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The fixed point property via dual space properties

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Abstract

A Banach space has the weak fixed point property if its dual space has a weak* sequentially compact unit ball and the dual space satisfies the weak* uniform Kadec–Klee property; and it has the fixed point property if there exists $\varepsilon > 0$ such that, for every infinite subset A of the unit sphere of the dual space, $A \cup (-A)$ fails to be $(2-\varepsilon)$-separated. In particular, E-convex Banach spaces, a class of spaces that includes the uniformly nonsquare spaces, have the fixed point property.

Keywords: Fixed point property; E-convexity

Determining conditions on a Banach space X so that every nonexpansive mapping from a nonempty, closed, bounded, convex subset of X into itself has a fixed point has been of considerable interest for many years. A Banach space has the fixed point property if, for each nonempty, closed, bounded, convex subset C of X, every nonexpansive mapping of C into itself has a fixed point. A Banach space is said to have the weak fixed point property if the class of sets C above is restricted to the set of weakly compact convex sets; and a Banach space is said to have the weak* fixed point property if X is a dual space and the class of sets C is restricted to the set of weak* compact convex subsets of X.

A well-known open problem in Banach spaces is whether every reflexive Banach space has the fixed point property for nonexpansive mappings. The question of whether more restrictive

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classes of reflexive spaces, such as the class of superreflexive Banach spaces or Banach spaces isomorphic to the Hilbert space ℓ^2, have the fixed point property has also long been open and has been investigated by many authors [8,14,15,17]. Recently, J. García-Falset, E. Llorens-Fuster, and E.M. Mazcuñan-Navarro [7] proved that uniformly nonsquare Banach spaces, a sub-class of the superreflexive spaces, have the fixed point property. In this article, it is shown that the larger class of E-convex Banach spaces have the fixed point property. The E-convex Banach spaces, introduced by S.V.R. Naidu and K.P.R. Sastry [18], are a class of Banach spaces lying strictly between the uniformly nonsquare Banach spaces and the superreflexive spaces (see also [1]).

The second geometric property of Banach spaces that is considered in this article is the weak* uniform Kadec–Klee property in a dual Banach space. A dual space X^* has the weak* uniform Kadec–Klee property if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, if (x_n^*) is a sequence in the unit ball of X^* converging weak* to x^* and the separation constant $\mathrm{sep}(x_n^*) \stackrel{\text{def}}{=} \inf\{\|x_n^* - x_m^*\|: m \neq n\} > \varepsilon$, then $\|x^*\| < 1 - \delta$. It is well known [6] that, if X^* has weak* uniform Kadec–Klee property, then X^* has the weak* fixed point property. If, in addition, the unit ball of X^* is weak* sequentially compact, more is true: Theorem 3 notes that, if X^* has weak* uniform Kadec–Klee property and the unit ball of X^* is weak* sequentially compact, then X has the weak fixed point property. As a consequence of Theorem 3, it is noted that several nonreflexive Banach spaces such as quotients of c_0 and $C(T)/A_0$, the predual of H^1, have the weak fixed point property.

Since the proofs of the main theorems in this paper will require elements of the proof that uniformly nonsquare Banach spaces have the fixed point property, a complete proof of this known result is presented. The proof presented here is a distillation of the original proof and combines elements from [5, Theorem 2.2] and [7, Theorem 3.3]. Recall that a Banach space X is uniformly nonsquare [10] if there exists $\delta > 0$ such that, if x and y are in the unit ball of X, then either $\|(x + y)/2\| < 1 - \delta$ or $\|(x - y)/2\| < 1 - \delta$.

The general set-up in proving that a Banach space has the weak fixed point property has, by now, become standard fare. If a Banach space X fails to have the weak fixed point property, there exists a nonempty, weakly compact, convex set C in X and a nonexpansive mapping $T : C \to C$ without a fixed point. Since C is weakly compact, it is possible by Zorn's lemma to find a minimal T-invariant, weakly compact, convex subset K of C such that T has no fixed point in K. Since the diameter of K is positive (otherwise K would be a singleton and T would have a fixed point in K), it can be assumed that the diameter of K is 1. It is well known that there exists an approximate fixed point sequence (x_n) for T in K and, without loss of generality, we may assume that (x_n) converges weakly to 0. For details on this general set-up, see [8, Chapter 3].

