On the nullity of bicyclic graphs

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Abstract

The nullity of a graph is the multiplicity of the eigenvalue zero in its spectrum. In this paper, we obtain
the nullity set of bicyclic graphs of order \( n \), and determine the bicyclic graphs with maximum nullity.

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1. Introduction

Let \( G \) be a simple graph with vertex set \( V \) and edge set \( E \). Let \( W \subseteq V \), then the subgraph
induced by \( W \) is the subgraph of \( G \) obtained by taking the vertices in \( W \) and joining those pairs
of vertices in \( W \) which are joined in \( G \). We write \( G - \{v_1, \ldots, v_k\} \) for the graph obtained from
\( G \) by removing the vertices \( v_1, \ldots, v_k \) and all edges incident to any of them. We define that the
union of \( G_1 \) and \( G_2 \), denoted by \( G_1 \cup G_2 \), is the graph with vertex-set \( V(G_1) \cup V(G_2) \) and
edge-set \( E(G_1) \cup E(G_2) \). The disjoint union of \( k \) copies of \( G \) is often written \( kG \). The null graph
of order \( n \) is the graph with \( n \) vertices and no edges. As usual, the complete graph and cycle of
order \( n \) are denoted by \( K_n \) and \( C_n \), respectively. An isolated vertex is sometimes denoted by \( K_1 \). The adjacency matrix \( A(G) \) of graph \( G \) of order \( n \), having vertex-set \( V(G) = \{v_1, v_2, \ldots, v_n\} \) is the \( n \times n \) symmetric matrix \([a_{ij}]\), such that \( a_{ij} = 1 \) if \( v_i \) and \( v_j \) are adjacent and 0, otherwise. A graph is said to be singular (non-singular) if its adjacency matrix is a singular (non-singular) matrix. The eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n \) of \( A(G) \) are said to be the eigenvalues of the graph \( G \), and to form the spectrum of this graph. The number of zero eigenvalues in the spectrum of the graph \( G \) is called its nullity and is denoted by \( \eta(G) \). Let \( r(A(G)) \) be the rank of \( A(G) \), clearly, \( \eta(G) = n - r(A(G)) \). The rank of a graph \( G \) is the rank of its adjacency matrix \( A(G) \), denoted by \( r(G) \). The following results are clear but useful.

**Proposition 1.1**

(i) Let \( H \) be an induced subgraph of \( G \). Then \( r(H) \leq r(G) \).

(ii) Let \( G = G_1 \cup G_2 \). Then \( r(G) = r(G_1) + r(G_2) \).

(iii) [9]

\[
r(C_p) = \begin{cases} p - 2, & \text{if } p \equiv 0 \pmod{4}; \\ p, & \text{if } p \not\equiv 0 \pmod{4}. \end{cases}
\]

In [1], Collatz and Sinogowitz first posed the problem of characterizing all graphs which satisfying \( \eta(G) > 0 \). This question is of great interest in chemistry, because, as has been shown in [2], for a bipartite graph \( G \) (corresponding to an alternant hydrocarbon), if \( \eta(G) > 0 \), then it indicates the molecule which such a graph represents is unstable. The nullity of a graph is also important in mathematics, since it is related to the singularity of \( A(G) \). The problem has not yet been solved completely, only for trees and bipartite graph some particular results are known (see [2,3]). In recent years, this problem has been payed much attention by many researchers ([4–9]). In [8,9], the nullity of trees and unicyclic graphs are discussed, respectively. This paper considers the same problem of bicyclic graphs.

A bicyclic graph is a simple connected graph in which the number of edges equals the number of vertices plus one. Throughout the paper, denote by \( \mathcal{B}_n \) the set of all bicyclic graphs of order \( n \).

Let \( C_p \) and \( C_q \) be two vertex-disjoint cycles. Suppose that \( v_1 \in C_1, v_l \in C_q \). Joining \( v_1 \) and \( v_l \) by a path \( v_1 v_2 \cdots v_l \) of length \( l - 1 \), where \( l \geq 1 \) and \( l = 1 \) means identifying \( v_1 \) with \( v_l \), the resultant graph, denoted by \( \infty(p, l, q) \), is called an \( \infty \)-graph. Let \( P_{l+1} \), \( P_{p+1} \), and \( P_{q+1} \) be three vertex-disjoint paths, where \( \min\{p, l, q\} \geq 1 \) and at most one of them is 1. Identifying the three initial vertices and terminal vertices of them, respectively, the resultant graph, denoted by \( \theta(p, l, q) \), is called a \( \theta \)-graph. Then the bicyclic graphs can be partitioned into two classes: the

\[
\infty(p, l, q)
\]

\[
\theta(p, l, q)
\]

Fig. 1. \( \infty \)-graph and \( \theta \)-graph.
class of graphs which contain an $\infty$-graph as an induced subgraph and the class of graphs which contain a $\theta$-graph as an induced subgraph (Fig. 1).

