Simple and prime crossed products of C*-algebras by compact quantum group coactions

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Abstract

Let $G = (A, \Delta)$ be a compact quantum group and $\delta$ a coaction of $G$ on a C*-algebra $B$. We give necessary and sufficient conditions for the simplicity and primeness of the crossed product $B \times_\delta G$ in terms of certain fixed point algebras.

Keywords: Crossed product; Compact quantum group

1. Introduction

A coaction of a compact quantum group $G$ on a C*-algebra $B$ gives rise to the reduced crossed product $B \times_\delta G$. The aim of this paper is to study the ideal structure of this crossed product. We give necessary and sufficient conditions for the simplicity and primeness of the crossed product obtained by compact quantum group coactions.

The problem of the ideal structure of the crossed product has been extensively studied in the last decades. In his fundamental paper [3], Connes establishes a connection between the ideal structure of the fixed points of compact, abelian group actions on von Neumann algebras and that of the crossed product. With $\alpha$ an action of an abelian group $G$ on a von Neumann algebra $A$, he
obtained that the crossed product $A \times_\alpha G$ is a factor if and only if the fixed point algebra $A^\alpha$ is a factor and the Arveson spectrum, $\text{Sp}(\alpha)$ is equal to the dual group $\hat{G}$.

Subsequently, these topics have been considered in the framework of $C^*$-algebras by Olesen, Pedersen, Elliott and others for abelian groups (see for instance [10]), Rieffel for finite, non-abelian groups in [13], and Landstad in [7], Peligrad in [11] for compact, non-abelian groups. With $\alpha$ an action of a compact, not necessary abelian group $G$ on a $C^*$-algebra $A$, in [11] Peligrad finds necessary and sufficient conditions for the simplicity or primeness of the crossed product $A \times_\alpha G$ in terms of the Arveson Spectrum and the simplicity or primeness of all the fixed point algebras associated to the action, $(A \otimes \mathcal{B}(H_\pi))^{\alpha \otimes \text{Ad} \pi}$, for all irreducible representations $\pi$ of $G$.

He also shows that in the case of a compact, non-abelian group, it is not enough anymore to consider only the simplicity or primeness of the fixed point algebra $A^\alpha$, as in the abelian case. These results have been further extended to the case of actions of finite dimensional Hopf $*$-algebras on $C^*$-algebras by Szymanski and Peligrad in [14].

In the case of compact quantum groups, the simplicity of the crossed product was studied in the ergodic case by Landstad in [8] (see also [2]). In this paper we find conditions for the simplicity or primeness of the crossed product (obtained by compact quantum group coactions) in terms of the simplicity or primeness of all the fixed point algebras $(B \otimes \mathcal{M}_d)^{\delta \otimes \text{Ad}(u^\alpha)}$, associated to the coaction $\delta$.

The structure of this paper is as follows: in Section 2 we recall the necessary preliminaries and we set some notations. In Section 3 we construct certain subspaces of the crossed product associated to the irreducible representations of $G$, which we denote by $I_\alpha$, and discuss their properties. These subspaces of the crossed product constructed here are the quantum version of the subspaces of spherical functions introduced by Landstad in [7]. In Section 4 we state and prove the main results on the simplicity and primeness of the crossed product $B \times_\delta G$.

2. Preliminaries

2.1. Compact quantum groups

We will first recall the definition of a compact quantum group [17,18].

**Definition 2.1.** Let $A$ be a unital $C^*$-algebra and $\Delta : A \to A \otimes A$ a $*$-homomorphism such that

(a) $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$, where $\iota$ is the identity map, and

(b) $\Delta(A)(1 \otimes A) = A \otimes A$ and $\Delta(A)(A \otimes 1) = A \otimes A$.

Then the pair $(A, \Delta)$ is called a compact quantum group.

We recall [18, p. 12] that $u$ is a unitary representation of a compact quantum group $G$ on a Hilbert space $H$ if $u$ is a unitary element in the multiplier algebra $\mathcal{M}(K(H) \otimes A)$ such that $(\iota \otimes \Delta)u = u_{12}u_{13}$. Furthermore, a unitary representation is called irreducible if the operators intertwining $u$ are scalar multiples of the identity. Note that if $u$ is an irreducible unitary representation then the Hilbert space on which it acts is finite dimensional.

Let $\hat{G}$ denote the dual of $G$, i.e. the set of all unitary equivalence classes of irreducible representations of $G$ [17,18]. For each $\alpha \in \hat{G}$, denote by $u^\alpha$ a representative of each class and by $(u^\alpha_{ij})_{1 \leq i,j \leq d_\alpha} \in \mathcal{M}_{d_\alpha}(A)$ the matrix form of $u^\alpha$. Set $\chi_\alpha = \sum_{i=1}^{d_\alpha} u^\alpha_{ii}$. Then $\chi_\alpha$ is called the character of $\alpha$. By [17,18] there is a unique invertible operator $F_\alpha \in \mathcal{B}(H_\alpha)$, where $H_\alpha$ is the finite...
dimensional Hilbert space of the representation $u^\alpha$, that intertwines $u^\alpha$ with its double contra-gradient representation $(u^\alpha)^{cc}$ such that $\text{tr}(F^\alpha) = \text{tr}(F^{-1}_\alpha)$. Set $M_\alpha = \text{tr}(F^\alpha)$.

