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Journal of Functional Analysis 217 (2004) 142–170

**JOURNAL OF
Functional
Analysis**<http://www.elsevier.com/locate/jfa>

AF-embeddings into C^* -algebras of real rank zero[☆]

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Received 6 November 2003; revised 5 May 2004; accepted 5 May 2004

Available online 11 June 2004

Communicated by G. Pisier

Abstract

It is proved that every separable C^* -algebra of real rank zero contains an AF-sub- C^* -algebra such that the inclusion mapping induces an isomorphism of the ideal lattices of the two C^* -algebras and such that every projection in a matrix algebra over the large C^* -algebra is equivalent to a projection in a matrix algebra over the AF-sub- C^* -algebra. This result is proved at the level of monoids, using that the monoid of Murray–von Neumann equivalence classes of projections in a C^* -algebra of real rank zero has the refinement property. As an application of our result, we show that given a unital C^* -algebra A of real rank zero and a natural number n , then there is a unital $*$ -homomorphism $M_{n_1} \oplus \cdots \oplus M_{n_r} \rightarrow A$ for some natural numbers r, n_1, \dots, n_r with $n_j \geq n$ for all j if and only if A has no representation of dimension less than n .

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Keywords: Approximately divisible; Dimension monoid; Real rank zero; Refinement monoid; Weakly divisible

[☆]Research supported by the Danish Research Council, SNF, the Nuffield Foundation NUF-NAL 02, the DGI and European Regional Development Fund through Project BFM2002-01390, and the Comissionat per Universitats i Recerca de la Generalitat de Catalunya.

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1. Introduction

Many properties of C^* -algebras, that can be traced back to properties of high-dimensional spaces, have recently been found in simple C^* -algebras. These properties include having arbitrary stable rank [26]; perforation of the ordered K_0 group [25]; and existence of a finite projection that becomes infinite if doubled [21]. Examples of simple nuclear C^* -algebras with these properties appear to be in disagreement with the classification conjecture by Elliott.

In contrast to the situation of (arbitrary) simple C^* -algebras, no such exotic phenomena have so far been found among C^* -algebras of real rank zero (first studied by Brown and Pedersen [7]). At this juncture we do not know if this lack of examples is because no such examples exist or if it is because new constructions are needed to create them. With this paper we wish to add to the knowledge of C^* -algebras of real rank zero in the hope to shed some new light on their structure.

The AF-embedding theorem (explained in the abstract) extends a result by Lin [19]. He shows that if A is a C^* -algebra of real rank zero and stable rank one and if $K_0(A)$ is a dimension group, then A contains an AF-sub- C^* -algebra B such that the inclusion mapping induces an isomorphism $K_0(B) \rightarrow K_0(A)$ that maps the dimension range of B onto that of A . (Such an inclusion will necessarily induce an isomorphism between the ideal lattices of A and B .) In our result we do not require A to have stable rank one, nor do we require K_0 to be a dimension group.

The AF-embedding result shows in particular that the primitive ideal space of an arbitrary separable C^* -algebra of real rank zero is homeomorphic to that of the embedded AF-algebra. By a theorem of Kirchberg in which \mathcal{O}_2 -absorbing C^* -algebras are classified by their primitive ideal spaces one can conclude that the tensor product of an arbitrary separable, nuclear stable C^* -algebra of real rank zero by \mathcal{O}_2 is isomorphic to the tensor product of an AF-algebra by \mathcal{O}_2 (Corollary 4.8). It also follows from an older theorem by Bratteli and Elliott that the primitive ideal space of a separable real rank zero C^* -algebra is realized by an AF-algebra (see Theorem 4.5).

In Section 5 we use the AF-embedding result to derive some divisibility properties of real rank zero C^* -algebras (one of which is explained in the abstract). We say that a C^* -algebra A is weakly divisible if for every projection p in A there is a unital $*$ -homomorphism $M_{n_1} \oplus \cdots \oplus M_{n_r} \rightarrow pAp$ whenever $\gcd\{n_1, \dots, n_r\} = 1$. It is shown in Theorem 5.8 that a separable real rank zero algebra is weakly divisible if and only if no representation of A has non-zero intersection with the compact operators.

The proof of the main result has two parts. In the first part, treated in Section 3, one solves the problem at the algebraic level of monoids. In more detail, one associates to each C^* -algebra A a monoid $V(A)$, consisting of Murray–von Neumann equivalence classes of projections in $M_\infty(A)$ (the union of all matrix algebras over A). The monoid $V(A)$ has the so-called refinement property (see Section 3) when A is of real rank zero. It is shown in Section 3 that if M is a monoid with the refinement property then there is a dimension monoid Δ and a surjective monoid morphism $\alpha: \Delta \rightarrow M$ such that for all $x, y \in \Delta$, if $\alpha(x) \leq m\alpha(y)$ for some natural number m , then $x \leq ny$ for some natural number n . (A morphism α with this

property will induce an isomorphism of the ideal lattices of the two monoids Δ and M , see below for the precise definitions.) This result also allows us to recover a representation result of distributive lattices due to Goodearl and Wehrung [15].

The second part of the proof of the embedding result, treated in Section 4, is fairly standard and consists of lifting a monoid morphism $V(B) \rightarrow V(A)$, where B is an AF-algebra and A is an arbitrary (stable) C^* -algebra to a $*$ -homomorphism. (Note that we do not require A to have the cancellation property).

2. Isomorphism of ideal lattices

In this short section we will establish a basic result that motivates the purely monoid theoretic work carried out in the sequel.

For any C^* -algebra A , we will in this paper use the notation $L(A)$ to refer to the lattice of two-sided closed ideals of A . Given a $*$ -homomorphism $\varphi : A \rightarrow B$ between C^* -algebras A and B , there is a natural way of relating the ideal lattice of A with that of B . This is given by the set map $\varphi^{-1} : L(B) \rightarrow L(A)$, defined by taking pre-images of ideals.

We shall be interested in the situation when the map φ^{-1} is an isomorphism, and we refer to that by saying that φ induces an isomorphism of ideal lattices. Note that such a $*$ -homomorphism φ will always be injective (as the zero ideal is contained in its kernel).

For any C^* -algebra A , let us denote by $\text{Proj}(A)$ the set of its projections. If $p, q \in \text{Proj}(A)$, we denote as usual $p \sim q$ to mean that they are Murray–von Neumann equivalent, that is, $p = vv^*$ and $q = v^*v$ for a (partial isometry) v in A . Let $M_\infty(A) = \varinjlim M_n(A)$, under the mappings $M_n(A) \rightarrow M_{n+1}(A)$, $x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$. Denote by $V(A)$ the set of Murray–von Neumann equivalence classes $[p]$ of projections p coming from $M_\infty(A)$. This is an abelian monoid that admits a natural preordering, namely $x \leq y$ if there is a z in $V(A)$ such that $x + z = y$. If A is unital, then the Grothendieck group of $V(A)$ is $K_0(A)$. Any $*$ -homomorphism $\varphi : A \rightarrow B$ induces a monoid morphism $V(\varphi) : V(A) \rightarrow V(B)$ by $V(\varphi)([p]) = [\varphi(p)]$.

Recall that a unital C^* -algebra A has *real rank zero* provided that every self-adjoint element can be approximated arbitrarily well by self-adjoint, invertible elements. If A is non-unital, then A is said to have real rank zero if its minimal unitization \tilde{A} has real rank zero (see [7]). Hereditary subalgebras of C^* -algebras of real rank zero (in particular ideals) have also real rank zero by Brown and Pedersen [7, Corollary 2.8], and hence they can be written as the closed linear span of their projections. It follows from this that two closed ideals in a C^* -algebra of real rank zero will be equal precisely when they have the same projections.

Now let M be an abelian monoid, equipped with its natural (algebraic) ordering. We say that a submonoid I of M is an *order-ideal* (or *o-ideal*) if $x + y \in I$ if and only if $x, y \in I$. (Equivalently, if $x \leq y$, and $y \in I$, then $x \in I$). Denote the ideal lattice of a monoid M by $L(M)$. If $f : M \rightarrow N$ is a monoid morphism, then we say that f induces

an isomorphism of ideal lattices if the set map $f^{-1} : L(N) \rightarrow L(M)$ is an isomorphism. In contrast to the C^* -algebra case, a morphism that induces an isomorphism of ideal lattices need not be injective, due to the absence of kernels in the category of abelian monoids. Let M be an abelian monoid, and let $x, y \in M$. In order to ease the notation in this and the next sections, we shall denote $x \propto y$ to mean that $x \leq ny$ for some n in \mathbb{N} , or, equivalently, x belongs to the order-ideal in M generated by y .

Lemma 2.1. *Let M and N be abelian monoids and let $\alpha : M \rightarrow N$ be a surjective monoid morphism. Suppose that $x \propto y$ for any x, y in M for which $\alpha(x) \propto \alpha(y)$. Then α induces an isomorphism of ideal lattices.*

Proof. Assume that, for order-ideals I and J of N , we have $\alpha^{-1}(I) = \alpha^{-1}(J)$. Let $x \in I$. Then there is y in M such that $\alpha(y) = x$, and so $y \in \alpha^{-1}(I) = \alpha^{-1}(J)$. This says $x = \alpha(y) \in J$. Thus $J \subseteq I$ and by symmetry $J = I$. This proves that $\alpha^{-1} : L(N) \rightarrow L(M)$ is injective.

Next, if I is an order-ideal of M , let $J = \{y \in N \mid y \propto \alpha(x) \text{ for some } x \text{ in } I\}$, which is an order-ideal in N . Clearly $I \subseteq \alpha^{-1}(J)$. If $x \in \alpha^{-1}(J)$, then $\alpha(x) \in J$, so $\alpha(x) \propto \alpha(y)$ for some y in I . Our assumption on α implies that $x \propto y$, hence $x \in I$. \square

Given a C^* -algebra A , there is a natural mapping $L(A) \rightarrow L(V(A))$, given by the rule $I \mapsto V(I)$. If A has real rank zero, then this mapping is a lattice isomorphism, as proved by Zhang [30]. We shall use this fact in the following lemma.

Lemma 2.2. *Let A and B be C^* -algebras with real rank zero, and let $\varphi : A \rightarrow B$ be a $*$ -homomorphism. Then φ induces an isomorphism of ideal lattices if and only if $V(\varphi)$ does.*

Proof. First we note that, if I is an ideal of B , then

$$V(\varphi)^{-1}(V(I)) = V(\varphi^{-1}(I)).$$

Indeed, let $x \in V(\varphi)^{-1}(V(I))$. Write $x = [p]$, where $p \in M_n(A)$, for some n . Then $V(\varphi)(x) = [\varphi(p)] \in V(I)$, so that $\varphi(p) \in M_n(I)$. This means that $p \in \varphi^{-1}(M_n(I))$, hence $x = [p] \in V(\varphi^{-1}(I))$. The converse inclusion is similar.

This says that the diagram

$$\begin{array}{ccc} L(A) & \xrightarrow{\cong} & L(V(A)) \\ \varphi^{-1} \uparrow & & \uparrow V(\varphi)^{-1} \\ L(B) & \xrightarrow{\cong} & L(V(B)) \end{array}$$

is commutative. Therefore the conclusion of the lemma follows at once. \square

Let us summarize our observations in the following:

Corollary 2.3. *Let A and B be C^* -algebras with real rank zero, and let $\varphi : A \rightarrow B$ be a $*$ -homomorphism. Suppose that $V(\varphi) : V(A) \rightarrow V(B)$ is onto and that $x \oslash y$ for any x, y in $V(A)$ for which $V(\varphi)(x) \oslash V(\varphi)(y)$. Then φ is injective and induces an isomorphism of ideal lattices.*

3. Mapping dimension monoids onto refinement monoids

In order to use Corollary 2.3 effectively, we need to gain insight into surjective mappings between monoids that arise as equivalence classes of projections of C^* -algebras with real rank zero. As indicated in the introduction, one of them will actually come from an AF -algebra, and hence will be a dimension monoid (see below).