Theorem 1. *(See García-Falset, Llorens-Fuster, and E.M. Mazcuñan-Navarro [7].) Uniformly nonsquare Banach spaces have the fixed point property for nonexpansive mappings.*

Proof. Assume that a Banach space X fails to have the fixed point property. Since uniformly nonsquare spaces are reflexive [10], the fixed point property and the weak fixed point property coincide for X. Therefore there exists a nonexpansive map $T : K \to K$ without a fixed point where K is a minimal T-invariant set in X with diameter 1. Let (x_n) be an approximate fixed point sequence for T in K and assume that (x_n) converges weakly to 0.

Consider the space $\ell^\infty(X)/c_0(X)$ endowed with the quotient norm given by $\|[w_n]\| = \limsup_n \|w_n\|$ where $[w_n]$ denotes the equivalence class of $(w_n) \in \ell^\infty(X)$. For a bounded set

C in X, the set $[C]$ in $\ell^\infty(X)/c_0(X)$ is defined by $[C]=\{[w_n]\colon\ w_n\in C \text{ for all } n\in\mathbb{N}\}$. Using the notation in [5, p. 840], let

$$[W]=\Bigl\{[z_n]\in[K]\colon\ \bigl\|[z_n]-[x_n]\bigr\|\leqslant 1/2 \text{ and } \limsup_n \limsup_m \|z_m-z_n\|\leqslant 1/2\Bigr\}.$$

It is easy to check that $[W]$ is a closed, bounded, convex, nonempty (since $[\frac{1}{2}x_n]$ is in $[W]$) subset of $[K]$, and is $[T]$-invariant where $[T]:[K]\to[K]$ is defined by $[T][z_n]=[T(z_n)]$. So, by a result of Lin [13], $\sup_{[z_n]\in[W]}\|[z_n]-x\|=1$ for each $x\in K$. In particular, with $x=0$, $\sup_{[z_n]\in[W]}\|[z_n]\|=1$.

Let $\varepsilon>0$ and choose $[z_n]\in[W]$ with $\|[z_n]\|>1-\varepsilon$. Let $(y_j)=(z_{n_j})$ be a subsequence of (z_n) such that $\lim\|y_n\|=\|[z_n]\|$ and (y_n) converges weakly to an element y in K. There is no loss in generality in assuming that $\|y_n\|>1-\varepsilon$ for all $n\in\mathbb{N}$ and in choosing $y_n^*\in X^*$ so that $\|y_n^*\|=1$, $y_n^*(y_n)=\|y_n\|$, and (y_n^*) converges weak* to y^*. (This is possible because the fixed point property is separably-determined [8, p. 35]; so there is no loss in generality in assuming that B_{X^*} is weak*-sequentially compact.)

From the definition of $[W]$ and the weak lower semicontinuity of the norm, it follows that, if n is large enough,

$$\|y_n-y\|\leqslant \liminf_m\|y_n-y_m\|<\frac{1+\varepsilon}{2} \quad\text{and}\quad \|y\|\leqslant\liminf_j\|y_j-x_{n_j}\|\leqslant\frac{1}{2}.$$

Therefore, with $u_n=\frac{2}{1+\varepsilon}(y_n-y)$ and $u=\frac{2}{1+\varepsilon}y$,

$$\|u_n+u\|=\left\|\frac{2}{1+\varepsilon}(y_n-y)+\frac{2}{1+\varepsilon}y\right\|=\frac{2}{1+\varepsilon}\|y_n\|>2\frac{1-\varepsilon}{1+\varepsilon}>2(1-2\varepsilon)$$

if $n\in\mathbb{N}$ is large enough. Applying the weak lower semicontinuity of the norm again, it follows that

$$\liminf_m\bigl\|(u_n-u_m)+u\bigr\|\geqslant\|u_n+u\|>2(1-2\varepsilon)$$

if $n\in\mathbb{N}$ is large enough. So, by taking another subsequence if necessary, we can assume that $\|u_n+u\|>2(1-2\varepsilon)$ and $\|(u_n-u_m)+u\|>2(1-2\varepsilon)$ for all n and all $m>n$.

