JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **169**, 92–110 (1992)

Finite-Dimensional Compensators for Linear Distributed Control Systems with Delays in Outputs*

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Submitted by George Leitmann

Received November 26, 1990

In this paper we are concerned with linear control systems of infinite dimension with delays in outputs. An asymptotic compensator of finite order is proposed. The compensator provides a control law that stabilizes a wide class of distributed systems. Moreover, the compensator is invariant with respect to small bounded perturbations of the system parameters. © 1992 Academic Press, Inc.

1. INTRODUCTION

The problem of stabilizing a linear invariant control system by a dynamic output feedback has a very extensive literature. Today, the theory for control systems of finite dimension is well established (see O'Reilly [6] and Wonham [12]), while the theory for infinite-dimensional control systems has been the subject of much activity recently. Several methods have been proposed for the design of stabilizing compensators. In particular, for a wide class of infinite-dimensional systems, which includes many systems of practical interest, it is possible to construct a finite order compensator. We refer to the works of Curtain [2], Schumacher [8], and Sakawa [7] and also to their references.

The question of stabilizing a finite-dimensional control system with delay in controls and outputs was discussed by Klamka [4] and Watanabe and Ito [11]. In these papers the existence and implementation of a reduced order observer were studied.

The purpose of this paper is to extend these results to infinite-dimensional control systems with delay in output and bounded observation.

First we begin by reviewing the results for finite-dimensional systems and introducing the notations that we will need in the next sections.

* This work was supported in part by FONDECYT, Project 89-0749, and by DICYT, Project 04-8433 HM.

0022-247X/92 $5.00

Let us consider the following linear invariant systems

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad t \geqslant 0 \tag{1.1}$$

$$y(t) = \Lambda(x_t) \tag{1.2}$$

$$x_0 \in C([-h, 0]; \mathbb{R}^n), \tag{1.3}$$

where $x(t) \in \mathbb{R}^n$ denotes the state of the system, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$ represent the input and output of the system, respectively, and A and B are $n \times n$ and $n \times m$ matrices, respectively. The constant $h > 0$ represents the time delay of the observation and Λ is a bounded linear operator from $C([-h, 0]; \mathbb{R}^n)$ into $\mathbb{R}^p$. Furthermore, as usual in the theory of functional equations, x_t denotes the function defined by $x_t(\Theta) := x(t + \Theta)$, for $-h \leqslant \Theta \leqslant 0$. In the sequel we shall abbreviate our notations and we use I to denote the interval $[-h, 0]$.

We assume that the input $u(t)$ and the output $y(t)$ are known but the state $x(t)$ and the initial function x_0 are unknown.

By the Riesz representation theorem there exists a normalized bounded variation matricial function $\eta: I \to M_{p,n}(\mathbb{R})$ such that

$$\Lambda(\varphi) = \int_{-h}^{0} [d\eta(\Theta)]\, \varphi(\Theta) \tag{1.4}$$

for every $\varphi \in C(I; \mathbb{R}^n)$. We shall denote by C the matrix

$$C = \int_{-h}^{0} [d\eta(\Theta)]\, e^{A\Theta} \tag{1.5}$$

and by $K: L^1(I; \mathbb{R}^n) \to \mathbb{R}^p$ the operator

$$K(\psi) = \int_{-h}^{0} [d\eta(\Theta)]\, e^{A\Theta} \int_{\Theta}^{0} e^{-\tau A} \psi(\tau)\, d\tau. \tag{1.6}$$

It is clear from this expression that K is a bounded linear operator.

Let $u(\cdot)$ be a locally integrable function. Henceforth we will assume that $u(s) = 0$, for $s < 0$. Since the solution of Eq. (1.1) is given by the formula

$$x(t) = e^{At} x(0) + \int_0^t e^{A(t-s)} Bu(s)\, ds, \qquad t \geqslant 0, \tag{1.7}$$

then we obtain for $\Theta \in I$ and $t + \Theta \geqslant 0$,

$$x(t + \Theta) = e^{A\Theta} x(t) - e^{A\Theta} \int_{t+\Theta}^{t} e^{A(t-s)} Bu(s)\, ds \tag{1.8}$$

and substituting (1.8) into (1.2), we obtain that

$$y(t) = Cx(t) - K(B \otimes u_t) + q(t), \qquad t \geq 0, \tag{1.9}$$

where $q(t)$ is a continuous function that depends on x_0 and that it vanishes for $t > h$. Furthermore, in the expression (1.9), and thereafter, we have used the symbol $B \otimes \varphi$ to denote the function defined by $(B \otimes \varphi)(\Theta) = B\varphi(\Theta)$, for $\Theta \in I$.

The dynamical system

$$\dot{z}(t) = Az(t) + GCz(t) - Gy(t) - GK(B \otimes u_t) + Bu(t) \tag{1.10}$$

with the control function given by

$$u(t) = Fz(t) \tag{1.11}$$

was considered in Klamka [4] and Watanabe and Ito [11]. Combining the equations of the control system with the equations (1.10)–(1.11) of the compensator we obtain a non-homogeneous augmented system

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} A & BF \\ -GC & A + GC + BF \end{bmatrix}\begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -Gq(t) \end{bmatrix}.$$

The next result is clear.

THEOREM 1. *If the pair* (A, B) *is stablizable and the pair* (C, A) *is detectable, then there exist matrices* F *and* G *for which the control systems* (1.10)–(1.11) *is an asymptotic compensator of system* (1.1)–(1.2). *Furthermore, if the pair* (A, B) *is controllable and the pair* (C, A) *is observable then it is possible to choose* F *and* G *such that the eigenvalues of the matrix of the augmented system* (1.1)–(1.10) *can be arbitrarily assigned.*

In the next section we generalize these ideas to construct an asymptotic compensator of finite order for a wide class of distributed control systems. In Section 3 we study some robustness properties of this compensator.