Using an orthonormal basis in $H_\alpha$, the Hilbert space of the representation $u^\alpha$, the operator $F^\alpha$ can be represented as a $(d_\alpha \times d_\alpha)$ matrix, $F^\alpha = (f^\alpha_{ij})_{1 \leq i,j \leq d_\alpha}$.

With $h$ the Haar state on $G$ (see [17, 18]), let $(H_h, \pi_h)$ be the GNS representation of $A$ associated to $h$, and denote by $\xi_h$ its cyclic vector. We will denote by $v_r$ the right regular representation of $G = (A, \Delta)$ (see [9, 18]), which is defined as follows: $v_r$ is a unitary operator in the multiplier algebra $\mathcal{M}(K(H_h) \otimes A)$ defined by

$$v_r(\pi_h(a)\xi_h \otimes \eta) = (\pi_h \otimes \pi_h)(\Delta(a))(\xi_h \otimes \eta)$$

for all $a \in A$ and $\eta \in H$.

We denote by $v_l$ the left regular representation of the compact quantum group $G = (A, \Delta)$, i.e. $v_l \in \mathcal{M}(K(H_h) \otimes A)$ is a unitary such that

$$v_l^*(\pi_h(a)\xi_h \otimes \eta) = (\pi_h \otimes \pi_h)(\Delta^{op}(a))(\xi_h \otimes \eta),$$

for all $a \in A$, $\eta \in H_h$.

Here $\Delta^{op} = \sigma \circ \Delta$ denotes the co-multiplication of $G^{op} = (A, \Delta^{op})$, the opposite group of $G$, and $\sigma : A \otimes A \to A \otimes A$, $\sigma(a_1 \otimes a_2) = a_2 \otimes a_1$, $\forall a_1, a_2 \in A$ denotes the flip operation (see for instance [9]).

We will use the following notations (see [17, 18]):

$$a \star \xi = (\xi \otimes \iota)(\Delta(a)); \quad \xi \star a = (\iota \otimes \xi)(\Delta(a)),$$

$$(h \cdot a)(b) = h(ba); \quad (a \cdot h)(b) = h(ab),$$

$$A = \text{lin}\{u^\alpha_{ij} \mid \alpha \in \hat{G}, i, j = 1, \ldots, d_\alpha\}$$

for all $a, b \in A$ and for all linear functionals $\xi$ on $A$.

For each $\alpha \in \hat{G}$ denote:

$$A_\alpha = \text{lin}\{u^\alpha_{ij} \mid 1 \leq i, j \leq d_\alpha\},$$

$$h_\alpha = M_\alpha h \cdot (\chi_\alpha \star f_1)^*; \quad a_\alpha = M_\alpha (\chi_\alpha \star f_1)^* \quad (1)$$

**Remark 2.2.** Recall [18, Theorem 3.1] that the Fourier transform is defined by:

$$\widehat{a} = \mathcal{F}_{v_r}(a) = (id \otimes h)a)(v_r^*), \quad \forall a \in A.$$  

Denote by $\widehat{A}$ the norm closure of the set of all operators of the form $\mathcal{F}_{v_r}(a)$, where $a \in A$.

As noticed by Tomatsu in [15], one can choose a representative of the class $\alpha$ such that $F_\alpha$ is a diagonal matrix. Choose such a representative, let $i, j \in \{1, \ldots, d_\alpha\}$ and define

$$E^\alpha_{ij} = \frac{1}{f_{-1}(u_{ii})}M_\alpha\mathcal{F}_{v_r}(u^\alpha_{ij})^*$$

Then $E^\alpha_{ij}$, $i, j = 1, \ldots, d_\alpha$ are operators on $\pi_h(A_\alpha)\xi_h$. 

Remark 2.3. Using standard computations one can show that the operators $E_{ij}^\alpha$ satisfy the following properties:

1. $E_{ij}^\alpha u_{kl}^\alpha \xi_h = \delta_{jl} u_{ki}^\alpha \xi_h$, for all $i, j, k, l \in \{1, \ldots, d_\alpha\}$;
2. $E_{ij}^\alpha E_{kl}^\alpha = \delta_{jk} E_{il}^\alpha$, for all $i, j, k, l \in \{1, \ldots, d_\alpha\}$.

Using the remark above, $E_{ij}^\alpha$ can be represented as a $(d_\alpha^2 \times d_\alpha^2)$ block diagonal matrix with equal diagonal entries given by $m_{ij}$ (where $m_{ij}$ is the $(d_\alpha \times d_\alpha)$ matrix with 1, as the only non-zero entry, in position $(i, j)$).

For any $\alpha \in \hat{G}$ consider

$$p_\alpha = (id \otimes h_\alpha)(v_r^*).$$

Note that $p_\alpha = F_{v_r}(a_\alpha)$ and that $p_\alpha$ are pairwise orthogonal central projections in $\hat{A}$. Furthermore, $p_\alpha$ is the orthogonal projection onto the finite dimensional subspace $A_\alpha \xi_h \subset H_h$.

