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On the number of unique expansions in non-integer bases

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ABSTRACT

Let $q>1$ be a real number and let $m=m(q)$ be the largest integer smaller than q. It is well known that each number $x\in J_q:=[0,\sum_{i=1}^{\infty} mq^{-i}]$ can be written as $x=\sum_{i=1}^{\infty} c_iq^{-i}$ with integer coefficients $0\leqslant c_i<q$. If q is a non-integer, then almost every $x\in J_q$ has continuum many expansions of this form. In this note we consider some properties of the set $\mathcal{U}_q$ consisting of numbers $x\in J_q$ having a unique representation of this form. More specifically, we compare the size of the sets $\mathcal{U}_q$ and $\mathcal{U}_r$ for values q and r satisfying $1<q<r$ and $m(q)=m(r)$.

1. Introduction

Beginning with the pioneering works of Rényi [13] and Parry [12], expansions of real numbers in non-integer bases have been widely studied during the last fifty years.

In this paper we consider only sequences of nonnegative integers. Given a real number $q>1$, an *expansion in base* q (or simply *expansion*) of a real number x is a sequence $(c_i)=c_1c_2\ldots$ of integers satisfying

$$0\leqslant c_i<q \quad \text{for each } i\geqslant 1 \quad \text{and} \quad x=\sum_{i=1}^{\infty}\frac{c_i}{q^i}.$$

Note that this definition is only meaningful if x belongs to the interval

$$J_q:=\left[0,\frac{\lceil q\rceil-1}{q-1}\right],$$

where $\lceil q\rceil$ is the smallest integer larger than or equal to q. Note that $[0,1]\subset J_q$.

The *greedy* expansion of a number $x\in J_q$, denoted by $(b_i(x))$ or (b_i), can be obtained by performing the greedy algorithm [13]: if for some $n\in\mathbb{N}:=\mathbb{Z}_{\geqslant 1}$, $b_i=b_i(x)$ is already defined for $1\leqslant i<n$ (no condition if $n=1$), then $b_n=b_n(x)$ is the largest integer smaller than q such that

$$\sum_{i=1}^{n}\frac{b_i}{q^i}\leqslant x.$$

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If $x \in J_q \setminus \{0\}$, then the *quasi-greedy* expansion, denoted by $(a_i(x))$ or (a_i), is obtained by applying the quasi-greedy algorithm [4,11,1]: if for some $n \in \mathbb{N}$, $a_i = a_i(x)$ is already defined for $1 \leqslant i < n$ (no condition if $n = 1$), then $a_n = a_n(x)$ is the largest integer smaller than q such that

$$\sum_{i=1}^{n} \frac{a_i}{q^i} < x.$$

The quasi-greedy expansion of $x \in J_q \setminus \{0\}$ is always infinite (we call an expansion *infinite* if it contains infinitely many nonzero elements; otherwise it is called *finite*) and coincides with the greedy expansion $(b_i(x))$ if and only if the latter is infinite. If the greedy expansion of $x \in J_q \setminus \{0\}$ is finite and b_n is its last nonzero element, then $(a_i(x)) = b_1 \dots b_{n-1} b_n^- \alpha_1 \alpha_2 \dots$, where $b_n^- := b_n - 1$ and $\alpha_i = \alpha_i(q) := a_i(1)$, $i \geqslant 1$. For convenience, we set $(a_i(0)) := 0^\infty$ and refer to it as the quasi-greedy expansion of 0 in base q. We will also write $q \sim (\alpha_i)$ if the quasi-greedy expansion of 1 in base q is given by (α_i).

If $q > 1$ is an integer, then the greedy expansion of a number $x \in J_q = [0, 1]$ is in fact the only expansion of x in base q, except when $x = i/q^n$, where $1 \leqslant i \leqslant q^n - 1$ is an integer and $n \in \mathbb{N}$. However, if $q > 1$ is a non-integer, then almost every $x \in J_q$ has continuum many expansions in base q, see [2,14]. Starting with a discovery of Erdős, Horváth and Joó [6], many works during the last fifteen years were devoted to the study of the exceptional set $\mathcal{U}_q$ consisting of those numbers $x \in J_q$ with a unique expansion in base q. For instance, it was shown in [7] that if $1 < q < (1 + \sqrt{5})/2$, then *each* number in the interior of J_q has continuum many expansions. Hence, in this case, $\mathcal{U}_q = \{0, 1/(q-1)\}$. However, if $q > (1 + \sqrt{5})/2$, then the set $\mathcal{U}_q$ is infinite [3].

