Journal of Computational and Applied Mathematics 21 (1988) 189–202
North-Holland

A fourth-order spline method for singular two-point boundary-value problems

M.M. CHAWLA and R. SUBRAMANIAN
Department of Mathematics, Indian Institute of Technology, Hauz Khas, New Delhi-110016, India

H.L. SATHI
Department of Computer Science Technology, University of Southern Colorado, Pueblo, CO 81001, U.S.A.

Received 22 December 1986
Revised 12 August 1987

Abstract: This paper describes two methods for the solution of (weakly) singular two-point boundary-value problems:

$$x^{-\alpha}(x^{\alpha}y')' = f(x, y), \quad 0 < x < 1,$$
$$y(0) = A, \quad y(1) = B, \quad 0 < \alpha < 1.$$

Consider the uniform mesh $x_i = ih$, $h = 1/N$, $i = 0(1)N$. Define the linear functionals $L_i(y) = y(x_i)$ and $M_i(y) = (x^{-\alpha}(x^{\alpha}y')'|_{x=x_i}$. In both these methods a piecewise 'spline' solution is obtained in the form $s(x) = s_i(x)$, $x \in [x_{i-1}, x_i]$, $i = 1(1)N$, where in each subinterval $s_i(x)$ is in the linear span of a certain set of (non-polynomial) basis functions in the representation of the solution $y(x)$ of the two-point boundary value problem and satisfies the interpolation conditions: $L_{i-1}(s) = L_{i-1}(y)$, $L_i(s) = L_i(y)$, $M_{i-1}(s) = M_{i-1}(y)$, $M_i(s) = M_i(y)$. By construction s and $x^{-\alpha}(x^{\alpha}s')' \in C[0, 1]$. Conditions of continuity are derived to ensure that $x^{\alpha}s' \in C[0, 1]$. It follows that the unknown parameters y_i and $M_i(y)$, $i = 1(1)N-1$, must satisfy conditions of the form:

$$-\frac{1}{J_i}y_{i-1} + \left(\frac{1}{J_i} + \frac{1}{J_{i+1}}\right)y_i - \frac{1}{J_{i+1}}y_{i+1}$$
$$+ k_{i,i-1}M_{i-1}(y) + k_{i,i}M_i(y) + k_{i,i+1}M_{i+1}(y) = 0, \quad i = 1(1)N-1. \tag{$*$}$$

The first method consists in replacing $M_i(y)$ by $f(x_i, y_i)$ and solving $(*)$ to obtain the values y_i; this method is a generalization of the idea of Bickley [2] for the case of (weakly) singular two-point boundary-value problems and provides order h^2 uniformly convergent approximations over [0, 1]. As a modification of the above method, in the second method we generate the solution $\bar{y}_i$ at the nodal points by adapting the fourth-order method of Chawla [3] and then use the conditions of continuity $(*)$ to obtain the corresponding smoothed approximations for $M_i(y)$ needed for the construction of the spline solution. We show that the resulting *new* spline method provides order h^4 uniformly convergent approximations over [0,1]. The second-order and the fourth-order methods are illustrated computationally.

Keywords: Singular two-point boundary-value problems, spline solution, non-polynomial basis functions, interpolation conditions, continuity conditions.

1. Introduction

We consider the class of (weakly) singular two-point boundary value problems:

$$\mathcal{D}y \equiv x^{-\alpha}(x^{\alpha}y')' = f(x, y), \quad 0 < x < 1, \quad \alpha \in (0, 1),$$
$$y(0) = A, \quad y(1) = B, \quad A, B \text{ constants}. \tag{1}$$

We assume that, for $(x, y) \in \{[0, 1] \times \mathbb{R}\}$;

(A) $f(x, y)$ is continuous, $\partial f/\partial y$ exists and is continuous and $\partial f/\partial y \geqslant 0$.

Such problems arise in the study of generalized axially symmetric potentials after separation of variables has been employed [8,11]. The discrete variable numerical solution of the singular two-point boundary-value problems (1) by finite differences has been considered by many authors; see, for example, [11], [8], [4], [5], and the references given in these papers. The finite difference methods presented so far had order at most two. Recently, Chawla [3] described a fourth-order discrete variable finite difference method for the problem (1) for the case $f(x, y)$ replaced by $x^{-\alpha}f(x, y)$.

The use of cubic spline for the solution of (regular) linear two-point boundary-value problems was suggested by Bickley [2]. Later, Fyfe [10] discussed the application of deferred corrections to the method suggested by Bickley by considering again the case of (regular) linear boundary-value problems. In comparison with finite difference methods, spline solution has its own advantages. For example, once the solution has been computed, the information required for spline interpolation between mesh points is available. This is particularly significant when the solution of the boundary-value problem is required at various locations in the interval [0, 1]. An important instance also is the use of an automatic plotter that frequently requires interpolation at great many intermediate points. However, it is well known since then that the cubic spline method of Bickley gives only order h^2 convergent approximations. But cubic spline itself is a fourth-order process. Recently, for (regular) nonlinear two-point boundary-value problems Chawla and Subramanian [6], [7] have described methods based on cubic splines which provide order h^4 uniformly convergent approximations.

In the present paper we describe two 'spline' approximation methods suited to the two-point boundary-value problem (1). Consider the uniform mesh $x_i = ih$, $h = 1/N$, $i = 0(1)N$. Associated with the differential operator $\mathscr{D}$ in (1), we define the following functionals:

$$L_i(y) = y(x_i), \qquad Z_i(y) = (x^\alpha y')\,|_{x=x_i},$$

$$M_i(y) = x^{-\alpha}(x^\alpha y')'\,|_{x=x_i}, \qquad M_i'(y) = (x^{-\alpha}(x^\alpha y')')'\,|_{x=x_i}.$$

Writing the differential equation in (1) as $(x^\alpha y')' = x^\alpha f(x, y)$, integrating from x_i to x, dividing by x^α, then integrating from x_i to x and interchanging the order of integration, we obtain

$$y(x) = \phi_{0,i}(x)L_i(y) + \phi_{1,i}(x)Z_i(y) + \int_{x_i}^{x} \frac{t^\alpha(x^{1-\alpha} - t^{1-\alpha})}{(1-\alpha)} f(t)\, \mathrm{d}t, \tag{2}$$

where $f(t) = f(t, y(t))$ and

$$\phi_{0,i}(x) = 1, \qquad \phi_{1,i}(x) = (x^{1-\alpha} - x_i^{1-\alpha})/(1-\alpha).$$

