Let $B$ be a semiprime commutative unital Banach algebra with connected character space $\Phi_B$. For each $x \in \Phi_B$, let $\pi_B(x)$ be the collection of all closed primary ideals contained in the maximal ideal $M(x) = x^{-1}(0)$. The purpose of this paper is to illustrate how knowledge of the collection $\pi_B(x)$ at each $x \in \Phi_B$ can be used in describing the outer spectrum of a quasi-compact unital endomorphism of $B$. Among other things, our results lead to the observation that when $B$ is strongly regular, every Riesz endomorphism of $B$ is quasi-nilpotent on an invariant maximal ideal. Some of the implications of our work for various other types of function algebra are explored at the end of the paper.

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Let \( B \) be a commutative unital Banach algebra. In this paper, an endomorphism of \( B \) is a bounded linear operator \( T : B \rightarrow B \) which is multiplicative and preserves the multiplicative identity \( 1 \in B \). Operators of this type have received a great deal of attention in recent years, and their properties are well understood in certain cases. As the following theorem of Feinstein and Kamowitz makes clear, quasi-compact endomorphisms are rather special.

**Theorem 1.** (See [7], Lemma 2.1 and Theorem 2.2.) Let \( B \) be a semi-prime commutative unital Banach algebra with connected character space \( \Phi_B \). Let \( T \) be a quasi-compact endomorphism of \( B \). Then:

(i) \( \sigma(T) \subseteq \{ \lambda : |\lambda| < 1 \} \cup \{ 1 \} \);
(ii) the eigenvalue 1 has (algebraic) multiplicity 1 and eigenspace \( \mathbb{C} \cdot 1 \);
(iii) there is a character \( x_0 \in \Phi_B \) such that the sequence \( (T^n)_{n=1}^{\infty} \) converges in norm to the rank 1 projection \( b \mapsto x_0(b) \cdot 1 \).

With this as our starting point, one of our main aims will be to prove the following result.

**Theorem 2.** Let \( B \) be a semi-prime commutative unital Banach algebra with connected character space \( \Phi_B \). Let \( T \) be a quasi-compact endomorphism of \( B \), and suppose that \( T^*x_0 = x_0 \). Then there is a family \( J \) of \( T \)-invariant closed primary ideals of finite codimension in \( B \) and hull \( \{ x_0 \} \), for which

\[
\sigma(T) \cap \{ \lambda : |\lambda| > r_e(T) \} = \bigcup_{I \in J} \sigma(T/I). \tag{1}
\]

The maximal ideal \( M(x_0) = \{ b \in B : x_0(b) = 0 \} \) always belongs to \( J \).

Here

\[
r_e(T) = \inf \{ r > 0 : |\lambda| \leq r \text{ for all } \lambda \in \sigma_e(T) \}
\]

is the essential spectral radius of \( T \), and (for each \( I \in J \)) \( T/I \) is the endomorphism of \( B/I \) which satisfies

\[
(T/I)(b + I) = Tb + I \quad (b \in B)
\]

for every \( b \in B \).

The reader will recall that an ideal in a commutative Banach algebra is said to be primary if it is simultaneously modular and contained in only one maximal ideal. We will later exploit the fact that, for many algebras of differentiable functions, these ideals are often of a very particular form. For example, if \( \Omega \) is a convex bounded domain in \( \mathbb{R}^d \),
every closed primary ideal in $C^k(\Omega)$ contains an ideal of the form
\[
\{u \in B : (D^\alpha u)(x) = 0 \text{ for all } |\alpha| \leq k\}
\]
for some $x \in \Omega$. We will see later that this implies that every Riesz endomorphism $T$ of $C^k(\Omega)$ satisfies $\sigma(T) = \{0, 1\}$. This conclusion will, in fact, be shown to apply throughout a large class of Shilov-regular function algebras on the closed unit ball in $\mathbb{R}^d$.

The situation for algebras of infinitely differentiable functions is much more interesting. In this direction, Theorem 2 can be used to reproduce a large proportion of the results in the literature concerning the spectra of Riesz endomorphisms on algebras of holomorphic functions on domains in $\mathbb{C}$. Among these is the following (now well-known) Theorem of Kamowitz. Below, $\mathbb{D}$ is the open unit disk, and $z : \mathbb{D} \to \mathbb{C}$ is the associated inclusion map.

**Theorem 3.** (See [11], Theorem 9.) Let $T$ be a Riesz endomorphism of the disk algebra $A(\mathbb{D})$, and suppose that the function $\phi = Tz$ fixes a point $p = \phi(p)$ in the open unit disk. Then the spectrum of $T$ is given by
\[
\sigma(T) = \{0, 1\} \cup \{\phi'(p)^k : k \in \mathbb{N}\}.
\]

As has been (rather comprehensively) established in the literature, a similar conclusion applies for a large number of other algebras of holomorphic functions on $\mathbb{D}$. Theorem 2 will allow us to prove a significant generalisation of Theorem 3, one which subsumes a large number of existing results in this area. In particular, we will show that the same conclusion applies in every unital Banach algebra obtained by completing the algebra of polynomials in $z$ with respect to a norm for which the multiplication operators $f \mapsto (z-p)f$ are bounded below (for $p \in \mathbb{D}$).

The material in this paper is organised as follows. After this brief introduction, we begin with a review of some preliminary material. All of this is standard, and is included mainly to provide the reader with an opportunity to familiarise himself with our notation.