In order to mention some more sophisticated properties of the set $\mathcal{U}_q$ for various values of q, we introduce the set of *univoque numbers* $\mathcal{U}$, defined by

$$\mathcal{U} := \{q > 1: \ 1 \text{ has a unique expansion in base } q\}.$$

It was shown in [6] that the set $\mathcal{U} \cap (1, 2)$ has continuum many elements. Subsequently, the set $\mathcal{U}$ was characterized lexicographically in [7,8,11], its smallest element $q_1 \approx 1.787$ was determined in [10], and its topological structure was described in [11]. It was also shown in [10] that the unique expansion of 1 in base q_1 is given by the truncated Thue–Morse sequence $(\tau_i) = 11010011\dots$, which can be defined recursively by setting $\tau_{2^N} = 1$ for $N = 0, 1, 2, \dots$ and

$$\tau_{2^N + i} = 1 - \tau_i \quad \text{for } 1 \leqslant i < 2^N, \ N = 1, 2, \dots.$$

Using the structure of this expansion, Glendinning and Sidorov [9] proved that $\mathcal{U}_q$ is countable if $1 < q < q_1$ and has the cardinality of the continuum if $q_1 \leqslant q < 2$ (see also [5]). They also proved that if $1 < q < q_1$, then the (unique) expansion in base q of a number $x \in \mathcal{U}_q$ is ultimately periodic. Finally, the topological structure of the sets $\mathcal{U}_q$ $(q > 1)$ was established in [5].

Let us call a sequence $(c_i) = c_1 c_2 \dots$ with integers $0 \leqslant c_i < q$ *univoque in base* q (or simply *univoque* if q is understood) if

$$x = \sum_{i=1}^{\infty} \frac{c_i}{q^i}$$

belongs to $\mathcal{U}_q$. Let $\mathcal{U}'_q$ denote the set of all univoque sequences in base q. Clearly, there is a natural bijection between $\mathcal{U}_q$ and $\mathcal{U}'_q$. In what follows we use systematically the lexicographical order between sequences: we write $(a_i) < (b_i)$ or $a_1 a_2 \dots < b_1 b_2 \dots$ if there is an integer $n \in \mathbb{N}$ such that $a_i = b_i$ for $i < n$ and $a_n < b_n$. We recall the following theorem which is essentially due to Parry [12]:

Theorem 1.1. *Let $q > 1$ be a real number and let m be the largest integer smaller than q.*

(i) *A sequence $(b_i) = b_1 b_2 \dots \in \{0, \dots, m\}^{\mathbb{N}}$ is the greedy expansion of a number $x \in J_q$ if and only if*

$$b_{n+1} b_{n+2} \dots < \alpha_1 \alpha_2 \dots \quad \textit{whenever } b_n < m.$$

(ii) *A sequence $(c_i) = c_1 c_2 \dots \in \{0, \dots, m\}^{\mathbb{N}}$ is univoque if and only if*

$$c_{n+1} c_{n+2} \dots < \alpha_1 \alpha_2 \dots \quad \textit{whenever } c_n < m$$

and

$$\overline{c_{n+1} c_{n+2} \dots} < \alpha_1 \alpha_2 \dots \quad \textit{whenever } c_n > 0,$$

where $\overline{c_i} := m - c_1 = \alpha_1 - c_1$, $i \in \mathbb{N}$, and $\overline{c_1 c_2 \dots} = \overline{c_1}\,\overline{c_2} \dots$.