The terminology and notations are those generally used in functional analysis. If X and Y denote Banach spaces, we indicate by $\mathscr{B}(X, Y)$ the Banach space of bounded linear operators from X into Y. Moreover, $\mathscr{B}(X)$ denotes $\mathscr{B}(X, X)$. If $A: D(A) \subseteq X \to X$ is a closed linear operator we indicate by $\sigma(A)$ and $\rho(A)$ the spectrum and resolvent set of A, respectively. If $\lambda \in \rho(A)$ then $R(\lambda, A)$ denotes the resolvent operator $(\lambda I - A)^{-1}$. If T is a strongly continuous operator semigroup with infinitesimal generator A then

$$\omega(T) = \lim_{t \to +\infty} \frac{\ln \|T(t)\|}{t}$$

is called the growth bound of T and

$$s(A) = \sup\{\mathrm{Re}(\lambda) : \lambda \in \sigma(A)\}$$

is called the spectral bound of A.

Moreover, in the sequel we consider the product $X_1 \times X_2 \times \cdots \times X_n$ of n Banach spaces $X_1, ..., X_n$ endowed with the norm

$$\|(x_1, ..., x_n)\| = \sum_{i=1}^{n} \|x_i\|_{X_i}.$$

2. Compensators for Distributed Control Systems

In this section we consider a first order infinite-dimensional control system described by equations

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad t \geqslant 0 \tag{2.1}$$

$$y(t) = \Lambda(x_t) \tag{2.2}$$

with states $x(t)$ in a Banach space X, controls $u(t)$ in a Banach space U, and outputs $y(t)$ in a Banach space Y (see Curtain and Pritchard [1]). We assume that A is the infinitesimal generator of a strongly continuous semigroup T defined on the Banach space X, B is a bounded linear operator from the Banach space U into X, and Λ is a bounded linear operator from $C(I; X)$ into Y. The initial function x_0 belongs to $C(I; X)$ and it is unknown. Moreover, we assume that U and Y are finite-dimensional spaces.

Next we introduce some notations. Let $0 < \alpha < \beta$ be constants such that the spectral sets

$$\sigma_1 = \{\lambda \in \sigma(A) : \mathrm{Re}(\lambda) > -\alpha\}$$

and

$$\sigma_2 = \{\lambda \in \sigma(A) : -\beta < \mathrm{Re}(\lambda) \leqslant -\alpha\}$$

are finite.

If we set $\sigma_3 = \{\lambda \in \sigma(A) : \mathrm{Re}(\lambda) \leqslant -\beta\}$ then the space X can be decomposed in the form $X = X_1 \oplus X_2 \oplus X_3$ corresponding to the decomposition of $\sigma(A)$ into the spectral sets σ_1, σ_2, and σ_3 (Nagel [5, Theorem A-III, 3.3.]).

In this decomposition each X_i is an invariant space under A. We will denote by P_i the projection associated to σ_i and by A_i the restriction of A to X_i, $i = 1, 2, 3$. Then each operator A_i is the infinitesimal generator of a strongly continuous semigroup T_i on X_i.

Clearly, we may consider the space $C(I; X_i)$ as a closed subspace of $C(I; X)$, $i = 1, 2, 3$, and, from the decomposition of X it follows that

$$C(I; X) = C(I; X_1) \oplus C(I; X_2) \oplus C(I; X_3).$$

We will represent by Λ_i the restriction of Λ to $C(I; X_i)$, $i = 1, 2, 3$.

Since we want to construct a finite order compensator, we restrict our focus to a class of systems already studied by several authors. Specifically we consider control systems which satisfy the following hypotheses.

Assumption I. (a) There exists a constant $\alpha > 0$ such that the set $\sigma_1 = \{\lambda \in \sigma(A) : \mathrm{Re}(\lambda) > -\alpha\}$ is finite.

(b) For every $\varepsilon > 0$, there exists $\beta > \alpha$ such that the set $\sigma_2 = \{\lambda \in \sigma(A) : -\beta < \mathrm{Re}(\lambda) \leqslant -\alpha\}$ is finite and the following conditions hold:

(b1) The spaces X_1 and X_2 have finite dimension.

(b2) The operator A_3 satisfies the Spectrum Determined Growth Assumption. This means that the growth bound $\omega(T_3)$ and the spectral bound $s(A_3)$ coincide (see Curtain and Pritchard [1] and Triggiani [10]).

(b3) The operator Λ satisfies $\|\Lambda_3\| \leqslant \varepsilon$.

There exists a large class of systems that satisfy the assumptions (a), (b1), and (b2). This class includes the systems described by parabolic partial differential equations on bounded domains and the systems described by retarded functional differential equations. The condition (b3) will allow us to design a compensator in which the component Λ_3 of the observation does not appear (this is known as the spillover observation).

An example is provided by any compact self-adjoint semigroup T defined on an infinite-dimensional Hilbert space X, endowed with an inner product $\langle\ ,\ \rangle$. We know that T has the representation

$$T(t)x = \sum_{k=1}^{\infty} e^{\lambda_k t} < x, x_k > x_k,$$

where $\{x_k : k \in \mathbb{N}\}$ is an orthonormal basis of X and λ_k are real numbers such that $\lambda_k \to -\infty$, as $k \to \infty$. We assume that $(\lambda_k)_k$ is a sequence strictly decreasing. Furthermore, each λ_k is an eigenvalue of A with eigenvector x_k and the spectrum $\sigma(A)$ coincides with the point spectrum $\sigma_p(A) = \{\lambda_k : k \in \mathbb{N}\}$. Let $P: X \to \mathbb{C}^p$ be a bounded linear operator and let us define $\Lambda: C(I; X) \to \mathbb{C}^p$ by

$$\Lambda(\varphi) = \sum_{k=1}^{\infty} \mu_k < \varphi(-h), x_k > Px_k,$$

where $(\mu_k)_k$ is a sequence convergent to zero.

Let us fix $\alpha > 0$. We choose $n_1 \in \mathbb{N}$ such that $\lambda_n < -\alpha$, for $n \geqslant n_1$ (the number n_1 must be chosen so that the hypotheses of Theorem 2 hold). For each $\varepsilon > 0$, let n_2 be a natural number such that $|\mu_k| \leqslant \varepsilon$, for $k \geqslant n_2$. We choose $\beta = |\lambda_{n_2}|$. Then X_1, X_2, and X_3 are the subspaces generated (in the sense of orthonormal basis) by $\{x_1, ..., x_{n_1}\}$, $\{x_{n_1+1}, ..., x_{n_2-1}\}$, and $\{x_k : k \geqslant n_2\}$, respectively. Therefore,

$$\begin{aligned}\|A_3\varphi\|^2 &\leqslant \|P\|^2 \sum_{k \geqslant n_2} |\mu_k|^2 \, |\langle \varphi(-h), x_k \rangle|^2 \\ &\leqslant \varepsilon^2 \, \|P\|^2 \, \|\varphi(-h)\|^2 \\ &\leqslant \varepsilon^2 \, \|P\|^2 \, \|\varphi\|^2\end{aligned}$$

for each $\varphi \in C(I; X_3)$. Thus $\|A_3\| \leqslant \varepsilon \, \|P\|$, which shows that this system satisfies Assumption I.