The machinery described is then used to prove a version of Theorem 2. Recovery of this central result is followed by a brief examination of some of the implications of our work for determining the spectra of Riesz endomorphisms of algebras possessing particular primary ideal structures. The final section is devoted to examining some of the implications of our results for Riesz endomorphisms in two rather different classes of function algebra: one class contains only regular algebras, and the other consists solely of algebras of functions which are holomorphic in the unit disk.

2. Preliminaries and notation

Given any linear map $T : X \to Y$ between vector spaces $X$ and $Y$, we will henceforth write
\[ \text{Ker}(T) = T^{-1}(0) = \{ x \in X : Tx = 0 \} \quad \text{and} \quad \text{Im}(T) = \{ Tx : x \in X \} \]

for the kernel and range of \( T \).

Let \( X \) be a non-zero complex Banach space. In this paper, the symbol \( L(X) \) denotes the unital Banach algebra of all bounded linear operators on \( X \). Given any \( T \in L(X) \), we set

\[
\rho(T) = \{ \lambda \in \mathbb{C} : \lambda - T \text{ is invertible in } L(X) \} , \quad \sigma(T) = \mathbb{C} \setminus \rho(T),
\]

\[
\rho_e(T) = \{ \lambda \in \mathbb{C} : \lambda - T \text{ is a Fredholm operator} \} , \quad \sigma_e(T) = \mathbb{C} \setminus \rho_e(T),
\]

These are referred to as the resolvent, spectrum, essential resolvent and essential spectrum of \( T \) respectively.

Suppose that \( X \) is infinite dimensional and let \( K(X) \) be the closed, proper ideal in \( L(X) \) consisting of all \( T \in L(X) \) which are compact. It follows from Atkinson’s theorem ([1], Theorem 4.46, page 161) that, for each \( T \in L(X) \), \( \sigma_e(T) \) is precisely the spectrum of \( T + K(X) \) in the Calkin algebra \( L(X)/K(X) \). As such, \( \sigma_e(T) \) is a non-empty, compact subset of \( \mathbb{C} \) with \( \sigma_e(T) \subseteq \sigma(T) \). An important connection between the essential and actual spectra of an operator is provided by the so-called punctured neighbourhood theorem, which we now quickly recall.

**Theorem 4.** (See [13], Section 19, Theorem 4, page 197.) Let \( X \) be a complex Banach space, let \( T \in L(X) \) and let \( U \) be a component of \( \rho_e(T) \). Then either \( U \subseteq \sigma(T) \) or \( U \cap \sigma(T) \) is at most countable and each \( \lambda \in U \cap \sigma(T) \) is isolated in \( \sigma(T) \).

Now let \( \sigma \) be a non-empty compact subset of \( \mathbb{C} \). We write \( \mathcal{O}(\sigma) \) for the algebra of germs of holomorphic functions over \( \sigma \), equipped with the inductive compact open topology (cf. [3], page 795). We use the symbol \( z_\sigma \) for the germ over \( \sigma \) of the complex co-ordinate functional \( z \) on \( \mathbb{C} \). Given disjoint compact subsets \( \sigma \) and \( \sigma' \) of \( \mathbb{C} \), we denote by \( 1_{\sigma,\sigma'} \), the germ over \( \sigma \cup \sigma' \) obtained in the following manner. Let \( U, U' \) be disjoint open neighbourhoods of \( \sigma \) and \( \sigma' \) respectively. Now let \( h \) be the holomorphic function on \( U \cup U' \) which is 1 on \( U \) and 0 on \( U' \) and let \( 1_{\sigma,\sigma'} \) be the germ of \( h \) over \( \sigma \cup \sigma' \). It is clear that \( 1_{\sigma,\sigma'} \) is an idempotent in \( \mathcal{O}(\sigma \cup \sigma') \) which does not depend on the particular choice of \( U \) and \( U' \).

For an element \( b \in B \) of a unital Banach algebra \( B \), the main single-variable holomorphic functional calculus theorem (cf. [3], Theorem 2.4.4) asserts that there is a unique continuous unital algebra homomorphism \( \Theta_b : \mathcal{O}(\sigma(b)) \rightarrow B \) satisfying the condition \( \Theta_b(z_{\sigma(b)}) = b \). It is standard that elements of \( \text{Im}(\Theta_b) \) commute with \( b \). If \( \sigma \subseteq \sigma(b) \) and both \( \sigma \) and its complement \( \sigma' = \sigma(b) \setminus \sigma \) are compact, we define \( P_b(\sigma) = \Theta_b(1_{\sigma,\sigma'}) \), and refer to this as the spectral projection of \( a \) over \( \sigma \). Each \( 1_{\sigma,\sigma'} \) is an idempotent in \( \mathcal{O}(\sigma(b)) \) and \( \Theta_b \) is multiplicative, so each spectral projection \( P_b(\sigma) \) is an idempotent in \( B \) which commutes with \( b \).
The next lemma summarises some pertinent properties of these idempotents when \( B = L(X) \) for a Banach space \( X \). The three assertions made therein can be more or less directly lifted from the results of Chapters 6 and 7 of [1].

