follows from the methods of [4] and list some open problems related to the combinatorics of $\mathscr{S}_n$.

Note. While revising this paper, we observed that if simplified and bijectified, Headley's proof of Theorem 1.1, which we mentioned earlier, could lead to the proof in Section 2. Due to its simplicity and various generalizations and consequences, we still find this bijective formulation interesting in its own right.

2. The bijection

We first describe our bijection in terms of parking functions. A *parking function* on $[n]$ is a map $f:[n] \to [n]$ such that for all $1 \leqslant j \leqslant n$, the cardinality of the set $f^{-1}([j])$ is at least j. We also use the notation $f = (a_1, a_2, \ldots, a_n)$, where $a_i = f(i)$ for $1 \leqslant i \leqslant n$. Parking functions were first studied by Konheim and Weiss [11]. For the reason for the terminology 'parking function' see [7, Section 2.6; 21, Section 5]. An extensive literature is given in [23].

In order to describe the bijection we index the regions of $\mathscr{S}_n$ as follows. First, given a region R, we consider only the $x_i - x_j = 0$ hyperplanes and let $w = w_1 w_2 \cdots w_n$ be the unique permutation of $[n]$ such that $x_{w_1} > x_{w_2} > \cdots > x_{w_n}$ holds on R. Second, we draw an arc (i,j) in w from i to j if $i<j$ and $x_i - x_j > 1$ holds on R. Third, we remove any arcs 'containing' another arc. In other words, if there is an arc (j,k) then we

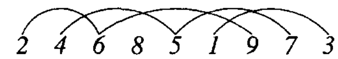

Fig. 2. The diagram of a region of $\mathscr{S}_9$.

remove any arcs (i,l), (i,k) or (j,l) if $x_i > x_j > x_k > x_l$, $x_i > x_j$ or $x_k > x_l$ holds on R respectively. Clearly, these arcs are forced by the arc (j,k) and hence are redundant. The diagram of R is the resulting permutation of $[n]$ with arcs going rightwards from smaller to larger integers, with no arc containing another.

Example. The region of $\mathscr{S}_9$ indexed by the diagram of Fig. 2 is defined by the inequalities $x_2 > x_4 > x_6 > \cdots > x_7 > x_3$ and $x_2 - x_4 < 1$, $x_2 - x_6 > 1$, $x_4 - x_6 < 1$, $x_2 - x_8 > 1$, etc.

Note that for each diagram ρ, the arcs naturally determine a partition $\pi = \pi_\rho$ of $[n]$ into chains of increasing integers. In the example above, this partition is $\pi = 269/457/8/13$. We say that the position of m in ρ is j if $m = w_j$, i.e. if m is the jth integer from the left which appears in ρ.

Definition 2.1. Let σ_n be the map from the regions of $\mathscr{S}_n$ to parking functions on $[n]$ which sends the region with diagram ρ to the function

$f(i) =$ the position in ρ of the leftmost element in the chain containing i.

This is clearly a parking function, so σ_n is well defined. The region in our example is mapped by σ_9 to the parking function $(6,1,6,2,2,1,2,4,1)$.

Theorem 2.2 (Main Theorem). *The map σ_n is a bijection between the regions of $\mathscr{S}_n$ and parking functions on $[n]$.*

Proof. We describe the inverse of σ_n explicitly. Given a parking function f, we get the partition π simply by placing i and j in the same block if $f(i) = f(j)$. The chains are obtained by listing the elements of each block in increasing order, from left to right. It remains to determine the permutation. To do so, we place the chains relative to each other one at a time, in increasing order of their values under f. Assume that we have already placed the chains with values less than j and are to place the chain with value j. Since f is a parking function, there are at least $j-1$ elements already placed. We insert the leftmost element of the chain in position j, counting from the left. There is a unique way to place the other elements of the chain to the right without forming any pair of arcs with one containing the other. This braiding defines a diagram ρ and hence a region R of $\mathscr{S}_n$. We leave it to the reader to check that this map is indeed the inverse of σ_n. $\square$

Fig. 3 illustrates the procedure to get back the region of $\mathscr{S}_9$ from the parking function for our example.

