On the number of unique expansions in non-integer bases

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Abstract

Let q > 1 be a real number and let m = m(q) be the largest integer smaller than q. It is well known that each number x ∈ J_q := [0, ⌈q⌉ − 1] can be written as x = ∑_{i=1}^{∞} c_i q^{-i} with integer coefficients 0 ≤ c_i < q. If q is a non-integer, then almost every x ∈ J_q has continuum many expansions of this form. In this note we consider some properties of the set U_q consisting of numbers x ∈ J_q having a unique representation of this form. More specifically, we compare the size of the sets U_q and U_r for values q and r satisfying 1 < q < r and m(q) = m(r).

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1. Introduction

Beginning with the pioneering works of Rényi [13] and Parry [12], expansions of real numbers in non-integer bases have been widely studied during the last fifty years.

In this paper we consider only sequences of nonnegative integers. Given a real number q > 1, an expansion in base q (or simply expansion) of a real number x is a sequence (c_i) = c_1 c_2 ... of integers satisfying

0 ≤ c_i < q for each i ≥ 1 and x = ∑_{i=1}^{∞} c_i q^{-i}.

Note that this definition is only meaningful if x belongs to the interval

J_q := [0, ⌈q⌉ − 1],

where ⌈q⌉ is the smallest integer larger than or equal to q. Note that [0, 1] ⊂ J_q.

The greedy expansion of a number x ∈ J_q, denoted by (b_i(x)) or (b_i), can be obtained by performing the greedy algorithm [13]: if for some n ∈ N := Z_{≥1}, b_i = b_i(x) is already defined for 1 ≤ i < n (no condition if n = 1), then b_n = b_n(x) is the largest integer smaller than q such that

∑_{i=1}^{n} b_i q^{-i} ≤ x.
If \( x \in J_q \setminus \{0\} \), then the quasi-greedy expansion, denoted by \((a_i(x))\) or \((a_i)\), is obtained by applying the quasi-greedy algorithm \([4,11,1]\): if for some \( n \in \mathbb{N} \), \( a_i = a_i(x) \) is already defined for \( 1 \leq i < n \) (no condition if \( n = 1 \)), then \( a_n = a_n(x) \) is the largest integer smaller than \( x \) such that
\[
\sum_{i=1}^{n} \frac{a_i}{q^i} < x.
\]

The quasi-greedy expansion of \( x \in J_q \setminus \{0\} \) is always infinite (we call an expansion infinite if it contains infinitely many nonzero elements; otherwise it is called finite) and coincides with the greedy expansion \((b_i(x))\) if and only if the latter is finite. If the greedy expansion of \( x \in J_q \setminus \{0\} \) is finite and \( b_n \) is its last nonzero element, then \((a_i(x)) = b_1 \ldots b_{n-1} b_n \alpha_1 \alpha_2 \ldots \) where \( b_n := b_n - 1 \) and \( \alpha_i = \alpha_i(q) := a_i(1) \), \( i \geq 1 \). For convenience, we set \( (a_i(0)) := 0^\infty \) and refer to it as the quasi-greedy expansion of \( 0 \) in base \( q \). We will also write \( x \sim (a_i) \) if the quasi-greedy expansion of \( x \) in base \( q \) is given by \((a_i)\).

If \( q > 1 \) is an integer, then the greedy expansion of a number \( x \in J_q = [0,1] \) is in fact the only expansion of \( x \) in base \( q \), except when \( x = i/q^k \), where \( 1 \leq i < q^k - 1 \) is an integer and \( n \in \mathbb{N} \). However, if \( q > 1 \) is a non-integer, then almost every \( x \in J_q \) has continuum many expansions in base \( q \), see \([2,14]\). Starting with a discovery of Erdős, Horváth and Joó \([6]\), many works during the last fifteen years were devoted to the study of the exceptional set \( \mathcal{U}_q \) consisting of those numbers \( x \in J_q \) with a unique expansion in base \( q \). For instance, it was shown in \([7]\) that if \( 1 < q < (1 + \sqrt{5})/2 \), then each number in the interior of \( J_q \) has continuum many expansions. Hence, in this case, \( \mathcal{U}_q = [0,1/(q-1)) \). However, if \( q > (1 + \sqrt{5})/2 \), then the set \( \mathcal{U}_q \) is infinite \([3]\).