**Lemma 5.** (Cf. [1], Theorems 6.34 and 7.44 and Exercise 14 on page 270.) Let \( X \) be a non-zero complex Banach space and let \( T \in L(X) \). Let \( \sigma \subseteq \sigma(T) \) and suppose that both \( \sigma \) and \( \sigma(T) \setminus \sigma \) are compact when considered as subsets of \( \mathbb{C} \). Then \( P_T(\sigma) \) has the following properties:

(i) The subspaces \( \text{Im}(P_T(\sigma)) \) and \( \text{Ker}(P_T(\sigma)) \) are \( T \)-invariant closed subspaces of \( X \) such that \( X = \text{Im}(P_T(\sigma)) \oplus \text{Ker}(P_T(\sigma)) \). If \( \sigma \) is a non-empty proper subset of \( \sigma(T) \) then the spectra of the restrictions of \( T \) to \( \text{Im}(P_T(\sigma)) \) and \( \text{Ker}(P_T(\sigma)) \) are \( \sigma \) and \( \sigma(T) \setminus \sigma \) respectively. The projection \( P_T(\sigma) \) is zero if and only if \( \sigma \) is empty and is the identity operator if and only if \( \sigma = \sigma(T) \).

(ii) If there is a number \( t > 0 \) such that \( |\lambda| < t \) for all \( \lambda \in \sigma \) and \( |\lambda| > t \) for all \( \lambda \in \sigma(T) \setminus \sigma \) then \( \text{Im}(P_T(\sigma)) = \{ x \in X : \|T^n x\|/t^n \to 0 \text{ as } n \to \infty \} \).

(iii) If \( \sigma \) consists of a single isolated point \( \lambda \) of \( \sigma(T) \) and \( \lambda - T \) is Fredholm then \( P_T(\sigma) \) is a finite rank operator. In this case, there is a non-negative integer \( k \) for which \( \text{Im}(P_T(\sigma)) = \text{Ker}((\lambda - T)^k) \).

It follows from (iii) that if \( \sigma \) is a finite set of isolated points of \( \sigma(T) \) belonging to \( \rho_e(T) \) then \( P_T(\sigma) \) is a finite rank operator; it is simply the sum of the finite rank operators \( P_T(\{\lambda\}) \) for \( \lambda \in \sigma \) (this is easy to see by writing \( 1_{\sigma,\sigma(T)\setminus\sigma} \) as the sum of germs of the form \( 1_{(\lambda),\sigma(T)\setminus(\lambda)} \) for \( \lambda \in \sigma \)). Assertion (ii) appears as an exercise in [1], and relies on considerations approaching so-called local spectral theory. The explicit description of \( \text{Im}(P_T(\sigma)) \) provided to us by (ii) in the circumstances described will be very useful later.

Let \( B \) be a commutative unital Banach algebra, and let \( I \subseteq B \) be an ideal. We write

\[ \mathfrak{h}(I) = \Phi_B \cap I^\perp = \{ x \in \Phi_B : I \subseteq M(x) \}, \]

and refer to this as the hull of \( I \). Let \( x \in \Phi_B \). Having agreed to use the symbol \( \pi_B(x) \) to denote the set of all closed primary ideals in \( B \) contained in the maximal ideal \( M(x) = x^{-1}(0) \), we have

\[ \pi_B(x) = \{ I : I \text{ is a closed ideal in } B \text{ and } \{ x \} = \mathfrak{h}(I) \}. \]

Let \( A \) be a second commutative unital Banach algebra and let \( \text{Hom}(A,B) \) be the set of all bounded linear operators \( T : A \to B \) which are multiplicative and which send the multiplicative identity in \( A \) to that in \( B \). An operator of this type obviously induces a continuous map.
such that $T^\dagger(x) = T^* x$ for every $x \in \Phi_B$ (where $T^* : B^* \longrightarrow A^*$ is the usual Banach space adjoint of $T$). Indeed, it is through this map that homomorphisms of commutative Banach algebras are typically studied.

Let $T$ be an endomorphism of $B$, and let $I \subseteq B$ be a closed $T$-invariant ideal. Letting $Q_I : B \longrightarrow B/I$ be the quotient map, it is standard that $Q_I^\dagger$ maps $\Phi_{B/I}$ homeomorphically onto $\mathfrak{h}(I)$. In fact, it does us no particular harm to identify $Q_I^\dagger$ with the inclusion map of $\mathfrak{h}(I)$ into $\Phi_B$. The fact that $I$ is $T$-invariant obviously makes $\mathfrak{h}(I)$ invariant under $T^\dagger$, so $T^\dagger$ restricts to give a map $\tau : \mathfrak{h}(I) \longrightarrow \mathfrak{h}(I)$ (given, of course, by $\tau(x) = T^* x$ for each $x \in \mathfrak{h}(I)$). Since the endomorphism $T/I$ is defined by the property that $Q_IT = (T/I)Q_I$, it is clear that $\tau$ is topologically conjugate to $(T/I)^\dagger$. It follows, in particular, that when $I$ is a closed ideal for which $T/I$ is invertible, $T^*$ maps $\mathfrak{h}(I)$ onto itself. This observation will turn out to be very important.

Given a closed $T$-invariant ideal $I$ for which $T/I$ is invertible in $L(B/I)$, the hull $\mathfrak{h}(I)$ now clearly belongs to the collection

$$\mathcal{C} = \{ E \subseteq \Phi_B : T^* E = E \}.$$ 

The union $F(T^\dagger) = \bigcup_{E \in \mathcal{C}} E$ is the so-called fixed set of $T^\dagger$, and is known to coincide with the intersection $\bigcap_{n=1}^\infty T^n \Phi_B$. The following lemma is now a consequence of part (iii) of Theorem 1.

Lemma 6. Let $B$ be a semi-prime commutative unital Banach algebra with connected character space $\Phi_B$, and let $T$ be a quasi-compact endomorphism of $B$. Suppose that $T^* x_0 = x_0$ for some $x_0 \in \Phi_B$, and that $I \subseteq B$ is a closed $T$-invariant ideal for which $T/I$ is invertible. Then $I \in \pi_B(x_0)$.