2.2. Coactions of compact quantum groups on $C^*$-algebras

Definition 2.4. (See [1].) Let $G = (A, \Delta)$ be a compact quantum group, let $B$ be a $C^*$-algebra and let $\delta : B \to B \otimes A$ be a one-to-one $*$-homomorphism of $B$ into the minimal tensor product $B \otimes A$. If:

(a) $(\delta \otimes \iota)\delta = (\iota \otimes \Delta)\delta$ and
(b) $\delta(B)(1 \otimes A) = B \otimes A$

then $\delta$ is called a coaction of $G$ on $B$.

For any $\alpha \in \hat{G}$ consider $P_\alpha(x) = (id \otimes h_\alpha)(\delta(x))$. Note that $P_\alpha : B \to B$ is a linear map which becomes a conditional expectation from $B$ to $B_\Delta = \{x \in B \mid \delta(x) = x \otimes 1\}$, in the case when $\alpha$ is the trivial representation. The spectral subspace corresponding to $\alpha \in \hat{G}$ in the quantum case has been defined in [2] to be the closed subspace $P_\alpha(B)$ of $B$, and it was denoted by $B_\alpha$. Recall that the algebraic direct sum of the spectral subspaces is a dense $*$-subalgebra of $B$ (see for instance [2] or [5, Lemma 2.3]).

Let $\delta$ be a coaction of $G$ on a $C^*$-algebra $B$ and let $u$ be a representation of $G$ on a Hilbert space $H$. It is straightforward to check that the following is a coaction of $G$ on $B \otimes K(H)$:

$$\delta_u(a \otimes k) = u_{23}\delta(a)_{13}(1 \otimes k \otimes 1)(u_{23}^*).$$

Denote by $B \times_\delta G$ the crossed product between $B$ and $G$ via a coaction $\delta$, as defined in [1]. Recall that $B \times_\delta G$ is the $C^*$-algebra generated by elements of the form

$$(\pi_u \otimes \pi_h)(\delta(b))(1 \otimes F_{v_r}(a))$$

with $a \in A$ and $b \in B$, where $\pi_u : B \to B(H_u)$ is the universal representation of the $C^*$-algebra $B$, and $\pi_h : A \to B(H_h)$ is the GNS representation of $A$ associated to the Haar state $h$. 
Remark that the crossed product is a fixed point subalgebra of $B \otimes K(H_h)$ (see [2, Remark 20]), more precisely:

$$B \times_\delta G = (B \otimes K(H_h))^{\delta_\iota}$$

(4)

3. Subspaces of the crossed product

Let $\alpha_1, \alpha_2 \in \hat{G}$ be two classes of irreducible representations of the compact quantum group $G$. Using Eq. (2), note that $1 \otimes p_\alpha \in M(B \times_\delta G)$, for each $\alpha \in \hat{G}$. We make the following notations:

$$S_{\alpha_1, \alpha_2} = (1 \otimes p_{\alpha_1})(B \times_\delta G)(1 \otimes p_{\alpha_2}).$$

When $\alpha = \alpha_1 = \alpha_2$, we define $S_\alpha = S_{\alpha, \alpha}$.

Then $S_{\alpha_1, \alpha_2}$ are closed subspaces of the crossed product. We collect some of their properties in the following lemma:

**Lemma 3.1.**

1. For all $\alpha \in \hat{G}$, $S_\alpha$ is a hereditary $C^*$-subalgebra of $B \times_\delta G$;
2. If $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \hat{G}$ and $\alpha_2 \neq \alpha_3$ then $S_{\alpha_1, \alpha_2}S_{\alpha_3, \alpha_4} = 0$;
3. If $\alpha_1, \alpha_2 \in \hat{G}$ then $(S_{\alpha_1, \alpha_2})^* = S_{\alpha_2, \alpha_1}$;
4. If $\alpha_1, \alpha_2 \in \hat{G}$ then $S_{\alpha_1, \alpha_2}S_{\alpha_2, \alpha_1}$ is a two-sided ideal of $S_{\alpha_1}$.

**Proof.** (1) follows using the following property: If $B$ is a $C^*$-subalgebra of $A$ then $B$ is hereditary if and only if $bab' \in B$ for all $b, b' \in B$ and $a \in A$.

Properties (2), (3), and (4) follow from the definitions and the relation $p_\alpha p_\alpha = \delta_\alpha p_\alpha$. This relation holds since $p_\alpha$ and $p_\alpha$ are projections onto orthogonal spaces if $\alpha_1 \neq \alpha_2$. □

Let $\iota$ be the trivial representation of the compact quantum group $G$. The next proposition establishes a connection between the spectral subspaces $B_\alpha$ and the subspaces of the crossed product, $S_{\alpha_1, \alpha_2}$ introduced above.