Again, in (2) using the Taylor expansion for $f(t)$:

$$f(t) = f(x_i) + (t - x_i)f'(x_i) + \int_{x_i}^{t} (t-u)f''(u)\, \mathrm{d}u, \tag{3}$$

we obtain the following representation for the solution of the two-point boundary value problem (1):

$$y(x)=\phi_{0,i}(x)L_i(y)+\phi_{1,i}(x)Z_i(y)+\phi_{2,i}(x)M_i(y)+\phi_{3,i}(x)M_i'(y)+R_i(y;\,x), \tag{4}$$

where

$$\phi_{2,i}(x)=\frac{x^2-x_i^2}{2(1+\alpha)}-\frac{x_i^{\alpha+1}}{\alpha+1}\phi_{1,i}(x),$$

$$\phi_{3,i}(x)=\frac{x^2(x-x_i)}{2(1+\alpha)}-\frac{(x^3-x_i^3)(4+\alpha)}{6(\alpha+1)(\alpha+2)}+\frac{x_i^{\alpha+2}}{(\alpha+1)(\alpha+2)}\phi_{1,i}(x),$$

and the remainder $R_i(y;\,x)$ is given by

$$R_i(y;\,x)=\int_{x_i}^{x}G(x;\,u)f''(u)\,\mathrm{d}u, \tag{5}$$

where

$$G(x;\,u)=\frac{x^2(x-u)}{2(1+\alpha)}-\frac{(x^3-u^3)(4+\alpha)}{6(1+\alpha)(2+\alpha)}+\frac{u^{\alpha+2}}{(\alpha+1)(\alpha+2)}\,\frac{x^{1-\alpha}-u^{1-\alpha}}{1-\alpha}.$$

The representation (4) for the solution of the two-point boundary-value problem (1) in terms of the linear functionals $L_i(y)$, $Z_i(y)$, $M_i(y)$, $M_i'(y)$ is fundamental for the construction of our spline approximation for the solution.

Let $\mathscr{L}\{\phi_{0,i},\,\phi_{1,i},\,\phi_{2,i},\,\phi_{3,i}\}$ denote the linear span of $\phi_{0,i}$, $\phi_{1,i}$, $\phi_{2,i}$, $\phi_{3,i}$. Note that $\phi_{0,i}$, $\phi_{1,i}$, $\phi_{2,i}$, $\phi_{3,i}$ are biorthonormal with respect to the linear functionals $L_i(y)$, $Z_i(y)$, $M_i(y)$ and $M_i'(y)$; that is, for $j=0(1)3$,

$$L_i(\phi_{j,i})=\delta_{0,j},\qquad Z_i(\phi_{j,i})=\delta_{1,j},$$

$$M_i(\phi_{j,i})=\delta_{2,j},\qquad M_i'(\phi_{j,i})=\delta_{3,j},$$

($\delta_{j,i}=1$, if $j=i$ and 0 otherwise). This implies that (4) represents an interpolation formula for the solution $y(x)$ of the two-point boundary-value problem (1) form $\mathscr{L}\{\phi_{0,i},\,\phi_{1,i},\,\phi_{2,i},\,\phi_{3,i}\}$ in terms of the linear functionals $L_i(y)$, $Z_i(y)$, $M_i(y)$ and $M_i'(y)$, together with a remainder. Note also that for any $\phi\in\mathscr{L}\{\phi_{0,i},\,\phi_{1,i},\,\phi_{2,i},\,\phi_{3,i}\}$, we have $(x^{-\alpha}(x^{\alpha}\phi')')''=0$.

For each $i=1(1)N$, on the subinterval $[x_{i-1},\,x_i]$ we construct a spline approximation $s_i(x)$ to the solution $y(x)$ of the singular two-point boundary value problem (1) as follows:

(i) $s_i(x)\in\mathscr{L}\{\phi_{0,i},\,\phi_{1,i},\,\phi_{2,i},\,\phi_{3,i}\}$,

(ii) $s_i(x)$ satisfies the interpolation conditions

$$L_i(s_i)=L_i(y),\qquad L_{i-1}(s_i)=L_{i-1}(y),$$

$$M_i(s_i)=M_i(y),\qquad M_{i-1}(s_i)=M_{i-1}(y). \tag{6}$$

The global spline approximation $s(x)$ for the solution $y(x)$ of the two-point boundary-value problem (1) may now be defined as follows:

(i) s, $x^{\alpha}s'$ and $x^{-\alpha}(x^{\alpha}s')'\in C[0,\,1]$,

(ii) for $i=1(1)N$, on each subinterval $[x_{i-1},\,x_i]$, $s(x)$ coincides with $s_i(x)$.

Note that by construction s and $x^{-\alpha}(x^\alpha s')'$ are continuous at the nodes and, hence, s and $x^{-\alpha}(x^\alpha s')' \in C[0, 1]$. Now, in order that $x^\alpha s' \in C[0, 1]$, $x^\alpha s'$ must be continuous at the nodes and, hence, s must satisfy the following 'continuity conditions':

$$Z_i(s_i) = Z_i(s_{i+1}), \quad i = 1(1)N-1. \tag{7}$$

In Section 2 we show that these conditions of continuity imply that the unknown parameters y_i and $M_i(y)$, $i = 1(1)N-1$, must satisfy the conditions of equation (13).

This paper describes two methods for the construction of spline solution of (weakly) singular two-point boundary-value problems (1). The first method described in Section 2 consists in replacing $M_i(y)$ by $f(x_i, y_i)$ and solving the 'conditions of continuity' given by equation (13) to obtain the values y_i; this method is a generalization of the idea of Bickley [2] for the case of (weakly) singular two-point boundary value problems (1). In Section 3 we show, under appropriate conditions (see Theorem 3.1), that this method provides order h^2 uniformly convergent approximations for the solution.