An abelian monoid M is said to be *conical* provided $x + y = 0$ precisely when $x = y = 0$. Note that if A is a C^* -algebra, then $V(A)$ is a conical monoid. Therefore, in this section, all monoids are assumed to be conical.

Given a natural number r , we shall refer to the monoid $(\mathbb{Z}^+)^r$ as a *simplicial monoid* (of rank r). Observe that such a monoid has a *canonical basis*, namely the one obtained by setting $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ (with 1 in the i th position), for $1 \leq i \leq r$. These basis elements are precisely the minimal non-zero elements of $(\mathbb{Z}^+)^r$. A monoid that can be written as an inductive limit of a sequence of (finitely generated) simplicial monoids and monoid morphisms will be called a *dimension monoid*. (The reason for this terminology is that dimension monoids are precisely the positive cones of dimension groups.)

The structure of $V(A)$ for a general C^* -algebra A with real rank zero can be more intricate than just being a dimension monoid. Still, any such $V(A)$ will enjoy the following important property:

An abelian monoid M is termed a *refinement monoid* (see e.g. [27] or [10]) if whenever $x_1 + x_2 = y_1 + y_2$ in M , then there exist elements z_{ij} in M such that $x_i = z_{i1} + z_{i2}$ and $y_i = z_{1i} + z_{2i}$, for $i = 1, 2$. That $V(A)$ is a refinement monoid for any C^* -algebra A of real rank zero is proved in [1, Lemma 2.3], based on work of Zhang [30].

The next few lemmas are of a technical nature and will be used to assemble the proof of the main monoid-theoretical result (Theorem 3.9). Applications to operator algebras will be given in the next sections.

Let Δ be a simplicial monoid of rank r and canonical basis e_1, \dots, e_r . If I is a finite subset of $\{1, \dots, r\}$, we denote $e_I = \sum_{i \in I} e_i$. If $I = \{1, \dots, r\}$, then we shall denote $e_\Delta = e_I$, and if $I = \emptyset$, then $e_I = 0$.

In the next lemma we shall make a critical use of the refinement property. One key step in the proof is provided by a result of Wehrung [27]. Recall that if M is an abelian monoid and $x, y \in M$, we use the notation $x \oslash y$ to mean $x \leq ny$ for some n in \mathbb{N} .

Lemma 3.1. *Let Δ be a simplicial monoid of rank r . Let M be a refinement monoid and $\alpha : \Delta \rightarrow M$ be a monoid morphism. Suppose that*

$$\alpha(e_1) \propto \alpha(e_2 + \dots + e_r)$$

in M . Then there exist a simplicial monoid Δ' (with $\text{rank}(\Delta') \geq 2(r - 1)$), monoid morphisms $\beta : \Delta \rightarrow \Delta'$ and $\alpha' : \Delta' \rightarrow M$ such that $\alpha(\Delta) \subseteq \alpha'(\Delta')$, the diagram

$$\begin{array}{ccc} \Delta & \xrightarrow{\alpha} & \alpha(\Delta) \\ \beta \downarrow & & \downarrow \iota \\ \Delta' & \xrightarrow{\alpha'} & \alpha'(\Delta') \end{array}$$

is commutative, and $\beta(e_1) \propto \beta(e_2 + \dots + e_r) = e_{\Delta'}$ in Δ' .

Proof. Our assumption means that $\alpha(e_1) \leq n\alpha(e_2 + \dots + e_r)$ for some n in \mathbb{N} .

Since M is a refinement monoid, we can use [27, Lemma 1.9] in order to find elements y_0, \dots, y_n in M such that

$$\alpha(e_1) = \sum_{j=1}^n jy_j, \quad \text{and} \quad \alpha(e_2 + \dots + e_r) = \sum_{j=0}^n y_j.$$

Another use of the refinement property (applied to the second identity above) yields elements x_{ij} in M , for $2 \leq i \leq r$ and $0 \leq j \leq n$ such that

$$\alpha(e_i) = \sum_{j=0}^n x_{ij}, \quad \text{for } i = 2, \dots, r$$

and

$$y_j = \sum_{i=2}^r x_{ij}, \quad \text{for } j = 0, \dots, n.$$

Next, let Δ' be the simplicial monoid of rank $(n + 1)(r - 1)$, and denote its canonical basis by (e_{ij}) , where $2 \leq i \leq r$ and $0 \leq j \leq n$. Now we can define $\alpha' : \Delta' \rightarrow M$ by $\alpha'(e_{ij}) = x_{ij}$. Define $\beta : \Delta \rightarrow \Delta'$ by

$$\beta(e_1) = \sum_{i=2}^r \sum_{j=1}^n je_{ij}, \quad \text{and} \quad \beta(e_i) = \sum_{j=0}^n e_{ij} \quad \text{for } 2 \leq i \leq r.$$

Then

$$\alpha'(\beta(e_1)) = \sum_{i=2}^r \sum_{j=1}^n jx_{ij} = \sum_{j=1}^n jy_j = \alpha(e_1)$$

and

$$\alpha'(\beta(e_i)) = \sum_{j=0}^n x_{ij} = \alpha(e_i).$$

Thus $\alpha' \circ \beta = \alpha$, and in particular we see that $\alpha(\Delta) = (\alpha' \circ \beta)(\Delta) \subseteq \alpha'(\Delta')$. Finally,

$$\beta(e_1) = \sum_{i=2}^r \sum_{j=0}^n j e_{ij} \leq n \sum_{i=2}^r \sum_{j=0}^n e_{ij} = n\beta(e_2 + \dots + e_r),$$

so $\beta(e_1) \propto \beta(e_2 + \dots + e_r)$. Also $\beta(e_2 + \dots + e_r) = \sum_{i=2}^r \sum_{j=0}^n e_{ij} = e_{\Delta'}$. \square

Of course, the sets of indices $\{1\}$ and $\{2, \dots, r\}$ in the above lemma can be replaced by $\{j\}$ and $\{1, \dots, r\} \setminus \{j\}$ to achieve a similar conclusion.

The following easy fact will be used tacitly a number of times in what follows.

Remark 3.2. Let Δ be a simplicial monoid of rank r and canonical basis $\{e_j\}_{1 \leq j \leq r}$, and let $\alpha: \Delta \rightarrow M$ be a monoid morphism, where M is an abelian monoid. If $I \subseteq \{1, \dots, r\}$ and $y \in M$, then $\alpha(e_I) \propto y$ if and only if $\alpha(e_i) \propto y$ for all i in I . Indeed, to prove the less trivial part, assume that there are natural numbers n_i , for i in I , such that $\alpha(e_i) \leq n_i y$ for all i . Then $\alpha(e_I) = \sum_{i \in I} \alpha(e_i) \leq (\sum_i n_i) y \propto y$.

Lemma 3.3. Let Δ be a simplicial monoid of rank r . Let I, J be non-empty subsets of $\{1, \dots, r\}$ such that $I \cap J = \emptyset$. Suppose that there is a refinement monoid M and a monoid morphism $\alpha: \Delta \rightarrow M$ such that $\alpha(e_J) \propto \alpha(e_I)$.

Let $j \in J$. Then there are a simplicial monoid Δ' with $\text{rank}(\Delta') = s \geq 2|I| + |J| - 1$ (and canonical basis denoted by $\{f_k\}_{k=1}^s$), monoid morphisms $\alpha': \Delta' \rightarrow M$ and $\beta: \Delta \rightarrow \Delta'$ such that the following conditions hold:

- (i) $\alpha(\Delta) \subseteq \alpha'(\Delta')$.
- (ii) The diagram

$$\begin{array}{ccc} \Delta & \xrightarrow{\alpha} & \alpha(\Delta) \\ \beta \downarrow & & \downarrow \iota \\ \Delta' & \xrightarrow{\alpha'} & \alpha'(\Delta') \end{array}$$

is commutative.

- (iii) $\beta(e_k) = f_k$ for all $k \in J \setminus \{j\}$.
- (iv) There is a subset K of $\{1, \dots, s\}$ such that $K \cap (J \setminus \{j\}) = \emptyset$, $\beta(e_j) \propto \beta(e_I) = f_K$, and $\alpha'(f_{J \setminus \{j\}}) \propto \alpha'(f_K)$.

Proof. Write $\Delta = \Delta_1 \oplus \Delta_2$ (i.e. $\Delta = \Delta_1 + \Delta_2$ and $\Delta_1 \cap \Delta_2 = 0$). Here Δ_1 is the simplicial submonoid of Δ with basis elements indexed by $\{1, \dots, r\} \setminus (I \cup \{j\})$, so it has rank at least $|J| - 1$, and Δ_2 is spanned by the remaining basis elements. Define $\alpha_2: \Delta_2 \rightarrow M$ by restriction of α . By assumption, we have that $\alpha_2(e_j) \propto \alpha_2(e_I)$.

By Lemma 3.1, there are a simplicial monoid Δ'_2 (of rank at least $2|I|$), monoid morphisms $\alpha'_2 : \Delta'_2 \rightarrow M$, $\beta_2 : \Delta_2 \rightarrow \Delta'_2$ such that $\alpha_2(\Delta_2) \subseteq \alpha'_2(\Delta'_2)$, the diagram

$$\begin{array}{ccc} \Delta_2 & \xrightarrow{\alpha_2} & \alpha_2(\Delta_2) \\ \beta_2 \downarrow & & \downarrow \iota \\ \Delta'_2 & \xrightarrow{\alpha'_2} & \alpha'_2(\Delta'_2) \end{array}$$

is commutative, and $\beta_2(e_j) \propto \beta_2(e_I)$.

Set $\Delta' = \Delta_1 \oplus \Delta'_2$ (this is an external direct sum). The canonical basis $\{f_k\}$ of Δ' is obtained by taking the union of the canonical bases of Δ_1 and Δ'_2 .

Note that $\text{rank}(\Delta') = \text{rank}(\Delta_1) + \text{rank}(\Delta'_2) \geq 2|I| + |J| - 1$. Define $\beta : \Delta \rightarrow \Delta'$ by $\beta|_{\Delta_1} = \text{id}$, and by $\beta|_{\Delta_2} = \beta_2$. Define $\alpha' : \Delta' \rightarrow M$ by $\alpha'|_{\Delta_1} = \alpha$ and $\alpha'|_{\Delta'_2} = \alpha'_2$. Observe that we get

$$\alpha(\Delta) = \alpha(\Delta_1) + \alpha(\Delta_2) = \alpha(\Delta_1) + \alpha_2(\Delta_2) \subseteq \alpha(\Delta_1) + \alpha'_2(\Delta'_2) = \alpha'(\Delta_1) + \alpha'(\Delta'_2) = \alpha'(\Delta'),$$

and also the following commutative diagram:

$$\begin{array}{ccc} \Delta = \Delta_1 \oplus \Delta_2 & \xrightarrow{\alpha} & \alpha(\Delta) \\ \beta \downarrow & & \downarrow \iota \\ \Delta' = \Delta_1 \oplus \Delta'_2 & \xrightarrow{\alpha'} & \alpha'(\Delta') \end{array}$$

Hence conditions (i) and (ii) are fulfilled. By construction $\beta(e_k) = f_k$ for any k in $J \setminus \{j\}$ (as they belong to Δ_1). Thus condition (iii) also holds.

Note that $\beta(e_j) = \beta_2(e_j) \propto \beta_2(e_I) = \beta(e_I)$.

Let K be the set of indices corresponding to a basis of Δ'_2 . Clearly, $K \cap (J \setminus \{j\}) = \emptyset$. By definition of β and Lemma 3.1, we see that $\beta(e_I) = \beta_2(e_I) = f_{\Delta'_2} = f_K$.