Using the fact that the map $q \mapsto (\alpha_i(q))$ is strictly increasing, it follows at once from this theorem that $\mathcal{U}'_q \subset \mathcal{U}'_r$ if $1 < q < r$ and $\lceil q \rceil = \lceil r \rceil$. It is the aim of this note to generalize the above mentioned result of Glendinning and Sidorov [9] by considering the difference of the sets $\mathcal{U}'_q$ and $\mathcal{U}'_r$, $1 < q < r$, $\lceil q \rceil = \lceil r \rceil$:

Theorem 1.2. *Let $1 < q < r$ be real numbers such that $\lceil q \rceil = \lceil r \rceil$. The following statements are equivalent.*

(i) $(q, r] \cap \mathcal{U} = \emptyset$.
(ii) $(q, r] \cap \overline{\mathcal{U}} = \emptyset$.
(iii) *Each sequence $(c_i) \in \mathcal{U}'_r \setminus \mathcal{U}'_q$ is ultimately periodic.*
(iv) *$\mathcal{U}'_r \setminus \mathcal{U}'_q$ is countable.*

Incidentally, we will also obtain new characterizations of the set of univoque numbers $\mathcal{U}$ and its closure $\overline{\mathcal{U}}$ (for other characterizations, see [5,11]).

2. Proof of Theorem 1.2

Recently, Baiocchi and Komornik [1] reformulated and extended some classical results of Rényi, Parry, Daróczy and Kátai [13,12,4] by characterizing the quasi-greedy expansions of numbers $x \in J_q \setminus \{0\}$ in a fixed base $q > 1$ (see Proposition 2.2 below).

Proposition 2.1. *The map $q \mapsto (\alpha_i(q))$ is a strictly increasing bijection from the open interval $(1, \infty)$ onto the set of all* infinite *sequences (α_i) satisfying*

$$\alpha_{k+1}\alpha_{k+2}\ldots \leqslant \alpha_1\alpha_2\ldots \quad \textit{for all } k \geqslant 1.$$

Proposition 2.2. *For each $q > 1$, the map $x \mapsto (a_i(x))$ is a strictly increasing bijection from $(0, \alpha_1/(q-1)]$ onto the set of all* infinite *sequences (a_i), satisfying*

$$0 \leqslant a_n \leqslant \alpha_1 \quad \textit{for all } n \geqslant 1$$

and

$$a_{n+1}a_{n+2}\ldots \leqslant \alpha_1\alpha_2\ldots \quad \textit{whenever } a_n < \alpha_1.$$

For any fixed $q > 1$, we introduce the sets

$$\mathcal{V}_q := \left\{x \in J_q\colon\ \overline{a_{n+1}(x)a_{n+2}(x)\ldots} \leqslant \alpha_1(q)\alpha_2(q)\ldots \text{ whenever } a_n > 0\right\}$$

and

$$\mathcal{V}'_q := \left\{\left(a_i(x)\right)\colon\ x \in \mathcal{V}_q\right\}.$$

It follows from Theorem 1.1 that $\mathcal{U}_q \subset \mathcal{V}_q$ for each $q > 1$. Moreover, $\mathcal{V}'_q \subset \mathcal{U}'_r$ if $1 < q < r$ and $\lceil q \rceil = \lceil r \rceil$. The precise relationship between the sets $\mathcal{U}_q$, its closure $\overline{\mathcal{U}_q}$ and $\mathcal{V}_q$ for each $q > 1$ was described in [5]. For instance, it was shown that $\mathcal{U}_q$ is closed if and only if $q \notin \overline{\mathcal{U}}$. Moreover, $\mathcal{U}_q = \overline{\mathcal{U}_q} = \mathcal{V}_q$, except when q belongs to the closed null set $\mathcal{V}$ consisting of those bases $q > 1$ such that

$$\overline{\alpha_{k+1}(q)\alpha_{k+2}(q)\ldots} \leqslant \alpha_1(q)\alpha_2(q)\ldots \quad \text{for each } k \geqslant 1.$$

If $q \in \mathcal{V}$, then the set $\mathcal{V}_q \setminus \mathcal{U}_q$ is countably infinite.

The relationship between the sets $\mathcal{U}$, $\overline{\mathcal{U}}$, and $\mathcal{V}$ has been investigated in [11]. In particular it was shown that

- $\mathcal{U} \subsetneq \overline{\mathcal{U}} \subsetneq \mathcal{V}$.
- $\overline{\mathcal{U}} \setminus \mathcal{U}$ is countable and dense in $\overline{\mathcal{U}}$.
- $\mathcal{V} \setminus \overline{\mathcal{U}}$ is a discrete set, dense in $\mathcal{V}$.
- $q \in \overline{\mathcal{U}}$ if and only if $\overline{\alpha_{k+1}(q)\alpha_{k+2}(q)\ldots} < \alpha_1(q)\alpha_2(q)\ldots$ for each $k \geqslant 1$.

