Variational inequality for the rotating Navier–Stokes equations with subdifferential boundary conditions

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Abstract

A steady viscous incompressible fluid through a rotating channel satisfies the rotating Navier–Stokes equations. Subdifferential boundary conditions are imposed. A variational inequality is derived for this system. Furthermore, under some assumptions, the weak solution of the variational inequality exists and is unique.

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1. Introduction

When the Navier–Stokes equations for a steady viscous incompressible fluid are written in a rotating frame of reference, two new terms appear. One of them is a centrifugal force which can be written as \( \mathbf{\omega} \times (\mathbf{\omega} \times \mathbf{r}) \), where \( \mathbf{\omega} \) is the velocity of rotation of the frame of reference and \( \mathbf{r} \) is the vector of position of the particles referred to this system. This term can be included in the body force. The other term is a Coriolis force which can be expressed as \( 2 \mathbf{\omega} \times \mathbf{u} \). In this case, the steady rotating Navier–Stokes equations can be written as

\[
\begin{align*}
-\nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + 2\mathbf{\omega} \times \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega, \\
\text{div } \mathbf{u} &= 0 \quad \text{in } \Omega,
\end{align*}
\]

(1)

where \( \mathbf{u} \) is the relative velocity to the rotating frame of reference, \( p \) is the pressure, \( \mathbf{f} \) is the vector of the body force containing the centrifugal force and \( \nu \) is the kinematic viscous coefficient. The domain is denoted by \( \Omega \subset \mathbb{R}^3 \). We suppose that the boundary \( \partial \Omega \) is composed of three components \( \Gamma', S_1 \) and \( S_2 \) in which \( S_1 \) and \( S_2 \) are, respectively, the inflow and outflow boundaries and \( \Gamma' \) is the solid wall. For this system, the whole Dirichlet boundary conditions are invalid. The following subdifferential boundary conditions are imposed:

\[
\begin{align*}
\mathbf{u} &= 0 \quad \text{on } \Gamma', \\
\mathbf{u}_n &= q_1, \quad -\sigma \tau \in g_1 \partial |\mathbf{u}_\tau| \quad \text{on } S_1, \\
\mathbf{u}_n &= q_2, \quad -\sigma \tau \in g_2 \partial |\mathbf{u}_\tau| \quad \text{on } S_2,
\end{align*}
\]

(2)

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where \( q_i \) and \( g_i \) are vector and scalar functions, respectively; \( u_n = u \cdot \vec{n} \) and \( u_\tau = u - \vec{n}u_n \) are the normal and tangential components of the velocity, where \( \vec{n} \) stands for the unit vector of the external normal to \( S_1 \) or \( S_2 \); \( \sigma_\tau = \sigma_\tau(u) \) is the tangential components of the stress vector (the precise definition will be recalled in Section 2); finally, \( \partial[z] \) denotes a graph (cf. [1])

\[
\partial[z] = \begin{cases} 
\frac{z}{|z|} & (z \neq 0, z \in \mathbb{R}^3), \\
\{w \in \mathbb{R}^3 ||w| \leq 1\}, & (z = 0, z \in \mathbb{R}^3).
\end{cases}
\]

There are some results on the problem (1) with other types of boundary conditions. For mixed boundary conditions, the authors in [2] have proved that if the initial data were sufficiently small, the problem (1) had a unique steady smooth solution. The periodic boundary conditions were considered by Babin in [3].

However, for boundary conditions (2), some scholars considered other problems, such as Stokes problems in [4] which were the model of blood flow in a vein of an arterial sclerosis patient and the model of avalanche of water and rocks, and generalized Stokes problems in [1] when \( q_i \equiv 0 \), \( i = 1, 2 \).

In this paper, the main idea comes from the Refs. [5,6], which is based on investigating the variational inequality of problem (1) with boundary conditions (2).