In order to mention some more sophisticated properties of the set \( \mathcal{U}_q \) for various values of \( q \), we introduce the set of univoque numbers \( \mathcal{U} \), defined by
\[
\mathcal{U} := \{ q > 1 \colon \text{1 has a unique expansion in base } q \}.
\]

It was shown in \([6]\) that the set \( \mathcal{U} \cap (1,2) \) has continuum many elements. Subsequently, the set \( \mathcal{U} \) was characterized lexicographically in \([7,8,11]\), its smallest element \( q_1 \approx 1.787 \) was determined in \([10]\), and its topological structure was described in \([11]\). It was also shown in \([10]\) that the unique expansion of \( 1 \) in base \( q_1 \) is given by the truncated Thue–Morse sequence \((\tau_n) = 110100111 \ldots \), which can be defined recursively by setting \( \tau_2N+1 = 1 - \tau_1 \) for \( 1 \leq i < 2^N \), \( N = 1,2, \ldots \).

Using the structure of this expansion, Glendinning and Sidorov \([9]\) proved that \( \mathcal{U}_q \) is countable if \( 1 < q < q_1 \) and has the cardinality of the continuum if \( q_1 \leq q < 2 \) (see also \([5]\)). They also proved that if \( 1 < q < q_1 \), then the (unique) expansion in base \( q \) of a number \( x \in \mathcal{U}_q \) is ultimately periodic. Finally, the topological structure of the sets \( \mathcal{U}_q \) \((q > 1)\) was established in \([5]\).

Let us call a sequence \((c_i) = c_1c_2 \ldots \) with integers \( 0 \leq c_i < q \) univoque in base \( q \) (or simply univoque if \( q \) is understood) if
\[
x = \sum_{i=1}^{\infty} \frac{c_i}{q^i}
\]
belongs to \( \mathcal{U}_q \). Let \( \mathcal{U}_q' \) denote the set of all univoque sequences in base \( q \). Clearly, there is a natural bijection between \( \mathcal{U}_q \) and \( \mathcal{U}_q' \). In what follows we use systematically the lexicographical order between sequences: we write \((a_i) < (b_i)\) or \(a_1a_2 \ldots < b_1b_2 \ldots\) if there is an integer \( n \in \mathbb{N} \) such that \( a_i = b_i \) for \( i < n \) and \( a_n < b_n \). We recall the following theorem which is essentially due to Parry \([12]\): 

**Theorem 1.1.** Let \( q > 1 \) be a real number and let \( m \) be the largest integer smaller than \( q \).

(i) A sequence \((b_i) = b_1b_2 \ldots \in \{0, \ldots, m\}^\mathbb{N}\) is the greedy expansion of a number \( x \in J_q \) if and only if
\[
b_{n+1} \ldots < \alpha_1 \alpha_2 \ldots \quad \text{whenever } b_n < m.
\]

(ii) A sequence \((c_i) = c_1c_2 \ldots \in \{0, \ldots, m\}^\mathbb{N}\) is univoque if and only if
\[
c_{n+1} \ldots < \alpha_1 \alpha_2 \ldots \quad \text{whenever } c_n < m
\]
and
\[
c_{n+1} \ldots < \alpha_1 \alpha_2 \ldots \quad \text{whenever } c_n > 0,
\]
where \( \bar{c}_i := m - c_i = \alpha_1 - c_i \), \( i \in \mathbb{N} \), and \( \bar{c}_1 \bar{c}_2 \ldots = \bar{c}_1 \bar{c}_2 \ldots \).

Using the fact that the map \( q \mapsto (a_i(q)) \) is strictly increasing, it follows at once from this theorem that \( \mathcal{U}_q' \subset \mathcal{U}_r' \) if \( 1 < q < r \) and \(|q| = |r|\). It is the aim of this note to generalize the above mentioned result of Glendinning and Sidorov \([9]\) by considering the difference of the sets \( \mathcal{U}_q' \) and \( \mathcal{U}_r' \), \( 1 < q < r \), \(|q| = |r|\).
Theorem 1.2. Let $1 < q < r$ be real numbers such that $[q] = [r]$. The following statements are equivalent.

(i) $(q, r) \cap \mathcal{U} = \emptyset$.
(ii) $(q, r) \cap \overline{\mathcal{U}} = \emptyset$.
(iii) Each sequence $(c_i) \in \mathcal{U}_q' \setminus \mathcal{U}_q$ is ultimately periodic.
(iv) $\mathcal{U}_q' \setminus \mathcal{U}_q$ is countable.

Incidentally, we will also obtain new characterizations of the set of univoque numbers $\mathcal{U}$ and its closure $\overline{\mathcal{U}}$ (for other characterizations, see [5,11]).

