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Some new sequence spaces derived by the domain of generalized difference matrix

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ABSTRACT

Let λ denote any one of the classical spaces ℓ_∞, c, c_0 and ℓ_p of bounded, convergent, null and absolutely p-summable sequences, respectively, and $\widehat{\lambda}$ also be the domain of the generalized difference matrix $B(r,s)$ in the sequence space λ, where $1 \le p < \infty$. The present paper is devoted to studying on the sequence space $\widehat{\lambda}$. Furthermore, the β- and γ-duals of the space $\widehat{\lambda}$ are determined, and the Schauder bases for the spaces $\widehat{c}$, $\widehat{c}_0$ and $\widehat{\ell}_p$ are given, and some topological properties of the spaces $\widehat{c}_0$, $\widehat{\ell}_1$ and $\widehat{\ell}_p$ are examined. Finally, the classes $(\widehat{\lambda}_1 : \lambda_2)$ and $(\widehat{\lambda}_1 : \widehat{\lambda}_2)$ of infinite matrices are characterized, where $\lambda_1 \in \{\ell_\infty, c, c_0, \ell_p, \ell_1\}$ and $\lambda_2 \in \{\ell_\infty, c, c_0, \ell_1\}$.

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1. Preliminaries, background and notation

By a *sequence space*, we understand a linear subspace of the space $\omega = \mathbb{C}^{\mathbb{N}}$ of all complex sequences which contains ϕ, the set of all finitely non-zero sequences, where $\mathbb{C}$ denotes the complex field and $\mathbb{N} = \{0, 1, 2, \ldots\}$. We write ℓ_∞, c, c_0 and ℓ_p for the classical sequence spaces of all bounded, convergent, null and absolutely p-summable sequences, respectively, where $1 \le p < \infty$. Also by bs and cs, we denote the spaces of all bounded and convergent series, respectively. bv is the space consisting of all sequences (x_k) such that $(x_k - x_{k+1})$ in ℓ_1 and bv_0 is the intersection of the spaces bv and c_0. We assume throughout unless stated otherwise that $p, q > 1$ with $p^{-1} + q^{-1} = 1$ and use the convention that any term with negative subscript is equal to zero.

Let $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk}, where $n, k \in \mathbb{N}$, and write

$$(Ax)_n := \sum_k a_{nk} x_k, \quad (n \in \mathbb{N},\ x \in D_{00}(A)), \tag{1.1}$$

where $D_{00}(A)$ denotes the subspace of ω consisting of $x \in \omega$ for which the sum exists as a finite sum. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. More generally, if μ is a normed sequence space, we can write $D_\mu(A)$ for $x \in \omega$ for which the sum in (1.1) converges in the norm of μ. We write

$$(\lambda : \mu) := \{A : \lambda \subseteq D_\mu(A)\}$$

for the space of those matrices which send the whole of the sequence space λ into the sequence space μ in this sense.

A matrix $A = (a_{nk})$ is called a *triangle* if $a_{nk} = 0$ for $k > n$ and $a_{nn} \ne 0$ for all $n \in \mathbb{N}$. It is trivial that $A(Bx) = (AB)x$ holds for the triangle matrices A, B and a sequence x. Further, a triangle matrix U uniquely has an inverse $U^{-1} = V$ which is also a triangle matrix. Then, $x = U(Vx) = V(Ux)$ holds for all $x \in \omega$.

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Let us give the definition of some triangle limitation matrices which are needed in the text. Let $t = (t_k)$ be a sequence of positive reals and write

$$T_n := \sum_{k=0}^{n} t_k, \quad (n \in \mathbb{N}).$$

Then the Cesàro mean of order one, Riesz mean with respect to the sequence $t = (t_k)$ and Euler mean of order r are respectively defined by the matrices $C = (c_{nk})$, $R^t = (r^t_{nk})$ and $E^r = (e^r_{nk})$; where