Furthermore, since $y_m^*\xrightarrow{w^*}y^*$,

$$\begin{aligned}\liminf_m\bigl\|(u_n-u_m)-u\bigr\|&=\liminf_m\bigl\|(u_m+u)-u_n\bigr\|\\&\geqslant\liminf_m y_m^*\bigl((u_m+u)-u_n\bigr)\\&=\liminf_m\bigl(\|u_m+u\|-y_m^*(u_n)\bigr)\\&\geqslant 2(1-2\varepsilon)-y^*(u_n).\end{aligned}$$

Then, since $u_n\xrightarrow{w}0$, it follows that $\liminf_m\|(u_n-u_m)-u\|>2(1-3\varepsilon)$ if n is large enough. Therefore, for n large enough and $m>n$ also large enough, both

$$\bigl\|(u_n-u_m)+u\bigr\|>2(1-3\varepsilon)\quad\text{and}\quad\bigl\|(u_n-u_m)-u\bigr\|>2(1-3\varepsilon)$$

hold. Since $\varepsilon > 0$ is arbitrary, $\|u_n - u_m\| < 1$, and $\|u\| < 1$, the above inequalities imply that X fails to be uniformly nonsquare, a contradiction which finishes the proof. □

We want to refine the sequences (x_{n_j}), (y_j), and (y_j^*) that appear in the proof of Theorem 1. Recall the result of Goebel and Karlovitz [8, p. 124]: If K is a minimal T-invariant, weakly compact, convex set for the nonexpansive map T and (x_n) is an approximate fixed point sequence for T in K, then the sequence $(\|x_n - x\|)$ converges to the diameter of K for every x in K.

Fix $\varepsilon > 0$ and set $\tilde{x}_1 = x_{n_1}$, $\tilde{y}_1 = y_1$, and $\tilde{y}_1^* = y_1^*$. Then, by the Goebel–Karlovitz Lemma, there exists $j_1 > 1$ such that $\min\{\|\tilde{x}_1 - x_{n_{j_1}}\|, \|\tilde{y}_1 - x_{n_{j_1}}\|\} > 1 - \varepsilon$. Set $\tilde{x}_2 = x_{n_{j_1}}$, $\tilde{y}_2 = y_{j_1}$, and $\tilde{y}_2^* = y_{j_1}^*$. Another application of the Goebel–Karlovitz Lemma yields $j_2 > j_1$ such that $\min_{i=1,2}\{\|\tilde{x}_i - x_{n_{j_2}}\|, \|\tilde{y}_i - x_{n_{j_2}}\|\} > 1 - \varepsilon$. Set $\tilde{x}_3 = x_{n_{j_2}}$, $\tilde{y}_3 = y_{j_2}$, and $\tilde{y}_3^* = y_{j_2}^*$. Continuing in this manner, we obtain sequences $(\tilde{x}_n)$ and $(\tilde{y}_n)$ in K (and B_X) and a sequence $(\tilde{y}_n^*)$ in S_{X^*} satisfying $\min_{i<n}\{\|\tilde{x}_i - \tilde{x}_n\|, \|\tilde{y}_i - \tilde{x}_n\|\} > 1 - \varepsilon$ for all $n \in \mathbb{N}$ and $\tilde{y}_n^*(\tilde{y}_n) = \|\tilde{y}_n\| > 1 - \varepsilon$ for all $n \in \mathbb{N}$. In the following, these sequences are renamed by omitting the tildes. The following result is a summary of several easy computations.