2. Proof of Theorem 1.2

Recently, Biaocchi and Komornik [1] reformulated and extended some classical results of Rényi, Parry, Daróczy and Kátai [13,12,4] by characterizing the quasi-greedy expansions of numbers $x \in J_q \setminus \{0\}$ in a fixed base $q > 1$ (see Proposition 2.2 below).

Proposition 2.1. The map $q \mapsto (\alpha_i(q))$ is a strictly increasing bijection from the open interval $(1, \infty)$ onto the set of all infinite sequences $(\alpha_i)$ satisfying

$$\alpha_{k+1}\alpha_{k+2}\ldots \leq \alpha_1\alpha_2\ldots \text{ for all } k \geq 1.$$ 

Proposition 2.2. For each $q > 1$, the map $x \mapsto (a_i(x))$ is a strictly increasing bijection from $(0, \alpha_1/(q - 1)]$ onto the set of all infinite sequences $(a_i)$, satisfying

$$0 \leq a_n \leq \alpha_1 \text{ for all } n \geq 1$$

and

$$a_{n+1}a_{n+2}\ldots \leq \alpha_1\alpha_2\ldots \text{ whenever } a_n < \alpha_1.$$ 

For any fixed $q > 1$, we introduce the sets

$$\mathcal{V}_q := \{x \in J_q; \ a_{n+1}(x)a_{n+2}(x)\ldots \leq \alpha_1(q)\alpha_2(q)\ldots \text{ whenever } a_n > 0\}$$

and

$$\mathcal{V}_q' := \{(a_i(x)); \ x \in \mathcal{V}_q\}.$$ 

It follows from Theorem 1.1 that $\mathcal{U}_q \subset \mathcal{V}_q$ for each $q > 1$. Moreover, $\mathcal{V}_q' \subset \mathcal{U}_q'$ if $1 < q < r$ and $[q] = [r]$. The precise relationship between the sets $\mathcal{U}_q$, its closure $\overline{\mathcal{U}_q}$ and $\mathcal{V}_q$ for each $q > 1$ was described in [5]. For instance, it was shown that $\mathcal{U}_q$ is closed if and only if $q \notin \overline{\mathcal{U}}$. Moreover, $\mathcal{U}_q = \overline{\mathcal{U}_q} = \mathcal{V}_q$, except when $q$ belongs to the closed null set $\mathcal{V}$ consisting of those bases $q > 1$ such that

$$\alpha_{k+1}(q)\alpha_{k+2}(q)\ldots \leq \alpha_1(q)\alpha_2(q)\ldots \text{ for each } k \geq 1.$$ 

If $q \in \mathcal{V}$, then the set $\mathcal{V}_q' \setminus \mathcal{U}_q$ is countably infinite.

The relationship between the sets $\mathcal{U}$, $\overline{\mathcal{U}}$, and $\mathcal{V}$ has been investigated in [11]. In particular it was shown that

- $\mathcal{U} \subseteq \overline{\mathcal{U}} \subseteq \mathcal{V}$.
- $\overline{\mathcal{U}} \setminus \mathcal{U}$ is countable and dense in $\overline{\mathcal{U}}$.
- $\mathcal{V} \setminus \overline{\mathcal{U}}$ is a discrete set, dense in $\mathcal{V}$.
- $q \in \overline{\mathcal{U}}$ if and only if $\alpha_{k+1}(\alpha)\alpha_{k+2}(\alpha)\ldots < \alpha_1(\alpha)\alpha_2(\alpha)\ldots \text{ for each } k \geq 1$.

Applying the above mentioned results, one can easily verify the statements in the following examples.

Examples.

- The smallest element of $\mathcal{V}$ is given by $G := (1 + \sqrt{5})/2$. Moreover, $G \sim (10)^\infty$ and $\mathcal{V}_q'$ is the set of all sequences in $[0, 1]^\mathbb{N}$ such that a one is never followed by two zeros and a one is never followed by two ones. Hence $\mathcal{U}_q$ is infinite if $q < q \leq 2$.
- Define the numbers $q_n$ ($n \in \mathbb{N}$) by setting $q_n \sim (110)^n(10)^\infty$. It follows from Theorem 1.1 that all these numbers belong to $\mathcal{U}$. However, if we set $q^* := \lim_{n \to \infty} q_n$, then $q^* \sim (110)^\infty$. Note that $q^* \notin \mathcal{U}$ because $111(0)^\infty$ is another expansion of 1 in base $q^*$. Hence $q^* \in \overline{\mathcal{U}} \setminus \mathcal{U}$.
Without further comment, we use frequently in the proof below some of the main results in [5], and in particular the analysis of one of the final remarks at the end of [5] which is concerned with the endpoints of the connected components of \((1, \infty) \setminus \overline{U}\); if we write \((1, \infty) \setminus \overline{U}\) as the union of countably many disjoint open intervals (its connected components), then the set \(L\) of left endpoints of these intervals is given by \(L = \mathbb{N} \cup (\overline{U} \setminus U)\) and the set \(R\) of right endpoints of these intervals satisfies the relationship \(R \subset U\).