Applying the above mentioned results, one can easily verify the statements in the following examples.

Examples.

- The smallest element of $\mathcal{V}$ is given by $G := (1 + \sqrt{5})/2$. Moreover, $G \sim (10)^\infty$ and $\mathcal{V}'_G$ is the set of all sequences in $\{0, 1\}^{\mathbb{N}}$ such that a one is never followed by two zeros and a zero is never followed by two ones. Hence $\mathcal{U}_q$ is infinite if $G < q \leqslant 2$.
- Define the numbers q_n $(n \in \mathbb{N})$ by setting $q_n \sim (110)^n(10)^\infty$. It follows from Theorem 1.1 that all these numbers belong to $\mathcal{U}$. However, if we set $q^* := \lim_{n\to\infty} q_n$, then $q^* \sim (110)^\infty$. Note that $q^* \notin \mathcal{U}$ because $111(0)^\infty$ is another expansion of 1 in base q^*. Hence $q^* \in \overline{\mathcal{U}} \setminus \mathcal{U}$.

Without further comment, we use frequently in the proof below some of the main results in [5], and in particular the analysis of one of the final remarks at the end of [5] which is concerned with the endpoints of the connected components of $(1,\infty)\setminus\overline{\mathcal{U}}$: if we write $(1,\infty)\setminus\overline{\mathcal{U}}$ as the union of countably many disjoint open intervals (its connected components), then the set L of left endpoints of these intervals is given by $L=\mathbb{N}\cup(\overline{\mathcal{U}}\setminus\mathcal{U})$ and the set R of right endpoints of these intervals satisfies the relationship $R\subset\mathcal{U}$.

Proof of Theorem 1.2. (i) $\Rightarrow$ (ii). Suppose that $(q,r]\cap\mathcal{U}=\emptyset$. Then $(q,r+\delta)\cap\mathcal{U}=\emptyset$ for some $\delta>0$ because $\mathcal{U}$ is closed from above [11] and (ii) follows.

(ii) $\Rightarrow$ (iii). If $(q,r]\cap\overline{\mathcal{U}}=\emptyset$, then $(q,r]$ is a subset of a connected component of $(1,\infty)\setminus\overline{\mathcal{U}}$. Moreover, $[q,r)\cap\mathcal{V}$ is a finite subset $\{r_1,\dots,r_m\}$ of $\mathcal{V}\setminus\mathcal{U}$, where $r_1<\cdots<r_m$. Although it is not important in the remainder of the proof, we recall from [5] that $r_2,\dots,r_m\in\mathcal{V}\setminus\overline{\mathcal{U}}$, but r_1 might belong to $\overline{\mathcal{U}}\setminus\mathcal{U}$. We may write

$$\mathcal{U}'_r=\mathcal{U}'_q\cup\bigcup_{\ell=1}^{m}\left(\mathcal{V}'_{r_\ell}\setminus\mathcal{U}'_{r_\ell}\right). \tag{2.1}$$

Fix $\ell\in\{1,\dots,m\}$ and let $x\in\mathcal{V}_{r_\ell}\setminus\mathcal{U}_{r_\ell}$. If the greedy expansion (b_i) of x in base r_ℓ is finite, then $(a_i(x))$ ends with $\alpha_1\alpha_2\dots$. Suppose now that (b_i) is infinite. Since $x\notin\mathcal{U}_{r_\ell}$, there exists an index n such that $b_n>0$, and $\overline{b_{n+1}b_{n+2}\dots}\geqslant\alpha_1\alpha_2\dots$. Since $x\in\mathcal{V}_{r_\ell}$ and $(a_i(x))=(b_i(x))$, the last inequality is in fact an equality. Hence the quasi-greedy expansion $(a_i(x))$ of x in base r_ℓ either ends with (α_i) or $(\overline{\alpha_i})$. Since $(\alpha_i(q))$ is periodic if $q\in\mathcal{V}\setminus\mathcal{U}$ [11], the implication follows from (2.1).

(iii) $\Rightarrow$ (iv) is clear.