Lemma 2. *Let X be a Banach space whose dual unit ball is weak* sequentially compact and assume that X fails the weak fixed point property. Given $\varepsilon > 0$, there exist sequences (y_n) in B_X and (y_n^*) in S_{X^*} and elements $y \in B_X$ and $y^* \in B_{X^*}$ satisfying*:

(1) $y_n \xrightarrow{w} y$ *and* $y_n^* \xrightarrow{w^*} y^*$;
(2) *for every* $n \in \mathbb{N}$, $1 - \varepsilon < \|y_n\| = y_n^*(y_n) \leqslant 1$;
(3) *for every* $n \in \mathbb{N}$, $\frac{1-3\varepsilon}{2} < y_n^*(y) \leqslant \|y\| < \frac{1+\varepsilon}{2}$;
(4) $\frac{1-3\varepsilon}{2} < \|y_n - y\| < \frac{1+\varepsilon}{2}$;
(5) *if* $n \neq m$, *then* $\frac{1-3\varepsilon}{2} < \|y_n - y_m\| < \frac{1+\varepsilon}{2}$;
(6) *if* $n \neq m$, *then* $\frac{1-3\varepsilon}{2} < y_n^*(y_m) < \frac{1+2\varepsilon}{2}$;
(7) $\frac{1-3\varepsilon}{2} \leqslant y^*(y) \leqslant \frac{1+\varepsilon}{2}$.

Proof. Claims (1) and (2), the third inequality in (3), and the second inequalities in (4) and (5) are immediate from the proof of Theorem 1. Then

$$\|y\| \geqslant y_n^*(y) = y_n^*(y_n) - y_n^*(y_n - y) > (1 - \varepsilon) - \frac{1+\varepsilon}{2} = \frac{1-3\varepsilon}{2}$$

proving (3).

Also

$$\|y_n - y\| \geqslant y_n^*(y_n - y) = \|y_n\| - y_n^*(y) > (1 - \varepsilon) - \frac{1+\varepsilon}{2} = \frac{1-3\varepsilon}{2}$$

which finishes the proof of (4).

From our refinement of (x_n) and (y_n) done just prior to the lemma and the definition of $[W]$ in the proof of Theorem 1, if $n > m$,

$$\begin{aligned}\|y_n - y_m\| &= \big\|(y_n - x_n) + (x_n - y_m)\big\| \\ &\geqslant \|x_n - y_m\| - \|y_n - x_n\|\end{aligned}$$

$$> (1-\varepsilon) - \frac{1+\varepsilon}{2}$$
$$= \frac{1-3\varepsilon}{2}$$

showing that (5) holds.

The lower inequality in (6) follows from (2) and (5): If $n \neq m$,

$$1-\varepsilon < y_n^*(y_n) = y_n^*(y_n - y_m) + y_n^*(y_m) \leqslant \|y_n - y_m\| + y_n^*(y_m) < \frac{1+\varepsilon}{2} + y_n^*(y_m).$$

Therefore, if $n \neq m$, $\frac{1-3\varepsilon}{2} < y_n^*(y_m)$.

In order to show the upper inequality in (6), we consider subsequences of the sequences (y_n) and (y_n^*) obtained so far. Note that all of the previous conditions will remain true for subsequences of the current (y_n) and (y_n^*). Let $\tilde{y}_1 = y_1$ and $\tilde{y}_1^* = y_1^*$. Since (y_n) converges weakly to y and $\frac{1-3\varepsilon}{2} < y_1^*(y) < \frac{1+\varepsilon}{2}$, there exists $n_1 \in \mathbb{N}$ such that, if $n \geqslant n_1$, $\frac{1-3\varepsilon}{2} < y_1^*(y_n) < \frac{1+\varepsilon}{2}$. Set $\tilde{y}_2 = y_{n_1}$ and $\tilde{y}_2^* = y_{n_1}^*$. Since (y_n) converges weakly to y and $\frac{1-3\varepsilon}{2} < y_{n_1}^*(y) < \frac{1+\varepsilon}{2}$, there exists $n_2 \in \mathbb{N}$ such that, if $n \geqslant n_2$, $\frac{1-3\varepsilon}{2} < y_{n_1}^*(y_n) < \frac{1+\varepsilon}{2}$. Set $\tilde{y}_3 = y_{n_2}$ and $\tilde{y}_3^* = y_{n_2}^*$. Continuing in this manner generates sequences $(\tilde{y}_n)$ and $(\tilde{y}_n^*)$ satisfying conditions (1)–(5) and satisfying $\tilde{y}_n^*(\tilde{y}_m) < \frac{1+\varepsilon}{2}$ if $n < m$. Again, we simplify the notation by considering these new sequences but omitting the tildes in the notation.