In the sequel we consider a system that satisfies Assumption I. From these assumptions it follows that the control system (2.1)–(2.2) can be decomposed as

$$\dot{x}_i(t) = A_i x_i(t) + B_i u(t), \tag{2.3}$$

$$y(t) = A_1(x_{1,t}) + A_2(x_{2,t}) + A_3(x_{3,t}) \tag{2.4}$$

$$x_{i,0} = P_i x_0, \tag{2.5}$$

where $x_i(t) = P_i x(t)$; $B_i = P_i B$, and $x_{i,t}$ denotes the function $x_{i,t}(\Theta) = x_i(t + \Theta)$, $i = 1, 2, 3$.

Since the spaces X_1 and Y have finite dimension, according to Eq. (1.9) we may write

$$A_1(x_{1,t}) = C_1 x_1(t) - K_1(B_1 \otimes u_t) + q_1(t), \tag{2.6}$$

where $C_1 \in \mathscr{B}(X_1, Y)$, $K_1 \in \mathscr{B}(L^1(I; X_1); Y)$ and q_1 is a continuous function that vanishes for $t \geqslant h$.

In order to design an asymptotic compensator of system (2.1)–(2.2) we choose two finite-dimensional spaces Z_i, $i = 1, 2$, such that $\dim Z_i = \dim X_i$. Let $R_i : X_i \to Z_i$ be an isomorphism. We introduce the following finite-dimensional systems

$$\begin{aligned}\dot{z}_1(t) = R_1 A_1 R_1^{-1} z_1(t) + G C_1 R_1^{-1} z_1(t) - G y(t) \\ - G K_1(B_1 \otimes u_t) + G A_2(R_2^{-1} \otimes z_{2,t}) + R_1 B_1 u(t)\end{aligned} \tag{2.7}$$

$$\dot{z}_2(t) = R_2 A_2 R_2^{-1} z_2(t) + R_2 B_2 u(t) \tag{2.8}$$

with variables $z_i(t) \in Z_i$, $i = 1, 2$, and initial function $z_{2,0} \in C(I; Z_2)$. We define the control law u in the form

$$u(t) = F \cdot z_1(t). \tag{2.9}$$

In these expressions, $F \in \mathscr{B}(Z_1, U)$ and $G \in \mathscr{B}(Y, Z_1)$ are unknown operators that we must determine so that the closed-loop system, constituted by the control system interconnected with the dynamical observer (2.7)–(2.8) through the control law (2.9), is uniformly stable. If we write $e_i(t)$ for

$$e_i(t) = z_i(t) - R_i x_i(t) \tag{2.10}$$

for $i = 1, 2$, then we obtain the following closed-loop system in the product space $W = X_1 \times Z_1 \times X_2 \times Z_2 \times X_3$

$$\dot{x}_1(t) = (A_1 + B_1 F R_1)\, x_1(t) + B_1 F e_1(t) \tag{2.11}$$

$$\begin{aligned}\dot{e}_1(t) = {} & (R_1 A_1 R_1^{-1} + G C_1 R_1^{-1})\, e_1(t) + G A_2(R_2^{-1} \otimes e_{2,t}) \\ & - G A_3(x_{3,t}) - G q_1(t)\end{aligned} \tag{2.12}$$

$$\dot{x}_2(t) = A_2 x_2(t) + B_2 F R_1 x_1(t) + B_2 F e_1(t) \tag{2.13}$$

$$\dot{e}_2(t) = R_2 A_2 R_2^{-1} e_2(t) \tag{2.14}$$

$$\dot{x}_3(t) = A_3 x_3(t) + B_3 F R_1 x_1(t) + B_3 F e_1(t). \tag{2.15}$$

From (2.14) it is clear that

$$e_2(t) = R_2 e^{A_2 t} R_2^{-1} e_2(0), \qquad t \geqslant 0, \tag{2.16}$$

and since A_2 is a bounded operator then

$$\|e_2(t)\| \leqslant M_2 e^{-\alpha t}\, \|e_2(0)\|, \qquad t \geqslant 0. \tag{2.17}$$

On the other hand, defining the vector $w_1 = (x_1, e_1, x_3)^T$, Eqs. (2.11), (2.12), and (2.15) may be rewritten in the space $W_1 = X_1 \times Z_1 \times X_3$ as

$$\dot{w}_1(t) = D w_1(t) + L(w_{1,t}) + f(t), \qquad t \geqslant 0 \tag{2.18}$$

$$w_{1,0} = \varphi \in C(I; W_1), \tag{2.19}$$

where the operators D and L have the block form

$$D = \begin{bmatrix} A_1 + B_1 F R_1 & B_1 F & 0 \\ 0 & R_1(A_1 + R_1^{-1} G C_1)\, R_1^{-1} & 0 \\ B_3 F R_1 & B_3 F & A_3 \end{bmatrix} \tag{2.20}$$

$$L = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -G A_3 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.21}$$

and

$$f(t) = (0, GA_2(R_2^{-1} \otimes e_{2,t}) - Gq_1(t), 0)^T. \tag{2.22}$$

It is clear that L is a bounded linear operator from $C(I; W_1)$ into W_1 and that D is the infinitesimal generator of a strongly continuous semigroup. We refer to Travis and Webb [9] for the properties of abstract Cauchy problems of type (2.18)–(2.19). Furthermore, in the Appendix we state other stability properties of these systems, which we shall need later.

THEOREM 2. *Suppose that the control system* (2.1)–(2.2) *satisfies Assumption* I. *If the pair* (A_1, B_1) *is controllable and the pair* (C_1, A_1) *is observable then there exist operators* F *and* G *so that the dynamical system* (2.7)–(2.9) *is a finite-dimensional asymptotic compensator of system* (2.1)–(2.2).