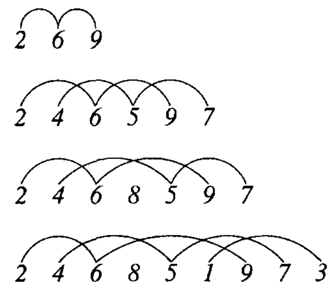

Fig. 3. Constructing the region $\sigma_g^{-1}(f)$.

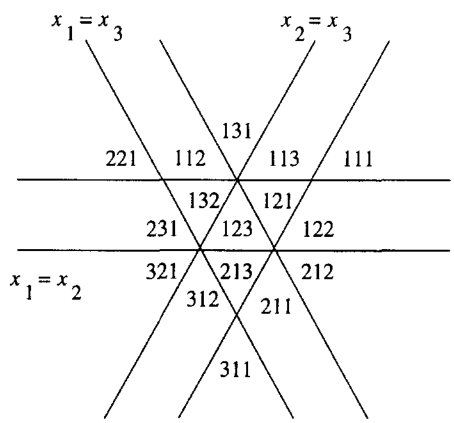

Fig. 4. The bijection σ_3.

Fig. 4 shows the 16 parking functions of length 3 associated to the regions of $\mathcal{S}_3$, according to σ_3.

The fact that there are $(n+1)^{n-1}$ parking functions on $[n]$ follows from the observation, due to Pollack [6, p. 13] and repeated by Haiman [7, pp. 28, 33] and Stanley [22, Section 2], that every coset in $\mathbb{Z}_{n+1}^n/H$ contains exactly one parking function. The following corollary proves Theorem 1.1.

Corollary 2.3. *The map σ_n induces a bijection between the regions of $\mathcal{S}_n$ and elements of $\mathbb{Z}_{n+1}^n/H$.*

The bounded regions. One can associate a region R of $\mathcal{S}_n$ to a diagram ρ in another obvious way. If $w = w_1 w_2 \cdots w_n$ is the permutation of $[n]$ defined by ρ, then we require that $x_{w_1} > x_{w_2} > \cdots > x_{w_n}$ holds on R, as before. If there is an arc in ρ going rightwards from i to j, so that $i<j$, we require instead that $x_i - x_j < 1$ holds on R. These rules define uniquely a region R of $\mathcal{S}_n$. For example, the diagram in Fig. 2 now corresponds to the region defined by the inequalities $x_2 > x_4 > x_6 > \cdots > x_7 > x_3$ and $x_2 - x_4 < 1$,

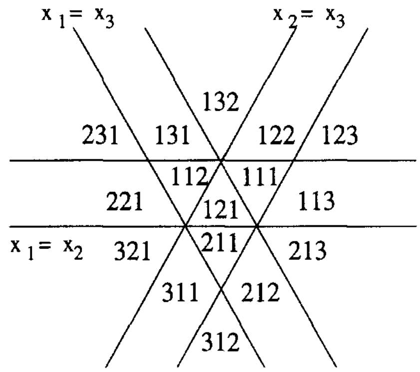

Fig. 5. The bijection τ_3.

$x_2 - x_6 < 1$, $x_4 - x_6 < 1$, $x_2 - x_8 > 1$, etc. We call ρ the *dual diagram* of R. The bijection between diagrams and parking functions, described in Definition 2.1, yields another bijection between regions of $\mathcal{S}_n$ and parking functions on $[n]$, which we denote by τ_n and illustrate in Fig. 5 for $n=3$.

A *prime parking function* on $[n]$ is a map $f:[n]\to[n]$ such that for all $1\leqslant j\leqslant n-1$, the cardinality of the set $f^{-1}([j])$ is at least $j+1$. In particular, f is a parking function on $[n]$ which does not contain n in its image. Prime parking functions on $[n]$ were recently defined by Gessel who showed that there are $(n-1)^{n-1}$ of them, using standard generating function techniques (unpublished). He also observed that they are in bijection with the forests of labeled rooted trees on $[n]$ in which the root with the smallest label has no descendants.

A region of $\mathcal{S}_n$ is relatively bounded, or simply bounded, if its intersection with the hyperplane $x_1+x_2+\cdots+x_n=0$ is bounded as a subset of Euclidean space.