### 2. Variational inequality

Before obtaining the variational inequality, we describe some symbols in this paper. We will use Sobolev spaces \( W^{m,p}(\Omega)^3 \) consisting of all \( \mathbb{R}^3 \)-valued functions which belong to \( L^p(\Omega) \) and possess generalized derivatives up to order \( m \) in \( L^p(\Omega) \). \( \| \cdot \|_{m,p} \) and \( \| \cdot \|_{m,p,\partial \Omega} \) denote the \( W^{m,p} \)-norm on \( \Omega \) and on \( \partial \Omega \), respectively. Especially, when \( p = 2 \), \( H^m(\Omega)^3 \neq W^{m,2}(\Omega)^3 \), \( \| \cdot \| \neq \| \cdot \|_{L^2(\Omega)} \), \( \| \cdot \|_{L^2(\partial \Omega)} = \| \cdot \|_{\partial \Omega} \), \( \| \cdot \|_{m,2} = \| \cdot \|_{m,2,\partial \Omega} \). Let \((\cdot, \cdot)\) and \((\cdot, \cdot)_1\), respectively, denote the inner product in \( L^2(\Omega)^3 \) and \( H^1(\Omega)^3 \). The following forms are needed:

\[
\begin{align*}
& a_0(u,v) = \frac{v}{2} \int_\Omega e_{ij}(u)v_{ij}(v)dx \quad \forall u, v \in H^1(\Omega)^3, \\
& a_1(u,v,w) = \int_\Omega u \cdot \nabla v_1w dx \quad \forall u, v, w \in H^1(\Omega)^3, \\
& C(u,v) = 2(\bar{\omega} \times u, v) \quad \forall u, v \in L^2(\Omega)^3,
\end{align*}
\]

where \( e_{ij}(u) = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \). Introduce the following space:

\[
V_\sigma(\Omega) = \{u \in H^1(\Omega)^3, \text{div}u = 0, u|_\Gamma = 0\}.
\]

The stress vector \( \sigma(u, p) \) is defined by

\[
\sigma(u, p) = [-p\delta_{ij} + e_{ij}(u)] \cdot \vec{n}.
\]

Then its normal component and tangential component can be written as

\[
\sigma_n(u, p) = \sigma(u, p) \cdot \vec{n}, \quad \sigma_\tau(u) = \sigma(u, p) - \sigma_n(u, p)\vec{n}.
\]

If there is no possibility of confusion, we simply use \( \sigma, \sigma_n \) and \( \sigma_\tau \) to express \( \sigma(u, p) \), \( \sigma_n(u, p) \) and \( \sigma_\tau(u) \), respectively.

Let \( \psi : \mathbb{R}^3 \to \vec{R} = (-\infty, +\infty) \) be a given function possessing the properties of convexity and weak semi-continuity from below (\( \psi \) is not identically equal to \( +\infty \)). The set \( \partial[\psi(\bar{a})] \) is a subdifferential of the function \( \psi \) at the point \( \bar{a} \):

\[
\partial[\psi(\bar{a})] = \{\bar{b} \in \mathbb{R}^3 : \psi(\bar{h}) - \psi(\bar{a}) \geq \bar{b} \cdot (\bar{h} - \bar{a}), \forall \bar{h} \in \mathbb{R}^3\}.
\]

It is easy to see \( \psi(z) = |z| \) for \( z \in \mathbb{R}^3 \) in (2) (cf. [1]).

In this paper, the backward flows are neglected; then we can easily know

\[
q_1 \leq 0 \quad \text{on } S_1; \quad q_2 \geq 0 \quad \text{on } S_2.
\]
Moreover, $q_1$ and $q_2$ satisfy the compatibility condition
\[
\int_{S_1 \cup S_2} (q_1 + q_2) ds = 0.
\] (3)

**Definition 1.** For $f \in L^2(\Omega)^3$, $g_i \in L^2(S_i)^3$, $q_i \in L^2(S_i)^3$, $i = 1, 2$, we say that $u \in K_\sigma$ is a weak solution to problem (1) and (2) if it satisfies the following variational inequality:
\[
\begin{align*}
  & \text{Find } u \in K_\sigma \text{ such that } \\
  & \forall v \in K_\sigma,
  \end{align*}
\]
\[
\begin{align*}
  & \left\{ \begin{array}{l}
    a_0(u, v - u) + a_1(u, u, v - u) + C(u, v - u) + j_1(v_\tau) - j_1(u_\tau) + j_2(v_\tau) - j_2(u_\tau) - (f, v - u) \\
    \int_S \sigma \cdot (v - u) ds = (f, v - u).
  \end{array} \right. 
\end{align*}
\] (4)

where
\[
\begin{align*}
  & j_i(\eta) = \int_{S_i} g_i |\eta| ds \quad i = 1, 2; \\
  & K_\sigma = \{ v \in V_\sigma(\Omega), v \cdot n = q_i \text{ on } S_i, i = 1, 2 \}.
\end{align*}
\]