Note that for $\alpha = 0$, the above method reduces to the well known procedure of the cubic spline solution of (regular) two-point boundary-value problems $y'' = f(x, y)$, $y(0) = A$, $y(1) = B$. However, as pointed out above, cubic spline interpolation is a fourth-order process (see (27)) and a suitable modification of the original idea of Bickley should produce order h^4 cubic spline approximations. This has been successfully demonstrated by Chawla and Subramanian [6,7] for the case of regular two-point boundary-value problems.

Accordingly, as a modification of the method described in Section 2, in the second method described in Section 4 we generate the solution $\bar{y}_i$ at the nodal points by adapting the fourth order method of Chawla [3]. These values of the solution are then used in the conditions of continuity given by equation (13) to obtain the corresponding smoothed approximations for $M_i(y)$ needed for the construction of the spline solution. We show, under appropriate conditions (see Theorem 4.1), that the resulting *new* spline method provides order h^4 uniformly convergent approximations over [0, 1]. Both the spline methods are illustrated computationally in Section 5.

2. The first spline method: generalization of Bickley's idea

In this section we describe the construction of the spline solution as defined above. For this purpose, we first consider the construction of $s_i(x)$ over $[x_{i-1}, x_i]$ for $i = 1(1)N$.

For each $i = 1(1)N$, $x \in [x_{i-1}, x_i]$, we may represent $s_i(x)$ as

$$s_i(x) = \psi_{0,i}(x)L_i(y) + \psi_{1,i}(x)L_{i-1}(y) + \psi_{2,i}(x)M_i(y) + \psi_{3,i}(x)M_{i-1}(y), \tag{8}$$

where $\psi_{0,i}(x)$, $\psi_{1,i}(x)$, $\psi_{2,i}(x)$, $\psi_{3,i}(x) \in \mathscr{L}\{\phi_{0,i}, \phi_{1,i}, \phi_{2,i}, \phi_{3,i}\}$.
In order that this $s_i(x)$ satisfies the interpolation conditions (6) it is clear that the functions $\psi_{0,i}(x)$, $\psi_{1,i}(x)$, $\psi_{2,i}(x)$, and $\psi_{3,i}(x)$ must be biorthonormal with respect to the linear functionals L_i, L_{i-1}, M_i and M_{i-1}. Noting that

$$M_{i-1}(\phi_{j,i}) = 0, \quad j = 0, 1, \qquad M_{i-1}(\phi_{2,i}) = 1, \qquad M_{i-1}(\phi_{3,i}) = -h,$$

and following the arguments given in Davis [9, pp. 35–36], we obtain

$$\begin{aligned}
\psi_{0,i}(x) &= \frac{1}{L_{i-1}(\phi_{1,i})}(L_{i-1}(\phi_{1,i}) - \phi_{1,i}(x)),\\
\psi_{1,i}(x) &= \frac{\phi_{1,i}(x)}{L_{i-1}(\phi_{1,i})},\\
\psi_{2,i}(x) &= \frac{1}{hL_{i-1}(\phi_{1,i})}[-\phi_{1,i}(x)(L_{i-1}(\phi_{3,i}) + hL_{i-1}(\phi_{2,i}))\\
&\qquad + L_{i-1}(\phi_{1,i})(h\phi_{2,i}(x) + \phi_{3,i}(x))],\\
\psi_{3,i}(x) &= \frac{1}{hL_{i-1}(\phi_{1,i})}[-\phi_{3,i}(x)L_{i-1}(\phi_{1,i}) + \phi_{1,i}(x)L_{i-1}(\phi_{3,i})].
\end{aligned} \tag{9}$$

While on each subinterval $[x_{i-1}, x_i]$, $i = 1(1)N$, $s(x)$ coincides with $s_i(x)$, now in order to ensure that $x^\alpha s' \in C[0, 1]$, the continuity conditions (7) must be satisfied. With the formula for $s_i(x)$, and the corresponding formula for $s_{i+1}(x)$, given by (8) the continuity conditions (7) become

$$\begin{aligned}
&Z_i(\psi_{1,i})L_{i-1}(y) + (Z_i(\psi_{0,i}) - Z_i(\psi_{1,i+1}))L_i(y) - Z_i(\psi_{0,i+1})L_{i+1}(y)\\
&\quad = -Z_i(\psi_{3,i})M_{i-1}(y) + (Z_i(\psi_{3,i+1}) - Z_i(\psi_{2,i}))M_i(y)\\
&\quad + Z_i(\psi_{2,i+1})M_{i+1}(y), \quad i=1(1)N-1.
\end{aligned} \tag{10}$$

These conditions ensure the continuity of $x^\alpha s'$ at the nodes.

In the following it will be convenient to set

$$J_i = (x_i^{1-\alpha} - x_{i-1}^{1-\alpha})/(1-\alpha), \quad i = 1(1)N. \tag{11}$$

Since $Z_i(\phi_{1,i}) = 1$ and $Z_i(\phi_{1,i+1}) = 1$, with the help of (9) it is easy to see that

$$\begin{aligned}
&Z_i(\psi_{0,i}) = 1/J_i, \qquad Z_i(\psi_{1,i}) = -1/J_i,\\
&Z_i(\psi_{0,i+1}) = 1/J_{i+1}, \qquad Z_i(\psi_{1,i+1}) = -1/J_{i+1}.
\end{aligned} \tag{12}$$

Now, with the help of (12), the conditions of continuity (10) can be written as:

$$\begin{aligned}
&-\frac{1}{J_i}L_{i-1}(y) + \left(\frac{1}{J_i} + \frac{1}{J_{i+1}}\right)L_i(y) - \frac{1}{J_{i+1}}L_{i+1}(y)\\
&\quad + k_{i,i-1}M_{i-1}(y) + k_{i,i}M_i(y) + k_{i,i+1}M_{i+1}(y) = 0, \quad i = 1(1)N-1,
\end{aligned} \tag{13}$$

where we have set

$$k_{i,i-1} = Z_i(\psi_{3,i}), \qquad k_{i,i} = Z_i(\psi_{2,i}) - Z_i(\psi_{3,i+1}), \qquad k_{i,i+1} = -Z_i(\psi_{2,i+1}). \tag{14}$$

It may be interesting to note here that for the case $\alpha = 0$, $J_i = h$, $k_{i,i\pm1} = \frac{1}{6}h$, $k_{i,i} = \frac{4}{6}h$ and the conditions (13) reduce to those for the usual cubic spline solution ensuring the continuity of y' across the interior nodes for the two-point boundary-value problem $y'' = f(x, y)$, $y(0) = A$, $y(1) = B$; see Ahlberg et al. [1].