To show the upper inequality in (6) for $n > m$, first combine (3) with the weak* convergence of (y_n^*) to y^* to obtain (7). Then, since (y_n) converges weakly to y, there is no loss in generality in assuming that $y^*(y_n) < \frac{1+2\varepsilon}{2}$ for all $n \in \mathbb{N}$. In particular, $y^*(y_1) < \frac{1+2\varepsilon}{2}$. Therefore, since (y_n^*) converges weak* to y^*, there exists $n_1 \in \mathbb{N}$ such that, if $n \geqslant n_1$, $y_n^*(y_1) < \frac{1+2\varepsilon}{2}$. Setting $\tilde{y}_1 = y_1$, $\tilde{y}_1^* = y_1^*$, $\tilde{y}_2 = y_{n_1}$, and $\tilde{y}_2^* = y_{n_1}^*$ gives $\tilde{y}_2^*(\tilde{y}_1) < \frac{1+2\varepsilon}{2}$. Then, since $y^*(\tilde{y}_2) < \frac{1+2\varepsilon}{2}$ and (y_n^*) converges weak* to y^*, there exists $n_2 \in \mathbb{N}$ such that $n_2 > n_1$, and, if $n \geqslant n_2$, $y_n^*(\tilde{y}_2) < \frac{1+2\varepsilon}{2}$. Set $\tilde{y}_3 = y_{n_2}$ and $\tilde{y}_3^* = y_{n_2}^*$. Continuing in this manner generates sequences $(\tilde{y}_n)$ and $(\tilde{y}_n^*)$ satisfying all of the conditions of the lemma. □

As a consequence of these computations, we have the following

Theorem 3. *Let X be a Banach space such that B_{X^*} is weak* sequentially compact. If X^* has the weak* uniform Kadec–Klee property, then X has the weak fixed point property for nonexpansive mappings.*

Proof. If X fails to have the weak fixed point property, consider the sequences (y_n) and (y_n^*) determined in Lemma 2. In particular, note that $\|y_n^*\| = 1$ for all $n \in \mathbb{N}$ and that (y_n^*) converges weak* to y^*. Note also that, if $n \neq m$,

$$\left(y_n^* - y_m^*\right)(y_n - y_m) = y_n^*(y_n) - y_n^*(y_m) - y_m^*(y_n) + y_m^*(y_m)$$
$$> (1-\varepsilon) - \frac{1+2\varepsilon}{2} - \frac{1+2\varepsilon}{2} + (1-\varepsilon)$$
$$= 1 - 4\varepsilon.$$

It follows that

$$2 \geqslant \|y_n^* - y_m^*\| \geqslant (y_n^* - y_m^*)\left(\frac{y_n - y_m}{\|y_n - y_m\|}\right) > \frac{1-4\varepsilon}{(1+\varepsilon)/2} = 2\,\frac{1-4\varepsilon}{1+\varepsilon} > 2 - 10\varepsilon.$$

Thus, if $\varepsilon < \frac{1}{10}$, (y_n^*) is a sequence in the unit sphere of X^*, (y_n^*) converges weak* to y^*, and $\operatorname{sep}\{y_n^*\} > 1$. Therefore, by the weak* uniform Kadec–Klee property of X^*, there exists $\delta > 0$ such that

$$\|y^*\| < 1 - \delta. \tag{$*$}$$

But, by (3) and (7),

$$\frac{1-3\varepsilon}{2} \leqslant y^*(y) \leqslant \|y\|\|y^*\| < \frac{1+\varepsilon}{2}\|y^*\|.$$

Therefore, if $\varepsilon < \min\{\frac{1}{10}, \frac{\delta}{4}\}$

$$\|y^*\| > \frac{1-3\varepsilon}{1+\varepsilon} > 1 - 4\varepsilon > 1 - \delta,$$

a contradiction to $(*)$. Therefore X has the weak fixed point property for nonexpansive mappings. □