**Theorem 1.** If $u$ is a smooth solution to problem (1) and (2), then it also satisfies the variational inequality (4). Conversely, it is also true.

**Proof.** Actually, if $u \in K_\sigma$ is a smooth solution to the problem (1) and (2), multiplying the equation by $v - u$ for $v \in K_\sigma$ and integrating over $\Omega$, we obtain
\[
a_0(u, v - u) + a_1(u, u, v - u) + C(u, v - u) - \int_S \sigma \cdot (v - u) ds = (f, v - u).
\] (5)

Since $\sigma = \sigma_n + \sigma_\tau$ and $v - u = (v_n - u_n) + (v_\tau - u_\tau)$, we have
\[
\int_S \sigma \cdot (v - u) ds = \sum_{i=1}^{2} \int_{S_i} \sigma_n \cdot (v - u) ds + \sum_{i=1}^{2} \int_{S_i} \sigma_\tau \cdot (v - u) ds
\]
\[
= \sum_{i=1}^{2} \int_{S_i} \sigma_n \cdot (v_n - u_n) ds + \sum_{i=1}^{2} \int_{S_i} \sigma_\tau \cdot (v_\tau - u_\tau) ds.
\]

Because $u$ and $v$ both belong to $K_\sigma$, then
\[
u_n - v_n = 0 \quad \text{on } S_i, i = 1, 2.
\]

Hence
\[
\int_S \sigma \cdot (v - u) ds = \sum_{i=1}^{2} \int_{S_i} \sigma_\tau \cdot (v - u) ds = \sum_{i=1}^{2} \int_{S_i} \sigma_\tau \cdot (v_\tau - u_\tau) ds.
\]

According to the definition of subdifferential property of $\partial |u_\tau|$ and $\partial |v_\tau|$, we have
\[
g_i|v_\tau| - g_i|u_\tau| \geq -\sigma_\tau(v_\tau - u_\tau) \quad \text{on } S_i, i = 1, 2.
\]

Then we obtain (4). Next, we prove that the reverse is also correct. Let
\[
\hat{K} = \{ v \in V_\sigma(\Omega), v \cdot n = 0 \text{ on } S_i, i = 1, 2 \};
\]
then $C_0^\infty(\Omega)^3$ is dense in $\hat{K}$. Hence substituting
\[
v = u \pm w, \quad \forall w \in C_0^\infty(\Omega)^3
\]
into (4), we obtain
\[
a_0(u, w) + a_1(u, u, w) + C(u, w) = (f, w).
\]
Let \( w = v - u \in \tilde{K} \); then the result is

\[
(-v \Delta u + u \cdot \nabla u + 2\tilde{\omega} \times u, v - u) + \sum_{i=1}^{2} \int_{S_i} \sigma_i \cdot (v_\tau - u_\tau) ds = 0.
\]

Comparing with (4), we have the following inequality:

\[
\int_{S_i} g_i |v_\tau| - g_i |u_\tau| ds \geq - \int_{S_i} \sigma_i \cdot (v_\tau - u_\tau) ds, \quad i = 1, 2,
\]

which confirms the validity of the subdifferential boundary conditions (2).

\[\Box\]

3. Main results

Following [5], let \( e^k (k = 1, 2, \ldots) \) form a basis of \( V_\sigma (\Omega) \) and be orthonormal in \( V_\sigma (\Omega) \) and \( L^2 (\Omega)^3 \). We can choose a solution \( u^n = \sum_{k=1}^{n} \theta_k^n e^k \) of the solution

\[
\begin{align*}
a_0(u^n, e^k) + a_1(P u^n, u^n, e^k) + C(u^n, e^k) + j_1(e^k_\Omega) + j_2(e^k_\tau) - (f, e^k) + n(\phi(u^n), e^k)_1 &= 0, \quad k = 1, \ldots, n,
\end{align*}
\]

where \( P \) is the projection of \( V_\sigma (\Omega) \) into \( K_\sigma \) and \( \phi(h) = h - Ph \) for \( h \in V_\sigma (\Omega) \).