Theorem 2.4. *The map τ_n is a bijection between the bounded regions of $\mathcal{S}_n$ and prime parking functions on $[n]$.*

Proof. A region R of $\mathcal{S}_n$ is unbounded if and only if there is a $j\in[n-1]$ such that no arc is directed from the first j integers to the last $n-j$ integers in its dual diagram. In terms of the parking function $f=\tau_n(R)$, this means that the cardinality of $f^{-1}([j])$ is exactly j. $\square$

A similar observation to that of Pollack for parking functions was made by Kalikow (personal communication) for prime parking functions. This can be stated as follows: every coset of the cyclic subgroup of $\mathbb{Z}_{n-1}^n$ generated by $(1,1,\ldots,1)$ contains exactly one prime parking function. The following corollary gives a bijective proof of the fact that the number of bounded regions of $\mathcal{S}_n$ is $(n-1)^{n-1}$.

Corollary 2.5. *The map τ_n induces a bijection between the bounded regions of $\mathcal{S}_n$ and elements of $\mathbb{Z}_{n-1}^n/(1,1,\ldots,1)$.*

3. Generalizations

In this section we generalize Theorem 2.2 to the arrangements between $\mathscr{A}_n$ and $\mathscr{S}_n$ and derive some special cases, previously obtained by other methods. Also, in a different direction, we give a generalization to the extended Shi arrangements.

Arrangements between $\mathscr{A}_n$ and $\mathscr{S}_n$. Recall the definition of $\mathscr{S}_{n,G}$ given in (2). A region R of $\mathscr{S}_{n,G}$ can be represented as a permutation w of $[n]$ together with a set of arcs, as in the case of $\mathscr{S}_n$. We now draw an arc (i,j) in w from i to j if $i<j$, $ij \in G$ and $x_i - x_j > 1$ holds on R. We remove all redundant arcs, as before, to get the *diagram* of R. We define the map $\sigma_{n,G}$ from the regions of $\mathscr{S}_{n,G}$ to parking functions as in Definition 2.1.

Theorem 3.1. *The map $\sigma_{n,G}$ is a bijection between the regions of $\mathscr{S}_{n,G}$ and the parking functions $f=(a_1,a_2,\ldots,a_n)$ which satisfy the following condition: given i, if j is the smallest integer such that $i<j$ and $a_i=a_j$, then $ij\in G$.*

Proof. Let R be a region of $\mathscr{S}_{n,G}$ with diagram ρ. For a chain $i_1<i_2<\cdots<i_r$ in ρ we have $i_{k-1}i_k \in G$ for all $1<k\leqslant r$ by construction. Hence the associated parking function $f=(a_1,a_2,\ldots,a_n)$, for which $a_{i_1}=a_{i_2}=\cdots=a_{i_r}$, has the property stated in the theorem. The inverse of $\sigma_{n,G}$ is as in the special case of Theorem 2.2. □

Theorem 1.2 follows immediately if we interpret parking functions as elements of $\mathbb{Z}^n_{n+1}/H$, as in Section 2. As an application of Theorem 1.2 we obtain bijective proofs for two simple results from [1,2]. The next theorem follows also from [4, Corollary 3.6].

Theorem 3.2 ([1, Theorem 3.4; 2, Theorem 6.2.2]). *Suppose that the graph G has the following property: if $1\leqslant i<j<k\leqslant n$ and $ij\in G$ then $ik\in G$. Then the number of regions of $\mathscr{S}_{n,G}$ is the product*

$$\prod_{1<j\leqslant n} (n-d_j+1), \tag{4}$$

where $d_j = \#\{i<j \mid ij \text{ is not in } G\}$ for $1<j\leqslant n$.

Proof. Under the given assumption on G, the cosets (3) of Theorem 1.2 are exactly the ones that satisfy the following condition: if $i<j$ and $a_i=a_j$ then $ij\in G$. It suffices to show that the number of such cosets is the product (4). Indeed, fix a value for a_1 to break the cyclic symmetry and suppose we have chosen values $a_2,\ldots,a_{j-1}$ satisfying the condition. We want to choose $a_j\in\mathbb{Z}_{n+1}$ so that $a_j\neq a_i$ whenever $i<j$ and ij is not in G. These values a_i are all distinct, since for two such $i_1<i_2<j$, i_1i_2 is not in G by the assumption on G and hence $a_{i_1}\neq a_{i_2}$ by the choice of a_{i_2}. It follows that there are d_j forbidden values for a_j and hence $n-d_j+1$ allowable ones. □

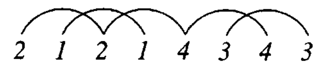

Fig. 6. The diagram of a region of $\mathscr{S}_4^2$.