Proof. Let $X = X_1 \oplus X_0$ be the decomposition of X associated to the decomposition of $\sigma(A)$ in the sets

$$\sigma_1 = \{\lambda \in \sigma(A) : \mathrm{Re}(\lambda) > -\alpha\},$$
$$\sigma_0 = \{\lambda \in \sigma(A) : \mathrm{Re}(\lambda) \leqslant -\alpha\}.$$

Since the semigroup $T_0 = T\,|_{X_0}$ satisfies the spectrum determined growth assumption then

$$\|T(t)\| \leqslant M_0 e^{-\alpha t}, \qquad t \geqslant 0 \tag{2.23}$$

for some constant $M_0 \geqslant 1$.

On the other hand, as the finite-dimensional pairs (A_1, B_1) and (C_1, A_1) are controllable and observable, respectively, then there exist linear operators F and G for which the spectrum of matrix

$$D_0 = \begin{bmatrix} A_1 + B_1 F R_1 & B_1 F \\ 0 & R_1(A_1 + R_1^{-1} G C_1) R_1^{-1} \end{bmatrix} \tag{2.24}$$

is included in the set $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \leqslant -\mu\}$, for some constant $\mu > \alpha$. Therefore, there exists $N_0 \geqslant 1$ such that

$$\|e^{D_0 t}\| \leqslant N_0 e^{-\mu t}, \qquad t \geqslant 0. \tag{2.25}$$

Let $\varepsilon > 0$ and $\beta > \alpha$ be constants such that

$$\varepsilon < \frac{\alpha \cdot e^{-\alpha h}}{N\,\|G\|}, \tag{2.26}$$

where we introduce the constant

$$N = \max\left\{N_0,\ M_0,\ \frac{N_0 \cdot M_0\ \|[B_3FR_1,\ B_3F]\|}{\mu - \alpha}\right\}. \tag{2.27}$$

Let $X = X_1 \oplus X_2 \oplus X_3$ be the decomposition of X associated to the decomposition of the spectrum set of A in the sets $\sigma_1 = \{\lambda \in \sigma(A):\ \mathrm{Re}(\lambda) > -\alpha\}$, $\sigma_2 = \{\lambda \in \sigma(A):\ -\beta < \mathrm{Re}(\lambda) \leqslant -\alpha\}$, and $\sigma_3 = \{\lambda \in \sigma(A):\ \mathrm{Re}(\lambda) \leqslant -\beta\}$. Now using that $X_3 \subseteq X_0$ and the inequality (2.23) we obtain that

$$\|T_3(t)\| \leqslant M_0 e^{-\alpha t}, \qquad t \geqslant 0. \tag{2.28}$$

On the other hand, from Schumacher [8, Proposition 4.7], it is known that the semigroup $S(t)$ generated by D has the following block triangular form

$$S(t) = \begin{bmatrix} S_0(t) & 0 \\ S_{21}(t) & T_3(t) \end{bmatrix}, \tag{2.29}$$

where the operator $S_{21}(t): X_1 \times Z_1 \to X_3$ is defined by

$$S_{21}(t) \begin{bmatrix} x_1(t) \\ z_1(t) \end{bmatrix} = \int_0^t T_3(t-s)[B_3FR_1,\ B_3F]\, e^{D_0 s} \begin{bmatrix} x_1(t) \\ z_1(t) \end{bmatrix} ds \tag{2.30}$$

and the growth bound $\omega(S) \leqslant -\alpha$. Actually, from (2.29) and (2.30) we obtain that

$$\|S(t)\| \leqslant N e^{-\alpha t}, \qquad t \geqslant 0. \tag{2.31}$$

Since $q_1(t)$ vanishes for $t \geqslant h$ and the function $e_2(t)$ satisfies (2.17) then the function $f(t)$ defined in Eq. (2.22) also verifies an inequality of type

$$\|f(t)\| \leqslant a e^{-\alpha t}, \qquad t \geqslant 0.$$

As $\|L\| \leqslant \|G\| \cdot \varepsilon$, from (2.26), (2.31), and Lemma 1 in the Appendix we obtain that the solution of problem (2.18), (2.19) satisfies an inequality of type

$$\|w_1(t)\| \leqslant b e^{-\gamma t},$$

for some constants $b > 0$ and $\gamma > 0$. Finally, from (2.13) and the variation of constants formula we may write that

$$x_2(t) = e^{A_2 t} x_2(0) + \int_0^t e^{A_2(t-s)}[B_2FR_1\ \ B_2F\ \ 0]\, w_1(s)\, ds$$

which implies that $\|x_2(t)\|$ also decreases exponentially.

3. Robustness of the Compensator

In this section we will study two properties of robustness of the compensator constructed in Theorem 2.

Initially we consider the control systems described by the equation

$$\dot{x}(t) = Ax(t) + Bu(t) + v(t) \tag{3.1}$$

$$y(t) = \Lambda(x_t), \tag{3.2}$$

where A, B, and Λ are operators that satisfy all the assumptions of Section 2 and v is an unknown perturbation.

We say that a locally integrable function $v: [0, +\infty) \to X$ vanishes at infinity in the Stepanov sense if

$$\lim_{t \to +\infty} \int_t^{t+1} \|v(s)\| \, ds = 0.$$

Next we will assume that the perturbation v vanishes at infinity in the Stepanov sense. Under this assumption we can show the following property.

Proposition 1. *If the function $v: [0, +\infty) \to X$ vanishes at infinity in the Stepanov sense and $\mu > 0$ then the function*

$$g(t) = \int_0^t e^{-\mu(t-s)} \|v(s)\| \, ds$$

converges to zero as $t \to +\infty$.

Proof. Let n be the greatest integer $\leqslant t$, then

$$g(t) = e^{-\mu t} \left(\sum_{k=0}^{n-1} \int_k^{k+1} e^{\mu s} \|v(s)\| \, ds + \int_n^t e^{\mu s} \|v(s)\| \, ds \right)$$

$$\leqslant e^{-\mu t} \sum_{k=0}^{n-1} e^{\mu(k+1)} \int_k^{k+1} \|v(s)\| \, ds + \int_n^{n+1} \|v(s)\| \, ds.$$

For each $\varepsilon > 0$, let $n_0 \in \mathbb{N}$ be such that

$$\int_k^{k+1} \|v(s)\| \, ds \leqslant \varepsilon$$

for every $k \geqslant n_0$. Then, if $t > n_0$, we obtain that

$$g(t) \leqslant e^{-\mu t} \sum_{k=0}^{n_0-1} e^{\mu(k+1)} \int_k^{k+1} \|v(s)\| \, ds + \varepsilon \left(\frac{e^{\mu}}{e^{\mu} - 1} + 1 \right)$$

which completes the proof.