(iv) $\Rightarrow$ (i). We prove the contraposition. Suppose that $(q,r]\cap\mathcal{U}\neq\emptyset$. We distinguish between two cases.

If $(q,r)\cap\mathcal{U}\neq\emptyset$, then $|(q,r)\cap\overline{\mathcal{U}}|=2^{\aleph_0}$ because $\overline{\mathcal{U}}$ is a nonempty perfect set [11] and thus each neighborhood of a number $t\in\overline{\mathcal{U}}$ contains uncountably many elements of $\overline{\mathcal{U}}$. Now

$$\mathcal{U}'_r\setminus\mathcal{U}'_q\supset\bigcup_{t\in(q,r)\cap\overline{\mathcal{U}}}\left(\mathcal{V}'_t\setminus\mathcal{U}'_t\right).$$

Hence $\mathcal{U}'_r\setminus\mathcal{U}'_q$ contains an uncountable union of nonempty disjoint sets and is therefore uncountable.

If $(q,r]\cap\mathcal{U}=\{r\}$, then $(q,r)\cap\overline{\mathcal{U}}=\emptyset$. Hence by enlarging q if necessary, we may assume that $q\notin\overline{\mathcal{U}}$. Let

$$\mathcal{W}_r=\left\{x\in\mathcal{U}_r\colon\ \text{the unique expansion of } x \text{ in base } r \text{ belongs to } \mathcal{U}'_q\right\}.$$

We claim that $\mathcal{W}_r$ is closed. The set $\mathcal{W}_r$ is a symmetric subset of J_r, so it suffices to show that $\mathcal{W}_r$ is closed from below. Let $x_i\in\mathcal{W}_r$ $(i\geqslant 1)$, and suppose that $x_i\uparrow x$. Let (c^i_j) be the unique expansion of x_i in base r, and let

$$y_i=\sum_{j=1}^{\infty}\frac{c^i_j}{q^j}.$$

Then the increasing sequence (y_i) converges to some $y\in\mathcal{U}_q$ because $\mathcal{U}_q$ is a compact set. Since $(c^1_j)\leqslant(c^2_j)\leqslant\cdots$, (c^i_j) converges coordinate-wise to the unique expansion (d_j) of y in base q as $i\to\infty$, and

$$x=\sum_{j=1}^{\infty}\frac{d_j}{r^j}.$$

Since $\mathcal{U}'_q\subset\mathcal{U}'_r$ we have $x\in\mathcal{U}_r$, and thus $x\in\mathcal{W}_r$. Now suppose that $\mathcal{U}'_r\setminus\mathcal{U}'_q$ is countable. Then $\mathcal{U}_r\setminus\mathcal{W}_r$ is countable. Note that $\mathcal{W}_r\subsetneq\mathcal{U}_r$ because $\mathcal{W}_r$ is closed and $\mathcal{U}_r$ is not. Let $x\in\mathcal{U}_r\setminus\mathcal{W}_r$. Since $\overline{\mathcal{U}_r}\setminus\mathcal{U}_r$ is a countable dense subset of $\overline{\mathcal{U}_r}$ (Theorem 1.3 in [5]), the latter set is perfect, and each neighborhood of x contains uncountably many elements of $\mathcal{U}_r$ and thus of $\mathcal{W}_r$. This contradicts the fact that $\mathcal{W}_r$ is closed. □

The above result yields new characterizations of $\mathcal{U}$ and $\overline{\mathcal{U}}$:

Corollary 2.3. *A real number $q>1$ belongs to $\mathcal{U}$ if and only if $\mathcal{U}'_r\setminus\mathcal{U}'_q$ is uncountable for each $r>q$ such that $\lceil q\rceil=\lceil r\rceil$.*

Proof. Note that the integers $2,3,\dots$ belong to $\mathcal{U}$. For these values of q the condition in the statement is also vacuously satisfied. Hence we may assume that $q\notin\mathbb{N}$. Suppose that $q\in\mathcal{U}\setminus\mathbb{N}$. For each $r>q$, $(q,r)\cap\mathcal{U}\neq\emptyset$ because elements of $\mathcal{U}\setminus\mathbb{N}$ do not belong to the set of left endpoints of the connected components of $(1,\infty)\setminus\overline{\mathcal{U}}$. Hence if, in addition, $\lceil q\rceil=\lceil r\rceil$, then $\mathcal{U}'_r\setminus\mathcal{U}'_q$ is uncountable by Theorem 1.2. Conversely, if the latter set is uncountable for each $r>q$ such that $\lceil q\rceil=\lceil r\rceil$, then $(q,r]\cap\mathcal{U}\neq\emptyset$ for each $r>q$ by Theorem 1.2, and the result follows because $\mathcal{U}$ is closed from above. □

For a fixed $r>1$, let $\mathcal{F}'_r=\bigcup\mathcal{U}'_q$, where the union runs over all $q<r$ for which $\lceil q\rceil=\lceil r\rceil$.