Theorem 3.3 ([1, Theorem 5.6; 2, Corollary 7.1.6]). *Let G be the path $\{12, 23, \ldots, (n-1)n\}$. The number of regions of $\mathscr{S}_{n,G}$ is the sum*

$$\sum_{k=1}^{n} \frac{n!}{k!}\binom{n-1}{k-1}.$$

Proof. Now a coset (1.2) satisfies the condition of Theorem 1.2 if and only if the entries which take any fixed value of $\mathbb{Z}_{n+1}$ form a string $a_i = a_{i+1} = \cdots = a_j$. There are $\binom{n-1}{n-k} = \binom{n-1}{k-1}$ ways to form $n-k+1$ such strings and $n!/k!$ ways to assign distinct values to them, modulo cyclic symmetry. □

This sum is also the number of ways to partition the set $[n]$ and linearly order the elements within each block. A simple bijection, as was observed by Stanley (private communication), shows that the same quantity is the total number of elements in the intersection poset of $\mathscr{S}_n$, i.e. the number of affine subspaces of $\mathbb{R}^n$ which are intersections of some of the hyperplanes of $\mathscr{S}_n$.

The extended Shi arrangements. Following [23], we denote by $\mathscr{S}_n^k$ the extended Shi arrangement (3). Stanley [23] has defined a *k-parking function* on $[n]$ to be a sequence of positive integers $f = (a_1, a_2, \ldots, a_n)$ such that the unique increasing rearrangement $b_1 \leqslant b_2 \leqslant \cdots \leqslant b_n$ of the terms of f satisfies $b_i \leqslant 1 + k(i-1)$ for all i. Thus a 1-parking function is an ordinary parking function. He generalized the correspondence of [21, Section 5] to a bijection between the regions of $\mathscr{S}_n^k$ and k-parking functions on $[n]$. He also noted that, in agreement to the $k=1$ case, k-parking functions on $[n]$ are in bijection with the cosets of the cyclic subgroup of $\mathbb{Z}_{kn+1}^n$ generated by $(1, 1, \ldots, 1)$, where $\mathbb{Z}_{kn+1}$ is the abelian group of integers modulo $kn+1$. Hence there are exactly $(kn+1)^{n-1}$ k-parking functions on $[n]$. Additional work on $\mathscr{S}_n^k$ is included in [2, Section 7.1; 4, Section 3; 15, Section 1.5; 16, Section 9].

We now generalize the bijection σ_n to treat the arrangements $\mathscr{S}_n^k$. We associate a diagram to a region R of $\mathscr{S}_n^k$ as follows. First, we consider only the hyperplanes $x_i - x_j = l$ for $-k+1 \leqslant l \leqslant k-1$ and let $y = y_1 y_2 \cdots y_{kn}$ be the unique permutation of the variables $x_i + m$, where $1 \leqslant i \leqslant n$ and $0 \leqslant m \leqslant k-1$, such that $y_1 > y_2 > \cdots > y_{kn}$ holds on R. We draw arcs in y going rightwards from $x_i + m$ to $x_i + m - 1$ for all i and $m > 0$. Second, we draw an arc from x_i to $x_j + k - 1$ if $i < j$ and $x_i - x_j > k$ holds on R. Finally, we remove all arcs containing another arc and replace each variable $x_i + m$ by i. The arcs determine naturally a partition of the multiset $M_n^k = \{1^k, 2^k, \ldots, n^k\}$ into chains of weakly increasing integers such that the elements of M_n^k equal to i all appear in the same chain.

Example. The diagram of Fig. 6 represents the region of $\mathscr{S}_4^2$ defined by the inequalities $x_2+1>x_1+1>x_2>x_1>x_4+1>x_3+1>x_4>x_3$, $x_2-x_4>2$ and $x_1-x_3<2$. The corresponding partition of $M_4^2=\{1,1,2,2,3,3,4,4\}$ into chains is 2244/11/33.