**Proposition 3.2.** For all $\alpha \in \hat{G}$, $S_{\alpha, \iota} \neq 0$ if and only if $B_\alpha \neq 0$.

**Proof.** Using straightforward computations and the fact that $p_\iota$ is a central projection in $\hat{A}$, one can check that $p_\iota F_{v_\iota}(a)^* p_\iota c \xi_h = h(a)\delta_\alpha p_\iota c \xi_h$, for all $a, c \in A$. Therefore $A p_\iota = \mathbb{C} p_\iota$. This implies that $S_{\alpha, \iota} \neq 0$ if and only if there exists $b \in B$ such that $(1 \otimes p_\alpha)\delta(b)(1 \otimes p_\iota) \neq 0$. On the other hand, using the definition of $B_\alpha$, it is clear that $B_\alpha \neq 0$ if and only if there exists a non-zero $b \in B$ such that $\delta(b) \in B \otimes A_\alpha$. Using that $p_\alpha$ is an orthogonal projection onto $A_\alpha \xi_h \in H_h$ the proof is completed. □

Although we are not going to use it in this paper, it is easy to see that the map $b \rightarrow (1 \otimes p_\alpha)\delta(b)(1 \otimes p_\iota)$ is a Banach space isomorphism between $B_\alpha$ and $S_{\alpha, \iota}$.

We now make the following notation:

$$ad(v_\iota) : B \times_\delta G \rightarrow (B \times_\delta G) \otimes A,$$

$$ad(v_\iota)(z) = (1 \otimes v_\iota)(z \otimes 1)(1 \otimes v_\iota^*), \quad \forall z \in B \times_\delta G.$$
We are going to show next that $ad(v_r)$ is a coaction of $G$ on the crossed product. In fact, this coaction corresponds at the level of compact groups to the following action considered by Landstad in [7] and Peligrad in [11]:

$$\beta : G \to \text{Aut}(B \times_\delta G)$$

defined by

$$\beta_k(f)(g) = \alpha_k(f(k^{-1}g)),$$

for all $k \in K$, $g \in G$, where $f : G \to B$ is a continuous function. The fixed points of the action $\beta$ in the crossed product are called $K$-central elements in [7,11], where $K$ is the compact group under consideration.

**Lemma 3.3.** The application $ad(v_r)$ defined above is a coaction of the compact quantum group $G$ on the crossed product $B \times_\delta G$.

**Proof.** Indeed, it is straightforward to check that $ad(v_r)$ is a unital $*$-homomorphism and that

$$\left( ad(v_r) \otimes id \right) (ad(v_r)(z)) = (id \otimes \Delta)(ad(v_r)(z))$$

for all $z \in B \times_\delta G$.

The second condition in Definition 2.4,

$$ad(v_r)(B \times_\delta G)(1 \otimes A) = (B \times_\delta G) \otimes A$$

follows from [16], Section 2, remarks and using that $v_r$ is the sum of all irreducible representations of $G$. \qed

Let now $(B \times_\delta G)^{ad(v_r)}$ be the fixed point subalgebra of the crossed product corresponding to the coaction $ad(v_r)$, i.e.

$$(B \times_\delta G)^{ad(v_r)} = \{ z \in B \times_\delta G \mid (1 \otimes v_r)(z \otimes 1)(1 \otimes v_r^*) = z \otimes 1 \}.$$ 

**Lemma 3.4.** The fixed point algebra above is the relative commutant of $1 \otimes \hat{A}$ in the crossed product. That is,

$$(B \times_\delta G)^{ad(v_r)} = (1 \otimes \hat{A})' \cap (B \times_\delta G).$$

**Proof.** We have $z \in (B \times_\delta G)^{ad(v_r)}$ if and only if $(z \otimes 1)(1 \otimes v_r) = (1 \otimes v_r)(z \otimes 1)$, and the conclusion follows since $(id \otimes B(H)_a)(v_r)$ is dense in $\hat{A}$. \qed

Let $\alpha \in \hat{G}$. We make the following notations

$$I_\alpha = (B \times_\delta G)^{ad(v_r)} \cap S_\alpha \quad \text{and} \quad I(\alpha) = \hat{A} p_\alpha.$$ 

**Remark 3.5.** For every $\alpha \in \hat{G}$, there is a $*$-algebra isomorphism

$$S_\alpha \simeq I(\alpha) \otimes I_\alpha.$$ 

Indeed, using Lemma 3.4, $I_\alpha \simeq (p_\alpha \hat{A})' = I(\alpha)'$, where the commutant is taken in $S_\alpha$. Since $I(\alpha)$ is a matrix algebra, using a result of Jacobson (see [6]), we get

$$S_\alpha \simeq I(\alpha) \otimes I(\alpha)' \simeq I(\alpha) \otimes I_\alpha.$$
Using Remark 3.5, we obtain that $I_\alpha$ and $S_\alpha$ are strongly Morita equivalent. Furthermore, using [12, Theorem 3.1], we obtain:

**Lemma 3.6.** $I_\alpha$ is strongly Morita equivalent with $S_\alpha$. Therefore

1. $I_\alpha$ is simple ⇔ $S_\alpha$ is simple;
2. $I_\alpha$ is prime ⇔ $S_\alpha$ is prime.

**Proposition 3.7.** Let $\delta$ be a coaction of a compact quantum group $G$ on a $C^*$-algebra $B$. Then $S_\iota$ is full in $B \times_\delta G$ (i.e. the two-sided ideal generated by $S_\iota$ is equal to $B \times_\delta G$) if and only if $S_\alpha S_{\iota} S_{\iota} S_{\alpha} = S_\alpha$, for all $\alpha \in \hat{G}$, where $\iota$ is the trivial representation of $G$.