**Definition 1.1.** Call a graph $\theta(p, l, q)$ (or $\infty(p, l, q)$) the base of the corresponding bicyclic graph $B$ which contain it. Denote the base of $B$ by $\chi_B$. Let $\mathcal{P} = B - V(\chi_B)$. $\mathcal{P}$ is said to be the periphery of $B$.

In Section 2, we give an upper bound of $\eta(B)$, and determine all the graphs with maximum nullity. In Section 3, we give the nullity set of $B_n$.

2. Main results

The following lemmas are needed.

**Lemma 2.1** [3]. For a graph $G$ containing a vertex of degree 1, if the induced subgraph $H$ (of $G$) is obtained by deleting this vertex together with the vertex adjacent to it, then the relation $\eta(H) = \eta(G)$ holds.

Since $\chi_B$ is an induced subgraph of $B$, it is helpful to observe the rank of $\chi_B$ in order for getting an upper bound of $\eta(B)$. Also, we are interested in the class of $\chi_B$s with smaller rank ($\leq 4$).

**Lemma 2.2** [5]. Let $G$ be a connected graph of order $n$. Then $r(G) = 2$ if and only if $G = K_{r,n-r}$; $r(G) = 3$ if and only if $G = K_{a,b,c}$, where $a + b + c = n$.

**Lemma 2.3.** Let $B$ be a bicyclic graph of order $n$. Then $r(B) = 2$ if and only if $B = K_2,3$; $r(B) = 3$ if and only if $B = K_4 - e, e \in E(K_4)$.

**Proof.** This is a direct result of Lemma 2.2. □

**Corollary 2.1.** Let $B \in \mathcal{B}_n$, and $B \notin \{K_2,3, K_4 - e\}$. Then $\eta(B) \leq n - 4$.

**Lemma 2.4.** The bicyclic graphs with rank 4 are $\theta(1, 2, 3)$ or $\infty(4, 1, 4)$.

**Proof.** Obviously, a graph of rank 4 cannot contain $K_3 \cup K_2$ or $P_6$ as an induced subgraph, since $r(K_3 \cup K_2) = 5$ and $r(P_6) = 6$.

**Case 1.** If $\chi_B = \infty(p, l, q)$, then $p \leq 4$ and $q \leq 4$. Otherwise, $r(C_p) \geq 5$ or $r(C_q) \geq 5$. If $p = 3, q = 4$ or $p = 4, q = 3$, then $K_3 \cup K_2$ is an induced subgraph of $B$, which is impossible. If $p = q = 4$, then $l < 2$. Otherwise, $P_6$ is an induced subgraph of $B$. Direct calculation shows that $r(\infty(4, 1, 4)) = 4$.

**Case 2.** $\chi_B = \theta(p, l, q)$. In this case, the length of the two induced cycles must be less or equal to 4. The bicyclic graphs satisfying this condition are $\theta(1, 2, 3), \theta(1, 3, 3)$ and $K_{2,3}$. Among these graphs, only $r(\theta(1, 2, 3)) = 4$. □

**Theorem 2.1.** Let $B \in \mathcal{B}_n$.

1. $\eta(B) = n - 2$ if and only if $B = K_{2,3}$;
2. $\eta(B) = n - 3$ if and only if $B = K_{4} - e$;
3. $\eta(B) = n - 4$ if and only if $B = B_i(1 \leq i \leq 7)$ (Fig. 2).
Proof. (1) and (2) hold because of Lemma 2.3.

(3) By a direct calculation, it’s easy to show that $\eta(B_i) = n - 4$ ($1 \leq i \leq 7$). Now assume that $\eta(B) = n - 4$. Then $r(B) = n - \eta(B) = 4$. By Proposition 1.1, for any induced subgraph $H$ of $B$, $r(H) \leq 4$ must hold. Together with Lemma 2.1 and Lemma 2.3–2.4, we claim the following arguments:

Arg. 1. If $\eta(B) = n - 4$, then $\chi_B$ must be one of the graphs in $\{K_{2,3}$, $K_4 - e$, $\emptyset(1, 2, 3)$, $\infty(4, 1, 4)\}$ (Fig. 3);

Arg. 2. A bicyclic graph $B$ with nullity $n - 4$ cannot contain $C_3 \cup K_2$ or $P_6$ as an induced subgraph;

Arg. 3. If Operation in Lemma 2.1 can be applied twice on a graph $B$ of nullity $n - 4$, then the resultant graph must be a null graph.