**Proof of Theorem 1.2.** (i) \(\Rightarrow\) (ii). Suppose that \((q, r) \cap U = \emptyset\). Then \((q, r + \delta) \cap U = \emptyset\) for some \(\delta > 0\) because \(U\) is closed from above [11] and (ii) follows.

(ii) \(\Rightarrow\) (iii). If \((q, r) \cap \overline{U} = \emptyset\), then \((q, r)\) is a subset of a connected component of \((1, \infty) \setminus \overline{U}\). Moreover, \((q, r) \cap V\) is a finite subset \([r_1, \ldots, r_m]\) of \(V \setminus U\), where \(r_1 < \cdots < r_m\). Although it is not important in the remainder of the proof, we recall from [5] that \(r_2, \ldots, r_m \in V \setminus U\), but \(r_1\) might belong to \(U \setminus U\). We may write
\[
U_r' = U_r' \cup \bigcup_{i=1}^m (V_{r_i} \setminus U_{r_i}). \tag{2.1}
\]

Fix \(\ell \in \{1, \ldots, m\}\) and let \(x \in V_{r_\ell} \setminus U_{r_\ell}\). If the greedy expansion \((b_1)\) of \(x\) in base \(r_\ell\) is finite, then \((a_1(x))\) ends with \(\alpha_1\alpha_2\ldots\).

Suppose now that \((b_1)\) is infinite. Since \(x \notin U_{r_\ell}\), there exists an index \(n\) such that \(b_n > 0\), and \(b_{n+1}b_{n+2} \cdots \geq \alpha_1\alpha_2\ldots\). Since \(x \in V_{r_\ell}\) and \((a_1(x)) = (b_1(x))\), the last inequality is in fact an equality. Hence the quasi-greedy expansion \((a_1(x))\) of \(x\) in base \(r_\ell\) either ends with \((\alpha_1)\) or \((\alpha_1)\). Since \((\alpha_1,q))\) is periodic if \(q \in V \setminus U\) [11], the implication follows from (2.1).

(iii) \(\Rightarrow\) (iv) is clear.

(iv) \(\Rightarrow\) (i). We prove the contraposition. Suppose that \((q, r) \cap U \neq \emptyset\). We distinguish between two cases.

If \((q, r) \cap U \neq \emptyset\), then \((q, r) \cap \overline{U} = 2^{\aleph_0}\) because \(\overline{U}\) is a nonempty perfect set [11] and thus each neighborhood of \(x\) contains uncountably many elements of \(\overline{U}\).

Hence \(U_r' \setminus U_q' \supset \bigcup_{t \in [q, r] \cap U} (V_t \setminus U_t')\).

If \((q, r) \cap U = \emptyset\), then \((q, r) \cap \overline{U} = \emptyset\). Hence by enlarging \(q\) if necessary, we may assume that \(q \notin U\). Let
\[
\mathcal{W}_r = \{x \in U_r : \text{the unique expansion of } x \text{ in base } r \text{ belongs to } U_q'\}.
\]

We claim that \(\mathcal{W}_r\) is closed. The set \(\mathcal{W}_r\) is a symmetric subset of \(J_r\), so it suffices to show that \(\mathcal{W}_r\) is closed from below. Let \(x_i \in \mathcal{W}_r\) \((i \geq 1)\), and suppose that \(x_i \uparrow x\). Let \((c_j)\) be the unique expansion of \(x_i\) in base \(r\), and let
\[
y_i = \sum_{j=1}^{\infty} \frac{c_j}{q^j}.
\]

Then the increasing sequence \((y_i)\) converges to some \(y \in U_q\) because \(U_q\) is a compact set. Since \((c_j) \leq (c_j) \leq \cdots\), \((c_j)\) converges coordinate-wise to the unique expansion \((d_j)\) of \(y\) in base \(q\) as \(i \to \infty\), and
\[
x = \sum_{j=1}^{\infty} \frac{d_j}{r^j}.
\]

Since \(U_q' \subset U_q\), we have \(x \in U_q\), and thus \(x \in \mathcal{W}_r\). Now suppose that \(U_r' \setminus U_q'\) is countable. Then \(U_r \setminus \mathcal{W}_r\) is countable. Note that \(\mathcal{W}_r \subseteq U_r\) because \(\mathcal{W}_r\) is closed and \(U_r\) is not. Let \(x \in U_r \setminus \mathcal{W}_r\). Since \(U_r \setminus \mathcal{W}_r\) is a countable dense subset of \(U_r\) (Theorem 1.3 in [5]), the latter set is perfect, and each neighborhood of \(x\) contains uncountably many elements of \(U_r\) and thus of \(\mathcal{W}_r\). This contradicts the fact that \(\mathcal{W}_r\) is closed.