THEOREM 3. *Assume that the hypotheses of Theorem 1 hold and suppose that v vanishes at infinity in the Stepanov sense. Then the dynamical system (2.7)–(2.9) is a finite-dimensional asymptotic compensator of system (3.1)–(3.2).*

Proof. The demonstration of this result is similar to the proof of Theorem 2. With the notations introduced in Section 2, we obtain the following closed-loop system

$$\dot{w}_1(t) = Dw_1(t) + L(w_{2,t}) + f(t) + g(t) \tag{3.3}$$

$$\dot{x}_2(t) = A_2 x_2(t) + B_2 F R_1 x_1(t) + B_2 F e_1(t) + P_2 v(t) \tag{3.4}$$

$$\dot{e}_2(t) = R_2 A_2 R_2^{-1} e_2(t) - R_2 P_2 v(t), \tag{3.5}$$

where D, L, and f are defined by (2.20), (2.21), and (2.22), respectively, and

$$g(t) = (P_1 v(t), -R_1 P_1 v(t), P_3 v(t))^T, \tag{3.6}$$

From (2.17) and Proposition 1 we infer that $e_2(t) \to 0$, as $t \to +\infty$. Hence, it follows that $f(t) \to 0$, $t \to \infty$ and since g vanishes at infinity in the Stepanov sense, a slight extension of Lemma 1 in the Appendix allows us to conclude that $w_1(t) \to 0$, as $t \to \infty$. Finally, proceeding in the same way we obtain that $x_2(t)$ converges to zero when $t \to +\infty$.

Next we consider the problem of robustness of the stability of the augmented system that results from the interconnection of the control system (2.1)–(2.2) with the compensator (2.7)–(2.9), when there exist small perturbations of the system parameters A, B, and Λ.

In the sequel we assume that the operators A, B, and Λ satisfy all the conditions established in Section 2. Let us consider $A_r \in \mathscr{B}(X)$, $B_r \in \mathscr{B}(U; X)$, and $\Lambda_r \in \mathscr{B}(C(I; X); Y)$ as perturbations of operators A, B, and Λ, respectively. In order to reduce the notations we shall write $\tilde{A} = A + A_r$, $\tilde{B} = B + B_r$, and $\tilde{\Lambda} = \Lambda + \Lambda_r$.

It is well know that $\tilde{A}$ generates a strongly continuous semigroup (Nagel [5, Theorem A – II, 1.29]) that we will denote by $\tilde{T}$. Consequently the control system

$$\dot{x}(A) = \tilde{A}x(t) + \tilde{B}u(t) \tag{3.7}$$

$$y(t) = \tilde{\Lambda}(x_t) \tag{3.8}$$

is well defined.

Next we will show that if the norms of the perturbations are sufficiently small then the compensator constructed in Theorem 2 also serves to stabilize the perturbed system (3.7)–(3.8).

Let us begin by observing that if $\sigma(A)=\sigma_1\cup\sigma_2\cup\sigma_3$ is the descomposition of the spectrum set of A considered in the proof of Theorem 2 and that if Γ_i, $i=1, 2$, are rectifiable closed curves which contain σ_i in their interior, then there exists $\delta>0$ with the following properties: if $\|A_r\|<\delta$ then the spectrum $\sigma(\tilde{A})$ can be decomposed into parts $\tilde{\sigma}_1$, $\tilde{\sigma}_2$, and $\tilde{\sigma}_3$ where $\tilde{\sigma}_i$, $i=1, 2$, is contained in the interior of Γ_i (Kato [3, Theorem 3.16]). Moreover, in the decomposition $X=\tilde{X}_1\oplus\tilde{X}_1\oplus\tilde{X}_3$ associated to $\tilde{A}$, $\dim(X_i)=\dim(\tilde{X}_i)$, $i=1, 2$, and if we denote by $\tilde{P}_i$, $i=1, 2, 3$, the projections induced by this decomposition of X then $\|P_i-\tilde{P}_i\|\to 0$, as $\delta\to 0$. Therefore, we may conclude that the restrictions $P_i\,|_{\tilde{X}_i}: \tilde{X}_i\to X_i$, $i=1, 2, 3$, which we shall designate with the same symbol P_i, are isomorphisms.

Proceeding as in Section 2, we will denote $\tilde{A}_i$ to the restriction of $\tilde{A}$ to $\tilde{X}_i$ and $\tilde{\Lambda}_i$ to the restriction of $\tilde{\Lambda}$ to $C(I; \tilde{X}_i)$, $i=1, 2, 3$. Also we shall write $\tilde{B}_i$ for $\tilde{P}_i\tilde{B}$, $i=1, 2, 3$, and we define $\tilde{R}_i=R_iP_i$, $i=1, 2$. It is clear that $\tilde{R}_i$ is an isomorphism from $\tilde{X}_i$ on Z_i.

Therefore, the control system (3.7)–(3.8) can be decomposed in the form

$$\dot{x}_i(t)=\tilde{A}_ix_i(t)+\tilde{B}_iu(t), \qquad i=1, 2, 3. \tag{3.9}$$

$$y(t)=\tilde{\Lambda}_1(x_{1,t})+\tilde{\Lambda}_2(x_{2,t})+\tilde{\Lambda}_3(x_{3,t}). \tag{3.10}$$