Of course the theorem implies that c_0 with its usual norm has the weak fixed point property which was a result first proven by Maurey [16]. Since H^1 has the weak* uniform Kadec–Klee property [2], its predual $C(T)/A_0$ has the weak fixed point property by this theorem. In the same manner, since $\mathcal{C}_1(H)$, the ideal of trace class operators on a Hilbert space H, has the weak* uniform Kadec–Klee property [12], its predual $\mathcal{C}_\infty(H)$, the ideal of compact operators in $B(H)$, has the weak fixed point property. Since quotients of Banach spaces with weak* sequentially compact dual unit balls have weak* sequentially compact dual unit balls [4, p. 227], it is easy to check that the following holds.

Corollary 4. *Let X be a Banach space such that B_{X^*} is weak* sequentially compact. If X^* has the weak* uniform Kadec–Klee property and Y is a closed subspace of X, then X/Y has the weak fixed point property for nonexpansive mappings.*

We note that the corollary implies that the quotients of c_0 have the weak fixed point property. This is implicit in the work of Borwein and Sims [3].

The authors had hoped that the sequences identified in Lemma 2 would be useful in establishing connections between superreflexive Banach spaces and the fixed point property for nonexpansive mappings. Consider the sequences generated in Lemma 2 for each $\varepsilon = \frac{1}{k}$, $k \in \mathbb{N}$. That is, for each $k \in \mathbb{N}$, let $(y_{k,n})$ and $(y_{k,n}^*)$ denote the sequences (y_n) and (y_n^*) constructed in Lemma 2 with $\varepsilon = \frac{1}{k}$; let $y_{k,\infty}$ denote the weak limit of the sequence $(y_{k,n})$; and let $y_{k,\infty}^*$ denote the weak* limit of the sequence $(y_{k,n}^*)$. For a non-trivial ultrafilter $\mathcal{U}$ on $\mathbb{N}$, let $X_{\mathcal{U}}$ denote the ultrapower of X with respect to $\mathcal{U}$. (For information on ultraproducts in Banach space theory, see [9] or [20].) Define sequences $(\mathbf{y}_n)$ in $X_{\mathcal{U}}$, $(\mathbf{y}_n^*)$ in $(X^*)_{\mathcal{U}}$, and the point $\mathbf{y}$ in $X_{\mathcal{U}}$ by

$$\mathbf{y}_n = (y_{1,n}, y_{2,n}, y_{3,n}, \ldots)_{\mathcal{U}},$$
$$\mathbf{y}_n^* = \left(y_{1,n}^*, y_{2,n}^*, y_{3,n}^*, \ldots\right)_{\mathcal{U}}, \quad \text{and}$$
$$\mathbf{y} = (y_{1,\infty}, y_{2,\infty}, y_{3,\infty}, \ldots)_{\mathcal{U}}.$$

The pair of sequences $(2(\mathbf{y}_n - \mathbf{y}))$ and $(\mathbf{y}_n^*)$ forms a biorthogonal system of norm-one elements in $X_{\mathcal{U}}$ and $(X^*)_{\mathcal{U}}$ and, for each sequence (α_n) of nonnegative real numbers, $\|\sum_{n=1}^{\infty} \alpha_n \mathbf{y}_n^*\| = \sum_{n=1}^{\infty} \alpha_n$. Moreover, as is clear from the proof of Theorem 3, $\|\mathbf{y}_m^* - \mathbf{y}_n^*\| = 2$ for all $m \neq n$. Initially, the authors felt that, if X was a renorming of ℓ^2, this "positive ℓ^1-type behavior" should not occur in $(X^*)_{\mathcal{U}}$ since $(X^*)_{\mathcal{U}}$ would also be a renorming of a Hilbert space. However, as first pointed out to us by Professor V.D. Milman, there do exist renormings of ℓ^2 with this behavior. A second example resulted from a discussion with Professors A. Pełczyński and M. Wojciechowski. In fact, every infinite-dimensional Banach space can be renormed to exhibit this ℓ^1-type behavior for nonnegative linear combinations. To see this, let (x_i, x_i^*) in $X \times X^*$ be a biorthogonal system with $\|x_i\| = 1$ and $\|x_i^*\| \leqslant 2$. (Such a biorthogonal system exists by applying a theorem of Ovsepian and Pełczyński [4, p. 56] to a separable subspace of X and then extending to functionals on all of X via the Hahn–Banach theorem.) Then $|||x||| = \max\{|x_1^*(x)|, \frac{1}{2}\|x\|, \sup_{i \neq j;\, i,j \geqslant 2}(|x_i^*(x)| + |x_j^*(x)|)\}$ defines an equivalent norm on X with $|||x_1 + x_n||| = 1$ and $|||\sum_{n=1}^{\infty} \alpha_n (x_1 + x_n)||| = \sum_{n=1}^{\infty} \alpha_n$ if $\alpha_n \geqslant 0$. (For related examples, see Example 3.13 in [18] and Theorem 7 in [11].)