**Proof.** Assume $S_\iota$ is full in $B \times_\delta G$ and let $\alpha \in \hat{G}$. Then

$$(B \times_\delta G)(1 \otimes p_\iota)(B \times_\delta G) = B \times_\delta G,$$

and, by multiplying both sides with $(1 \otimes p_\alpha)$,

$$(1 \otimes p_\alpha)(B \times_\delta G)(1 \otimes p_\iota)(B \times_\delta G)(1 \otimes p_\alpha) = (1 \otimes p_\alpha)(B \times_\delta G)(1 \otimes p_\alpha).$$

Hence $S_{\alpha S_{\iota} S_{\iota} S_{\alpha}} = S_\alpha$.

Conversely, assume that $S_{\alpha S_{\iota} S_{\iota} S_{\alpha}} = S_\alpha$, for all $\alpha \in \hat{G}$. Let $\alpha \in \hat{G}$ and $z \in B \times_\delta G$ such that $0 \neq z \in (1 \otimes p_\alpha)(B \times_\delta G)$. Note that $(1 \otimes p_\alpha)(B \times_\delta G) \neq (0)$ since it contains at least the element $1 \otimes p_\alpha$.

Since $S_{\alpha S_{\iota} S_{\iota} S_{\alpha}}$ is a dense ideal in $S_\alpha$, there exists an approximate identity $(e_\lambda)$ of $S_\alpha$ contained in $S_{\alpha S_{\iota} S_{\iota} S_{\alpha}}$. Then, since $zz^* \in S_\alpha$, we get

$$\lim_\lambda \|e_\lambda z - z\|^2 = \lim_\lambda \|e_\lambda z - z)(z^* e_\lambda^* - z^*)\|$$

$$= \lim_\lambda \|e_\lambda z z^* e_\lambda - e_\lambda z z^* - z z^* e_\lambda + z z^*\| = 0.$$

Since $e_\lambda \in S_{\alpha S_{\iota} S_{\iota} S_{\alpha}}$, then $e_\lambda z \in (B \times_\delta G)(1 \otimes p_\iota)(B \times_\delta G)$. Therefore

$$(B \times_\delta G)(1 \otimes p_\iota)(B \times_\delta G).$$

On the other hand, since $\sum_{\alpha \in \hat{G}} p_\alpha = 1$, then

$$\sum_{\alpha \in \hat{G}} (1 \otimes p_\alpha)(B \times_\delta G) = B \times_\delta G,$$

and the conclusion follows. □

**Remark 3.8.** The elements $z$ of $I_\alpha$ have the form $\Lambda \otimes I_{d_\alpha}$, where $\Lambda \in M_{d_\alpha} (B)$.

Note that $S_\alpha \subset B \otimes B(p_\alpha H_h)$ and since $B(p_\alpha H_h) \simeq M_{d_\alpha^2} (\mathbb{C})$, then any element $z \in S_\alpha$ can be represented as a $(d_\alpha^2 \times d_\alpha^2)$ matrix over $B$, $z = [z_{kl}]$. Using moreover the appropriate
identification $\mathcal{M}_{d^2} \simeq \mathcal{M}_d \times \mathcal{M}_d$, then the elements $z$ of $I_\alpha \subset S_\alpha$ have the form $\Lambda \otimes I_d$, where $\Lambda \in \mathcal{M}_d(B)$. Indeed, let $z \in I_\alpha = (B \times G)^{ad(v_r)} \cap S_\alpha$. In particular,

$$(1 \otimes v_r)(z \otimes 1) = (z \otimes 1)(1 \otimes v_r)$$

and hence, by applying $(id \otimes u_{ij}^* \cdot h)$ to both sides, we get

$$(1 \otimes E_{ij}^\alpha)(z \otimes 1) = (z \otimes 1)(1 \otimes E_{ij}^\alpha)$$

for all $i, j = 1, \ldots, d_\alpha$. Using now Remark 2.3, the identity above implies that $z = \Lambda \otimes I_d$, for some $\Lambda \in \mathcal{M}_d(B)$.

4. Simplicity and primeness of the crossed product

We begin this section by giving necessary and sufficient conditions for the simplicity of the crossed product $B \times G$. Theorems 4.1 and 4.4 are extensions of [11, Theorems 3.4 and 3.11], respectively, to the case of compact quantum groups.

**Theorem 4.1.** Let $\delta$ be a coaction of a compact quantum group $G$ on a $C^*$-algebra $B$. The following are equivalent:

1. $B \times G$ is simple;
2. (a) $S_{\alpha, \iota} \neq (0)$, $\forall \alpha \in \hat{G}$, and
   (b) $I_\alpha$ is simple, $\forall \alpha \in \hat{G}$ (equivalently, $S_\alpha$ is simple, $\forall \alpha \in \hat{G}$).

In this case, $B \times G$ is strongly Morita equivalent with $S_\alpha$, $\forall \alpha \in \hat{G}$.