Proof. By Theorem 1, the sequence $(T^n)_{n=1}^\infty$ converges in norm to a projection of the form $b \longmapsto y_0(b) \cdot 1$ for some $y_0 \in \Phi_B$. As Feinstein and Kamowitz note in [6], this implies that $F(T^\dagger) = \{ y_0 \}$. The result now follows by noting that any fixed point of $T^\dagger$ belongs to $F(T^\dagger)$; our insistence that $T^* x_0 = x_0$ is simply to ensure that $x_0 = y_0$. \Box

3. From spectral projection to primary ideals

The following Lemma reveals the source of the ideals mentioned in Theorem 2; they will be the kernels of certain spectral projections.

Lemma 7. Let $B$ be a semi-prime commutative unital Banach algebra with connected character space, and let $T$ be a quasi-compact endomorphism of $B$. Let

$$\sigma_r = \sigma(T) \cap \{ \lambda : |\lambda| \geq r \}$$

for any $r_0(T) < r \leq 1$. Then $\sigma_r$ is finite, and $I_r = \text{Ker}(P_T(\sigma_r))$ is a closed $T$-invariant ideal of finite codimension in $B$. 
Proof. It is clear that $\sigma_r$ is a compact subset of $\mathbb{C}$. Applying Theorem 4 with $U$ equal to the unbounded component of $\rho_e(T)$, we observe that, in addition, every $\lambda \in \sigma_r$ is isolated in $\sigma(T)$. As a compact subset of $\mathbb{C}$ with no accumulation points, $\sigma_r$ is finite. It is clear from Lemma 5 (and the remarks immediately following it) that $I_r = \text{Ker}(P_T(\sigma_r))$ is a closed $T$-invariant subspace of finite codimension in $B$.

We have not yet established that $I_r$ is an ideal. To do this, we first observe that since $\sigma(T)$ is compact and every point of $\sigma_r$ is isolated in $\sigma(T)$, the complement $\sigma(T) \setminus \sigma_r$ is also a compact subset of $\mathbb{C}$. This implies, in particular, that there is a $0 < t(r) < r$ for such that

$$\sigma_r = \sigma(T) \cap \{\lambda : |\lambda| > t(r)\}$$

and $\sigma(T) \setminus \sigma_r = \sigma(T) \cap \{\lambda : |\lambda| < t(r)\}$.

Invoking part (ii) of Lemma 5, we see that

$$I_r = \{b \in B : ||T^n b|| / t(r)^n \to 0 \text{ as } n \to \infty\}$$

(2)

The fact that $I_r$ is an ideal now follows from Theorem 1 which ensures, of course, that $T$ is power bounded. ∎

It is clear from part (i) of Theorem 1 that

$$\text{Im}(P_T(\sigma_1)) = \text{Im}(P_T(\{1\})).$$

By part (ii) of the same theorem, this subspace has dimension 1. It follows that $I_1 = \text{Ker}(P_T(\sigma_1))$ is a maximal ideal. It is already clear that the ideal $I_r$ is contained in the maximal ideal $I_1$ for each $r_e(T) < r \leq 1$. With Lemma 6 at our disposal, the following observation helps us show (among other things) that $I_1$ is the only maximal ideal with this property.

Lemma 8. Let $B$ and $T$ be as in Lemma 7 and define $\sigma_r$ and $I_r$ as before for $r_e(T) < r \leq 1$. Then $\sigma(T/I_r) = \sigma_r$ for all such $r$.

Proof. Let $r_e(T) < r \leq 1$, and let

$$i_r : \text{Im}(P_T(\sigma_r)) \to B \text{ and } Q_r : B \to B/I_r$$

be the appropriate inclusion and quotient maps. Composing these in the obvious fashion, we obtain a Banach space isomorphism

$$U_r = Q_r i_r : \text{Im}(P_T(\sigma_r)) \to B/I_r$$

Letting $T_r$ be the restriction of $T$ to $\text{Im}(P_T(\sigma_r))$, we have the equations $i_r T_r = T i_r$ and $Q_r T = (T/I) Q_r$. 

Taken together, these lead to the intertwining relation \((T/I)U_r = U_r T_r\). Since this implies that \(T_r\) and \(T/I\) have the same spectrum, the result now follows from part (i) of Lemma 5. \(\Box\)

We have now assembled everything we need in order to prove Theorem 2. However, before we do so, it is perhaps worth making explicit the respective rôles of Lemmas 6, 7 and 8. Under the hypotheses of Theorem 2, Lemma 7 provides us with a family of closed \(T\)-invariant ideals of finite codimension in \(B\); these are simply the kernels of the spectral projections \(P_T(\sigma_r)\) for \(r_o(T) < r \leq 1\). Lemma 8 does two things. Unsurprisingly, it tells us that

\[
\sigma(T) \cap \{\lambda : |\lambda| > r_o(T)\} = \bigcup_{r_o(T) < r \leq 1} \sigma(T/I_r).
\]

However, this is not its only rôle. At this stage in the proof, we do not yet know anything about the hulls \(\mathfrak{h}(I_r)\), except that they all contain \(x_0\). It is here that Lemma 8 really comes to our rescue; the fact that \(0 \notin \sigma_r\) means that each of the endomorphisms \(T/I_r\) is invertible, and it is this information which (via Lemma 6) allows us to show that each \(I_r\) belongs to \(\pi_B(x_0)\).