Finally, $\alpha'(f_{J \setminus \{j\}}) = \alpha'(\beta((e_{J \setminus \{j\}}))) = \alpha(e_{J \setminus \{j\}}) \propto \alpha(e_I) = \alpha'(\beta(e_I)) = \alpha'(f_K)$. This verifies condition (iv). \square

Lemma 3.4. *Let Δ be a simplicial monoid of rank r . Let I, J be non-empty subsets of $\{1, \dots, r\}$ such that $I \cap J = \emptyset$. Suppose there is a refinement monoid M and a monoid morphism $\alpha : \Delta \rightarrow M$ such that $\alpha(e_J) \propto \alpha(e_I)$.*

Then there are a simplicial monoid Δ' , monoid morphisms $\alpha' : \Delta' \rightarrow M$ and $\beta : \Delta \rightarrow \Delta'$ such that $\alpha(\Delta) \subseteq \alpha'(\Delta')$, the diagram

$$\begin{array}{ccc} \Delta & \xrightarrow{\alpha} & \alpha(\Delta) \\ \beta \downarrow & & \downarrow \iota \\ \Delta' & \xrightarrow{\alpha'} & \alpha'(\Delta') \end{array}$$

is commutative, and $\beta(e_J) \propto \beta(e_I)$.

Proof. We proceed by induction on $|J|$. If $|J| = 1$, the conclusion is provided by (the argument of) Lemma 3.1.

Assume that $|J| \geq 2$ and that the conclusion holds for cardinals smaller than $|J|$. Let j be the first element in J (in the natural ordering). By Lemma 3.3, there are a simplicial monoid Δ'_1 with rank $(\Delta'_1) = s \geq 2|I| + |J| - 1$ (and canonical basis denoted by $\{f_k\}_{k=1}^s$), monoid morphisms $\alpha'_1 : \Delta'_1 \rightarrow M$ and $\beta_1 : \Delta \rightarrow \Delta'_1$ such that $\alpha(\Delta) \subseteq \alpha'_1(\Delta'_1)$, the diagram

$$\begin{array}{ccc} \Delta & \xrightarrow{\alpha} & \alpha(\Delta) \\ \beta_1 \downarrow & & \downarrow \iota \\ \Delta'_1 & \xrightarrow{\alpha'_1} & \alpha'_1(\Delta'_1) \end{array}$$

is commutative, $\beta_1(e_k) = f_k$ for all $k \in J \setminus \{j\}$, $\beta_1(e_j) \circ \beta_1(e_I) = f_K$, and $\alpha'_1(f_{J \setminus \{j\}}) \circ \alpha'_1(f_K)$, for some subset K of $\{1, \dots, s\}$ such that $K \cap (J \setminus \{j\}) = \emptyset$.

By the induction hypothesis applied to $\Delta'_1, K, J \setminus \{j\}, \alpha'_1$, there are a simplicial monoid Δ' , monoid morphisms $\alpha' : \Delta' \rightarrow M$ and $\beta_2 : \Delta'_1 \rightarrow \Delta'$ such that $\alpha'_1(\Delta'_1) \subseteq \alpha'(\Delta')$, the diagram

$$\begin{array}{ccc} \Delta'_1 & \xrightarrow{\alpha'_1} & \alpha'_1(\Delta'_1) \\ \beta_2 \downarrow & & \downarrow \iota \\ \Delta' & \xrightarrow{\alpha'} & \alpha'(\Delta') \end{array}$$

is commutative, and $\beta_2(f_{J \setminus \{j\}}) \circ \beta_2(f_K)$.

Next, define $\beta = \beta_2 \circ \beta_1$. We have that

$$\begin{aligned} \beta(e_J) &= \beta_2(\beta_1(e_J)) = \beta_2(\beta_1(e_j)) + \beta_2(\beta_1(e_{J \setminus \{j\}})) \\ &\circ \beta_2(\beta_1(e_I)) + \beta_2(f_{J \setminus \{j\}}) \\ &\circ \beta_2(\beta_1(e_I)) + \beta_2(f_K) \\ &= \beta_2(\beta_1(e_I)) + \beta_2(\beta_1(e_I)) = 2\beta_2(\beta_1(e_I)) \circ \beta(e_I), \end{aligned}$$

as desired. \square

Let Δ be a simplicial monoid of rank r with canonical basis denoted by $\{e_i\}_{1 \leq i \leq r}$. For any j , we have the coordinate monoid morphism $\pi_j : \Delta \rightarrow \mathbb{Z}^+$, i.e. $\pi_j(x) = \alpha_j$ provided that $x = \sum_{i=1}^r \alpha_i e_i$ (with all α_i in \mathbb{Z}^+).

For any element x in Δ , we define its *support* as the following subset of $\{1, \dots, r\}$

$$\text{supp}(x) = \{j \mid \pi_j(x) \neq 0\}.$$

Lemma 3.5. *Let Δ be a simplicial monoid of rank r and canonical basis $\{e_j\}_{1 \leq j \leq r}$. Suppose that M is an abelian monoid and that $\alpha : \Delta \rightarrow M$ is a monoid morphism. For elements x and y in Δ with $J = \text{supp}(x)$ and $I = \text{supp}(y)$, the following conditions are equivalent:*

- (i) $\alpha(x) \propto \alpha(y)$.
- (ii) $\alpha(e_J) \propto \alpha(e_I)$.
- (iii) $\alpha(e_{J \setminus I}) \propto \alpha(e_I)$.

Proof. Write $x = \sum_{j \in J} \alpha_j e_j$ and $y = \sum_{i \in I} \beta_i e_i$, where $\alpha_j \geq 1$ and $\beta_i \geq 1$ for all j and i .

(i) \Rightarrow (ii): Assume $\alpha(x) \propto \alpha(y)$. Then, for any j in J we have that

$$\alpha(e_j) \leq \alpha_j \alpha(e_j) \leq \alpha(x) \propto \alpha(y) = \sum_{i \in I} \beta_i \alpha(e_i) \leq \max\{\beta_i \mid i \in I\} \alpha(e_I) \propto \alpha(e_I).$$

Hence $\alpha(e_J) \propto \alpha(e_I)$.

(ii) \Rightarrow (iii): Trivial.

(iii) \Rightarrow (i): Assume that $\alpha(e_{J \setminus I}) \propto \alpha(e_I)$. Then

$$\begin{aligned} \alpha(x) &= \sum_{j \in J} \alpha_j \alpha(e_j) = \sum_{j \in J \setminus I} \alpha_j \alpha(e_j) + \sum_{j \in J \cap I} \alpha_j \alpha(e_j) \\ &\leq \max\{\alpha_j \mid j \in J \setminus I\} \alpha(e_{J \setminus I}) + \max\{\alpha_j \mid j \in J \cap I\} \alpha(e_{J \cap I}) \\ &\propto 2\alpha(e_I) \propto \sum_{i \in I} \beta_i \alpha(e_i) = \alpha(y). \quad \square \end{aligned}$$

Lemma 3.6. *Let Δ be a simplicial monoid of rank r . Let M be a refinement monoid and $\alpha : \Delta \rightarrow M$ be a monoid morphism. Suppose that x and y in Δ satisfy $\alpha(x) \propto \alpha(y)$.*

Then there are a simplicial monoid Δ' , monoid morphisms $\alpha' : \Delta' \rightarrow M$ and $\beta : \Delta \rightarrow \Delta'$ such that $\alpha(\Delta) \subseteq \alpha'(\Delta')$, the diagram

$$\begin{array}{ccc} \Delta & \xrightarrow{\alpha} & \alpha(\Delta) \\ \beta \downarrow & & \downarrow \iota \\ \Delta' & \xrightarrow{\alpha'} & \alpha'(\Delta') \end{array}$$

is commutative, and $\beta(x) \propto \beta(y)$.

Proof. Write $J = \text{supp}(x)$ and $I = \text{supp}(y)$. We may clearly assume that $J \neq \emptyset$. If $I = \emptyset$, then $y = 0$ and our assumption means that $\alpha(x) \leq 0$. By conicality, $\alpha(x) = 0$. If $x = \sum_{j \in J} \alpha_j e_j$, then $\alpha(e_j) = 0$ for every j in J . Take $\Delta' = \Delta$, and set $\beta(e_j) = e_j$ if $j \notin J$, $\beta(e_j) = 0$ otherwise. Take $\alpha' = \alpha$. Clearly, $\alpha \circ \beta = \alpha$ and $\beta(x) = 0$.

So, we may assume that I and J are both non-empty. By Lemma 3.5, our assumption means that $\alpha(e_{J \setminus I}) \propto \alpha(e_I)$. We can therefore use Lemma 3.4 to find a

simplicial monoid Δ' , monoid morphisms $\alpha': \Delta' \rightarrow M$ and $\beta: \Delta \rightarrow \Delta'$ such that $\alpha(\Delta) \subseteq \alpha'(\Delta')$, the diagram

$$\begin{array}{ccc} \Delta & \xrightarrow{\alpha} & \alpha(\Delta) \\ \beta \downarrow & & \downarrow \iota \\ \Delta' & \xrightarrow{\alpha'} & \alpha'(\Delta') \end{array}$$

is commutative, and $\beta(e_{JI}) \in \beta(e_I)$. Now Lemma 3.5 implies that $\beta(x) \in \beta(y)$, as desired. \square

Lemma 3.7. *Let Δ be a simplicial monoid of rank r , and let $\alpha: \Delta \rightarrow M$ be a monoid morphism, where M is a refinement monoid. Then there are a simplicial monoid Δ' , monoid morphisms $\beta: \Delta \rightarrow \Delta'$ and $\alpha': \Delta' \rightarrow M$ such that $\alpha(\Delta) \subseteq \alpha'(\Delta')$, the diagram*

$$\begin{array}{ccc} \Delta & \xrightarrow{\alpha} & \alpha(\Delta) \\ \beta \downarrow & & \downarrow \iota \\ \Delta' & \xrightarrow{\alpha'} & \alpha'(\Delta') \end{array}$$

is commutative, and such that $\beta(x) \in \beta(y)$ whenever x and y are elements in Δ with $\alpha(x) \in \alpha(y)$.

Proof. Our Lemma 3.5 reduces the comparison of $\alpha(x)$ and $\alpha(y)$ (with respect to the relation \in) to the comparison of their respective supports. Therefore it suffices to arrange the construction for a finite set of pairs written as $\{(x_i, y_i) \mid 1 \leq i \leq n\}$ (for some n in \mathbb{N}) such that $\alpha(x_i) \in \alpha(y_i)$ in M for every i .

This is done by iteration of Lemma 3.6, starting with the pair (x_1, y_1) . Notice that in this first step, we get the following diagram

$$\begin{array}{ccc} \Delta & \xrightarrow{\alpha} & \alpha(\Delta) \\ \beta_1 \downarrow & & \downarrow \iota \\ \Delta'_1 & \xrightarrow{\alpha'_1} & \alpha'_1(\Delta'_1) \end{array}$$

For the next pair (x_2, y_2) , just observe that $\alpha'_1(\beta_1(x_2)) = \alpha(x_2) \in \alpha(y_2) = \alpha'_1(\beta_1(x_2))$, hence the process can be reiterated. \square

Proposition 3.8. *Let $M = \{x_0, x_1, \dots\}$ be a countable refinement monoid. There is then a commutative diagram:*

$$\begin{array}{ccc}
 \Delta_0 & \xrightarrow{\alpha_0} & M_0 \\
 \beta_0 \downarrow & & \downarrow \iota \\
 \Delta_1 & \xrightarrow{\alpha_1} & M_1 \\
 \beta_1 \downarrow & & \downarrow \iota \\
 \Delta_2 & \xrightarrow{\alpha_2} & M_2 \\
 \beta_2 \downarrow & & \downarrow \iota \\
 \vdots & & \vdots
 \end{array} \tag{3.1}$$

such that $\{x_0, \dots, x_j\} \subseteq M_j$ for all j , each Δ_j is a simplicial monoid, the morphisms $\alpha_j : \Delta_j \rightarrow M_j$ are surjective, and for each j and each pair of elements x and y in Δ_j for which $\alpha_j(x) \propto \alpha_j(y)$ in M , it follows that $\beta_j(x) \propto \beta_j(y)$ in Δ_{j+1} .