From the boundary conditions for the problem (1) we obtain

$$M_0(y) = f(0, A), \qquad M_N(y) = f(1, B), \tag{15}$$

while from the differential equation we obtain

$$M_i(y) = f(x_i, y_i), \quad i = 1(1)N-1. \tag{16}$$

As pointed out in the Introduction in Section 1, Bickley's [2] main idea was to use the conditions of continuity as discretization equations for the solution of the two-point boundary-value problem. Now, following Bickley for the solution of the two-point boundary-value problem (1), we substitute (16) and (15) in equation (13) which results in the following system giving approximate values $\tilde{y}_1, \ldots, \tilde{y}_{N-1}$:

$$-\frac{1}{J_i}\tilde{y}_{i-1} + \left(\frac{1}{J_i} + \frac{1}{J_{i+1}}\right)\tilde{y}_i - \frac{1}{J_{i+1}}\tilde{y}_{i+1} + k_{i,i-1}\tilde{f}_{i-1}$$
$$+ k_{i,i}\tilde{f}_i + k_{i,i+1}\tilde{f}_{i+1} = 0, \quad i = 1(1)N-1, \tag{17}$$

where $\tilde{y}_0 = y_0 = A$, $\tilde{y}_N = y_N = B$, and we have set $\tilde{f}_i = f(x_i, \tilde{y}_i)$, $i = 0(1)N$. Once the $\tilde{y}_1, \ldots, \tilde{y}_{N-1}$ have been computed from (17), the corresponding approximate $\tilde{M}_i(y)$ are found from (16). With these values of $\tilde{y}_1, \ldots, \tilde{y}_{N-1}$ and $\tilde{M}_1(y), \ldots, \tilde{M}_{N-1}(y)$, the corresponding spline solution of the two-point boundary-value problem (1) is given by $\tilde{s}_i(x)$, where

$$\tilde{s}_i(x) = \psi_{0,i}(x)\tilde{y}_i + \psi_{1,i}(x)\tilde{y}_{i-1} + \psi_{2,i}(x)\tilde{M}_i(y) + \psi_{3,i}(x)\tilde{M}_{i-1}(y),$$
$$x_{i-1} \leqslant x \leqslant x_i, \quad i = 1(1)N. \tag{18}$$

The approximate spline solution thus obtained over [0, 1] will be denoted by $\tilde{s}(x)$.

3. Error analysis of the first spline method

In this section we show that the spline method for the solution of the singular two-point boundary-value problem (1) discussed in Section 2 gives order h^2 uniformly convergent approximations over [0, 1].

For the spline approximation $\tilde{s}(x)$ to the solution $y(x)$ of the boundary-value problem (1), let the error be denoted by

$$\tilde{e}(x) = y(x) - \tilde{s}(x), \quad 0 \leqslant x \leqslant 1. \tag{19}$$

It is clear that we may write

$$\tilde{e}(x) = e_{\mathrm{I}}(x) + e_{\mathrm{D}}(x), \tag{20}$$

where

$$e_{\mathrm{I}}(x) = y(x) - s(x), \tag{21}$$

is the error due to spline interpolation, and

$$e_{\mathrm{D}}(x) = s(x) - \tilde{s}(x), \tag{22}$$

is the error due to the use of the 'conditions of continuity' (13) as discretization equations (17). In the following we estimate these two errors separately.

We first estimate the error $e_{\mathrm{I}}(x)$. For this purpose calculating $L_{i-1}(y)$ and $M_{i-1}(y)$ from the representation for $y(x)$ in equation (4) and substituting these in equation (8) and simplifying, from (21) we finally obtain for $x_{i-1} \leqslant x \leqslant x_i$,

$$e_{\mathrm{I}}(x) = \int_{x_i}^{x_{i-1}} G^*(x;\, u) f''(u)\, \mathrm{d}u, \tag{23}$$

where

$$G^*(x;\, u) = \begin{cases} G(x;\, u) - \psi_{1,i}(x)G(x_{i-1};\, u) - \psi_{3,i}(x)(x_{i-1} - u), & x \leqslant u \leqslant x_i, \\ -\psi_{1,i}(x)G(x_{i-1};\, u) - \psi_{3,i}(x)(x_{i-1} - u), & x_{i-1} \leqslant u \leqslant x. \end{cases} \tag{24}$$

It can be shown that $G^*(x;\ u)$ is nonnegative for $x_{i-1} \leqslant u \leqslant x_i$, and therefore with the mean-value theorem, we obtain the following bound for $e_{\mathrm{I}}(x)$:

$$|e_{\mathrm{I}}(x)| \leqslant 3h^4 |f''(\xi_x)|, \qquad x_{i-1} \leqslant x, \quad \xi_x \leqslant x_i. \tag{25}$$

We assume that

$$|f''| \leqslant N_2, \quad 0 \leqslant x \leqslant 1, \tag{26}$$

for a suitable positive constant N_2. Then from (25) we obtain

$$\| e_{\mathrm{I}} \|_{\infty} \leqslant 3h^4 N_2. \tag{27}$$

Next, for the error $e_{\mathrm{D}}(x)$, we first note that we may write

$$f(x_i,\ y_i) - f(x_i,\ \tilde{y}_i) = u_i(y_i - \tilde{y}_i), \quad i = 1(1)N-1, \tag{28}$$

for some u_i. (Note that $u_i \geqslant 0$). Now, from (8) and the definition of $\tilde{s}(x)$ following (18), and making use of (26) we obtain

$$e_{\mathrm{D}}(x) = (\psi_{0,i}(x) + u_i \psi_{2,i}(x))(y_i - \tilde{y}_i) + (\psi_{1,i}(x) + u_{i-1}\psi_{3,i}(x))(y_{i-1} - \tilde{y}_{i-1}),$$
$$x_{i-1} \leqslant x \leqslant x_i. \tag{29}$$