**Proof.** Assume (1) holds and suppose there exists $\alpha \in \hat{G}$ such that $S_{\alpha, \iota} = (0)$. Then

$$(1 \otimes p_\alpha)(B \times G)(1 \otimes p_\iota) = (0)$$

and hence

$$(1 \otimes p_\alpha)(B \times G)(1 \otimes p_\iota)(B \times G) = (0).$$

The ideal $I = (B \times G)(1 \otimes p_\iota)(B \times G)$ is a proper ideal of $B \times G$, since $1 \otimes p_\iota$ is a non-zero element in the multiplier algebra $\mathcal{M}(B \times G)$, so $I \neq (0)$ and $I \neq B \times G$ since $(1 \otimes p_\alpha)I = (0)$.

Hence we obtain that $B \times G$ is not simple and, by contradiction, (2)(a) is proved.

Since $S_\alpha$ is a hereditary subalgebra of $B \times G$, then $S_\alpha$ is simple. Using Lemma 3.6, $I_\alpha$ is simple if and only if $S_\alpha$ is simple and hence (2)(b) follows.

Conversely, assume (2) holds. Since $S_\alpha$ is simple and $S_{\alpha, \iota}S_{\iota, \alpha}$ is a two-sided ideal of $S_\alpha$, we get $S_{\alpha, \iota}S_{\iota, \alpha} = S_\alpha$. By Proposition 3.7, $B \times G$ is strongly Morita equivalent with $S_\iota$, which is simple by hypothesis. Applying [12, Theorem 3.1], it follows that $B \times G$ is simple. □

Using Proposition 3.2 we obtain the following corollary:
Corollary 4.2. Let $\delta$ be a coaction of a compact quantum group $G$ on a $C^*$-algebra $B$. The following are equivalent:

1. $B \times_\delta G$ is simple;
2. (a) $B_\alpha \neq (0)$, $\forall \alpha \in \widehat{G}$, and
   (b) $S_\alpha$ is simple, $\forall \alpha \in \widehat{G}$.

The results we obtain next are related to the primeness of the crossed product $B \times_\delta G$. First, we need the following remark:

Remark 4.3.

1. Every hereditary $C^*$-subalgebra of a prime $C^*$-algebra is prime.
2. If a $C^*$-algebra $A$ contains a prime essential ideal then $A$ is prime.

Theorem 4.4. Let $\delta$ be a coaction of a compact quantum group $G$ on a $C^*$-algebra $B$. The following are equivalent:

1. $B \times_\delta G$ is prime;
2. (a) $S_{\alpha,\iota} \neq (0)$, $\forall \alpha \in \widehat{G}$, and
   (b) $S_\alpha$ is prime, $\forall \alpha \in \widehat{G}$.

Proof. Assume (1) holds. Since $0 \neq 1 \otimes p_\alpha \in B \times_\delta G$ and $0 \neq 1 \otimes p_\iota \in B \times_\delta G$, then

$$S_{\alpha,\iota} = (1 \otimes p_\alpha)(B \times_\delta G)(1 \otimes p_\iota) \neq (0).$$

Also, since $S_\alpha$ are hereditary $C^*$-subalgebras of the prime $C^*$-algebra $B \times_\delta G$, using Remark 4.3, it follows that $S_\alpha$ are prime for all $\alpha \in \widehat{G}$.

Conversely, assume (2) holds. Let

$$I = (B \times_\delta G)(1 \otimes p_\iota)(B \times_\delta G)$$

be the two-sided ideal of $B \times_\delta G$ generated by $1 \otimes p_\iota$.

Using Remark 4.3 it is enough to prove that $I$ is a prime, essential ideal of $B \times_\delta G$. We prove first that $I$ is an essential ideal, by showing that $zI \neq (0)$, for all $0 \neq z \in B \times_\delta G$.

Let $0 \neq z \in B \times_\delta G$. Since $p_\alpha$ are orthogonal projections with sum 1, then there exists $\alpha \in \widehat{G}$ such that $z(1 \otimes p_\alpha) \neq 0$. Then

$$z^*(z(1 \otimes p_\alpha)) = 0.$$ (5)

Using (2)(b) and since $S_{\alpha,\iota}S_{\iota,\alpha}$ is a two-sided ideal of $S_\alpha$, we obtain

$$(1 \otimes p_\alpha)I(1 \otimes p_\alpha) = (1 \otimes p_\alpha)(B \times_\delta G)(1 \otimes p_\iota)(B \times_\delta G)(1 \otimes p_\alpha)$$

$$(1 \otimes p_\alpha)I(1 \otimes p_\alpha) = S_{\alpha,\iota}S_{\iota,\alpha} \neq (0).$$

Therefore

$$(1 \otimes p_\alpha)I(1 \otimes p_\alpha) \neq (0).$$ (6)
Using relations (5) and (6) above we get
\[(1 \otimes p_\alpha)z^*z(I \otimes p_\alpha)I(1 \otimes p_\alpha) \neq (0)\]
and hence \(z(I \otimes p_\alpha)I \neq (0)\), so \(z \neq (0)\). Therefore \(I\) is an essential ideal of \(B \times_\delta G\).