We now prove the following strong form of Theorem 2.

**Theorem 9.** Let \(B\) be a semi-prime commutative unital Banach algebra with connected character space \(\Phi_B\), and let \(T\) be a quasi-compact endomorphism of \(B\). Let \(x_0 \in \Phi_B\) satisfy \(T^* x_0 = x_0\). Then there is a family \(\mathcal{J}\) of closed \(T\)-invariant ideals with the following properties

(i) \(\sigma(T) \cap \{\lambda : |\lambda| > r_o(T)\} = \bigcup_{I \in \mathcal{J}} \sigma(T/I);\)

(ii) \(\mathcal{J} \subseteq \pi_B(x_0);\)

(iii) the maximal ideal \(M(x_0)\) always belongs to \(\mathcal{J};\)

(iv) each \(I \in \mathcal{J}\) is the kernel of a finite rank spectral projection associated with \(T;\)

(v) \(\mathcal{J}\) is at-most-countable, and its elements form a chain, in the sense that if \(I, I' \in \mathcal{J}\), then either \(I \subseteq I'\) or \(I' \subseteq I;\)

(vi) for \(r_o(T) < r \leq 1\), there is some \(I \in \mathcal{J}\) for which \(\sigma(T/I)\) is contained in the disk \(\{\lambda : |\lambda| < r\} \).

**Proof.** Invoking Lemma 7, and setting

\[
\mathcal{J} = \{\text{Ker}(P_T(\sigma_r)) : r_o(T) < r \leq 1\},
\]

we have immediately that \(\mathcal{J}\) is a family of \(T\)-invariant closed ideals of finite codimension in \(B\). Assertions (i)–(vi) are now proved as follows.

(i) Since \(\sigma(T) \cap \{\lambda : |\lambda| > r_o(T)\}\) is the union of the sets
Showing Theorem 9.

\[ \sigma_r = \sigma(T) \cap \{ \lambda : |\lambda| \geq r \} \]

for \( r_0(T) < r \leq 1 \), this part is clear from Lemma 8.

(ii) Making a second appeal to Lemma 8, we observe that for each ideal \( I \in \mathcal{J} \), the spectrum \( \sigma(T/I) \) consists solely of points of modulus strictly larger than 0. This means, in particular, that each of the operators \( (T/I)_{I \in \mathcal{J}} \) is invertible. Lemma 6 therefore implies that \( \mathcal{J} \subseteq \pi_B(x_0) \).

(iii) Showing that \( I_1 = \text{Ker}(P_T(\sigma_1)) \) is a maximal ideal is achieved using the argument described in the comments before Lemma 8. Explicitly, a combination of Theorem 1 and Lemma 5 gives us

\[ B = \text{Im}(P_T(\{1\})) \oplus I_1. \]

Using part (ii) of Theorem 1, this implies that \( I_1 \) has codimension 1 in \( B \). Having already shown that \( I_1 \) is an ideal, this gives us what we need.

(iv) This is immediate from definition (3).

(v) The fact that \( \mathcal{J} \) is at-most-countable is a straightforward consequence of the punctured neighbourhood theorem (Theorem 4). That \( \mathcal{J} \) is a chain in the sense indicated is easily proved using description (2) from the proof of Lemma 7.

(vi) Let \( r_0(T) < r \leq 1 \). Setting \( I = \text{Ker}(P_T(\sigma_r)) \), Lemma 5 tells us that

\[ \sigma(T_I) = \sigma(T) \setminus \sigma_t = \sigma(T) \cap \{ \lambda : |\lambda| < r \}. \]

The proof is now complete. \( \square \)

4. Some useful corollaries

In this paper, a function algebra is a semisimple commutative unital Banach algebra, considered as an algebra of continuous functions on its character space. Let \( B \) be such an algebra. Then, as the reader will recall, \( B \) is said to be:

- regular if for each closed subset \( F \subseteq \Phi_B \) and each point \( x \in \Phi_B \setminus F \), there is some \( f \in B \) satisfying \( f(x) = 1 \) and \( f(F) \subseteq \{0\} \); and
- strongly regular if \( \pi_B(x) = \{ M(x) \} \) for every \( x \in \Phi_B \).

When \( B \) is regular and \( x \in \Phi_B \), we write \( J(x) \) for the closure of the ideal

\[ J_0(x) = \{ f \in B : f^{-1}(0) \text{ is a neighbourhood of } x \}. \]

A celebrated result of Shilov ensures that when \( B \) is regular, \( J(x) \) is the intersection of all the closed ideals \( I \in \pi_B(x) \). We exploit this to establish the following consequence of Theorem 9.
Theorem 10. Let $B$ be a regular function algebra with connected character space $\Phi_B$. Let $T$ be a Riesz endomorphism of $B$, and suppose that $T^*x_0 = x_0$. Then $J(x_0)$ is a $T$-invariant closed ideal, and
\[ \{0\} \cup \sigma(T) = \{0\} \cup \sigma(T/J(x_0)). \]
Proof. Showing that $J(x_0)$ is $T$-invariant is straightforward, and can be achieved by using the fact that $Tf = f \circ T^\dagger$ for each $f \in B$. Let $J \subseteq \pi_B(x_0)$ be the family of ideals supplied by Theorem 9, and fix any $\epsilon$. Then, by part (vi), there is some $I \in J$ such that
\[ \sigma(T|_I) \subseteq \{\lambda: |\lambda| < \epsilon\}. \]
As the restriction of a Riesz operator to a closed $T$-invariant subspace, $T|_I$ is also a Riesz operator. This implies, in particular, that $\rho(T|_I)$ is connected, hence that $\sigma(T|_N) \subseteq \sigma(T|_I)$ for every $T|_I$-invariant closed subspace $N \subseteq I$. Using Shilov’s result, we therefore see that
\[ \sigma(T|_{J(x_0)}) \subseteq \{\lambda: |\lambda| < \epsilon\}. \]
Since an identical conclusion is available for every $\epsilon > 0$, $T|_{J(x_0)}$ is quasi-nilpotent. The result now follows from the standard spectral inclusions $\sigma(T) \subseteq \sigma(T|_N) \cup \sigma(T/N)$ and $\sigma(T/N) \subseteq \sigma(T) \cup \sigma(T|_N)$, which hold for any closed $T$-invariant subspace $N$ of $B$. The proofs associated with the latest inclusions can be found in Section 3.11 of [10].