Let $Y = (y_1, \ldots, y_{N-1})^{\mathrm{T}}$ and $\tilde{Y} = (\tilde{y}_1, \ldots, \tilde{y}_{N-1})^{\mathrm{T}}$. (Here, and in the following, for a vector $V = (v_1, \ldots, v_{N-1})^{\mathrm{T}}$, we shall denote $\| V \|_{\infty} = \max_{1 \leqslant i \leqslant N-1} |v_i|$). Since for $x_{i-1} \leqslant x \leqslant x_i$,

$$|\psi_{0,i}(x)| \leqslant 1, \qquad |\psi_{1,i}(x)| \leqslant 1,$$
$$|\psi_{2,i}(x)| \leqslant 6\alpha h^2, \qquad |\psi_{3,i}(x)| \leqslant 3\alpha h^2,$$

and using the fact that $\tilde{y}_0 = y_0$, $\tilde{y}_N = y_N$, from (29) we obtain for $N \geqslant 2$,

$$\| e_{\mathrm{D}} \|_{\infty} \leqslant (2 + 3\alpha u) \| Y - \tilde{Y} \|_{\infty}, \tag{30}$$

where we have set

$$u = \sup_{0 \leqslant x \leqslant 1} \partial f / \partial y.$$

We next estimate $\| Y - \tilde{Y} \|_{\infty}$. For this purpose, we first write the system (17) in matrix form. Let $D = (d_{ij})$ denote the symmetric tridiagonal matrix with

$$d_{ii} = 1/J_i + 1/J_{i+1}, \quad i = 1(1)N-1,$$
$$d_{ii+1} = -1/J_{i+1}, \quad i = 1(1)N-2, \qquad d_{ii-1} = -1/J_i, \quad i = 2(1)N-1,$$

Let $K = (k_{i,j})$ denote the tridiagonal matrix with $k_{i,i}$, $k_{i,i\pm1}$ as defined in (14), $F(\tilde{Y}) = (\tilde{f}_1, \ldots, \tilde{f}_{N-1})^{\mathrm{T}}$ and let C denote the vector $(c_1, 0, \ldots, 0, c_{N-1})^{\mathrm{T}}$, where

$$c_1 = \tilde{y}_0 / J_1 - k_{1,0}\tilde{f}_0, \qquad c_{N-1} = \tilde{y}_N / J_N - k_{N-1,N}\tilde{f}_N.$$

Then the system (17) can be written as:

$$D\tilde{Y} + KF(\tilde{Y}) = C. \tag{31}$$

Let the discretization for the exact solution corresponding to (31) be written as:

$$DY + KF(Y) + \tilde{T}(h) = C. \tag{32}$$

Let $\tilde{T}(h)=(\tilde{t}_1(h),\ldots,\tilde{t}_{N-1}(h))^{\mathrm{T}}$. By definition,

$$\tilde{t}_i(h)=\frac{1}{J_i}y_{i-1}-\left(\frac{1}{J_i}+\frac{1}{J_{i+1}}\right)y_i+\frac{1}{J_{i+1}}y_{i+1}-k_{i,i-1}f_{i-1}-k_{i,i}f_i-k_{i,i+1}f_{i+1},$$
$$i=1(1)N-1. \tag{33}$$

Now, substituting in (33) for y_{i-1}, y_{i+1} and M_{i-1}, M_{i+1} as obtained from the representation (4) for $y(x)$, and simplifying we obtain

$$\tilde{t}_i(h)=\int_{x_i}^{x_{i-1}}\left(\frac{G(x_{i-1};\,u)}{J_i}+k_{i,i-1}(u-x_{i-1})\right)f''(u)\,\mathrm{d}u$$
$$+\int_{x_i}^{x_{i+1}}\left(\frac{G(x_{i+1};\,u)}{J_{i+1}}+k_{i,i+1}(x_{i+1}-u)\right)f''(u)\,\mathrm{d}u. \tag{34}$$

Since the part of the integrands multiplying $f''(u)$ in the above integrals are nonnegative in their respective ranges of integration, by the mean-value theorem and condition (26) we obtain the following bound:

$$|\tilde{t}_i(h)|\leqslant ch^3x_i^{\alpha}, \tag{35}$$

where

$$c=\tfrac{1}{6}(3-2\alpha)N_2.$$

Since $K\geqslant 0$, following arguments as given in Chawla and Katti [5, pp. 562–563], from (31), (32) and (35) it follows that there exists a constant c^* such that

$$\|Y-\tilde{Y}\|_{\infty}\leqslant c^*h^2. \tag{36}$$

Substituting (36) in (30) we obtain

$$\|e_{\mathrm{D}}\|_{\infty}\leqslant(2+3\alpha u)c^*h^2. \tag{37}$$

Finally, with the help of (27) and (37) from (20) we obtain the following result.

Theorem 3.1 *Assume that f satisfies* (A); *further, let f', $f''\in C[0,1]$. Then, the method described above provides order h^2 uniformly convergent spline approximations $\tilde{s}(x)$ for the solution $y(x)$ of the two-point boundary-value problem* (1); *that is, for sufficiently small h,*

$$\|\tilde{e}\|_{\infty}\leqslant c^{**}h^2, \tag{38}$$

where

$$c^{**}=(2+3\alpha u)c^*+3N_2.$$

The second-order convergence of the above spline method is illustrated in Section 5.

4. A fourth-order spline method

We next present a modification of the method discussed in Section 2 which provides fourth-order uniformly convergent spline approximations for the solution of the two-point boundary-value problem (1). The method is described as follows.