We introduce the auxiliary system defined by Eqs. (2.7)–(2.8) and we close the system with the feedback control law $u(t)=Fz_1(t)$. Now, we define the states reconstruction errors

$$e_i(t)=z_i(t)-\tilde{R}_ix_i(t), \qquad i=1, 2. \tag{3.11}$$

Proceeding as in Section 2, we obtain the following equations for the composed system

$$\begin{aligned}\dot{\bar{x}}_1(t)&=(A_1+B_1FR_1)\,\bar{x}_1(t)+B_1Fe_1(t)\\&\quad+(P_1\tilde{A}_1P_1^{-1}-A_1)\,\bar{x}_1(t)\\&\quad+P_1(\tilde{B}_1-B_1)\,FR_1\bar{x}_1(t)+P_1(\tilde{B}_1-B_1)\,Fe_1(t)\end{aligned} \tag{3.12}$$

$$\begin{aligned}\dot{e}_1(t)=&\,(R_1A_1R_1^{-1}+GC_1R_1^{-1})\,e_1(t)+G\Lambda_2(R_2^{-1}\otimes e_{2,t})-G\Lambda_3(x_{3,t})\\&+[R_1A_1-R_1P_1\tilde{A}_1P_1^{-1}+GC_1-G\tilde{C}_1P_1^{-1}\\&+R_1P_1(B_1-\tilde{B}_1)\,FR_1]\,\bar{x}_1(t)\\&+R_1P_1(B_1-\tilde{B}_1)\,Fe_1(t)+G[\tilde{Q}(R_1\otimes\bar{x}_{1,t})-Q(R_1\otimes\bar{x}_{1,t})]\\&+G[\tilde{Q}(e_{1,t})-Q(e_{1,t})]+G[\Lambda_2(\bar{x}_{2,t})-\tilde{\Lambda}_2(P_2^{-1}\otimes\bar{x}_{2,t})]\\&+G[\Lambda_3(x_{3,t})-\tilde{\Lambda}_3(x_{3,t})]-G\tilde{q}_1(t)\end{aligned} \tag{3.13}$$

$$\dot{\bar{x}}_2(t) = A_2\bar{x}_2(t) + B_2FR_1\bar{x}_1(t) + B_2Fe_1(t)$$
$$+ P_2(\tilde{B}_2 - B_2)\,FR_1\bar{x}_1(t) + P_2(\tilde{B}_2 - B_2)\,Fe_1(t)$$
$$+ (P_2\tilde{A}_2P_2^{-1} - A_2)\,\bar{x}_2(t) \tag{3.14}$$
$$\dot{e}_2(t) = R_2A_2R_2^{-1}e_2(t) + R_2P_2(B_2 - \tilde{B}_2)\,FR_1\bar{x}_1(t)$$
$$+ R_2P_2(B_2 - \tilde{B}_2)\,Fe_1(t) + R_2(A_2 - P_2\tilde{A}_2P_2^{-1})\,\bar{x}_2(t) \tag{3.15}$$
$$\dot{x}_3(t) = \tilde{A}_3x_3(t) + \tilde{B}_3FR_1\bar{x}_3(t) + \tilde{B}_3Fe_1(t). \tag{3.16}$$

In these equations we have introduced the notations $\bar{x}_i = P_ix_i$, $i = 1, 2$, and we use Q and $\tilde{Q}$ to designate the operators defined on $L^1(I; Z_1)$ by

$$Q(\varphi) = K_1(B_1F \otimes \varphi),$$
$$\tilde{Q}(\varphi) = \tilde{K}_1(\tilde{B}_1F \otimes \varphi).$$

Defining the vector $w = (\bar{x}_1, e_1, \bar{x}_2, e_2, x_3)$, Eqs. (3.12)–(3.16) may be rewritten in the space $W = X_1 \times Z_1 \times X_2 \times Z_2 \times X_3$ as

$$\dot{w}(t) = \tilde{D}w(t) + L(w_t) + D_rw(t) + L_r(w_t) + f(t), \tag{3.17}$$

where the operators $\tilde{D}$ and L have the following block form

$$\tilde{D} = \begin{bmatrix} A_1 + B_1FR_1 & B_1F & 0 & 0 & 0 \\ 0 & (R_1A_1 + GC_1)\,R_1^{-1} & 0 & 0 & 0 \\ B_2FR_1 & B_2F & A_2 & 0 & 0 \\ 0 & 0 & 0 & R_2A_2P_2^{-1} & 0 \\ B_3FR_1 & B_3F & 0 & 0 & \tilde{A}_3 \end{bmatrix} \tag{3.18}$$

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & GA_2(R_2^{-1} \otimes \cdot) & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.19}$$

The function f is given by

$$f(t) = (0, -G\tilde{q}_1(t), 0, 0, 0)^T \tag{3.20}$$

and so D_r as well as L_r are bounded linear operators such that $\|D_r\| \to 0$, as $\delta \to 0$, and $\|L_r\| \to 0$, as $\delta \to 0$.

THEOREM 4. *Assume that the hypotheses of Theorem* 1 *hold and that the semigroup T is continuous in the operator norm on* $(0, +\infty)$. *Then for δ sufficiently small the dynamical system* (2.7)–(2.9) *is a finite-dimensional asymptotic compensator of system* (3.9)–(3.10).

Proof. We must prove that the solutions of Eq. (3.17) converge to zero as $t \to +\infty$. But this is a straightforward consequence of Lemma 2 in the Appendix. In fact, it is clear from (3.18) that $\tilde{D}$ has block triangular form

$$\tilde{D} = \begin{bmatrix} \tilde{D}_{11} & 0 & 0 \\ \tilde{D}_{21} & \tilde{D}_{22} & 0 \\ \tilde{D}_{31} & 0 & \tilde{D}_{33} \end{bmatrix}$$

and that the semigroups generated by $\tilde{D}_{ii}$ are uniformly stable. Moreover, if the semigroup T generated by A is continuous in the operator norm then so is the semigroup $\tilde{T}$ generated by $\tilde{A}$ (Nagel [5, Theorem A-II, 1.30]), which in turn implies that the semigroup generated by $\tilde{D}_{33} = \tilde{A}_3$ is continuous in the operator norm. Since X_1 and X_2 are finite-dimensional spaces we may conclude that the semigroups $\tilde{S}_i$ generated by $\tilde{D}_{ii}$, $i = 1, 2, 3$, are continuous in the operator norm on $(0, +\infty)$. We complete the proof observing that

$$L_{12,\lambda} R(\lambda, D_{22})\, D_{21} = \begin{bmatrix} 0 & 0 \\ 0 & G_2(R_2^{-1} \otimes \cdot) \end{bmatrix} \begin{bmatrix} R(\lambda, A_2) & 0 \\ 0 & R_2 R(\lambda, A_2)\, R_2^{-1} \end{bmatrix} \times \begin{bmatrix} B_2 F R_1 & B_2 F \\ 0 & 0 \end{bmatrix} = 0.$$

4. CONCLUSION

In this paper a method to design a dynamical compensator for a linear distributed control system with retarded bounded observation is studied. The method is an extension of the design previously proposed by Klamka [4] and Watanabe and Ito [11] for linear finite-dimensional control systems. It is proved that for a large class of distributed control systems, there exists a compensator of finite order. This class of systems has been previously studied by Schumacher [8], Curtain [2], and Sakawa [7]. Moreover, the proposed compensator is invariant with respect to small parameter perturbations.