**Lemma 1** ([7, p. 9–10]). If \( x \in V_\sigma \), then there exists a unique \( Px \in K_\sigma \) such that

\[
\begin{align*}
Px &\in K_\sigma, \\
\| Px - x \|_1 &\leq \| y - x \|_1, \quad \forall y \in K_\sigma, \\
(x - Px, y - Px)_1 &\leq 0, \quad \forall y \in K_\sigma,
\end{align*}
\]

and

\[
\| Px_1 - Px_2 \|_1 \leq \| x_2 - x_1 \|_1, \quad \forall x_1, x_2 \in V_\sigma (\Omega).
\]

**Lemma 2** ([5]).

\[
(\phi(h), h)_1 \geq \| \phi(h) \|_1^2, \quad (\phi(h), h)_1 \geq \varepsilon \| \phi(h) \|_1, \quad \forall \varepsilon > 0.
\]

In addition to Lemmas 1 and 2, we also use the following trace inequalities and the well-known Korn–Poincare inequality:

**Trace Inequality**

\[
\| u \|_{L^2(S_i)}^2 \leq c_1(\Omega) \| u \|_1^2, \quad i = 1, 2, c_1 > 0, \forall u \in H^1(\Omega)^3.
\]

\[
\| u \|_{S_i} \leq c_2(\Omega) \| u \|_1, \quad i = 1, 2, c_2 > 0, \forall u \in H^1(\Omega)^3.
\]

**Korn–Poincare Inequality**

\[
\| u \|^2 \leq c_3(\Omega) \| \nabla u \|^2, \quad c_3 > 0, \forall u \in V_\sigma (\Omega).
\]

The main results are the following Theorems 2 and 3:

**Theorem 2.** Given \( f \in L^2 (\Omega)^3, g_i \in L^2(S_i)^3, q_i \in L^2(S_i)^3, i = 1, 2 \), which satisfy the compatibility condition (3).

If \( \| q_1 \|_{S_1} \) is sufficiently small such that

\[
Q := \frac{v}{1 + c_3} - \frac{c_1 \| q_1 \|_{S_1}}{2} > 0
\]

holds, then there exists a weak solution \( u \in K_\sigma \) to the variational inequality (4) and

\[
\| u \|_1 \leq \frac{\| f \| + c_2(\| g_1 \|_{S_1} + \| g_2 \|_{S_1})}{Q}.
\]
**Proof.** Multiplying (6) by $\theta_k^n$ and summing for $k = 1, \ldots, n$, we obtain

$$a_0(u^n, u^n) + a_1(Pu^n, u^n, u^n) + j_1(u^n) + j_2(u^n) - (f, u^n) + n(\phi(u^n), u^n)_1 \leq 0.$$  \hspace{1cm} (9)

Since

$$a_0(u^n, u^n) + a_1(Pu^n, u^n, u^n) + j_1(u^n) + j_2(u^n) + n(\phi(u^n), u^n)_1 \geq \frac{\nu}{1 + c_3} \|u\|_1^2 + \frac{1}{2} \int S_1 Pu^n \cdot \nabla u^n |u^n|^2 ds + \frac{1}{2} \int S_2 Pu^n \cdot \nabla u^n |u^n|^2 ds - \|g_1\|_{S_1} \|u^n\|_{S_1} - \|g_2\|_{S_2} \|u^n\|_{S_2}$$

$$\geq \left( \frac{\nu}{1 + c_2} - \frac{c_1 \|q_1\|_{S_1}}{2} \right) \|u^n\|_1^2 - c_2(\|g_1\|_{S_1} + \|g_2\|_{S_2}) \|u^n\|_1.$$  