Step 1

We compute the approximate solution at the nodal points $\bar{y}_1, \ldots, \bar{y}_{N-1}$ by the method of Chawla [3] adapted for the two-point boundary-value problem (1) as described in the following:

$$-\frac{1}{J_k} y_{k-1} + \left(\frac{1}{J_k} + \frac{1}{J_{k+1}}\right) y_k - \frac{1}{J_{k+1}} y_{k+1} + \frac{1}{2h}\left(-B_{1,k} + \frac{1}{h} B_{2,k}\right) f_{k-1}$$
$$+ \left(B_{0,k} - \frac{1}{h^2} B_{2,k}\right) f_k + \frac{1}{2h}\left(B_{1,k} + \frac{1}{h} B_{2,k}\right) f_{k+1} = 0, \quad k = 1(1)N-1, \tag{39}$$

where

$$B_{i,k} = A_{i,k}^{+}/J_{k+1} + A_{i,k}^{-}/J_k, \quad i = 0, 1, 2,$$

and

$$A_{i,k}^{\pm} = \frac{1}{1-\alpha}\left[\sum_{j=0}^{i} (-1)^j \binom{i}{j} \mu_{j+2} x_{k \pm 1}^{j+2} (\pm h)^{i-j}\right.$$
$$\left. + (-1)^{i+1} x_k^{\alpha+i+1}\left(\frac{x_{k\pm1}^{1-\alpha}}{(\alpha+1)\cdots(\alpha+i+1)} - \frac{x_k^{1-\alpha}}{(i+2)!}\right)\right], \quad i = 0, 1, 2,$$

where

$$\mu_j = \frac{1}{(\alpha+1)\cdots(\alpha+j)} - \frac{1}{(j+1)!}.$$

Step 2

With $\bar{y}_1, \ldots, \bar{y}_{N-1}$ we compute $M_1^*(y), \ldots, M_{N-1}^*(y)$ using the 'conditions of continuity' (13):

$$-\frac{1}{J_i} \bar{y}_{i-1} + \left(\frac{1}{J_i} + \frac{1}{J_{i+1}}\right) \bar{y}_i - \frac{1}{J_{i+1}} \bar{y}_{i+1}$$
$$+ k_{i,i-1} M_{i-1}^*(y) + k_{i,i} M_i^*(y) + k_{i,i+1} M_{i+1}^*(y) = 0, \quad i = 1(1)N-1, \tag{40}$$

with

$$M_0^*(y) = M_0(y), \qquad M_N^*(y) = M_N(y).$$

Step 3

With $\bar{y}_1, \ldots, \bar{y}_{N-1}$ and $M_1^*, \ldots, M_{N-1}^*$, from (8) we construct the spline approximation for the solution of the nonlinear two-point boundary-value problem (1):

$$s_i^*(x) = \psi_{0,i}(x) \bar{y}_i + \psi_{1,i}(x) \bar{y}_{i-1} + \psi_{2,i}(x) M_i^*(y) + \psi_{3,i}(x) M_{i-1}^*(y),$$
$$x_{i-1} \leqslant x \leqslant x_i. \tag{41}$$

The approximate spline solution thus obtained over [0, 1] will be denoted by $s^*(x)$. Note that s^* and $x^{-\alpha}(x^\alpha s^{*\prime})'$ are continuous at the nodes by construction, and (40) ensures continuity of $x^\alpha s^{*\prime}$ at the nodes. It follows that s^*, $x^\alpha s^{*\prime}$ and $x^{-\alpha}(x^\alpha s^{*\prime})' \in C[0, 1]$.

We next show that our method described above provides fourth-order approximations for the solution of the nonlinear two-point boundary-value problem (1). Let $e^*(x)$ denote the error

$$e^*(x) = y(x) - s^*(x), \quad x \in [0, 1]. \tag{42}$$

Define

$$\overline{M}_i(y) = f(x_i, \bar{y}_i), \quad i = 0(1)N, \tag{43}$$

with $\overline{M}_0(y) = M_0(y)$, $\overline{M}_N(y) = M_N(y)$. Let $\bar{s}_i(x)$ denote the spline constructed using the values $\bar{y}_1, \ldots, \bar{y}_{N-1}$ and $\overline{M}_1(y), \ldots, \overline{M}_{N-1}(y)$:

$$\bar{s}_i(x) = \psi_{0,i}(x)\bar{y}_i + \psi_{1,i}(x)\bar{y}_{i-1} + \psi_{2,i}(x)\overline{M}_i(y) + \psi_{3,i}(x)\overline{M}_{i-1}(y), \quad x_{i-1} \leqslant x \leqslant x_i. \tag{44}$$

The approximate spline thus constructed over [0, 1] will be denoted by $\bar{s}(x)$.

Now, we may write:

$$\begin{aligned} e^*(x) &= (y(x) - s(x)) + (s(x) - \bar{s}(x)) + (\bar{s}(x) - s^*(x)) \\ &= e_{\mathrm{I}}(x) + \bar{e}_{\mathrm{D}}(x) + e_{\mathrm{S}}(x), \end{aligned} \tag{45}$$

where $e_{\mathrm{I}}(x)$ is the error due to spline interpolation given by (21), $\bar{e}_{\mathrm{D}}(x)$ is the error due to the discretization of the differential equation and $e_{\mathrm{S}}(x)$ is the error due to the smoothing of $M_i(y)$ needed for the construction of the solution. In the following we shall estimate these errors separately. For fixed $\alpha \in (0, 1)$, let β be chosen such that $\alpha + \beta < 1$. In this section we shall assume that

$$x^{\alpha+\beta} \mid f'' \mid \leqslant N_3, \quad x^{\alpha+\beta+1} \mid f^{(4)} \mid \leqslant N_4, \qquad 0 < x \leqslant 1, \tag{46}$$

for suitable positive constants N_3 and N_4.

To estimate the error $\bar{e}_{\mathrm{D}}(x)$, we note that $\bar{e}_{\mathrm{D}}(x)$ is the same as the error $\tilde{e}_{\mathrm{D}}(x)$ defined in (22) with $\tilde{s}(x)$ replaced by $\bar{s}(x)$. Following arguments precisely similar to those following equation (22), as in (30) we can show that

$$\| \bar{e}_D \|_\infty \leqslant (2 + 3\alpha u) \| Y - \overline{Y} \|_\infty. \tag{47}$$

Now, following arguments similar to those given in Chawla [3, eqn. (27)] it can be shown for the finite-difference method described in Step 1 that there exists a constant $\bar{c}$ such that

$$\| Y - \overline{Y} \|_\infty \leqslant \bar{c}h^4. \tag{48}$$

From (47) and (48) it follows that

$$\| \bar{e}_D \|_\infty \leqslant (2 + 3\alpha u)\bar{c}h^4. \tag{49}$$

We next estimate the error $e_{\mathrm{S}}(x)$. From (41) and (42) we obtain

$$\begin{aligned} e_{\mathrm{S}}(x) &= \psi_{2,i}(x)\big(\overline{M}_i(y) - M_i^*(y)\big) + \psi_{3,i}(x)\big(\overline{M}_{i-1}(y) - M_{i-1}^*(y)\big), \\ &x_{i-1} \leqslant x \leqslant x_i. \end{aligned} \tag{50}$$