Appendix

In this Appendix we collect some stability properties of abstract retarded differential equations.

Let T be a strongly continuous semigroup defined on a Banach space X and let A be its infinitesimal generator. Let L be a bounded linear operator from $Y = C([-h, 0]; X)$, $h > 0$, into X and let f be a continuous function with values in X. The initial value problem

$$\dot{x}(t) = Ax(t) + L(x_t) + f(t), \qquad t \geqslant 0 \tag{A.1}$$

$$x_0 = \varphi \tag{A.2}$$

has been studied by several authors (see Nagel [5] and Travis and Webb [9]). It is known that there exists a unique continuous solution $x(t)$ of (A.1)–(A.2) on $[0, \infty)$, in the sense that x solves

$$x(t) = T(t)\,\varphi(0) + \int_0^t T(t-s)\,L(x_s)\,ds + \int_0^t T(t-s)\,f(s)\,ds \tag{A.3}$$

for $t \geqslant 0$ and $x_0 = \varphi$.

If $x(t, \varphi)$ denotes the solution to this equation with $f \equiv 0$ (homogeneous equation), then the operator $U(t)\colon Y \to Y$, $U(t)\varphi = x_t(\cdot, \varphi)$ defines a strongly continuous semigroup on Y. We will denote by A_U the infinitesimal generator of this semigroup.

Lemma 1. *Suppose that there exist constants* $N \geqslant 1$ *and* $\mu > 0$ *such that*

$$\|T(t)\| \leqslant Ne^{-\mu t}, \qquad t \geqslant 0 \tag{A.4}$$

and

$$\nu = \mu - N\,\|L\|\,e^{\mu h} > 0. \tag{A.5}$$

If f *is a continuous function that vanishes at infinity then* $x(t, \varphi) \to 0$, $t \to \infty$, *for every* $\varphi \in C([-h, 0]; X)$.

Furthermore, if $\|f(t)\| \leqslant ae^{-\alpha t}$, $t \geqslant 0$, *for some constants* $a \geqslant 0$ *and* $\alpha > 0$ *then*

$$\|x(t, \varphi)\| \leqslant Ne^{\mu h}\,\|\varphi\| e^{-\nu t} + aNe^{\mu h}e^{-\beta t} \tag{A.6}$$

for every constant β *such that* $0 < \beta < \min\{\alpha, \nu\}$.

Proof. This result is a straightforward consequence of the Gronwall–Bellman theorem. In fact, it follows from (A.3) that

$$x(t) = T(t)\,\varphi(0) + \int_0^t T(t-s)\,L(x_s)\,ds + \int_0^t T(t-s)\,f(s)\,ds,$$

where we shall abbreviate $x(t)$ instead of $x(t, \varphi)$. Hence,

$$\|x(t)\| \leqslant Ne^{-\mu t} \|\varphi\| + N \|L\| \int_0^t e^{-\mu(t-s)} \|x_s\| \, ds + N \int_0^t e^{-\mu(t-s)} \|f(s)\| \, ds$$

for every $t \geqslant 0$. Since $\|x(t+\Theta)\| \leqslant \|\varphi\|$ for all $t+\Theta \leqslant 0$, from the last inequality we infer that

$$\|x_t\| = \sup_{\Theta \in I} \|x(t+\Theta)\|$$

$$= Ne^{\mu h - \mu t} \left[\|\varphi\| + \|L\| \int_0^t e^{\mu s} \|x_s\| \, ds + \int_0^t e^{\mu s} \|f(s)\| \, ds \right]$$

and the Gronwall–Bellman theorem implies that

$$\|x_t\| \leqslant Ne^{\mu h} \|\varphi\| \, e^{(v-\mu)t} + Ne^{\mu h} \int_0^t e^{-\mu(t-s)} \|f(s)\| \, ds$$

$$+ vNe^{\mu h} \int_0^t e^{(v-\mu)(t-s)} \int_0^s e^{-\mu(s-\tau)} \|f(\tau)\| \, d\tau \, ds. \tag{A.7}$$

On the other hand, if g is a scalar function which vanishes at infinity it is well known that the function v defined by

$$v(t) = \int_0^t e^{-\varepsilon(t-s)} g(s) \, ds$$

also vanishes at infinity, for every $\varepsilon > 0$. Therefore the second and third terms of (A.7) converge to zero, as $t \to +\infty$, which proves the first assertion. Finally if $\|f(t)\| \leqslant ae^{-\alpha t}$ and we substitute this inequality into (A.7) we obtain (A.6).

LEMMA 2. *If the semigroup T is uniformly continuous on $(0, +\infty)$ then U is a uniformly continuous semigroup on $(h, +\infty)$.*

Proof. Arguing as in the proof of Proposition 2.1 in Travis and Webb [9], for each $\varphi \in Y$ we consider the sequence of functions $(u^n)_{n \geqslant 0}$ defined by $u^n(\Theta) = \varphi(\Theta)$ for $-h \leqslant \Theta \leqslant 0$ and $n \geqslant 0$, $u^0(t) = T(t)\,\varphi(0)$ for $t \geqslant 0$ and

$$u^n(t) = T(t)\,\varphi(0) + \int_0^t T(t-s)\, \Lambda(u_s^{n-1}) \, ds. \tag{A.8}$$

for $t \geqslant 0$ and $n \in \mathbb{N}$.

Let the operator functions U_n: $[0, \infty) \to \mathscr{B}(Y)$ be defined by

$$U_n(t)\varphi = u_t^n, \qquad t \geqslant 0,\ \varphi \in Y, \text{ and } n \geqslant 0. \tag{A.9}$$

It is easy to verify that U_n are strongly continuous and continuous in the operator norm on $(h, +\infty)$. Moreover the sequence $(U_n(t))_n$ converges in the operator norm uniformly on bounded intervals to $U(t)$. This proves the assertion

For $\lambda \in \mathbb{C}$, we use the notation

$$L_\lambda x = L(e^{\lambda \Theta} \cdot x), \qquad x \in X. \tag{A.10}$$

Clearly L_λ is a bounded linear operator on X and $\|L_\lambda\| \leqslant \|L\| \, e^{|\mathrm{Re}(\lambda)| \cdot h}$.