The above result yields new characterizations of \(U\) and \(\overline{U}\):

**Corollary 2.3.** A real number \(q > 1\) belongs to \(U\) if and only if \(U_q' \setminus U_q'\) is uncountable for each \(r > q\) such that \([q] = [r]\).

**Proof.** Note that the integers \(2, 3, \ldots\) belong to \(U\). For these values of \(q\) the condition in the statement is also vacuously satisfied. Hence we may assume that \(q \notin \mathbb{N}\). Suppose that \(q \in \mathbb{N} \setminus \mathbb{N}\). For each \(r > q\), \((q, r) \cap U \neq \emptyset\) because elements of \(U \setminus \mathbb{N}\) do not belong to the set of left endpoints of the connected components of \((1, \infty) \setminus \overline{U}\). Hence if, in addition, \([q] = [r]\), then \(U_q' \setminus U_q'\) is uncountable by Theorem 1.2. Conversely, if the latter set is uncountable for each \(r > q\) such that \([q] = [r]\), then \((q, r) \cap \overline{U} \neq \emptyset\) for each \(r > q\) by Theorem 1.2, and the result follows because \(U\) is closed from above.

For a fixed \(r > 1\), let \(F_r' = \bigcup U_q'\), where the union runs over all \(q < r\) for which \([q] = [r]\).
Corollary 2.4. Let \( r > 1 \) be a real number. The following statements are equivalent.

(i) \( r \in \overline{U} \).
(ii) \( U_1' \setminus F_q' \) is uncountable.
(iii) \( U_1' \setminus U_q' \) is uncountable for each \( q < r \) such that \( [q] = [r] \).
(iv) \( U_1' \setminus U_q' \) is nonempty for each \( q < r \) such that \( [q] = [r] \).

Proof. It is clear that (ii) \( \Rightarrow \) (iii) \( \Rightarrow \) (iv). It remains to show that (i) \( \Rightarrow \) (ii) and (iv) \( \Rightarrow \) (i).

(i) \( \Rightarrow \) (ii). Suppose \( r \in \overline{U} \). Let \( \{q_n\}_{n \geq 1} \) be an increasing sequence that converges to \( r \), such that \( q_n \notin \overline{U} \) and \( [q_n] = [r] \), \( n \in \mathbb{N} \). This can be done, since \( \overline{U} \) is a null set. Let

\[
W_q^n = \{x \in U_q': \text{the unique expansion of } x \text{ in base } r \text{ belongs to } U_q' \}
\]

and

\[
W_q = \bigcup_{n=1}^{\infty} W_q^n.
\]

It follows from the proof of Theorem 1.2 that \( W_q^n \) is closed for each \( n \in \mathbb{N} \). Moreover, \( |U_q \setminus W_q| = |U_q' \setminus F_q'| \). We know that \( \overline{U} \setminus U_q \) is countable. If \( U_q \setminus W_q \) were countable, then \( \overline{U} \) would be an \( F_{\sigma} \)-set:

\[
\overline{U} = \bigcup_{n=1}^{\infty} W_q^n \cup \left( \bigcup_{x \in U_q \setminus W_q} \{x\} \right).
\]

(2.2)

Note that \( \overline{U} \) is a complete metric space. By Baire’s theorem, one of the sets on the right-hand side of (2.2) has a nonempty interior. Since \( \overline{U} \) is a perfect set, each singleton belonging to it is not open. Hence one of the sets \( W_q^n \subset U_q \) has an interior point. But this contradicts the fact that \( \overline{U} \setminus U_q \) is dense in \( \overline{U} \) (Theorem 1.3 in [5]).

(iv) \( \Rightarrow \) (i). We prove the contraposition. Suppose \( r \notin \overline{U} \). We can choose \( q \in (1, r) \) close enough to \( r \) such that \( [q, r) \cap V = \emptyset \).

It follows from (2.1) that \( U_q' = U_q \). \qed

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