Definition 3.4. We define the map σ_n^k by sending the region R of $\mathscr{S}_n^k$ with diagram ρ to the function $f=(a_1,a_2,\dots,a_n)$ with

$$a_i = \text{the position in } \rho \text{ of the leftmost element of the chain containing all } i\text{'s.}$$

As before, it is easy to check that f is a k-parking function. For the region of Fig. 6 we have $f=(2,1,6,1)$.

Theorem 3.5. *The map σ_n^k is a bijection between the regions of $\mathscr{S}_n^k$ and k-parking functions on $[n]$.*

Proof. We describe the inverse map of σ_n^k, as in the proof of Theorem 2.2. Let $f=(a_1,a_2,\dots,a_n)$ be a k-parking function and $b_1\leqslant b_2\leqslant\cdots\leqslant b_n$ be the unique increasing rearrangement of its terms. For each value j of f, consider a chain C_j of positive integers, listed from left to right in increasing order. The chain C_j contains k copies of r if $a_r=j$ and none otherwise. Place the chains one at a time, in order of increasing value j. If $j=b_i>b_{i-1}$, then there are $k(i-1)\geqslant j-1$ elements listed before placing C_j, since f is a k-parking function. Insert the leftmost element of C_j in position j, counting from the left, and the other elements to the right so that no pair of arcs with one containing the other is formed. This defines the desired diagram, and hence region of $\mathscr{S}_n^k$. □

Fig. 7 illustrates the bijection σ_3^2.

4. Remarks and open problems

1. The computation of $\chi(\mathscr{S}_n,q)$ by deletion and restriction in [4, Theorem 3.1] can be carried out on the level of the number of regions. It results in a naive inductive proof of Theorem 1.1, suitably generalized, which we describe explicitly next. Let $\mathscr{A}$ be a hyperplane arrangement and $H\in\mathscr{A}$ a distinguished hyperplane. The crucial and well known fact that we use in what follows is that

$$r(\mathscr{A})=r(\mathscr{A}')+r(\mathscr{A}''), \tag{5}$$

where $\mathscr{A}'=\mathscr{A}-\{H\}$ is the corresponding *deleted arrangement* and $\mathscr{A}''=\{H'\cap H \mid H'\in\mathscr{A}'\}$ is the *restricted arrangement* to H. Note that $\mathscr{A}''$ is an arrangement in the affine space H.

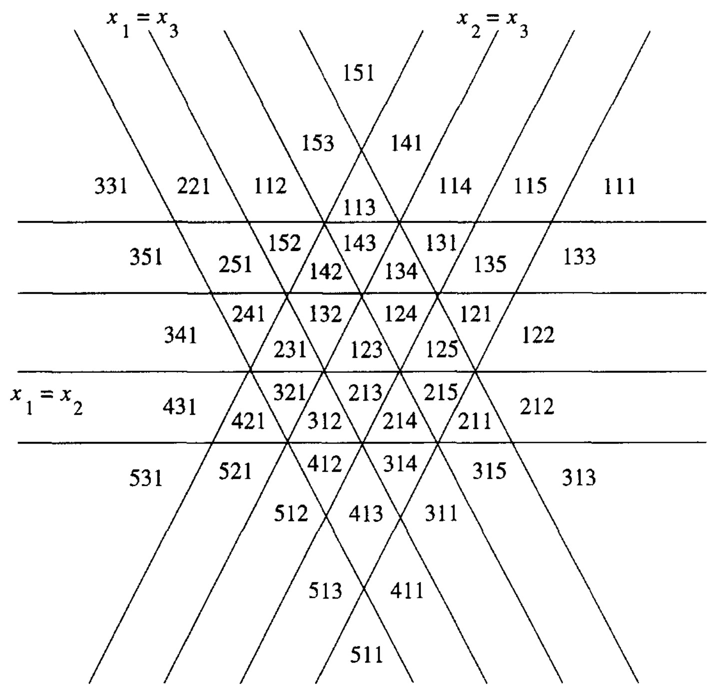

Fig. 7. The bijection σ_3^2.

Theorem 4.1. *For any integers $m \geqslant 0$ and $2 \leqslant k \leqslant n+1$, the arrangement*

$$x_1 - x_j = \begin{cases} 0,1,\ldots,m & \text{for } 2 \leqslant j < k, \\ 0,1,\ldots,m+1 & \text{for } k \leqslant j \leqslant n, \\ 0,1 & \text{for } 2 \leqslant i < j \leqslant n \end{cases} \tag{6}$$

has $(n+m)^{k-2}(n+m+1)^{n-k+1}$ regions. In particular, for $m=0$ and $k=2$, $\mathscr{S}_n$ has $(n+1)^{n-1}$ regions.