When the space X has product form $X = X_1 \times X_2 \times X_3$ and the operators A and L have an approximately triangular block matrix representation, we obtain the following result of stability.

LEMMA 3. *Suppose that A_{ii} are infinitesimal generators of semigroups T_i on the Banach spaces X_i, $i = 1, 2, 3$, respectively, and that $A_{21}: X_1 \to X_2$, $A_{31}: X_1 \to X_3$, and $L_{12}: C([-h, 0]; X_2) \to X_1$ are bounded linear operators such that the following conditions hold*:

(a) *The semigroups T_i, $i = 1, 2, 3$, are uniformly stable and continuous in the operator norm on $(0, +\infty)$.*

(b) *Let*

$$A = \begin{bmatrix} A_{11} & 0 & 0 \\ A_{21} & A_{22} & 0 \\ A_{31} & 0 & A_{33} \end{bmatrix}, \qquad L = \begin{bmatrix} 0 & L_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(c) *For every $\lambda \in \rho(A_{22})$,*

$$L_{12,\lambda} R(\lambda, A_{22}) A_{21} = 0. \tag{A.11}$$

(d) *The function f: $[0, \infty) \to X$ is continuous and it vanishes for $t > h$.*

Then there exists $\varepsilon > 0$ such that the solution to the problem

$$\dot{x}(t) = Ax(t) + A_r x(t) + L(x_t) + L_r(x_t) + f(t) \tag{A.12}$$

$$x_0 = \varphi \tag{A.13}$$

converges to zero, for every $A_r \in \mathscr{B}(X)$ and $L_r \in \mathscr{B}(Y, X)$ such that $\|A_r\| < \varepsilon$ and $\|L_r\| < \varepsilon$.

Proof. First we observe that by virtue of the hypotheses and Theorem A-II, 1.30 in Nagel [5] the semigroup generated by $A + A_r$ is continuous in the operator norm on $(0, +\infty)$. Applying Lemma 2 we infer that the semigroup U associated to Eq. (A.12) is continuous in the operator norm on $(h, +\infty)$. Therefore U satisfies the spectrum determined growth assumption (Nagel [5, Corollary A-III, 6.6]). Thus in order to show that U is uniformly stable it is sufficient to prove that $\sup\{\text{Re}(\lambda) : \lambda \in \sigma(A_U)\} < 0$.

Since the semigroups T_i, $i = 1, 2, 3$, are uniformly stable there exist constants $\alpha_i > 0$ and constants $N_i \geqslant 1$, $i = 1, 2, 3$, such that

$$\|T_i(t)\| \leqslant N_i e^{-\alpha_i t}, \qquad t \geqslant 0. \tag{A.14}$$

Let $\alpha = \min\{\alpha_1, \alpha_2, \alpha_3\}$. We will prove that for $0 < \delta < \alpha$ there exist $\varepsilon > 0$ so that $\|A_r\| < \varepsilon$ and $\|L_r\| < \varepsilon$ imply that $\sup \text{Re}\, \sigma(A_U) \leqslant -\alpha + \delta$. In fact, it follows from Nagel [5, Proposition B-IV, 3.4] that $\lambda \in \sigma(A_U)$ if, and only if, $\lambda \in \sigma(A + A_r + L_\lambda + L_{r,\lambda})$.

For $\lambda \in \mathbb{C}$, $\text{Re}(\lambda) > -\alpha + \delta$, we define the operators

$$D(\lambda) = \text{Diag}(\lambda I - A_{11}, \lambda I - A_{22}, \lambda I - A_{33}) \tag{A.15}$$

and

$$E(\lambda) = \begin{bmatrix} I & -L_{12,\lambda} R(\lambda, A_{22}) & 0 \\ -A_{21} R(\lambda, A_{11}) & I & 0 \\ -A_{31} R(\lambda, A_{11}) & 0 & I \end{bmatrix}. \tag{A.16}$$

It is clear from the semigroup properties that $E(\lambda)$ is well defined. Moreover, using condition (c) it follows that $E(\lambda)$ has bounded inverse and

$$E(\lambda)^{-1} = \begin{bmatrix} I & L_{12,\lambda} R(\lambda, A_{22}) & 0 \\ A_{21} R(\lambda, A_{11}) & I + A_{21} R(\lambda, A_{11}) L_{12,\lambda} R(\lambda, A_{23}) & 0 \\ A_{31} R(\lambda, A_{11}) & A_{31} R(\lambda, A_{11}) L_{12,\lambda} R(\lambda, A_{21}) & I \end{bmatrix}. \tag{A.17}$$

Therefore, we may write

$$\begin{aligned} \lambda I - A - A_r - L_\lambda - L_{r,\lambda} &= E(\lambda)\, D(\lambda) - A_r - L_{r,\lambda} \\ &= E(\lambda)[I - E(\lambda)^{-1} (A_r + L_{r,\lambda})\, D(\lambda)^{-1}]\, D(\lambda). \end{aligned} \tag{A.18}$$

But, from the semigroup theory we deduce that the operator functions $\lambda \to D(\lambda)^{-1}$ and $\lambda \to E(\lambda)^{-1}$ are bounded for $\text{Re}(\lambda) \geqslant -\alpha + \delta$. Hence we infer that there exists $\varepsilon > 0$ such that $\|A_r\| < \varepsilon$ and $\|L_r\| < \varepsilon$ imply that $\|E(\lambda)^{-1} (A_r + L_{r,\lambda})\ D(\lambda)^{-1}\| < 1$. Thus $\lambda \notin \sigma(A_U)$ and $\sigma(A_U) \subseteq \{\lambda \in \mathbb{C} : \text{Re}(\lambda) \leqslant -\alpha + \delta\}$.

Now, since the function f vanishes for $t > h$, Eq. (A.12) is reduced to a homogeneous equation for $t > h$. In view of the fact that U is uniformly stable then the solution to (A.12)–(A.13) converges to zero as $t \to +\infty$. This completes the proof.

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