Furthermore, \(I\) is also a prime ideal of \(B \times_\delta G\), since \(I\) is strongly Morita equivalent with \((1 \otimes p_\iota)(B \times_\delta G)(1 \otimes p_\iota)\) which is assumed prime. \(\square\)

The next corollary follows from the theorem above using Proposition 3.2.

**Corollary 4.5.** Let \(\delta\) be a coaction of a compact quantum group \(G\) on a \(C^*\)-algebra \(B\). The following are equivalent:

1. \(B \times_\delta G\) is prime;
2. (a) \(B_\alpha \neq (0), \forall \alpha \in \hat{G}\), and
   (2) \(S_\alpha\) is prime, \(\forall \alpha \in \hat{G}\).

We dedicate the rest of this section to a study of the form of the elements of \(I_\alpha\). We obtain that these elements are matrices over \(B\), fixed under a certain coaction. We start by making a few remarks.

In [4] we give an explicit decomposition of the right regular representation, \(v_r\), as a direct sum of irreducible representations. One can show the similar decomposition for \(v_l\) as well. Using the arguments of the proof of [4, Proposition 9], with \(v_r\) replaced by \(v_l\) one can show that \((p_\alpha \otimes 1)v_l(p_\alpha \otimes 1)\) is a multiple of \(u_\alpha\), in the sense that it can be seen as a \(d_\alpha^2\)-dimensional matrix over \(A\):

\[(p_\alpha \otimes 1)v_l(p_\alpha \otimes 1) = \sum_{i,j=1}^{d_\alpha} (m_{ij} \otimes I_{d_\alpha}) \otimes u_{ij}^{\alpha}.\]  

(7)

Let now \(z \in I_\alpha = (B \times_\delta G)^{ad(v_r)} \cap S_\alpha\). Then \(z\) has the form given in Remark 3.8, 
\[z = \sum_{i,j=1}^{d_\alpha} \lambda_{ij} \otimes (m_{ij} \otimes I_{d_\alpha}).\]

In what follows we will use the notation \(\delta_\alpha = \delta_{\alpha^a}\), defined as in Eq. (3). We are going to show that there exists a \(*\)-isomorphism between \(I_\alpha\) and the fixed point algebra \((B \otimes M_{d_\alpha}(\mathbb{C}))^{\delta_\alpha}\).

Define first the application \(\Phi : I_\alpha \rightarrow B \otimes M_{d_\alpha}(\mathbb{C})\),
\[\Phi(z) = \Lambda := [\lambda_{ij}] \in B \otimes M_{d_\alpha}(\mathbb{C}),\]

where by \(M_{d_\alpha}(\mathbb{C})\) we denote the \(d_\alpha\)-dimensional matrices over \(\mathbb{C}\).

Note that \(\Phi\) is a \(*\)-homomorphism. Furthermore, we obtain the following result:

**Proposition 4.6.** Let \(G\) be a compact quantum group and let \(\alpha \in \hat{G}\). Then
\[\Phi(I_\alpha) \subseteq (B \otimes M_{d_\alpha}(\mathbb{C}))^{\delta_\alpha}.\]

**Proof.** Let \(z \in I_\alpha\). Then \(z\) has the form in Remark 3.8 and using Eq. (4) it follows that:
\[ z \otimes 1 = \Lambda \otimes I_{d_\alpha} \otimes 1 \]

\[ = \sum_{i,j=1}^{d_\alpha} (1 \otimes v_l)(id \otimes \sigma)(\delta \otimes id)(\lambda_{ij} \otimes (m_{ij} \otimes I_{d_\alpha}))(1 \otimes v_l^*) \]

\[ = \sum_{i,j=1}^{d_\alpha} (1 \otimes v_l)(id \otimes \sigma)(\delta(\lambda_{ij}))(\otimes (m_{ij} \otimes I_{d_\alpha}))(1 \otimes v_l^*) \]

\[ = \sum_{i,j=1}^{d_\alpha} (1 \otimes v_l)(\delta(\lambda_{ij})_{13}(1 \otimes (m_{ij} \otimes I_{d_\alpha}) \otimes 1))(1 \otimes v_l^*) \]

\[ = \sum_{i,j=1}^{d_\alpha} (1 \otimes (m_{kl} \otimes I_{d_\alpha}) \otimes u_{kl}^\alpha)(\delta(\lambda_{ij})_{13}(1 \otimes (m_{ij} \otimes I_{d_\alpha}) \otimes 1)) \]

\[ \times (1 \otimes (m_{rs} \otimes I_{d_\alpha}) \otimes u_{sr}^\alpha). \]

Therefore,

\[ \Lambda \otimes 1 = \sum_{i,j,k,l,r,s=1}^{d_\alpha} (1 \otimes m_{kl} \otimes u_{kl}^\alpha)(\delta(\lambda_{ij})_{13}(1 \otimes m_{ij} \otimes I_{d_\alpha}) \otimes 1)) \]

\[ \times (1 \otimes m_{rs} \otimes I_{d_\alpha}) \otimes u_{sr}^\alpha. \]

where \( \delta_i \) is defined as in Eq. (3) with \( u = \iota \), the trivial representation. Hence \( \Phi(z) \otimes 1 = (1 \otimes u^\alpha)\delta_i(\Phi(z))(1 \otimes u^\alpha) \). Therefore

\[ \Phi(I_{\alpha}) \subseteq (B \otimes \mathcal{M}_{d_\alpha}(\mathbb{C}))^{\delta_{\alpha}}. \quad \square \]

Conversely, let \( \Lambda = [\lambda_{ij}] \in B \otimes \mathcal{M}_{d_\alpha}(\mathbb{C}) \) be a matrix with the property that

\[ \Lambda \otimes 1 = (1 \otimes u^\alpha)\delta_i(\Lambda)(1 \otimes u^\alpha), \]

and define \( \Psi(\Lambda) = \Lambda \otimes I_{d_\alpha} \).