We may write the conditions of continuity (40) as

$$\begin{aligned} &k_{i,i-1}\big(\overline{M}_{i-1}(y) - M_{i-1}^*(y)\big) + k_{i,i}\big(\overline{M}_i(y) - M_i^*(y)\big) + k_{i,i+1}\big(\overline{M}_{i+1}(y) - M_{i+1}^*(y)\big) \\ &\quad = -\frac{1}{J_i}\bar{y}_{i-1} + \left(\frac{1}{J_i} + \frac{1}{J_{i+1}}\right)\bar{y}_i - \frac{1}{J_{i+1}}\bar{y}_{i+1} + k_{i,i-1}\overline{M}_{i-1}(y) \\ &\qquad + k_{i,i}\overline{M}_i(y) + k_{i,i+1}\overline{M}_{i+1}(y), \\ &\quad i = 1(1)N-1. \end{aligned} \tag{51}$$

Let $\overline{M} = (\overline{M}_1(y), \ldots, \overline{M}_{N-1}(y))^{\mathrm{T}}$, $M^* = (M_1^*(y), \ldots, M_{N-1}^*(y))^{\mathrm{T}}$ and $M = (M_1(y), \ldots, M_{N-1}(y))^{\mathrm{T}}$.

Since $\overline{M}_0 = M_0$, $\overline{M}_N = M_N$, $\bar{y}_0 = y_0$, $\bar{y}_N = y_N$, and with the matrices D and K and the vector C as defined in (31) we may write the system (51) as:

$$K(\overline{M} - M^*) = D\overline{Y} + K\overline{M} - C. \tag{52}$$

Again, with the help of (32) we may write (52) as:

$$K(\overline{M} - M^*) = -\tilde{T}(h) + D(\overline{Y} - Y) + K(\overline{M} - M). \tag{53}$$

We may write

$$\overline{M} - M = \overline{U}(\overline{Y} - Y), \tag{54}$$

for some $\overline{U} = \mathrm{diag}\{\bar{u}_1, \ldots, \bar{u}_{N-1}\}$ and then (53) becomes

$$\begin{aligned} K(\overline{M} - M^*) &= -\tilde{T}(h) + (D + K\overline{U})(\overline{Y} - Y), \\ &= T^*(h), \quad \text{say}. \end{aligned} \tag{55}$$

We define $R = \mathrm{diag}\{x_1^\alpha, \ldots, x_{N-1}^\alpha\}$ and set $S = R^{-1}K$. Since for sufficiently small h, K is irreducible and monotone, therefore S^{-1} exists and (55) can be written in the equivalent form:

$$\overline{M} - M^* = S^{-1}\left(R^{-1}T^*(h)\right). \tag{56}$$

Now, from the definition of the matrix D, for fixed x_i and $h \to 0$, we can show that

$$\sum_{j=1}^{N-1} | d_{ij} | \leqslant \frac{4}{h} x_i^\alpha, \quad i = 1(1)N-1. \tag{57}$$

Since $\| K \|_\infty \leqslant h$, therefore with the help of (35) and (57), and from the definition of $T^*(h)$ in (55) we obtain

$$\| R^{-1}T^*(h) \|_\infty \leqslant c_1 h^3, \tag{58}$$

where

$$c_1 = c + \bar{c}(4 + 2^{\alpha-2}u).$$

We next show that $\| S^{-1} \|_\infty = \mathrm{O}(h^{-1})$. For this purpose, let $\boldsymbol{e} = (1, \ldots, 1)^\mathrm{T}$. Since $S^{-1}(S\boldsymbol{e}) = \boldsymbol{e}$, therefore

$$\sum_{j=1}^{N-1} S_{ij}^{-1}(S\boldsymbol{e})_j = 1, \tag{59}$$

and hence

$$\sum_{j=1}^{N-1} S_{ij}^{-1} \leqslant \left(\min_{1 \leqslant j \leqslant N-1} (S\boldsymbol{e})_j \right)^{-1}. \tag{60}$$

Now, it can be seen that for sufficiently small h,

$$(S\boldsymbol{e})_j \geqslant x_j^{-\alpha} k_{j,j}, \quad j = 1(1)N-1,$$

and hence

$$\| S^{-1} \|_\infty \leqslant \left(\min_{1 \leqslant j \leqslant N-1} \left(k_{j,j} x_j^{-\alpha}\right) \right)^{-1}. \tag{61}$$

Since $\lim_{h\to 0}(1/h)x_j^{-\alpha}k_{j,j}=\frac{2}{3}$, it follows that for sufficiently small h, $x_j^{-\alpha}k_{j,j}>\frac{1}{3}h$, and hence,

$$\| S^{-1}\|_\infty \leqslant 3/h. \tag{62}$$

With (58) and (62), from (56) we obtain

$$\| \overline{M}-M^*\|_\infty \leqslant 3c_1h^2. \tag{63}$$

Now, since

$$\lim_{h\to 0}(1/h^2)\psi_{2,i}(x)=\tfrac{1}{8},\qquad \lim_{h\to 0}(1/h^2)\psi_{3,i}(x)=\tfrac{1}{24}, \tag{64}$$

and with the help of (63), from (50) we obtain for sufficiently small h,

$$\| e_S\|_\infty \leqslant \tfrac{1}{2}c_1h^4. \tag{65}$$

Finally, with the help of (27), (49) and (65), from (45) we obtain the following result.