If (7) holds, then we have

$$\|u^n\|_1 \leq \|f\| + c_2(\|g_1\|_{S_1} + \|g_2\|_{S_1}) \frac{Q}{\nu}.$$  \hspace{1cm} (10)

Hence there exists a subsequence of $\{u^n\}_{n=1}^\infty$ which converges weakly in $H^1(\Omega)^3$. We denote the subsequence by $\{u^n\}$ again and its limit by $u$ which also satisfies (10). Meanwhile, we also have

$$n(\phi(u^n), u^n)_1 \leq (f, u^n) - a_0(u^n, u^n) - a_1(Pu^n, u^n, u^n) - j_1(u^n) - j_2(u^n) \leq (\|f\| + c_2 \|g_1\|_{S_1} + \|g_2\|_{S_2}) \|u^n\|_1 + \left( 1 + \frac{c_1 \|q_1\|_{S_1}}{2} \right) \|u^n\|_1^2.$$  

Since (10), as $n \to \infty$, we have

$$\phi(u^n) \to 0 \quad \text{in} \quad H^1(\Omega)^3.$$

With a similar manner in [5], we can show $u \in K_{\sigma}, Pu = u$. Hence, for any $v \in K_{\sigma}$, we have

$$a_0(u^n, Pu^n - v) + a_1(Pu^n, u^n, Pu^n - v) + C(u^n, Pu^n - v) + j_1(Pu^n - v) - j_2(Pu^n - v) \leq 0.$$  

Following **Lemma 1** and the properties of convexity and weak semicontinuity from below of $|\eta|$, the result is

$$a_0(u^n, Pu^n - v) + a_1(Pu^n, u^n, Pu^n - v) + C(u^n, Pu^n - v) + j_1(Pu^n - v) - j_1(v) \leq 0.$$  

Let $n$ go to infinity in (13); we need to prove the convergence of every term. For simplicity, we only prove the convergence of the trilinear term and $j_1(Pu^n)$.

$$|a_1(Pu^n, u^n, Pu^n) - a_1(u, u, u)| = \left| \int_\Omega (Pu^n \cdot \nabla)u^n \cdot Pu^n dx - \int_\Omega (u \cdot \nabla)u \cdot u dx \right|$$

$$= \left| \int_\Omega (Pu^n \cdot \nabla)u^n \cdot u dx - \int_\Omega (u \cdot \nabla)u \cdot u dx - \int_\Omega (Pu^n \cdot \nabla)u^n \cdot \phi(u^n) dx \right|$$

$$\leq \left| \int_{S_1} Pu^n \cdot \nabla u^n |u^n|^2 ds - \int_{S_1} u \cdot \nabla u |u|^2 ds \right|$$

$$+ \left| \int_{S_2} Pu^n \cdot \nabla u^n |u^n|^2 ds - \int_{S_2} u \cdot \nabla u |u|^2 ds \right| + \left| \int_\Omega (Pu^n \cdot \nabla)u^n \cdot \phi(u^n) dx \right|$$

$$= I_1 + I_2 + I_3.$$  

$$I_1 = \left| \int_{S_1} Pu^n \cdot \nabla u^n |u^n|^2 ds - \int_{S_1} u \cdot \nabla u |u|^2 ds \right|$$

$$\leq \left| \int_{S_1} Pu^n \cdot \nabla |u^n|^2 - |u|^2 ds \right| + \min \left| \int_{S_1} (Pu^n - Pu \cdot \nabla u^n |u^n|^2 ds \right|$$