Proof. Let $M_0 = \langle x_0 \rangle$ be the submonoid generated by x_0 , and let Δ_0 be the simplicial monoid of rank 1, with basis $\{e_0\}$. Define $\alpha_0 : \Delta_0 \rightarrow M$ by $\alpha_0(e_0) = x_0$. Clearly, $\alpha_0(\Delta_0) = M_0$.

We can now use Lemma 3.7 to find a simplicial monoid Δ'_1 , monoid morphisms $\beta'_0 : \Delta_0 \rightarrow \Delta'_1$ and $\alpha'_1 : \Delta'_1 \rightarrow M$ such that $M_0 \subseteq \alpha'_1(\Delta'_1)$, the diagram

$$\begin{array}{ccc}
 \Delta_0 & \xrightarrow{\alpha_0} & M_0 \\
 \beta'_0 \downarrow & & \downarrow \iota \\
 \Delta'_1 & \xrightarrow{\alpha'_1} & \alpha'_1(\Delta'_1)
 \end{array}$$

is commutative, and $\beta'_0(x) \propto \beta'_0(y)$ for any pair of elements x, y in Δ_0 such that $\alpha_0(x) \propto \alpha_0(y)$ (in M).

If $x_1 \in \alpha'_1(\Delta'_1)$, then set $\Delta_1 = \Delta'_1$, $\alpha_1 = \alpha'_1$, $\beta_0 = \beta'_0$ and $M_1 = \alpha_1(\Delta_1)$. Otherwise, set $\Delta_1 = \Delta'_1 \oplus \langle e \rangle$, where e is an additional basis element, so that Δ_1 is also a simplicial monoid. Define $\alpha_1 : \Delta_1 \rightarrow M$ by $\alpha_1|_{\Delta'_1} = \alpha'_1$ and $\alpha_1(e) = x_1$, and set $M_1 = \alpha_1(\Delta_1)$, a submonoid of M . Define also $\beta_0 : \Delta_0 \rightarrow \Delta_1$ as the composition of β'_0 with the natural inclusion of Δ'_1 in Δ_1 .

Suppose that $\Delta_j, M_j, \alpha_j : \Delta_j \rightarrow M_j$ and $\beta_{j-1} : \Delta_{j-1} \rightarrow \Delta_j$ have been constructed satisfying the requirements of our statement. Apply Lemma 3.7 to obtain Δ'_{j+1} , morphisms $\alpha'_{j+1} : \Delta'_{j+1} \rightarrow M$ and $\beta'_j : \Delta_j \rightarrow \Delta'_{j+1}$ such that $M_j \subseteq \alpha'_{j+1}(\Delta'_{j+1})$, the diagram

$$\begin{array}{ccc}
 \Delta_j & \xrightarrow{\alpha_j} & M_j \\
 \beta'_j \downarrow & & \downarrow \iota \\
 \Delta'_{j+1} & \xrightarrow{\alpha'_{j+1}} & \alpha'_{j+1}(\Delta'_{j+1})
 \end{array}$$

is commutative, and $\beta'_j(x) \circ \beta'_j(y)$ for every pair of elements x, y in Δ_j such that $\alpha_j(x) \circ \alpha_j(y)$ in M . As before, if $x_{j+1} \in \alpha'_{j+1}(\Delta'_{j+1})$, we set $M_{j+1} = \alpha'_{j+1}(\Delta'_{j+1})$, $\alpha_{j+1} = \alpha'_{j+1}$, $\beta_j = \beta'_j$. Otherwise, let $\Delta_{j+1} = \Delta'_{j+1} \oplus \langle e' \rangle$, where e' is an additional basis element. Define $\alpha_{j+1|_{\Delta'_{j+1}}} = \alpha'_{j+1}$, $\alpha_{j+1}(e') = x_{j+1}$ and β_j as the composition of β'_j with the natural inclusion of Δ'_{j+1} in Δ_{j+1} .

The proof follows then by induction. \square

We are now ready to prove the main result of this section.

Theorem 3.9. *Let M be a countable refinement monoid. Then there is a dimension monoid Δ and a surjective morphism $\alpha : \Delta \rightarrow M$ such that $x \circ y$ for every pair of elements x, y in Δ that satisfy $\alpha(x) \circ \alpha(y)$.*

Proof. Write $M = \{x_0, x_1, \dots\}$. We take a commutative diagram of simplicial monoids Δ_j and finitely generated submonoids M_j of M as diagram 3.1 in Proposition 3.8.

Define $\Delta = \varinjlim (\Delta_j, \beta_j)$, which is a dimension monoid. The diagram (3.1) induces a map $\alpha : \Delta \rightarrow M$ by the universal property of inductive limits. Denote by $\beta_{\infty, j} : \Delta_j \rightarrow \Delta$ the inductive limit maps, and observe that we have a commutative diagram

$$\begin{array}{ccc} \Delta_j & \xrightarrow{\alpha_j} & M_j \\ \beta_{\infty, j} \downarrow & & \downarrow \iota \\ \Delta & \xrightarrow{\alpha} & M \end{array}$$

Note that

$$\text{Im}(\alpha) = \bigcup_j \alpha_j(\Delta_j) = \bigcup_j M_j = M,$$

and so α is surjective.

Next, let $x, y \in \Delta$, and assume that $\alpha(x) \circ \alpha(y)$. There exist then $j \geq 0$ and elements x_j, y_j in Δ_j such that $\beta_{\infty, j}(x_j) = x$ and $\beta_{\infty, j}(y_j) = y$. Now, $\alpha_j(x_j) = \alpha(x) \circ \alpha(y) = \alpha_j(y_j)$, and hence $\beta_j(x_j) \circ \beta_j(y_j)$ in Δ_{j+1} , by Proposition 3.8. But then

$$x = \beta_{\infty, j}(x_j) = \beta_{\infty, j+1}(\beta_j(x_j)) \circ \beta_{\infty, j+1}(\beta_j(y_j)) = \beta_{\infty, j}(y_j) = y,$$

and this finishes the proof. \square

Corollary 3.10 (Cf. Goodearl and Wehrung [15]). *Let M be a countable refinement monoid. There exists then a dimension monoid Δ such that $L(\Delta) \cong L(M)$.*

Proof. Follows immediately from Theorem 3.9 and Lemma 2.1. \square

The preceding corollary was already known (although maybe not with this precise formulation) and is related to a result of Bergman about realizing distributive algebraic lattices as ideal lattices of von Neumann regular rings ([2], see also [15,28,29]). We make the connection more explicit as follows (see also the comments after Theorem 4.5).

Let M be a monoid. Define a congruence on M by writing $x \asymp y$ if $x \circ y$ and $y \circ x$. Set $\nabla(M) = M / \asymp$, which in the literature is referred to as the *maximal semilattice quotient* of M (see e.g. [8]). If M is a refinement monoid, then $\nabla(M)$ is also a refinement monoid, see [15, Lemma 2.4]. This is an example of a distributive 0-semilattice (which is, by definition, a commutative refinement monoid such that $x + x = x$ for all x).

It was asked in [15, Problem 10.1] (see also [24, Problem 3]) whether for a distributive 0-semilattice S there is a dimension group G such that $\nabla(G^+) \cong S$ (where G^+ is the positive cone of G). The cases where S is in fact a lattice or where S is countable were handled in [15], with positive solutions.

Let us note first that the countable case [15, Theorem 5.2] can also be obtained from our results above. Namely, given a countable distributive 0-semilattice S , observe first that $\nabla(S) \cong S$. Since S is a refinement monoid, there exists by Theorem 3.9 a dimension monoid N and a surjective monoid morphism $\alpha : N \rightarrow S$ such that $x \circ y$ whenever x and y in N satisfy $\alpha(x) \circ \alpha(y)$. This morphism clearly induces an isomorphism $\nabla(N) \cong \nabla(S)$, given by $[x] \mapsto [\alpha(x)]$. (Here $[x]$ denotes the class of the element x modulo the congruence \asymp).

We now indicate how to prove Corollary 3.10 from the methods in [15]. We use first that if M is a countable refinement monoid, then $\nabla(M)$ is a countable distributive 0-semilattice. By Goodearl and Wehrung [15, Theorem 5.2], there is a dimension monoid N such that $\nabla(N) \cong \nabla(M)$, and applying [15, Proposition 2.6 (iii)], we find that $L(N) \cong L(\nabla(N)) \cong L(\nabla(M)) \cong L(M)$, as desired.

In light of Theorem 3.9, it is also natural to ask whether a similar result is available if the refinement monoid M is not countable. The answer to this question is negative in general. Specifically, there are semilattices S of size \aleph_2 that cannot be isomorphic to $\nabla(N)$ for any dimension monoid N , as proved by Růžička [23]. Counterexamples for semilattices of size \aleph_1 were later obtained by Wehrung [29].

4. Applications to operator algebras

Let A be a C^* -algebra. Recall (from Section 2) that $V(A)$ is the monoid of Murray–von Neuman equivalence classes of projections in $M_\infty(A)$, and that $V(A)$ is a refinement monoid if A is of real rank zero. Define the *dimension range* of A to be the subset $D(A)$ of $V(A)$ given by

$$D(A) = \{[p] \in V(A) \mid p \text{ is a projection in } A\}.$$

The subset $D(A)$ is a *partial monoid* in the sense that addition in $D(A)$ is only partially defined. Any $*$ -homomorphism $\varphi: A \rightarrow B$ induces a monoid morphism $V(\varphi): V(A) \rightarrow V(B)$ defined by $[p] \mapsto [\varphi(p)]$, and it satisfies $V(\varphi)(D(A)) \subseteq D(B)$. If A is unital, then an element x in $V(A)$ belongs to $D(A)$ if and only if $x \leq [1_A]$. If A is stable, then $D(A) = V(A)$.

The following two lemmas are well-known in the case where the target algebra A has the cancellation property (in which case these statements are true with $V(A)$ replaced with $K_0(A)$); see for example [22, Lemma 7.3.2].

Lemma 4.1. *Let B be a finite dimensional C^* -algebra and let A be a unital C^* -algebra.*

- (i) *Suppose that $\varphi, \psi: B \rightarrow A$ are unital $*$ -homomorphisms that satisfy $V(\varphi) = V(\psi)$. Then there is a unitary u in A such that $u\varphi(x)u^* = \psi(x)$ for all x in B .*
- (ii) *Suppose that $\alpha: V(B) \rightarrow V(A)$ is a monoid morphism that satisfies $\alpha([1_B]) = [1_A]$. Then there is a unital $*$ -homomorphism $\varphi: B \rightarrow A$ for which $V(\varphi) = \alpha$. If $e \in B$ and $f \in A$ are projections such that $\alpha([e]) = [f]$ and $\alpha([1_B - e]) = [1_A - f]$, then φ above can be chosen satisfying $\varphi(e) = f$.*

Proof. Write $B = M_{n_1} \oplus M_{n_2} \oplus \dots \oplus M_{n_r}$ and let $\{e_{ij}^{(k)}\}$ be a system of matrix units for B (where $1 \leq k \leq r$ and $1 \leq i, j \leq n_k$).