Proof. We proceed by double induction on n and $n-k$, the result being clear for $n=2$. The case $m=0$ and $k=n+1$ follows easily from the result for $\mathscr{S}_{n-1}$. Indeed, each of the n^{n-2} regions of $\mathscr{S}_{n-1}$ in the space with coordinate functions $x_2, \ldots, x_n$ determines a linear order of these variables and there are n ways to form a region of (6) by inserting x_1 in this order. We can now assume $2 \leqslant k \leqslant n$, since the arrangement (6) having parameters $m \geqslant 1$ and $k = n+1$ coincides with (6) having parameters $m-1$ and $k=2$. Consider the hyperplane H of (6) with equation $x_1 - x_k = m+1$. The corresponding deleted arrangement has the same form as (6), with k replaced by $k+1$. The restricted arrangement to H has again the same form, with n replaced by $n-1$ and m replaced by $m+1$, once one replaces x_k by $x_1 - m - 1$ in the equations involving x_k. The result follows by the induction hypothesis on these two arrangements and (5). □

2. Let k be any integer satisfying $1 \leqslant k \leqslant n$. The number of faces of $\mathscr{S}_n$ of dimension k was shown [1, Theorem 6.5; 2, Corollary 8.2.1] to have the surprisingly simple combinatorial interpretation

$$f_k(\mathscr{S}_n) = \binom{n}{k} \# \{f\colon [n-1] \to [n+1] \mid [n-k] \subseteq \operatorname{Im} f\},$$

where $\operatorname{Im} f$ is the image of the map f. This formula reduces to Theorem 1.1 for $k=n$. The general case lacks a bijective proof and shows that the combinatorics of $\mathscr{S}_n$ is still not well understood. A similar interpretation was obtained for the extended Shi arrangements [2, Theorem 8.2.2].

3. Shi [19] has generalized Theorem 1.1 to any irreducible crystallographic root system Φ. The analogue of the Shi arrangement for the root system D_n has the additional hyperplanes $x_i + x_j = 0$ or 1 for $1 \leqslant i < j \leqslant n$. For B_n and C_n, one has to add to these the hyperplanes $x_i = 0$ or 1 and $2x_i = 0$ or 1, respectively, for $1 \leqslant i \leqslant n$. The number of regions in the general case is $(h+1)^l$, where h is the Coxeter number of Φ, which has the value $2n-2, 2n$ and $2n$ for $\Phi = D_n, B_n$ and C_n, respectively, and l is the rank of Φ, which has the value n for these three root systems.

Shi's proof [19] is long but uniform, while the proofs in [1, Theorem 3.13; 2, Theorem 6.3.5] are simple but are given case by case and use the characteristic polynomial. It would be interesting to find simple bijective proofs, similar to the one for type A in Section 2, at least for the infinite families of type B, C and D.

4. There is no uniform proof known for the analogue of Headley's formula [8, 9, Chapter VI, 10] for other irreducible crystallographic root systems (see also [1, Corollary 3.2, 2, Corollary 6.1.2] and the remarks that follow these corollaries). This formula expresses the characteristic polynomial as $(q-h)^l$, where h and l are as in Remark 3.

5. For a region R of $\mathscr{S}_n$, let $d(R)$ be the number of pairs (i,j) with $1 \leqslant i < j \leqslant n$ such that either $x_i < x_j$ or $x_i - x_j > 1$ holds on R. In other words, $d(R)$ is the number of hyperplanes of $\mathscr{S}_n$ which seperate R from the base region R_0, defined by the inequalities $x_1 > x_2 > \cdots > x_n$ and $x_1 - x_n < 1$. Recall that an *inversion* of a tree T on the vertex set $[0,n] := \{0,1,\ldots,n\}$ is a pair (i,j) with $1 \leqslant i < j \leqslant n$, such that the vertex j lies on the unique path in T from 0 to i. The bijection of Pak and Stanley [21, Section 5] combined with a bijection between parking functions and trees due to Kreweras [12] shows that the number of regions R of $\mathscr{S}_n$ with $d(R) = m$ is equal to the number of trees on $[0,n]$ with $\binom{n}{2} - m$ inversions. It would be interesting to find a simpler and more direct proof of this statement.