$$\leq \|q_1\|_{S_1} \|u^n\|_{S_1}^2 - \|u\|_{S_1}^2 \|u\|_{L^4(S_1)}^2 \to 0 \quad \text{as} \quad n \to \infty.$$
Similarly,
\[ I_2 \to 0 \quad \text{as } n \to \infty. \]
According to (11), we can easily prove
\[ I_3 \to 0 \quad \text{as } n \to \infty. \]
So
\[ a_1(Pu^n, u^n, Pu^n) \to a_1(u, u, u) \quad \text{as } n \to \infty. \]
In addition, we also have
\[
|f_1(Pu^n_\tau) - f_1(u_\tau)| = \left| \int_{S_1} g_1|Pu^n_\tau|ds - \int_{S_1} g_1|u_\tau|ds \right| \leq \int_{S_1} |g_1||Pu^n_\tau - u_\tau|ds
\]
\[
\leq \int_{S_1} |g_1||u^n_\tau - u_\tau|ds + \int_{S_1} |g_1||\phi(u^n_\tau)|ds
\]
\[
\leq \|g_1\|S_1\|u^n_\tau - u_\tau\|_{S_1} + \|g_1\|S_1\|\phi(u^n_\tau)\|_{S_1} \to 0 \quad \text{as } n \to \infty.
\]
Hence, when \( n \to \infty \) in (13), we have
\[
a_0(u, u - v) + a_1(u, u, u - v) + C(u, u - v) + f_1(u_\tau) - f_1(v_\tau)
\]
\[
+ j_2(u_\tau) - j_2(v_\tau) - (f, u - v) \leq 0, \quad \forall v \in K_{\sigma}.
\]
From (14), we know that \( u \) belongs to \( K_{\sigma} \) and satisfies the variational inequality (4). \( \square \)

**Theorem 3.** Under the same conditions as in Theorem 2, assume that \( u \in K_{\sigma} \) is a solution to the variational inequality (4). If
\[
\frac{v}{1 + c_3} - \frac{2c_4(\|f\| + c_2\|g_1\|_{S_1} + c_2\|g_2\|_{S_2})}{Q} > 0,
\]
then \( u \) is unique, where \( c_4 \) depends only on \( \Omega \).

**Proof.** Let \( u_1, u_2 \in K_{\sigma} \) both be weak solutions to (4); then
\[
a_0(u_1, v - u_1) + a_1(u_1, u_1, v - u_1) + C(u_1, v - u_1) + f_1(v_\tau) - f_1(u_\tau)
\]
\[
+ j_2(v_\tau) - j_2(u_\tau) - (f, v - u_1) \geq 0,
\]
(16)
\[
a_0(u_2, v - u_2) + a_1(u_2, u_2, v - u_2) + C(u_2, v - u_2) + f_1(v_\tau) - f_1(u_\tau)
\]
\[
+ j_2(v_\tau) - j_2(u_\tau) - (f, v - u_2) \geq 0.
\]
(17)
Taking \( v = u_2 \) in (16) and \( v = u_1 \) in (17) and adding them, we have
\[
a_0(u_1 - u_2, u_1 - u_2) + a_1(u_1, u_1, u_1 - u_2) - a_1(u_2, u_2, u_1 - u_2) = 0.
\]
(18)
By means of (8), we have
\[
|a_1(u_1, u_1, u_1 - u_2) - a_1(u_2, u_2, u_1 - u_2)| = |a_1(u_1, u_1 - u_2, u_1 - u_2) + a_1(u_1 - u_2, u_2, u_1 - u_2)|
\]
\[
\leq c_4(\|g_1\|_1\|u_1 - u_2\|_1^2 + \|g_2\|_2\|u_1 - u_2\|_2^2)
\]
\[
\leq \frac{2c_4(\|f\| + c_2\|g_1\|_{S_1} + c_2\|g_2\|_{S_2})}{Q}\|u_1 - u_2\|_1^2.
\]
Hence
\[
0 \geq a_0(u_1 - u_2, u_1 - u_2) - |a_1(u_1, u_1, u_1 - u_2) - a_1(u_2, u_2, u_1 - u_2)|
\]
\[
\geq \frac{v}{1 + c_3}\|u_1 - u_2\|_1^2 - \frac{2c_4(\|f\| + c_2\|g_1\|_{S_1} + c_2\|g_2\|_{S_2})}{Q}\|u_1 - u_2\|_1^2
\]
\[
= \left( \frac{v}{1 + c_3} - \frac{2c_4(\|f\| + c_2\|g_1\|_{S_1} + c_2\|g_2\|_{S_2})}{Q} \right)\|u_1 - u_2\|_1^2.
\]
Under the assumption (15), we have
\[ \|u_1 - u_2\|_1^2 \leq 0 \implies u_1 \equiv u_2. \]
Thus \( u \) is unique. \( \square \)

References