(i) Since $\varphi(e_{11}^{(k)}) \sim \psi(e_{11}^{(k)})$ for all k , we can find partial isometries v_k in A such that $v_k^* v_k = \varphi(e_{11}^{(k)})$ and $v_k v_k^* = \psi(e_{11}^{(k)})$. Set

$$u = \sum_{k=1}^r \sum_{i=1}^{n_k} \psi(e_{i1}^{(k)}) v_k \varphi(e_{1i}^{(k)}).$$

One can now check that u is a unitary element in A with the required properties.

(ii) The matrix units $\{e_{ij}^{(k)}\}$ for B can be chosen such that $e = \sum_{(i,k) \in \Gamma} e_{ii}^{(k)}$ for some subset Γ of

$$\Omega := \{(i, k) \mid k = 1, 2, \dots, r, i = 1, 2, \dots, n_k\}.$$

We then have $1_B - e = \sum_{(i,k) \in \Gamma'} e_{ii}^{(k)}$, when $\Gamma' = \Omega \setminus \Gamma$.

Find, for some large enough m , pairwise orthogonal projections $g_i^{(k)}$ in $M_m(A)$ such that $[g_i^{(k)}] = \alpha([e_{ii}^{(k)}])$ for all $(i, k) \in \Omega$. Put $g = \sum_{(i,k) \in \Gamma} g_i^{(k)}$ and $g' = \sum_{(i,k) \in \Gamma'} g_i^{(k)}$. Then

$$[g] = \sum_{(i,k) \in \Gamma} [g_i^{(k)}] = \sum_{(i,k) \in \Gamma} \alpha([e_{ii}^{(k)}]) = \alpha\left(\left[\sum_{(i,k) \in \Gamma} e_{ii}^{(k)}\right]\right) = \alpha([e]) = [f],$$

and, similarly, $[g'] = [1_A - f]$. It follows that

$$g = v^*v, \quad f = vv^*, \quad g' = w^*w, \quad 1_A - f = ww^*,$$

for some partial isometries v and w in $M_{1m}(A)$. Put $u = v + w$ and put $f_{ii}^{(k)} = ug_i^{(k)}u^*$ for all (i, k) in Ω . Then $\{f_{ii}^{(k)}\}$ are pairwise orthogonal projections in A ,

$$[f_{ii}^{(k)}] = [g_i^{(k)}] = \alpha([e_{ii}^{(k)}]), \quad \sum_{(i,k) \in \Gamma} f_{ii}^{(k)} = f, \quad \sum_{(i,k) \in \Gamma'} f_{ii}^{(k)} = 1_A - f.$$

Now, $[f_{ii}^{(k)}] = \alpha([e_{ii}^{(k)}]) = \alpha([e_{jj}^{(k)}]) = [f_{jj}^{(k)}]$, that is, $f_{ii}^{(k)} \sim f_{jj}^{(k)}$ for fixed k and for all i and j . It is standard (see for example [22, Lemma 7.1.2]) that the system $\{f_{ii}^{(k)}\}$ extends to a system of matrix units $\{f_{ij}^{(k)}\}$ for $B = M_{n_1} \oplus M_{n_2} \oplus \dots \oplus M_{n_r}$, and that there is a $*$ -homomorphism $\varphi : B \rightarrow A$ given by $\varphi(e_{ij}^{(k)}) = f_{ij}^{(k)}$.

We must check that φ has the desired properties, and note first

$$\varphi(e) = \varphi\left(\sum_{(i,k) \in \Gamma} e_{ii}^{(k)}\right) = \sum_{(i,k) \in \Gamma} f_{ii}^{(k)} = f,$$

and, similarly, $\varphi(1_B - e) = 1_A - f$. In particular, $\varphi(1_B) = 1_A$. Finally, as $[e_{ii}^{(k)}]$ generate $V(B)$ and

$$V(\varphi)([e_{ii}^{(k)}]) = [\varphi([e_{ii}^{(k)}])] = [f_{ij}^{(k)}] = \alpha([e_{ii}^{(k)}]),$$

we conclude that $V(\varphi) = \alpha$. \square

Lemma 4.2. *Let A be a stable C^* -algebra, let B be an AF-algebra, and let $\alpha : V(B) \rightarrow V(A)$ be a monoid morphism. Then there is a $*$ -homomorphism $\varphi : B \rightarrow A$ such that $V(\varphi) = \alpha$.*

Proof. The proof consists of constructing the commutative diagram:

$$\begin{array}{ccccccc}
 B_1 & \xrightarrow{\psi_1} & B_2 & \xrightarrow{\psi_2} & B_3 & \xrightarrow{\psi_3} & \dots \longrightarrow B \\
 \varphi_1 \downarrow & & \varphi_2 \downarrow & & \varphi_3 \downarrow & & \downarrow \varphi \\
 f_1 A f_1 & \xrightarrow{\iota} & f_2 A f_2 & \xrightarrow{\iota} & f_3 A f_3 & \xrightarrow{\iota} & \dots \longrightarrow A_0
 \end{array} \tag{4.1}$$

where B is written as an inductive limit of finite dimensional C^* -algebras B_n with connecting maps ψ_n (that are not necessarily unital). The projections f_1, f_2, \dots , constructed below, form an increasing sequence, and they satisfy $[f_n] = \alpha([\psi_{\infty, n}(1_{B_n})])$, where $\psi_{\infty, n} : B_n \rightarrow B$ is the inductive limit map. The algebra A_0

in (4.1) is the hereditary sub- C^* -algebra of A given by

$$A_0 = \overline{\bigcup_{n=1}^{\infty} f_n A f_n}.$$

We proceed to find the projections f_n . Put $e_n = \psi_{\infty,n}(1_{B_n})$. Then e_1, e_2, \dots is an increasing sequence of projections in B . Since A is stable we can find pairwise orthogonal projections g_1, g_2, \dots in A such that $[g_1] = \alpha([e_1])$ and $[g_n] = \alpha([e_n - e_{n-1}])$ for $n \geq 2$. Put $f_n = g_1 + g_2 + \dots + g_n$. Then f_1, f_2, \dots is an increasing sequence of projections in A and $[f_n] = \alpha([\psi_{\infty,n}(1_{B_n})]) = (\alpha \circ V(\psi_{\infty,n}))([1_{B_n}])$.

The next step is to find unital $*$ -homomorphisms $\varphi_n : B_n \rightarrow f_n A f_n$ making diagram (4.1) commutative and such that $V(\varphi_n) = \alpha \circ V(\psi_{\infty,n})$. The existence of φ_1 follows from Lemma 4.1(ii) (with $e = f = 0$). At step two, use again Lemma 4.1(ii) to find a unital $*$ -homomorphism $\psi_2 : B_2 \rightarrow f_2 A f_2$ such that $V(\psi_2) = \alpha \circ V(\psi_{\infty,2})$ and $\psi_2(\varphi_1(1_{B_1})) = f_1$. The two $*$ -homomorphisms $\psi_2 \circ \varphi_1$ and $\iota \circ \varphi_1$ co-restrict to unital $*$ -homomorphisms $B_1 \rightarrow f_1 A f_1$, and $V(\psi_2 \circ \varphi_1) = V(\iota \circ \varphi_1)$. Use Lemma 4.1(i) to find a unitary u_0 in $f_1 A f_1$ such that $\text{Ad}_{u_0} \circ \psi_2 \circ \varphi_1 = \iota \circ \varphi_1$. Put $u = u_0 + (f_2 - f_1)$ (which is a unitary in $f_2 A f_2$) and put $\varphi_2 = \text{Ad}_u \circ \psi_2$. Then φ_2 is a unital $*$ -homomorphism, $V(\varphi_2) = \alpha \circ V(\psi_{\infty,2})$, and the first square in the diagram (4.1) commutes.

Continuing in this way we find the remaining $*$ -homomorphisms φ_n making the successive squares in diagram (4.1) commutative. By the universal property of inductive limits there is a $*$ -homomorphism $\varphi : B \rightarrow A$ that makes (4.1) commutative. For x in $V(B_n)$ one has

$$V(\varphi)(V(\psi_{\infty,n})(x)) = V(\varphi \circ \psi_{\infty,n})(x) = V(\varphi_n)(x) = \alpha(V(\psi_{\infty,n})(x)).$$

The functor V is continuous, which entails that $V(B) = \bigcup_{n=1}^{\infty} V(\psi_{\infty,n})(V(B_n))$. We can now conclude that $V(\varphi) = \alpha$. \square

At first sight it is tempting to believe that Lemma 4.2 holds without the assumption that A is stable, but with the extra assumption $\alpha(D(B)) \subseteq D(A)$. At second thought, it appears plausible (to the authors) that there might exist a (non-stable) C^* -algebra A for which there is a monoid morphism $\alpha : V(\mathcal{K}) \rightarrow V(A)$, that satisfies $\alpha(D(\mathcal{K})) \subseteq D(A)$, and that does not lift to a $*$ -homomorphism $\varphi : \mathcal{K} \rightarrow A$.

Theorem 4.3. *Let A be a separable C^* -algebra of real rank zero. Then there exists an AF-algebra B and a $*$ -monomorphism $\varphi : B \rightarrow A$ such that*

- (i) φ induces an isomorphism of ideal lattices,
- (ii) for any two projections e, f in B one has $[e] \preceq [f]$ in $V(B)$ if and only if $[\varphi(e)] \preceq [\varphi(f)]$ in $V(A)$,
- (iii) $V(\varphi) : V(B) \rightarrow V(A)$ is onto.
- (iv) $A = \overline{\varphi(B)A\varphi(B)}$.

If A is unital, then the AF-algebra B is necessarily also unital, and φ is unit preserving.

Proof. Since A has real rank zero, we know that $V(A)$ is a refinement monoid [1, Lemma 2.3]. It is also countable because A is separable. Therefore we can apply Theorem 3.9 to find a dimension monoid Δ and a surjective monoid morphism $\alpha: \Delta \rightarrow V(A)$ such that $x \propto y$ for every pair of elements x, y in Δ for which $\alpha(x) \propto \alpha(y)$.

By the structure theory for AF-algebras (see e.g. [12,13]), there is a stable AF-algebra B_s such that $V(B_s)$ is isomorphic to Δ . Identifying Δ with $V(B_s)$ we will assume that the domain of α is $V(B_s)$. Let A_s denote the stabilization of A and identify A with a (full) hereditary sub- C^* -algebra of A_s . Identify $V(A)$ and $V(A_s)$, so that α is identified with a monoid morphism $V(B_s) \rightarrow V(A_s)$. Use Lemma 4.2 to lift α to a $*$ -homomorphism $\psi: B_s \rightarrow A_s$.

Choose an increasing approximate unit $\{f_n\}_{n=1}^\infty$ consisting of projections for A (so that $A = \overline{\bigcup_{n=1}^\infty f_n A_s f_n}$). By surjectivity of $V(\psi)$, and since B_s is stable, there are pairwise orthogonal projections g_1, g_2, \dots in B_s such that $[\psi(g_1)] = [f_1]$ and $[\psi(g_n)] = [f_n - f_{n-1}]$ for $n \geq 2$. Put $e_n = g_1 + g_2 + \dots + g_n$. Then e_1, e_2, \dots is an increasing sequence of projections in B_s and $[\psi(e_n)] = [f_n]$. Put $B = \overline{\bigcup_{n=1}^\infty e_n B_s e_n}$, so that B is a hereditary sub- C^* -algebra of B_s , and hence is an AF-algebra. We claim that B is full in B_s . Indeed, if p is a projection in B_s , then $\alpha([p]) \propto [f_n] = \alpha([e_n])$ for some n , because A is full in A_s , and this implies $[p] \propto [e_n]$ by the special property of α . Thus p belongs to the closed two-sided ideal in B_s generated by B ; and as p was arbitrary, B is full in B_s .