The same quantity was interpreted in [3] as the number of posets P on $[n]$ which avoid certain three-element induced subposets and consist of m relations of the form $a <_P b$.

6. Except for the type A [4], it is not known whether the Shi arrangements are free. This is a special case of a conjecture of Edelman and Reiner [5, Conjecture 3.3]. An indication for the validity of this statement is contained in recent work of Solomon and Terao [20].

References

[1] C.A. Athanasiadis, Characteristic polynomials of subspace arrangements and finite fields, Adv. Math. 122 (1996) 193–233.

[2] C.A. Athanasiadis, Algebraic combinatorics of graph spectra, subspace arrangements and Tutte polynomials, Ph.D. Thesis, MIT, 1996.

[3] C.A. Athanasiadis, A class of labeled posets and the Shi arrangement of hyperplanes, J. Combin. Theory Ser. A 80 (1997) 158–162.

[4] C.A. Athanasiadis, On free deformations of the braid arrangement, European J. Combin. 19 (1998) 7–18.

[5] P.H. Edelman, V. Reiner, Free arrangements and rhombic tilings, Discrete Comput. Geom. 15 (1996) 307–340.

[6] D. Foata, J. Riordan, Mappings of acyclic and parking functions, Aequationes Math. 10 (1974) 10–22.

[7] M.D. Haiman, Conjectures on the quotient ring by diagonal invariants, J. Algebraic Combin. 3 (1994) 17–76.

[8] P. Headley, Reduced expressions in infinite Coxeter groups, Ph.D. Thesis, University of Michigan, 1994.

[9] P. Headley, On reduced words in affine Weyl groups, in: L.J. Billera, C. Greene, R. Simion, R. Stanley (Eds.), Proc. Formal Power Series and Algebraic Combinatorics (FPSAC) 1994, DIMACS Series in Discrete Mathematics and Theoretical Computer Science, Amer. Math. Soc. 24 (1996) 225–232.

[10] P. Headley, On a family of hyperplane arrangements related to affine Weyl groups, J. Algebraic Combin. 6 (1997) 331–338.

[11] A.G. Konheim, B. Weiss, An occupancy discipline and applications, SIAM J. Appl. Math. 14 (1966) 1266–1274.

[12] G. Kreweras, Une famille de polynômes ayant plusieurs propriétés énumeratives, Period. Math. Hungar. 11 (1980) 309–320.

[13] C.L. Mallows, J. Riordan, The inversion enumerator for labeled trees, Bull. Amer. Math. Soc. 74 (1968) 92–94.

[14] P. Orlik, H. Terao, Arrangements of Hyperplanes, Grundlehren 300, Springer, New York, NY, 1992.

[15] A. Postnikov, Enumeration in algebra and geometry, Ph.D. Thesis, MIT, 1997.

[16] A. Postnikov, R. Stanley, Deformations of Coxeter hyperplane arrangements, Preprint, 1997.

[17] A. Ram, Standard Young tableaux for finite root systems, preprint, 1998.

[18] J.-Y. Shi, The Kazhdan-Lusztig cells in certain affine Weyl groups, Lecture Notes in Mathematics, No. 1179, Springer, Berlin, 1986.

[19] J.-Y. Shi, Sign types corresponding to an affine Weyl group, J. London Math. Soc. 35 (1987) 56–74.

[20] L. Solomon, H. Terao, The double Coxeter arrangement, Commentarii Math. Helv. 73 (1998) 237–258.

[21] R. Stanley, Hyperplane arrangements, interval orders and trees, Proc. Nat. Acad. Sci. 93 (1996) 2620–2625.

[22] R. Stanley, Parking functions and noncrossing partitions, Electronic J. Combin. 4 (1997) R20.

[23] R. Stanley, Hyperplane arrangements, parking functions and tree inversions, in: B.E. Sagan and R. Stanley, eds., Mathematical Essays in Honor of Gian-Carlo Rota, Progress in Math. 161 (1998), Birkhäuser, Boston, pp. 359–375.

[24] T. Zaslavsky, Facing up to arrangements: face-count formulas for partitions of space by hyperplanes, Mem. Amer. Math. Soc. 1 (1975) (154).