The next result indicates, among other things, that a Riesz endomorphism of a strongly regular function algebra can never have a nontrivial spectrum.

Corollary 11. Let $B$ be a regular function algebra with connected character space, and let $T$ be a Riesz endomorphism of $B$. Let $x_0$ be the element of $\Phi_B$ for which $T^*x_0 = x_0$, and suppose that $J(x_0)$ has finite codimension in $B$. Then $\sigma(T) = \{0, 1\}$.
Proof. By the previous result, the nonzero spectrum of $T$ coincides with that of $T/J(x_0)$ (an operator on a finite dimensional space). This means that $\sigma(T)$ is finite. Now choose any nonzero point $\lambda$ of $\sigma(T)$. Since $T$ is a Riesz operator, $\lambda$ is necessarily an eigenvalue. It follows that $\lambda^n \in \sigma(T)$ for every $n \in \mathbb{N}$, which leads to a contradiction unless $\lambda = 1$.

Although the class of algebras to which Corollary 11 applies is very large, quite a number of standard regular function algebras lie entirely beyond its reach. Examples include the ‘big’ Lipschitz algebras $\text{Lip}(Y)$ over compact metric spaces. However, even these algebras are not immune to the following theorem.

Theorem 12. Let $B$ be a semi-prime commutative unital Banach algebra with connected character space $\Phi_B$. Let $T$ be a quasi-compact endomorphism of $B$, and let $x_0$ be a character for which $T^*x_0 = x_0$. Suppose that
contains a point other than 1. Then there is a bounded point derivation \(d\) at \(x_0\) such that \(T^*d \neq 0\).

Before proceeding with the proof, we remind the reader that a **bounded point derivation** (on a commutative unital Banach algebra \(B\)) is bounded linear functional \(d \in B^*\) such that \(d(uv) = x(u)dv + x(v)du\) holds for some \(x \in \Phi_B\) and all \(u, v \in B\).

In this case, \(d\) is said to be a bounded point derivation at \(x\). It is standard that \(B\) supports a bounded point derivation at some \(x \in \Phi_B\) if and only if \(M(x) \neq M(x)^2\).

**Proof.** Our hypothesis on \(\sigma(T)\) means that there is some \(r > 0\) for which the set \(\sigma_r = \sigma(T) \cap \{\lambda : |\lambda| \geq r\}\) contains at least two points. The ideal \(I_r = \text{Ker}(P_T(\sigma_r))\) therefore has codimension at least 2. As a non-trivial finite dimensional commutative unital Banach algebra with exactly one maximal ideal, the quotient algebra \(B/I_r\) supports at least one nonzero bounded point derivation; call this \(d_0\). Letting \(Q : B \to B/I_r\) be the quotient map, set \(d = Q^*d_0\); it is straightforward to verify that this is a bounded point derivation on \(B\) at \(x_0\) with

\[
T^*d = Q^*(T/I_r)^*d_0.
\]

The result follows, since \(Q^*\) is injective and (by Lemma 8), \((T/I)^*\) is invertible. \(\square\)

This result complements an existing result of Udo Klein (cf. [8], Theorem 10), who proved an analogous assertion for compact endomorphisms of uniform algebras.

### 5. Applications for some concrete function algebras

It goes without saying that the results of Section 4 have some serious consequences for Riesz endomorphisms of strongly regular algebras such as \(C(X)\) and \(\text{lip}(Y)\) (where \(X\) is any connected compact Hausdorff space and \(Y\) is any connected compact metric space); it is immediate from any of the three previous results that such operators all have spectra equal to \(\{0, 1\}\). Our aim, at least for the first half of this section, is to describe another large class of function algebras with this property. For simplicity, we work with function algebras defined on (the closure of) the open ball \(\Omega\) in \(\mathbb{R}^d\). However much of what we have to say applies equally to algebras on \(\mathbb{T}^d\) and other connected compact smooth manifolds.

Adopting the standard notation, we write \(C^k(\overline{\Omega})\) for the algebra of \(k\)-times continuously differentiable functions \(u : \Omega \to \mathbb{C}\) which, together with their partial derivatives \(D^\alpha u\) of orders \(|\alpha| \leq k\), extend to be continuous on \(\overline{\Omega}\). This is a regular function algebra under the norm

\[
\|u\|_{k, \infty, \Omega} = \sum_{|\alpha| \leq k} \frac{1}{\alpha!} \sup_{x \in \Omega} |(D^\alpha u)(x)|, \quad (u \in C^k(\overline{\Omega})).
\]
Setting $C^\infty(\bar{\Omega}) = \bigcap_{k \geq 1} C^k(\bar{\Omega})$, we now recall a famous result of Shilov; a particularly straightforward proof of this theorem can be found on page 58 of Mirkill’s treatise, [12].