Choose partial isometries v_n in A such that

$$v_1 v_1^* = \psi(g_1), \quad v_1^* v_1 = f_1, \quad v_n v_n^* = \psi(g_n), \quad v_n^* v_n = f_n - f_{n-1}, \quad n \geq 2,$$

set $u_n = v_1 + v_2 + \dots + v_n$, so that $u_n u_n^* = \psi(e_n)$ and $u_n^* u_n = f_n$, and define $\varphi_n: e_n B_s e_n \rightarrow f_n A_s f_n$ by $\varphi_n(b) = u_n^* \psi(b) u_n$. Then we obtain a commutative diagram

$$\begin{array}{ccccccc}
 e_1 B_s e_1 & \xrightarrow{\iota} & e_2 B_s e_2 & \xrightarrow{\iota} & e_3 B_s e_3 & \xrightarrow{\iota} & \cdots \longrightarrow B \\
 \downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow \varphi_3 & & \downarrow \varphi \\
 f_1 A_s f_1 & \xrightarrow{\iota} & f_2 A_s f_2 & \xrightarrow{\iota} & f_3 A_s f_3 & \xrightarrow{\iota} & \cdots \longrightarrow A
 \end{array} \tag{4.2}$$

which by the universal property of inductive limits produces the $*$ -homomorphism φ . Since $\varphi(e_n) = \varphi_n(e_n) = f_n$ we see that (iv) holds.

We claim that $V(\varphi) = \alpha$. Indeed, $V(\varphi_n) = V(\psi|_{e_n B_s e_n})$. Hence, if we apply the continuous functor V to diagram (4.2), then the resulting diagram remains commutative if $V(\varphi)$ is replaced with $V(\psi|_B)$, whence $V(\varphi) = V(\psi|_B)$ by the universal property of inductive limits. Identifying $V(A)$ and $V(B)$ with $V(A_s)$ and $V(B_s)$, respectively, we obtain $V(\varphi) = V(\psi) = \alpha$.

Now, (ii) follows by the stipulated property of α , and (i) follows from Corollary 2.3, part (ii), and the assumption that A is of real rank zero. It follows in particular from (i) that φ is injective.

Surjectivity of $V(\varphi): V(B) \rightarrow V(A)$ follows from the surjectivity of α .

Suppose finally that A is unital. Then 1_A belongs to $\varphi(B)$ by (iv). This entails that $\varphi(B)$ is unital with unit 1_A . Because φ is injective, and hence an isomorphism from B

to $\varphi(B)$, it follows that B is unital and that φ maps the unit of B onto the unit of $\varphi(B)$, i.e. $\varphi(1_B) = 1_A$, as desired. \square

Corollary 4.4. *Let A be a simple, separable C^* -algebra of real rank zero. Then there is a simple AF-algebra B and an embedding $\varphi: B \rightarrow A$ such that $V(\varphi): V(B) \rightarrow V(A)$ is surjective, and $A = \overline{\varphi(B)A\varphi(B)}$.*

One of our motivations for pursuing Theorem 4.3 was to establish—what later turned out to be a well-known fact! (see Theorem 4.5)—that the primitive ideal space of an arbitrary, separable, real rank zero algebra is homeomorphic to the primitive ideal space of some AF-algebra. Combining this with Kirchberg’s seminal classification of \mathcal{O}_2 -absorbing separable, nuclear C^* -algebras (see Theorem 4.6) one obtains Corollary 4.8, that the tensor product of an arbitrary separable, stable, nuclear C^* -algebra of real rank zero with \mathcal{O}_2 is isomorphic to an AF-algebra tensor \mathcal{O}_2 .

Recall that the primitive ideal space, $\text{Prim}(A)$, of a C^* -algebra A is the set of all kernels of irreducible representations of A equipped with the hull-kernel topology (also known as the Jacobson topology). There is a bijective correspondence

$$I \mapsto \text{hull}(I) = \{P \in \text{Prim}(A) \mid I \subseteq P\}$$

between the set of closed ideals of A and the set of closed subsets of $\text{Prim}(A)$ (so $\text{Prim}(A)$ contains the same information as $L(A)$, the ideal lattice of A). It is known that $\text{Prim}(A)$ is a locally compact T_0 -space with the Baire property, but there is no description of which spaces with these properties arise as $\text{Prim}(A)$ for some C^* -algebra A .

In the real rank zero case, or more generally, in the case of C^* -algebras with property (IP) (each ideal in the C^* -algebra is generated by its projections) we have the following theorem of Bratteli and Elliott [6]. We write $X \cong Y$ when X and Y are homeomorphic topological spaces.

Theorem 4.5 (Bratteli–Elliott). *Let X be a locally compact T_0 -space with the Baire property and with a countable basis. Then the following conditions are equivalent.*

- (i) X has a basis of compact-open sets.
- (ii) $X \cong \text{Prim}(A)$ for some AF-algebra A .
- (iii) $X \cong \text{Prim}(A)$ for some C^* -algebra A with property (IP).

It follows either from Theorem 4.5 or from our Theorem 4.3 that if A is any separable C^* -algebra of real rank zero, then $\text{Prim}(A) \cong \text{Prim}(B)$ for some AF-algebra B . Alternatively, this fact follows from our Corollary 3.10 combined with Zhang’s theorem that the ideal lattice of a C^* -algebra A of real rank zero is (canonically) isomorphic to the ideal lattice of $V(A)$.

The theorem below was proved by Kirchberg [16].

Theorem 4.6 (Kirchberg). *Let A and B be separable, nuclear C^* -algebras. Then*

$$\text{Prim}(A) \cong \text{Prim}(B) \iff A \otimes \mathcal{O}_2 \otimes \mathcal{K} \cong B \otimes \mathcal{O}_2 \otimes \mathcal{K}.$$

Lemma 4.7. *Let B be a separable C^* -algebra of real rank zero. Then every projection in $B \otimes \mathcal{O}_2$ is equivalent to a projection in $B \otimes \mathbb{C}1 \subseteq B \otimes \mathcal{O}_2$.*

Proof. Let p be a projection in $B \otimes \mathcal{O}_2$, and let J be the closed two-sided ideal in $B \otimes \mathcal{O}_2$ generated by p . As \mathcal{O}_2 is nuclear it follows from [3, Theorem 3.3] that $J = J_0 \otimes \mathcal{O}_2$ for some closed two-sided ideal J_0 in B . We proceed to show that J_0 contains a full projection. As B is separable and of real rank zero, J_0 has an increasing approximate unit $\{e_n\}_{n=1}^\infty$ consisting of projections. Let I_n be the closed two-sided ideal in J_0 generated by e_n . Then

$$p \in J = \overline{\bigcup_{n=1}^\infty I_n \otimes \mathcal{O}_2},$$

whence p belongs to $I_n \otimes \mathcal{O}_2$ for some n (because p is a projection). Put $q = e_n \otimes 1 \in (B \otimes \mathbb{C}1) \cap J$. Then p and q are full projections in J ; and they are properly infinite because J is purely infinite (cf. [17, Proposition 4.5 and Theorem 4.16]). As $K_0(J) = 0$ (see e.g. [9, Theorem 2.3]) it follows from [9, Section 2] that p and q are equivalent. \square

Combining Kirchberg’s theorem with the result about the primitive ideal space of real rank zero algebras mentioned above yields:

Corollary 4.8. *Let A be a separable, nuclear C^* -algebra of real rank zero. Then $A \otimes \mathcal{O}_2$ is of real rank zero, there is an AF-algebra B , and there is a sequence of natural numbers $\{r_n\}_{n=1}^\infty$ such that*

$$A \otimes \mathcal{O}_2 \cong B \otimes \mathcal{O}_2 \cong \varinjlim_{j=1}^{r_n} \bigoplus_{j=1}^{r_n} \mathcal{O}_2.$$

Proof. The right-most C^* -algebra displayed above is of real rank zero, so the first claim follows from the last claim. If B is an AF-algebra, then $B = \varinjlim B_n$, where B_n is the direct sum of r_n full matrix algebras. As $M_k \otimes \mathcal{O}_2$ is isomorphic to \mathcal{O}_2 for all k , we see that $B_n \otimes \mathcal{O}_2$ is isomorphic to $\bigoplus_{j=1}^{r_n} \mathcal{O}_2$. This proves the second isomorphism.

Use Theorem 4.5 or Theorem 4.3 to find a stable AF-algebra B_s with $\text{Prim}(A) \cong \text{Prim}(B_s)$. Then $A \otimes \mathcal{O}_2 \otimes \mathcal{K}$ is isomorphic to $B_s \otimes \mathcal{O}_2$ by Kirchberg’s Theorem 4.6. This implies that $A \otimes \mathcal{O}_2$ is isomorphic to a hereditary sub- C^* -algebra D of $B_s \otimes \mathcal{O}_2$. As $B_s \otimes \mathcal{O}_2$ is separable and of real rank zero, there is an increasing approximate unit $\{q_n\}_{n=1}^\infty$ consisting of projections for D . Use Lemma 4.7 and

stability of B_s to find pairwise orthogonal projections f_1, f_2, \dots in B_s such that $f_1 \otimes 1 \sim q_1$ and $f_n \otimes 1 \sim q_n - q_{n-1}$ for $n \geq 2$. Choose partial isometries v_n in $B_s \otimes \mathcal{O}_2$ such that $v_n^* v_n = f_n \otimes 1$, $v_n v_n^* = q_n - q_{n-1}$, set $u_n = v_1 + v_2 + \dots + v_n$, set $p_n = f_1 + f_2 + \dots + f_n$ and set $B = \bigcup_{n=1}^{\infty} p_n B_s p_n$. Then B is an AF-algebra, $u_n^* u_n = p_n \otimes 1$, $u_n u_n^* = q_n$, and we have a commutative diagram

$$\begin{array}{ccccccc}
 q_1(\mathcal{O}_2 \otimes B_s)q_1 & \xrightarrow{\iota} & q_2(\mathcal{O}_2 \otimes B_s)q_2 & \xrightarrow{\iota} & q_3(\mathcal{O}_2 \otimes B_s)q_3 & \xrightarrow{\iota} & \dots \longrightarrow D \\
 \downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow \varphi_3 & & \downarrow \varphi \\
 (p_1 B_s p_1) \otimes \mathcal{O}_2 & \xrightarrow{\iota} & (p_2 B_s p_2) \otimes \mathcal{O}_2 & \xrightarrow{\iota} & (p_3 B_s p_3) \otimes \mathcal{O}_2 & \xrightarrow{\iota} & \dots \longrightarrow B \otimes \mathcal{O}_2
 \end{array}$$

where $\varphi_n(x) = u_n^* x u_n$. The $*$ -homomorphism φ induced by the diagram is an isomorphism because each φ_n is an isomorphism. It follows that $A \otimes \mathcal{O}_2 \cong D \cong B \otimes \mathcal{O}_2$. \square

We can improve Corollary 4.8 by replacing the condition that A is of real rank zero to the milder assumption that A has property (IP). In that case we must in the proof refer to Bratteli and Elliott’s Theorem 4.5 (instead of our Theorem 4.3).

In [20], Lin considered a class \mathcal{A} of separable C^* -algebras that have trivial K -theory. This class consists of those C^* -algebras A that can be written as inductive limits of finite direct sums of hereditary sub- C^* -algebras of \mathcal{O}_2 [20, Corollary 3.11], and can be completely classified by their monoids of equivalence classes of projections [20, Theorem 3.13]. Since every hereditary sub- C^* -algebra of \mathcal{O}_2 is isomorphic to \mathcal{O}_2 (in the unital case), or to $\mathcal{O}_2 \otimes \mathcal{K}$ (in the non-unital case), it follows from Corollary 4.8, that the C^* -algebras in the class \mathcal{A} are precisely those of the form $A \otimes \mathcal{O}_2$, where A is a (separable) AF-algebra. In the unital case, this was already observed in [15, Theorem 9.4]. The methods used there were based on the fact that for any $A \in \mathcal{A}$, the monoid $V(A)$ is a distributive 0-semilattice (cf. [20]).