Corollary 2.4. *Let $r>1$ be a real number. The following statements are equivalent.*

(i) $r \in \overline{\mathcal{U}}$.
(ii) $\mathcal{U}_r' \setminus \mathcal{F}_r'$ *is uncountable.*
(iii) $\mathcal{U}_r' \setminus \mathcal{U}_q'$ *is uncountable for each $q<r$ such that $\lceil q\rceil = \lceil r\rceil$.*
(iv) $\mathcal{U}_r' \setminus \mathcal{U}_q'$ *is nonempty for each $q<r$ such that $\lceil q\rceil = \lceil r\rceil$.*

Proof. It is clear that (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). It remains to show that (i) $\Rightarrow$ (ii) and (iv) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii). Suppose $r \in \overline{\mathcal{U}}$. Let $(q_n)_{n\geqslant 1}$ be an increasing sequence that converges to r, such that $q_n \notin \overline{\mathcal{U}}$ and $\lceil q_n\rceil = \lceil r\rceil$, $n \in \mathbb{N}$. This can be done, since $\overline{\mathcal{U}}$ is a null set. Let

$$\mathcal{W}_r^n = \{x \in \mathcal{U}_r\colon \text{ the unique expansion of } x \text{ in base } r \text{ belongs to } \mathcal{U}_{q_n}'\}$$

and

$$\mathcal{W}_r = \bigcup_{n=1}^{\infty} \mathcal{W}_r^n.$$

It follows from the proof of Theorem 1.2 that $\mathcal{W}_r^n$ is closed for each $n \in \mathbb{N}$. Moreover, $|\mathcal{U}_r \setminus \mathcal{W}_r| = |\mathcal{U}_r' \setminus \mathcal{F}_r'|$. We know that $\overline{\mathcal{U}_r} \setminus \mathcal{U}_r$ is countable. If $\mathcal{U}_r \setminus \mathcal{W}_r$ were countable, then $\overline{\mathcal{U}_r}$ would be an F_σ-set:

$$\overline{\mathcal{U}_r} = \bigcup_{n=1}^{\infty} \mathcal{W}_r^n \cup \Bigl(\bigcup_{x \in \overline{\mathcal{U}_r}\setminus \mathcal{W}_r} \{x\}\Bigr). \tag{2.2}$$

Note that $\overline{\mathcal{U}_r}$ is a complete metric space. By Baire's theorem, one of the sets on the right-hand side of (2.2) has a nonempty interior. Since $\overline{\mathcal{U}_r}$ is a perfect set, each singleton belonging to it is not open. Hence one of the sets $\mathcal{W}_r^n \subset \mathcal{U}_r$ has an interior point. But this contradicts the fact that $\overline{\mathcal{U}_r} \setminus \mathcal{U}_r$ is dense in $\overline{\mathcal{U}_r}$ (Theorem 1.3 in [5]).

(iv) $\Rightarrow$ (i). We prove the contraposition. Suppose $r \notin \overline{\mathcal{U}}$. We can choose $q \in (1,r)$ close enough to r such that $[q,r) \cap \mathcal{V} = \emptyset$. It follows from (2.1) that $\mathcal{U}_q' = \mathcal{U}_r'$. $\square$

Let $q>1$ be a non-integer, and let $\mathcal{G}_q' = \bigcap \mathcal{U}_r'$, where the intersection runs over all $r>q$ for which $\lceil q\rceil = \lceil r\rceil$. In view of Corollary 2.4 it is natural to ask whether the following variant of Corollary 2.3 holds: the number $q>1$ belongs to $\mathcal{U}$ if and only if $\mathcal{G}_q' \setminus \mathcal{U}_q'$ is uncountable. In order to show that this is *not* true, it is sufficient to prove that $\mathcal{G}_q' = \mathcal{V}_q'$, since $\mathcal{V}_q' \setminus \mathcal{U}_q'$ is known to be countable [5]. Let us recall Lemma 3.2 from [11]:

Lemma 2.5. *Let $q>1$ be a non-integer, and let $(\beta_i) = \beta_1\beta_2\ldots$ be the greedy expansion of 1 in base q. For each $n \in \mathbb{N}$, there exists a number $r = r_n > q$ such that the greedy expansion of 1 in base r starts with $\beta_1 \ldots \beta_n$.*

If $q \in \mathcal{U} \setminus \mathbb{N}$, then 1 has an infinite greedy expansion in base q, i.e., $(\alpha_i) = (\alpha_i(q)) = (\beta_i(q))$. If a sequence $(a_i) \in \{0,\ldots,\alpha_1\}^{\mathbb{N}}$ belonged to $\mathcal{G}_q' \setminus \mathcal{V}_q'$, then either there would exist indices n and m, such that

$$a_n < \alpha_1 \quad \text{and} \quad a_{n+1}\ldots a_{n+m} > \alpha_1 \ldots \alpha_m$$

or there would exist indices n and m, such that

$$a_n > 0 \quad \text{and} \quad \overline{a_{n+1}\ldots a_{n+m}} > \alpha_1 \ldots \alpha_m.$$

If $r_m > q$ is the number that is defined in Lemma 2.5, then $\alpha_i(q) = \alpha_i(r_m)$ for $1 \leqslant i \leqslant m$, and thus $(a_i) \notin \mathcal{U}_{r_m}'$ which is a contradiction. On the other hand, $\mathcal{V}_q' \subset \mathcal{U}_r'$ for each $r>q$ such that $\lceil q\rceil = \lceil r\rceil$, and therefore $\mathcal{G}_q' = \mathcal{V}_q'$.

If $q \in (1,\infty) \setminus \mathcal{U}$, then the equality $\mathcal{G}_q' = \mathcal{V}_q'$ easily follows from Theorem 1.7 in [5].

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References

[1] C. Baiocchi, V. Komornik, Greedy and quasi-greedy expansions in non-integer bases, arXiv: math/0710.3001.
[2] K. Dajani, M. de Vries, Invariant densities for random β-expansions, J. Eur. Math. Soc. 9 (1) (2007) 157–176.
[3] Z. Daróczy, I. Kátai, Univoque sequences, Publ. Math. Debrecen 42 (3–4) (1993) 397–407.
[4] Z. Daróczy, I. Kátai, On the structure of univoque numbers, Publ. Math. Debrecen 46 (3–4) (1995) 385–408.
[5] M. de Vries, V. Komornik, Unique expansions of real numbers, arXiv: math/0609708v3.
[6] P. Erdős, M. Horváth, I. Joó, On the uniqueness of the expansions $1 = \sum q^{-n_i}$, Acta Math. Hungar. 58 (3–4) (1991) 333–342.

[7] P. Erdős, I. Joó, V. Komornik, Characterization of the unique expansions $1 = \sum q^{-n_i}$ and related problems, Bull. Soc. Math. France 118 (3) (1990) 377–390.
[8] P. Erdős, I. Joó, V. Komornik, On the number of q-expansions, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. 37 (1994) 109–118.
[9] P. Glendinning, N. Sidorov, Unique representations of real numbers in non-integer bases, Math. Res. Lett. 8 (4) (2001) 535–543.
[10] V. Komornik, P. Loreti, Unique developments in non-integer bases, Amer. Math. Monthly 105 (7) (1998) 636–639.
[11] V. Komornik, P. Loreti, On the topological structure of univoque sets, J. Number Theory 122 (1) (2007) 157–183.
[12] W. Parry, On the β-expansion of real numbers, Acta Math. Acad. Sci. Hungar. 11 (1960) 401–416.
[13] A. Rényi, Representations for real numbers and their ergodic properties, Acta Math. Acad. Sci. Hungar. 8 (1957) 477–493.
[14] N. Sidorov, Almost every number has a continuum of β-expansions, Amer. Math. Monthly 110 (9) (2003) 838–842.