**Theorem 13** (Shilov, 1950). Let $B$ be a Banach space of functions on $\bar{\Omega}$ (or any other compact smooth manifold), equipped with pointwise addition and a topology stronger than that of pointwise convergence on $\bar{\Omega}$. Then $C^\infty(\bar{\Omega}) \subset B$ if and only if $C^k(\bar{\Omega}) \subset B$ for some $k \in \mathbb{N}$.

The following result is also due to Shilov, and can be found in the book, [9], of Gelfand, Shilov and Raikov.

**Theorem 14.** (See [9], Chapter VI, Theorem 39.1.) Let $B'$ and $B''$ be regular function algebras with the same space $X$ of maximal ideals; furthermore, let $B' \subset B''$ with $B'$ dense in $B''$. If $J' \subset B'$ and $J'' \subset B''$ are the minimal primary ideals corresponding to the same point $x_0 \in X$ and the quotient algebra $B'/J'$ is finite dimensional then $B''/J''$ is also finite dimensional, and its dimension is no greater than that of $B'/J'$.

With these two results at our disposal, we can give the following application of Corollary 11. The regularity of the domain $\Omega$ means that the next result applies, in particular, when $B$ is one of the Sobolev algebras

$$W^{s,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \text{ for } |\alpha| \leq s\}$$

with $1 \leq p, s < +\infty$ and $sp > d$.

**Theorem 15.** Let $B$ be Banach algebra of continuous functions on $\bar{\Omega}$ for which $C^\infty(\bar{\Omega})$ is dense in $B$. Let $T$ be a Riesz endomorphism of $B$. Then $\sigma(T) = \{0, 1\}$.

**Proof.** It is clear from Theorem 13 that, under our hypotheses, there is some $k \in \mathbb{N}$ for which $C^k(\bar{\Omega})$ is dense in $B$. Since the inclusion maps $C^k(\bar{\Omega}) \hookrightarrow B$ and $B \hookrightarrow C(\bar{\Omega})$ are both continuous, there are constants $C_1$ and $C_2$ such that

$$\|f\|_{0,\infty,\Omega} \leq C_1\|f\|_B \leq C_2\|f\|_{k,\infty,\Omega}$$

for every $f \in C^k(\bar{\Omega})$.

Since the spectral radius of any element of $C^k(\bar{\Omega})$ is therefore the same with respect to $B$ as it is with respect to $C^k(\bar{\Omega})$, it follows from Theorem 1 of [2] that these two algebras have ‘the same’ space of maximal ideals. This latest assertion is, of course, to be understood in the sense that the restriction map $x \mapsto x|_{C^k(\bar{\Omega})}$ is a homeomorphism of $\Phi_B$ onto $\Phi_{C^k(\bar{\Omega})}$. Recalling that every minimal primary ideal in $C^k(\bar{\Omega})$ is of the form

$$J' = \{u \in C^k(\bar{\Omega}) : (D^\alpha u)(x) = 0 \text{ for } |\alpha| \leq k\},$$
for some \( x \in \bar{\Omega} \), an application of Theorem 14 (with \( B' = C^k(\bar{\Omega}) \) and \( B'' = B \)) indicates that \( B/J(x) \) is finite dimensional for every \( x \in \Phi_B \). The result is now an obvious consequence of Corollary 11. \( \square \)

Our next application demonstrates a little of what can be achieved when we appeal to Theorem 9 more directly. For the remainder of this paper, \( B \) is a unital Banach algebra of functions which are continuous on the open unit disk and holomorphic on its interior. We will also assume that \( B \) contains the inclusion map \( z : \mathbb{D} \to \mathbb{C} \). The only other restriction we will impose is that

\[
(z - p)^kB = \{ f \in B : f(p) = f'(p) = \cdots = f^{(k-1)}(p) = 0 \} \tag{4}
\]

for each \( p \in \mathbb{D} \) (the open disk) and each \( k \in \mathbb{N} \). We note that this condition is automatically satisfied when the polynomials (in \( z \)) are dense in \( B \) and there is a \( c_p > 0 \) for each \( p \in \mathbb{D} \) such that

\[ \|f\| \leq c_p \|(z - p)f\| \]

for each \( f \in B \). The disk algebra \( A(\mathbb{D}) \) is easily seen to have this property.

Our insistence that (4) holds is purely to give us access to the following theorem of Domar from 1982.

**Theorem 16.** (See [5], Theorem 4.) Let \( B \) be a commutative unital Banach algebra, let \( f \in B \), and suppose that, for each \( n \in \mathbb{N} \), the principal ideal \( M_n = f^nB \) has codimension \( n \). Then the \( M_n \) are the only closed primary ideals of finite-codimension with \( M_n \subseteq M_1 \).

In light of our assumptions on \( B \), this implies that the only closed primary ideals at \( p \) of finite codimension are of the form (4).

We can now prove the following theorem; given the looseness of our assumptions on \( B \), it subsumes a large number of existing algebra-specific results already in the literature.