The range of the invariant for the C^* -algebras in the class \mathcal{A} was also described in [20, Section 4], as follows. Given any countable distributive 0-semilattice V with a largest element, there is a unital C^* -algebra A in \mathcal{A} such that $V(A) \cong V$ [20, Theorem 4.14]. We note that our methods developed in Section 3, combined with the observations above, allow us to recover this result. Namely, given a countable distributive 0-semilattice V , find by Theorem 3.9 a dimension monoid N and a surjective monoid morphism $\alpha : N \rightarrow V$ such that $\alpha(x) \alpha \alpha(y)$, whenever $x \alpha y$. As in the comments following Theorem 3.9, this induces an isomorphism $\nabla(N) \cong \nabla(V) \cong V$ (where $\nabla(\cdot)$ denotes the passage to the maximal semilattice quotient). By the structure theory of AF-algebras, there is a stable AF-algebra B such that $V(B) \cong N$. But now $V(B \otimes \mathcal{O}_2) \cong V(B) / \asymp = \nabla(V(B)) \cong \nabla(N)$, and from this it follows that $V(B \otimes \mathcal{O}_2) \cong \nabla(V) \cong V$.

In the case that V has a maximal element u , this is an order-unit of the monoid V , and the surjectivity of the morphism α in the paragraph above provides an element v in N such that $\alpha(v) = u$ and that will be an order-unit for N . In this case, the AF-algebra B can be chosen to be unital.

One can combine Bratteli and Elliott’s Theorem 4.5 with Kirchberg’s Theorem 4.6 to conclude that $A \otimes \mathcal{O}_2$ is stably isomorphic to an AF-algebra tensor \mathcal{O}_2 whenever A is a separable nuclear C^* -algebra with property (IP). In particular, for such C^* -algebras A , the tensor product $A \otimes \mathcal{O}_2$ is of real rank zero; a curious fact that we expect is derivable by more direct means. (Warning: There are separable nuclear C^* -algebras A with the ideal property where $A \otimes \mathcal{O}_\infty$ is not of real rank zero!).

5. Divisible C^* -algebras

Divisibility in the context of a monoid M refers to the property that the equation $nx = y$ has a solution $x \in M$ (in which case we say that n divides y in M) for all y in M and for all (or some) large natural numbers n . If A is a C^* -algebra and p is a projection in A , then $nx = [p]$ has a solution $x \in V(A)$ if and only if there is a unital embedding $M_n \rightarrow pAp$. Divisibility in this strong form is rare. A weaker form of divisibility, which is much more frequent—and still useful—is described in the definition below.

Definition 5.1. Let A be a C^* -algebra and let p be a non-zero projection in A . We say that A is *weakly divisible of degree n at p* if there is a unital $*$ -homomorphism

$$M_{n_1} \oplus M_{n_2} \oplus \dots \oplus M_{n_r} \rightarrow pAp \tag{5.1}$$

for some natural numbers r, n_1, n_2, \dots, n_r where $n_j \geq n$ for all j . If there is a unital $*$ -homomorphism as in (5.1) for *each* set r, n_1, n_2, \dots, n_r of natural numbers for which $\gcd(n_1, n_2, \dots, n_r)$ divides $[p]$ in $V(A)$, then we say that A is *weakly divisible at p* . Finally, if A is weakly divisible at p for every non-zero projection p in A , then A is called *weakly divisible*.

The notion of *approximate divisibility* was introduced in [5]. We recall the definition: A unital C^* -algebra A is *approximately divisible* if there is a sequence of unital $*$ -homomorphisms $\varphi_n : M_2 \oplus M_3 \rightarrow A$ such that $\varphi_n(x)a - a\varphi_n(x) \rightarrow 0$ for all $x \in M_2 \oplus M_3$ and all $a \in A$. Being approximately divisible implies being weakly divisible (at all projections p in A) (see [5]). The crucial difference between weak divisibility and approximate divisibility is the assumption of asymptotic centrality in the latter. Approximately divisible C^* -algebras are very well behaved. In particular, any simple, approximately divisible C^* -algebra is either stably finite or purely infinite, and its ordered K_0 -group is always weakly unperforated (see [5]).

We show here that weak divisibility is almost automatic for C^* -algebras of real rank zero. There are examples of non-nuclear, simple, unital C^* -algebras of real rank zero that are weakly divisible but not approximately divisible [11]. Perhaps the most fundamental—and optimistic!—open question concerning C^* -algebras of real rank zero is the following:

Question. Is every simple, unital, nuclear, non-type I C^* -algebra of real rank zero approximately divisible?

If A is weakly divisible at p , then there is a unital $*$ -homomorphism from $M_{n_1} \oplus M_{n_2} \oplus \dots \oplus M_{n_r}$ into pAp for every finite set n_1, n_2, \dots, n_r of natural numbers such that $\text{gcd}(n_1, n_2, \dots, n_r) = 1$.

There is a similar notion of weak divisibility in a monoid M : M is weakly divisible of degree n at an element $x \in M$ if there are natural numbers r, n_1, \dots, n_r and elements x_1, \dots, x_r in M such that $n_j \geq n$ for all j and $x = n_1x_1 + \dots + n_rx_r$. It is easily seen that a C^* -algebra A is weakly divisible of degree n at a projection $p \in A$ if and only if $V(A)$ is weakly divisible of degree n at $[p]$.

If A is weakly divisible, then so is any quotient, ideal, and corner of A .

Since $M_2 \oplus M_3$ maps unitaly (but not necessarily injectively) into any matrix algebra M_n , with $n \geq 2$, weak divisibility of degree 2 at a projection p means that there is a unital $*$ -homomorphism $M_2 \oplus M_3 \rightarrow pAp$.

Clearly, if A is weakly divisible at a projection p , then A is weakly divisible of degree 2 at p . In the converse direction, we have the following:

Lemma 5.2. *Let A be a C^* -algebra. If A is weakly divisible of degree 2 at all (non-zero) projections in A , then A is weakly divisible.*

Proof. We prove this in the monoid language. Hence we assume that for every x in $V(A)$ there are elements y, z with $x = 2y + 3z$. Let now x in $V(A)$ be fixed and let n_1, \dots, n_r be natural numbers such that $d = \text{gcd}(n_1, \dots, n_r)$ divides x . It suffices to consider the case where $d = 1$. Indeed, $x = dy$ for some y in $V(A)$. Put $m_j = n_j/d$. Then $\text{gcd}(m_1, \dots, m_r) = 1$, and if there are elements y_1, \dots, y_r in $V(A)$ with $y = m_1y_1 + \dots + m_ry_r$, then $x = n_1y_1 + \dots + n_ry_r$.

Assume now that $d = 1$. There is a natural number m_0 such that all natural numbers $m \geq m_0$ belong to the sub-semigroup of the natural numbers generated by n_1, \dots, n_r . Let k be a natural number such that $2^k \geq m_0$. By k successive applications of weak divisibility (first to $x = 2y + 3z$, second to y and z , and so forth) we obtain elements y_0, y_1, \dots, y_k in $V(A)$ such that

$$x = 2^k y_0 + 2^{k-1} 3 y_1 + \dots + 3^k y_k.$$

Since $2^{k-l} \cdot 3^l \geq m_0$ we can write $2^{k-l} \cdot 3^l = \sum_{j=1}^r d_{l,j} n_j$ for suitable non-negative integers $d_{l,j}$. Now,

$$x = \sum_{l=0}^k (2^{k-l} \cdot 3^l) y_l = \sum_{l=0}^k \left(\sum_{j=1}^r d_{l,j} n_j \right) y_l = \sum_{j=1}^r n_j \left(\sum_{l=0}^k d_{l,j} y_l \right) = \sum_{j=1}^r n_j x_j,$$

as desired, when $x_j = \sum_{l=0}^k d_{l,j} y_l$. \square

Our main results on divisibility in C^* -algebras of real rank zero are contained in Propositions 5.3 and 5.7 and in Theorem 5.8. Proposition 5.3 is actually a special case of Theorem 5.8, but is emphasized because of its independent interest and because it is used in the proof of the more general results.

A simple separable C^* -algebra A is of type I precisely when it is isomorphic to a matrix algebra M_n , for some n , or to the compact operators, \mathcal{K} , on some separable Hilbert space H . In other words, A is of type I precisely when it is isomorphic to a sub- C^* -algebra of \mathcal{K} .

Proposition 5.3. *Every simple, separable C^* -algebra of real rank zero, that is not of type I, is weakly divisible.*

Proof. By Lemma 5.2 it suffices to show that A is weakly divisible of degree 2 at every non-zero projection p in A . By Theorem 4.3 (see also Corollary 4.4) there is a simple unital AF-algebra B and a unital embedding $\varphi: B \rightarrow pAp$ such that $V(\varphi): V(B) \rightarrow V(pAp)$ is onto. If B is weakly divisible of degree 2 at its unit, then A is weakly divisible of degree 2 at p . This is well-known to be the case when B is infinite dimensional (see for example [5, Proposition 4.1]) and it is trivially true when $B \cong M_n(\mathbb{C})$ for $n \geq 2$.

Suppose now that $B = \mathbb{C}$ (the only case where B is not weakly divisible of degree 2 at its unit). We show that pAp is properly infinite, and this will imply that the Cuntz algebra \mathcal{O}_∞ , and hence $M_2 \oplus M_3$, embed unitaly into pAp , thus showing that A is weakly divisible of degree 2 at p also in this case.

The assumption that A is not of type I (and $p \neq 0$) implies that $pAp \neq \mathbb{C}p$. The corner pAp therefore contains a non-trivial projection q . By surjectivity of $V(\varphi)$, there are natural numbers n and m such that $[q] = nV(\varphi)([1_B]) = n[p]$ and $[p - q] = mV(\varphi)([1_B]) = m[p]$. In particular, $[p] \leq [q]$ and $[p] \leq [p - q]$, and so $2[p] \leq [q] + [p - q] = [p]$, which entails that p is properly infinite. \square

We shall also need the following lemmas for the proof of our main results on divisibility in C^* -algebras of real rank zero.

Lemma 5.4. *Let M be a conical refinement monoid (or a monoid with the Riesz decomposition property), let n be a natural number, and let x, y in M be such that $y \leq x$, $x \propto y$, and M is weakly divisible of degree n at y . It follows that M is weakly divisible of degree n at x .*

Proof. Find $u \in M$ such that $x = y + u$, and find natural numbers r, n_1, \dots, n_r , with $n_j \geq n$ for all j , and elements y_1, \dots, y_r in M such that $y = n_1y_1 + \dots + n_ry_r$. The assumption $x \propto y$ implies that $u \propto y_1 + \dots + y_r$, and so there is a natural number k such that $u \leq k(y_1 + \dots + y_r)$.

It suffices to show that whenever k is a natural number, $x, u, y, y_1, \dots, y_r \in M$, and $n_1, \dots, n_r \in \mathbb{N}$ are such that $n_j \geq n$ for all j , $x = y + u$, $y = n_1y_1 + \dots + n_ry_r$, and $u \leq k(y_1 + \dots + y_r)$, then there is a natural number s and there are $u', y', y'_1, \dots, y'_s \in M$ and $n'_1, \dots, n'_s \in \mathbb{N}$ such that $n'_i \geq n$ for all i , $x = y' + u'$, $y' = n'_1y'_1 + \dots + n'_sy'_s$, and $u' \leq (k - 1)(y'_1 + \dots + y'_s)$. The proof of the lemma is completed after k such reductions.