Denote by $T_i$ the set of vertices from $V(\mathcal{P})$ which are adjacent exactly to a vertex $v_i$ of $\mathcal{P}_B$. When we mention a pendant vertex and a pendant neighbour we mean a vertex with degree 1 and a neighbour of a pendant vertex, respectively. We distinguish the following four cases:

Case 1. $\chi_B = K_{2,3}$. Then $\mathcal{P}$ contains at most one pendant neighbour (in the opposite case, Arg. 3 is invalid). Hence, $\mathcal{P} = S_{1,s} \cup (n - s - 6)K_1$ or $\mathcal{P} = (n - 5)K_1$.

Subcase 1.1. $\mathcal{P}_B = S_{1,s} \cup (n - s - 6)K_1$. Obviously, there is exactly one edge between $K_{2,3}$ and $S_{1,s}$. Denote the centre of $S_{1,s}$ by $u$. The pendant vertex of $S_{1,s}$ cannot be adjacent to any vertex of $K_{2,3}(\mathcal{P}_B)$. Hence, there are only two ways to join an edge between the vertices of $K_{2,3}$ and $S_{1,s}$: (1) $uv_1 \in E(B)$ (or $uv_3 \in E(B)$). If for example, $uv_3 \in E(B)$, then $T_1 = T_2 = T_4 = T_5 = \Phi(\mathcal{P}_B)$. (2) $u$ is adjacent to $v_2$ (or $v_4$ or $v_5$). If for example, $uv_2 \in E(B)$, then $T_1 = T_3 = T_4 = T_5 = \Phi(\mathcal{P}_B)$.

Subcase 1.2. $\mathcal{P} = (n - 5)K_1$. If $T_1 \neq \Phi$, or $T_3 \neq \Phi$, then $T_2 = T_4 = T_5 = \Phi$. Also, if $T_2 \neq \Phi$, then $T_1 = T_3 = T_4 = T_5 = \Phi(\mathcal{P}_B)$. Thus, $B = B_4(s)$ ($s \geq 1$) or $B = B_7(s, t)$ $(\max\{s, t\} \geq 1)$.

Case 2. $\chi_B = K_4 - e$. Then $\mathcal{P}_B = (n - 4)K_1(\mathcal{P}_B)$. Also, $T_2 = T_4 = \Phi$ must hold (Arg. 3). Thus, $B = B_5(s, t)$ $(\max\{s, t\} \geq 1)$.

3 To be short, we shall often reduce the mentioned sentence simply by “Arg. 3”.
Case 3. $\chi_B = \theta(1, 2, 3)$. Then $\mathcal{P} = (n - 5)K_1$ and $T_1 = T_5 = \Phi(\mathrm{Arg.}2)$. Also, $T_2 = T_4 = \Phi(\mathrm{Arg.}3)$. Thus, $B = B_5(s)$ ($s \geq 0$).

Case 4. $\chi_B = \infty(4, 1, 4)$. In this case, $T_i = \Phi (1 \leq i \leq 7)$ must hold, otherwise, it contains $P_6$ as an induced subgraph. Thus, $B = B_1$. □

3. Other results

In this section, we give some miscellaneous results without proof.

Definition 3.1. Let $\mathcal{G}_n$ be the set of all n-vertex graphs, and let $[0, n] = \{0, 1, 2, \ldots, n\}$. A subset $N$ of $[0, n]$ is said to be the nullity set of $\mathcal{G}_n$ provided that for any $k \in N$, there exists at least one graph $G \in \mathcal{G}_n$ such that $\eta(G) = k$.

Definition 3.2. Call an n-vertex tree $T$ a PM-tree if $\eta(T) = 0$.

Theorem 3.1. The nullity set of $\mathcal{B}_n$ is $[0, n - 2]$.

Theorem 3.2. Let $B$ be a bicyclic graph satisfying the following conditions:

(i) $\eta(\chi_B) = 0$;
(ii) $\mathcal{P}$ is the union of PM-trees.

Then $B$ is a non-singular bicyclic graph.

References