Theorem 4.1. *Assume that f satisfies* (A) *and let f', $f''\in C[0,1]$. Further, let f''' and $f^{(4)}$ satisfy the conditions* (46). *Then the spline method described above in Steps* 1–3 *provides order h^4 uniformly convergent approximations $s^*(x)$ for the solution $y(x)$ of the two-point boundary-value problem* (1); *that is, for sufficiently small h,*

$$\| e^*\|_\infty \leqslant dh^4, \tag{66}$$

where

$$d=3N_2+\tfrac{1}{2}c_1+(2+3\alpha u)\bar{c}.$$

5. Numerical illustrations

In this section we illustrate the second- and the fourth-order of the spline methods described in Sections 2 and 4. For this purpose, we consider the following nonlinear singular two-point boundary-value problem:

$$\begin{aligned} &x^{-\alpha}(x^\alpha y')'=x^{5+\alpha}\ln x,\quad 0<x<1,\\ &y(0)=1,\qquad y(1)=1-\frac{13+3\alpha}{(6+2\alpha)^2(7+\alpha)^2}-\frac{A}{1-\alpha},\end{aligned} \tag{67}$$

with the exact solution

$$y(x)=1+A\frac{x^{1-\alpha}}{1-\alpha}+\frac{x^{7+\alpha}}{(6+2\alpha)(7+\alpha)}\ln x-\frac{(13+3\alpha)x^{7+\alpha}}{(6+2\alpha)^2(7+\alpha)^2}.$$

We solved the problem (67) with $A=1$ and a few selections of α by the second-order spline method described by (18) and by the fourth-order spline method described by (41). For the second-order spline approximations $\tilde{s}(x)$ the corresponding errors $\tilde{e}(x)$ at the nodal points as well as at the mid-points are shown in Table 1 for a few values of N. The errors $e^*(x)$ in the fourth-order spline approximation $s^*(x)$ are shown in Table 2 for a few values of N. Table 1 confirms the second order uniform convergence of the method in Section 2 and Table 2 confirms

Table 1

N	$\|Y-\tilde{Y}\|_\infty$	Order	$\max_{1\leqslant i\leqslant N}\|\tilde{e}(x_{i-1/2})\|$	Order
($\alpha=0.25$)				
32	2.17 (−4)		4.43 (−4)	
64	5.85 (−5)	1.89	1.21 (−4)	1.87
128	1.48 (−5)	1.98	3.08 (−5)	1.97
256	3.71 (−6)	2.00	7.70 (−6)	2.00
($\alpha=0.5$)				
32	4.32 (−4)		7.67 (−4)	
64	1.21 (−4)	1.83	2.14 (−4)	1.84
128	3.19 (−5)	1.92	5.62 (−5)	1.93
256	8.03 (−6)	1.99	1.42 (−5)	1.98
($\alpha=0.75$)				
64	3.74 (−4)		6.48 (−4)	
128	9.67 (−5)	1.87	1.70 (−4)	1.84
256	2.48 (−5)	1.94	4.67 (−5)	1.93
512	6.23 (−6)	1.99	1.17 (−5)	1.99
($\alpha=0.9$)				
128	2.89 (−4)		3.72 (−4)	
256	7.74 (−5)	1.85	9.74 (−5)	1.87
512	1.99 (−5)	1.95	2.50 (−5)	1.96
1024	4.98 (−6)	1.99	6.25 (−6)	2.00

Table 2

N	$\|Y-\bar{Y}\|_\infty$	Order	$\max_{1\leqslant i\leqslant N}\|e^*(x_{i-1/2})\|$	Order
($\alpha=0.25$)				
64	4.37 (−7)		6.41 (−7)	
128	2.84 (−8)	3.94	4.31 (−8)	3.93
256	1.77 (−9)	3.99	2.60 (−9)	3.97
512	1.11 (−10)	4.00	1.63 (−10)	3.99
($\alpha=0.5$)				
64	5.07 (−7)		8.17 (−7)	
128	3.47 (−8)	3.87	5.67 (−8)	3.85
256	2.27 (−9)	3.92	3.82 (−9)	3.92
512	1.44 (−10)	3.99	2.47 (−10)	3.97
($\alpha=0.75$)				
128	9.84 (−8)		1.27 (−7)	
256	6.82 (−9)	3.85	8.99 (−9)	3.82
512	4.47 (−10)	3.93	5.98 (−10)	3.91
1024	2.85 (−11)	3.97	3.78 (−11)	3.98
($\alpha=0.9$)				
128	2.57 (−7)		3.88 (−7)	
256	1.79 (−8)	3.84	2.71 (−8)	3.84
512	1.19 (−9)	3.90	1.80 (−9)	3.91
1024	7.65 (−11)	3.96	1.15 (−10)	3.96

the fourth order uniform convergence of our modified method as described in Section 4.

Acknowledgements

We are grateful to the referees for their very helpful comments and suggestions.

References

[1] J.H. Ahlberg, E.N. Nilson and J.L. Walsh, *The Theory of Splines and Their Applications* (Academic Press, New York, 1967).
[2] W.G. Bickley, Piecewise cubic interpolation and two-point boundary value problems, *Comput. J.* **11** (1968) 206–208.
[3] M.M. Chawla, A fourth-order finite difference method based on uniform mesh for singular two-point boundary-value problems, *J. Comput. Appl. Math.* **17** (1987) 359–364.
[4] M.M. Chawla and C.P. Katti, Finite difference methods and their convergence for a class of singular two-point boundary value problems, *Numer. Math.* **39** (1982) 341–350.
[5] M.M. Chawla and C.P. Katti, A uniform mesh finite difference method for a class of singular two-point boundary value problems, *SIAM J. Numer. Anal.* **22** (1985) 561–565.
[6] M.M. Chawla and R. Subramanian, A new fourth order cubic spline method for non-linear two-point boundary value problems, *Intern. J. Computer. Math.* **22** (1987) 321–341.
[7] M.M. Chawla and R. Subramanian, A new fourth-order cubic spline method for second order nonlinear two-point boundary-value problems, *J. Comput. Appl. Math.*, submitted for publication.
[8] P.G. Ciarlet, F. Natterer and R.S. Varga, Numerical methods of high-order accuracy for singular non-linear boundary value problems, *Numer. Math.* **15** (1970) 87–99.
[9] P.J. Davis, *Interpolation and Approximation* (Blaisdell, New York, 1963).
[10] P.J. Fyfe, The use of cubic splines in the solution of two-point boundary value problems, *Comput. J.* **12** (1969) 188–192.
[11] P. Jamet, On the convergence of finite-difference approximations to one-dimensional singular boundary-value problems, *Numer. Math.* **14** (1970) 355–378.