Despite the above disappointment, the sequence $(\mathbf{y}_n^*)$ in $(X_{\mathcal{U}})^*$ or the sequences (y_n^*) in X^* for a given ε in Lemma 2 can be used to generalize Theorem 1. A subset A of X is *symmetrically ε-separated* if the distance between any two distinct points of $A \cup (-A)$ is at least ε and a Banach space X is *O-convex* if the unit ball B_X contains no symmetrically $(2 - \varepsilon)$-separated subset of cardinality n for some $\varepsilon > 0$ and some $n \in \mathbb{N}$ [18]. O-convex spaces are superreflexive. Therefore the proof of Theorem 3 combines with property (3) in Lemma 2 to show that, if X fails to have the fixed point property, then, for every $\varepsilon > 0$, there exists a countably infinite set $A = \{y_1^*, y_2^*, \ldots\}$ in the unit sphere of X^* such that $A \cup (-A)$ is $(2 - \varepsilon)$-separated. In particular, this implies:

Theorem 5. *If X^* is O-convex, then the Banach space X has the fixed point property for nonexpansive mappings.*

Since uniformly nonsquare Banach spaces are O-convex, Theorem 5 is a generalization of Theorem 1. Naidu and Sastry [18] also characterized the dual property to O-convexity. For $\varepsilon > 0$, a convex subset A of B_X is an *ε-flat* if $A \cap (1 - \varepsilon)B_X = \emptyset$. Note that the convex hulls of the sets $\{y_1^*, y_2^*, \ldots\}$ from Lemma 2 are 3ε-flats. A collection $\mathcal{D}$ of ε-flats is *jointly complemented* if, for each distinct ε-flats A and B in $\mathcal{D}$, the sets $A \cap B$ and $A \cap (-B)$ are nonempty. Define

$$E(n, X) = \inf\{\varepsilon\colon\ B_X \text{ contains a jointly complemented collection of } \varepsilon\text{-flats of cardinality } n\}.$$

A Banach space X is *E-convex* if $E(n, X) > 0$ for some $n \in \mathbb{N}$. In [19], S. Saejung noted that E-convex spaces may fail to have normal structure and asked if E convex spaces have the fixed point property. Since a Banach space is E-convex if and only if its dual space is O-convex, Theorem 5 can be restated to give a positive answer to Saejung's question.

Theorem 6. *E-convex spaces have the fixed point property for nonexpansive mappings.*

For a detailed analysis of O-convex, E-convex, and related properties in the hierarchy between Hilbert spaces and reflexive spaces, see [1,18,19].

Note added in proof

The authors thank Jesús García-Falset for pointing out that X^* having the weak* uniform Kadec–Klee property implies that $R(X) < 2$ which, by Theorem 3 in [J. García-Falset, The fixed point property in Banach spaces with the NUS-property, J. Math. Anal. Appl. 215 (2) (1997) 532–542], shows that X has the weak fixed point property. Thus Theorem 3 in this article is a special case of Theorem 3 in the article of García-Falset.

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