**Theorem 17.** Let \( \phi \) be a continuous self-map of \( \mathbb{D} \) for which \( f \circ \phi \in B \) for every \( f \in B \). Suppose that the operator defined by

\[ Tf = f \circ \phi \quad (f \in B) \]

is Riesz, and that \( \phi \) has a fixed point \( p \) in \( \mathbb{D} \), the open unit disk. Then

\[ \sigma(T) = \{0,1\} \cup \{\phi'(p)^k : k \in \mathbb{N} \}. \]

**Proof.** Combining Domar’s theorem with assumption (4), we have

\[ \pi_B(p) = \{ M_k : k \in \mathbb{N} \}, \tag{5} \]

for some \( x \in \bar{\Omega} \), an application of Theorem 14 (with \( B' = C^k(\bar{\Omega}) \) and \( B'' = B \)) indicates that \( B/J(x) \) is finite dimensional for every \( x \in \Phi_B \). The result is now an obvious consequence of Corollary 11. \( \square \)

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\[ \pi_B(p) = \{ M_k : k \in \mathbb{N} \}, \tag{5} \]
where \( M_k = \{ f \in B : f(p) = f'(p) = \ldots = f^{(k-1)}(p) = 0 \} \) for each \( k \in \mathbb{N} \). It is clear that \( B \) cannot have any idempotents other than the functions \( f = 0 \) and \( f = 1 \) so, by the Shilov idempotent theorem, \( \Phi_B \) is connected. Invoking Theorem 9, there is a family \( \mathcal{J} \subseteq \pi_B(p) \) of closed, primary ideals of finite codimension in \( B \) such that

\[
\sigma(T) \setminus \{0\} = \bigcup_{I \in \mathcal{J}} \sigma(T/I).
\]

We know, of course, that each \( I \in \mathcal{J} \) is equal to \( M_k \) for some \( k \in \mathbb{N} \), and this makes the problem of determining \( \sigma(T) \) particularly tractable. Fixing any \( k \in \mathbb{N} \), let \( \mathbb{C}[y] \) be the algebra of formal polynomials with coefficients in \( \mathbb{C} \), and let \( j^k_p \) be the operator from \( B \) into \( \mathbb{C}[y]/y^k \mathbb{C}[y] \) given by

\[
j^k_p f = \sum_{l=0}^{k-1} \frac{1}{l!} f^{(l)}(p) Z^l, \quad (f \in B)
\]

where \( Z \) is the residue class of \( y \) in \( \mathbb{C}[y]/y^k \mathbb{C}[y] \). It is easy to see that given any \( I \in \mathcal{J} \), there is a \( k \in \mathbb{N} \) for which \( T/I \) is similar to an endomorphism \( T_k : \mathbb{C}[y]/y^k \mathbb{C}[y] \to \mathbb{C}[y]/y^k \mathbb{C}[y] \), where

\[
T_k \left( \sum_{l=0}^{k-1} a_l Z^l \right) = \sum_{l=0}^{k-1} a_l (j^k_p \phi - p)^l, \quad (k \in \mathbb{N})
\]

(6)

Here, we adopt the convention that, in all cases, \( (j^k_p \phi - p)^0 \) is the identity element in \( \mathbb{C}[y]/y^k \mathbb{C}[y] \). A routine calculation shows that the matrix of \( T_k \) (with respect to the basis \( 1, Z, Z^2, \ldots, Z^{k-1} \)) is lower triangular, with \( 1, \phi'(p), \phi'(p)^2, \ldots, \phi'(p)^{k-1} \) along the diagonal. Thus, given any \( I \in \mathcal{J} \),

\[
\sigma(T/I) = \{ \phi'(p)^j : j = 0, 1, \ldots, k - 1 \}
\]

for some \( k \in \mathbb{N} \). This is enough for us to be able to conclude that

\[
\sigma(T) \subseteq \{0, 1\} \cup \{ \phi'(p)^k : k \in \mathbb{N} \}.
\]

That \( \{0, 1\} \subseteq \sigma(T) \) is obvious, so it only remains to show that \( \phi'(p)^k \in \sigma(T) \) for every \( k \in \mathbb{N} \). If it were guaranteed that \( \mathcal{J} = \pi_B(p) \), this would already be clear. However, this is not what Theorem 9 tells us. To complete the proof, we consider the bounded linear functional on \( B \) given by

\[
\delta'_p(f) = f'(p), \quad (f \in B).
\]

The chain rule now gives \( T^* \delta'_p = \phi'(p) \delta'_p \). As \( T \) is Riesz, this means that either \( \phi'(p) = 0 \), or \( \phi'(p) \) is an eigenvalue of \( T \). The result now follows from the fact that the set of eigenvalues of \( T \) is closed under powers. \( \Box \)
The reader may wish to note that we have said nothing about the situation when $p \in \Phi_B \setminus \mathbb{D}$. Here, the result is much more dependent on the algebra under consideration. However, progress can still be made with the help of Theorem 12; the situation is particularly straightforward when $B$ has no non-zero bounded point derivations at points of $\Phi_B \setminus \mathbb{D}$.

**Remark.** The argument used to prove Theorem 17 is also available in quotients of the form

$$L^1(\mathbb{R}, \omega)/\{f : \hat{f}(\xi) = 0 \text{ for } \xi \in [0, 1]\},$$

where $\omega(x) = e^{|x|^\alpha}$ ($x \in \mathbb{R}$) for some $0 < \alpha < 1$, and $f \mapsto \hat{f}$ is the usual Fourier transform. In [4], Domar proves that these algebras have a similar ideal structure to those considered above. However, there is a danger (in light of the existence of theorems of Beurling–Helson type) that many of these algebras may not support any non-trivial Riesz endomorphisms at all; this matter does not seem to have been entirely settled.

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