Use the refinement (or the decomposition) property on the inequality $u \leq k(y_1 + \dots + y_r)$ to find elements $y_{ji} \in M$ with

$$u = \sum_{j=1}^r \sum_{i=1}^k y_{ji}, \quad y_{ji} \leq y_j, \quad j = 1, \dots, r, \quad i = 1, \dots, k.$$

Find $z_j \in M$ with $y_{j1} + z_j = y_j$ for $j = 1, \dots, r$. Put $s = 2r$, and put

$$\begin{aligned} y' &\stackrel{\text{def}}{=} y + y_{11} + y_{21} + \dots + y_{r1} \\ &= n_1(y_{11} + z_1) + n_2(y_{21} + z_2) + \dots + n_r(y_{r1} + z_r) + y_{11} + y_{21} + \dots + y_{r1} \\ &= (n_1 + 1)y_{11} + (n_2 + 1)y_{21} + \dots + (n_r + 1)y_{r1} + n_1z_1 + n_2z_2 + \dots + n_rz_r \\ &= n'_1y'_1 + n'_2y'_2 + \dots + n'_sy'_s, \end{aligned}$$

where

$$y'_j = \begin{cases} y_{j1}, & j = 1, \dots, r, \\ z_{j-r}, & j = r + 1, \dots, s, \end{cases} \quad n'_j = \begin{cases} n_j + 1, & j = 1, \dots, r, \\ n_{j-r}, & j = r + 1, \dots, s. \end{cases}$$

Moreover,

$$u' \stackrel{\text{def}}{=} \sum_{j=1}^r \sum_{i=2}^k y_{ji} \leq \sum_{j=1}^r \sum_{i=2}^k y_j = (k - 1) \sum_{j=1}^r y_j = (k - 1) \sum_{j=1}^s y'_j,$$

and $x = y' + u'$ as desired. \square

We shall a couple of times use the following lemma, whose proof is verbatim identical to the proof of [12, Lemma 9.8] by Effros, and which therefore is omitted. In the formulation of Effros’ lemma, it is required that the ideal I below is an AF-algebra, but inspection of the proof shows that we only need I to be of real rank zero.

Lemma 5.5. *Let A be a C^* -algebra, let I be a closed two-sided ideal in A , such that I is of real rank zero, and let $\pi : A \rightarrow A/I$ be the quotient mapping. Then each $*$ -homomorphism $\mu : B \rightarrow A/I$, where B is a finite dimensional C^* -algebra, lifts to a $*$ -homomorphism $\lambda : B \rightarrow A$, i.e. $\mu = \pi \circ \lambda$.*

Lemma 5.6. *Let A be a unital C^* -algebra of real rank zero, and let n be a natural number. Let \mathcal{P}_n be the set of projections $p \in A$ for which A is weakly divisible of degree n at p . Suppose \mathcal{P}_n is full in A (i.e., is not contained in any proper two-sided ideal in A). Then A is weakly divisible of degree n at its unit 1_A .*

Proof. We show that \mathcal{P}_n contains a full projection.

The assumptions imply that 1_A belongs to the closed—and hence to the algebraic—ideal in A generated by \mathcal{P}_n . It follows that 1_A belongs to the (closed) two-sided ideal generated by some finite set of projections in \mathcal{P}_n .

To prove that \mathcal{P}_n contains a full projection, it suffices to show that if $k \geq 2$ and p_1, \dots, p_k are projections in \mathcal{P}_n , then there are projections q_1, \dots, q_{k-1} in \mathcal{P}_n , such that the two sets $\{p_1, \dots, p_k\}$ and $\{q_1, \dots, q_{k-1}\}$ generate the same closed two-sided ideal in A .

Let I be the closed two-sided ideal in A generated by p_{k-1} . If p_k belongs to I , then we can take $q_j = p_j$ for $j = 1, \dots, k - 1$. Suppose that p_k does not belong to I . Let $\pi : A \rightarrow A/I$ be the quotient mapping. Then A/I is weakly divisible of degree n at $\pi(p_k)$ (and $\pi(p_k) \neq 0$). Hence there is a finite dimensional C^* -algebra $B = M_{n_1} \oplus \dots \oplus M_{n_r}$, with $n_j \geq n$ for all j , and a unital $*$ -homomorphism $\mu : B \rightarrow \pi(p_k)(A/I)\pi(p_k)$. Let $\pi' : (1 - p_{k-1})A(1 - p_{k-1}) \rightarrow A/I$ be the restriction of π , and notice that π' is surjective. It follows from Lemma 5.5 that μ lifts to a $*$ -homomorphism $\lambda : B \rightarrow (1 - p_{k-1})A(1 - p_{k-1})$. The two projections p_{k-1} and $\lambda(1_B)$ are mutually orthogonal and both belong to \mathcal{P}_n , so the projection $q_{k-1} = p_{k-1} + \lambda(1_B)$ belongs to \mathcal{P}_n . Let J and J' be the closed two-sided ideals in A generated by $\{p_{k-1}, p_k\}$ and q_{k-1} , respectively. Then p_{k-1} belongs to J' (and clearly also to J), so I is contained in both J and J' . The quotients J/I and J'/I are generated by $\pi(p_k)$ and $\pi(q_{k-1})$, respectively; but

$$\pi(q_{k-1}) = (\pi \circ \lambda)(1_B) = \mu(1_B) = \pi(p_k),$$

and this proves that $J/I = J'/I$, whence $J = J'$. We conclude that $\{p_1, \dots, p_{k-2}, q_{k-1}\}$ generates the same closed two-sided ideal in A as $\{p_1, \dots, p_k\}$.

We now have a full projection $p \in \mathcal{P}_n$. Phrased in the language of monoids, $[p] \leq [1_A] \circ [p]$, and $V(A)$ is weakly divisible of degree n at $[p]$. Lemma 5.4 implies that $V(A)$ is weakly divisible of degree n at $[1_A]$, and this in turn implies that A is weakly divisible of degree n at 1_A . \square

Proposition 5.7. *Let A be a separable C^* -algebra of real rank zero, let p be a projection in A , and let n be a natural number. Then A is weakly divisible of degree n at p if and only if the corner pAp has no (non-zero) representation of dimension less than n .*

Proof. Suppose first that pAp has a non-zero (possibly non-faithful) representation π on a Hilbert space of dimension m , and suppose that there is a unital $*$ -homomorphism from $B = M_{n_1} \oplus \dots \oplus M_{n_r}$ into pAp , where $n_j \geq n$ for all j . The representation π will then restrict to a non-zero representation of B , but this is possible only when $m \geq \min\{n_1, \dots, n_r\}$. This proves the “only if” part of the proposition.

Suppose now that pAp has no (non-zero) representation of dimension less than n . For ease of notation, and upon replacing A with pAp , we can assume that A is unital and that $p = 1_A$. Let \mathcal{P}_n be the set of all projections q in A for which A is weakly

divisible of degree n at q . Let I be the closed two-sided ideal in A generated by \mathcal{P}_n . We claim that $I = A$, and this will complete the proof by Lemma 5.6.

Suppose, to reach a contradiction, that $I \neq A$. Then I is contained in a maximal ideal J of A . Let $\pi : A \rightarrow A/J$ be the quotient mapping. The quotient A/J is separable, simple, unital, and of real rank zero. If A/J is finite dimensional, then it is isomorphic to M_m for some $m \geq n$ (by the assumption that A has no representation of dimension less than n). If A/J is infinite dimensional, then it is weakly divisible by Proposition 5.3. In either case there is a finite dimensional C^* -algebra $B = M_{n_1} \oplus \dots \oplus M_{n_r}$, with $n_j \geq n$ for all j , and a unital $*$ -homomorphism $\mu : B \rightarrow A/J$. By Lemma 5.5, μ lifts to a $*$ -homomorphism $\lambda : B \rightarrow A$. This entails that $q = \lambda(1_B)$ belongs to \mathcal{P}_n and hence to $I \subseteq J$, thus yielding the contradiction $0 = \pi(q) = (\pi \circ \lambda)(1_B) = \mu(1_B) = 1_{A/J} \neq 0$. \square

If A is weakly divisible of degree n at a projection p , then there is a full (but not necessarily unital) $*$ -homomorphism from M_n into pAp . (A $*$ -homomorphism is called *full* if the closed two-sided ideal generated by its image is the entire C^* -algebra). By Proposition 5.7 such a $*$ -homomorphism exists whenever pAp has no representation of dimension less than n , provided that A is of real rank zero. In particular, there is a full $*$ -homomorphism $M_2 \rightarrow pAp$ precisely when pAp has no character.

In the non-real rank zero case, there are infinite dimensional simple unital C^* -algebras with no projections other than 0 and 1, and there is no (full) embedding of M_2 into such a C^* -algebra.

It is not known for which unital, non-real rank zero algebras A there exists a full $*$ -homomorphism from $M_n \otimes C_0((0, 1])$ into A (even for $n = 2$). It is known, however, that absence of characters of A is not a sufficient condition to ensure the existence of a full $*$ -homomorphism from $M_2 \otimes C_0((0, 1])$ into A , but absence of finite dimensional representations could be sufficient. This problem is referred to as the *Global Glimm problem*, and it has been considered in the study of purely infinite C^* -algebras, see for example [4,18].

Theorem 5.8. *Let A be a separable C^* -algebra of real rank zero. Then the following are equivalent:*

- (i) A is weakly divisible (cf. Definition 5.1).
- (ii) No non-zero corner pAp (where p is a projection on A) admits a character.
- (iii) There is no representation π of A on a Hilbert space H for which $\pi(A) \cap \mathcal{K}(H) \neq \{0\}$ (where $\mathcal{K}(H)$ denotes the algebra of compact operators on H).

Proof. (i) \Rightarrow (iii): Suppose that π is a representation of A on a Hilbert space H such that $\pi(A) \cap \mathcal{K}(H) \neq \{0\}$. Let I be the kernel of π , and put $J = \pi^{-1}(\mathcal{K}(H))$. Then I and J are closed two-sided ideals, $I \subseteq J$, and J/I is isomorphic to $\pi(A) \cap \mathcal{K}(H)$. The property of having real rank zero passes to ideals and quotients, so J/I , and hence $\pi(A) \cap \mathcal{K}(H)$ are of real rank zero. Take a non-zero projection p in $\pi(A) \cap \mathcal{K}(H)$ and use again the real rank zero property of A to lift p to a projection q in A . There is

no unital $*$ -monomorphism $M_n \oplus M_{n+1} \rightarrow p\pi(A)p = \pi(qAq)$ for $n > \dim(p)$, and hence there is no unital $*$ -monomorphism $M_n \oplus M_{n+1} \rightarrow qAq$ for those n . This shows that A is not weakly divisible at q , and therefore A is not weakly divisible.

(iii) \Rightarrow (ii): Suppose that p is a non-zero projection in A and that $\rho : pAp \rightarrow \mathbb{C}$ is a character. Put $\bar{\rho}(a) = \rho(pap)$ for a in A . Then $\bar{\rho}$ is an extremal state on A which extends ρ . Let (π, H, ξ) be the GNS representation of A that arises from the state $\bar{\rho}$. Then π is irreducible, $\pi(p)\pi(A)\pi(p) = \pi(pAp)$ is abelian, and $\pi(p) \neq 0$ (the latter because $\langle \pi(p)\xi, \xi \rangle = \bar{\rho}(p) = 1$). This entails that $\pi(p)$ is a one-dimensional projection, and therefore $\pi(A) \cap \mathcal{K}(H) \neq \{0\}$.

(ii) \Rightarrow (i): This follows from Proposition 5.7 (with $n = 2$) together with Lemma 5.2. \square

Condition (iii) is equivalent to the statement that there is no pair of closed two-sided ideals $I \subset J \subseteq A$ such that J/I is isomorphic to a sub- C^* -algebra of the compact operators \mathcal{K} .

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