We are going to show next that, under these assumptions, \( \Psi(\Lambda) \in I_{\alpha} \). It is then clear that \( \Psi \) is a \( \ast \)-homomorphism and that \( \Psi \) and \( \Phi \) are inverse to each other, creating thus an isomorphism between \( I_{\alpha} \) and \( (B \otimes \mathcal{M}_{d_\alpha}(\mathbb{C}))^{\delta_{\alpha}} \).

**Proposition 4.7.** Let \( \Lambda \in (B \otimes \mathcal{M}_{d_\alpha}(\mathbb{C}))^{\delta_{\alpha}} \). Then \( \Psi(\Lambda) \in I_{\alpha} \).

**Proof.** Let \( \Lambda = [\lambda_{ij}] \in (B \otimes \mathcal{M}_{d_\alpha}(\mathbb{C})) \) with the property \( \delta_{\alpha}(\Lambda) = \Lambda \otimes 1_A \). We will first show that \( \Psi(\Lambda) \in B \times_\delta G \).

Using Eq. (4), it enough to check that \( \Psi(\Lambda) \in (B \otimes K(H_h))^{\delta_{\eta}} \). We have

\[ \delta_{\eta}(\Psi(\Lambda)) = (1 \otimes v_l)(\delta(\lambda_{ij})_{13})(1 \otimes (m_{ij} \otimes I_{d_\alpha}) \otimes 1)(1 \otimes v_l^*). \quad (8) \]
Since \( \Lambda \otimes 1 = (1 \otimes u^\alpha)\delta_\eta(\Lambda)(1 \otimes u^{\alpha^*}) \), using the proof of Proposition 4.6 we obtain

\[
\Psi(\Lambda) \otimes 1 = (1 \otimes v_l)(id \otimes \sigma)(\delta \otimes id)(\Psi(\Lambda))(1 \otimes v_l^*),
\]

and hence

\[
\Psi(\Lambda) \otimes 1 = (1 \otimes v_l)(\delta(\lambda_{ij}) \otimes (m_{ij} \otimes I_{d_\eta}))(1 \otimes v_l^*).
\]

Using the relation above together with Eq. (8) we obtain \( \delta v_l(\Psi(\Lambda)) = \Psi(\Lambda) \otimes 1 \), and hence \( \Psi(\Lambda) \in B \times_\delta G \).

We show next that \( \Psi(\Lambda) \in I_\alpha \). It is easy to check that \( \Psi(\Lambda) \) commutes with \( E_{ij} \) in the following sense:

\[
(1 \otimes E_{ij})\Psi(\Lambda) = \Psi(\Lambda)(1 \otimes E_{ij}),
\]

for all \( i, j = 1, \ldots, d_\alpha \).

In particular, this means that \( (1 \otimes p_\alpha)\Psi(\Lambda) = \Psi(\Lambda)(1 \otimes p_\alpha) \) and hence \( \Psi(\Lambda) \in S_\alpha \). Furthermore, relation (9) implies that \( \Psi(\Lambda) \in (1 \otimes \hat{A})' \). Using Lemma 3.4 it follows that \( \Psi(\Lambda) \in (B \times_\delta G)^{ad(v_r)} \) and since \( \Psi(\Lambda) \in S_\alpha \) we obtain \( \Psi(\Lambda) \in I_\alpha \). \( \square \)

Using Propositions 4.6 and 4.7 and the discussion between them we obtain the following result:

**Proposition 4.8.** Let \( G \) be a compact quantum group. For every \( \alpha \in \hat{G} \), there is an isomorphism between \( I_\alpha \) and \( (B \otimes M_{d_\alpha}(\mathbb{C}))^{\delta_\alpha} \).

Finally, using the proposition above we obtain the following corollary of Theorems 4.1 and 4.4, which gives a description of the simplicity and primeness of the crossed product in terms of the simplicity and primeness of fixed point subalgebras of matrices over \( B \).

**Corollary 4.9.** Let \( \delta \) be a coaction of a compact quantum group \( G \) on a \( C^* \)-algebra \( B \). The following are equivalent:

1. \( B \times_\delta G \) is simple (prime);
2. (a) \( B_\alpha \neq (0), \forall \alpha \in \hat{G}, \) and
   (b) \( (B \otimes M_{d_\alpha}(\mathbb{C}))^{\delta_\alpha} \) is simple (prime), \( \forall \alpha \in \hat{G} \).

**Proof.** The result follows from Theorems 4.1 and 4.4 and Proposition 4.8. \( \square \)

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References