$$c_{nk} := \begin{cases} \dfrac{1}{n+1}, & (0 \le k \le n), \\ 0, & (k > n), \end{cases} \qquad r^t_{nk} := \begin{cases} \dfrac{t_k}{T_n}, & (0 \le k \le n), \\ 0, & (k > n) \end{cases}$$

and

$$e^r_{nk} := \begin{cases} \dbinom{n}{k}(1-r)^{n-k} r^k, & (0 \le k \le n), \\ 0, & (k > n), \end{cases}$$

for all $k, n \in \mathbb{N}$. We write $\mathcal{U}$ for the set of all sequences $u = (u_k)$ such that $u_k \ne 0$ for all $k \in \mathbb{N}$. For $u \in \mathcal{U}$, let $1/u = (1/u_k)$. Let $z, u, v \in \mathcal{U}$ and define the summation matrix $S = (s_{nk})$, the difference matrix $\Delta = (\delta_{nk})$, the generalized weighted mean or factorable matrix $G(u, v) = (g_{nk})$, $\Delta^{(m)} = (\Delta^{(m)}_{nk})$, $A^r_u = \{a_{nk}(r)\}$ and $A^z = (a^z_{nk})$ by

$$s_{nk} := \begin{cases} 1, & (0 \le k \le n), \\ 0, & (k > n), \end{cases} \qquad \delta_{nk} := \begin{cases} (-1)^{n-k}, & (n-1 \le k \le n), \\ 0, & (0 \le k < n-1 \text{ or } k > n), \end{cases}$$

$$g_{nk} := \begin{cases} u_n v_k, & (0 \le k \le n), \\ 0, & (k > n), \end{cases} \qquad \Delta^{(m)}_{nk} := \begin{cases} (-1)^{n-k}\dbinom{m}{n-k}, & (\max\{0, n-m\} \le k \le n), \\ 0, & (0 \le k < \max\{0, n-m\} \text{ or } k > n), \end{cases}$$

$$a_{nk}(r) := \begin{cases} \dfrac{1+r^k}{n+1} u_k, & (0 \le k \le n), \\ 0, & (k > n) \end{cases} \quad \text{and} \quad a^z_{nk} := \begin{cases} (-1)^{n-k} z_k, & (n-1 \le k \le n), \\ 0, & (0 \le k < n-1 \text{ or } k > n), \end{cases}$$

for all $k, m, n \in \mathbb{N}$; where u_n depends only on n and v_k only on k.

The *domain* λ_A of an infinite matrix A in a sequence space λ is defined by

$$\lambda_A := \{x = (x_k) \in \omega : Ax \in \lambda\}, \tag{1.2}$$

which is a sequence space. If A is triangle, then one can easily observe that the sequence spaces λ_A and λ are linearly isomorphic, i.e., $\lambda_A \cong \lambda$. If λ is a sequence space, then the continuous dual λ^*_A of the space λ_A is defined by

$$\lambda^*_A := \{f : f = g \circ A,\ g \in \lambda^*\}.$$

Although in most cases the new sequence space λ_A generated by the limitation matrix A from a sequence space λ is the expansion or the contraction of the original space λ, it may be observed in some cases that those spaces overlap. Indeed, one can easily see that the inclusion $\lambda_S \subset \lambda$ strictly holds for $\lambda \in \{\ell_\infty, c, c_0\}$. As this, one can deduce that the inclusion $\lambda \subset \lambda_{\Delta^{(1)}}$ also strictly holds for $\lambda \in \left\{\ell_\infty, c, c_0, \ell_p\right\}$. However, if we define $\lambda := c_0 \oplus \operatorname{span}\{z\}$ with $z = ((-1)^k)$, i.e. $x \in \lambda$ if and only if $x := s + \alpha z$ for some $s \in c_0$ and some $\alpha \in \mathbb{C}$, and consider the matrix A with the rows A_n defined by $A_n := (-1)^n e^{(n)}$ for all $n \in \mathbb{N}$, we have $Ae = z \in \lambda$ but $Az = e \notin \lambda$ which lead us to the consequences that $z \in \lambda \setminus \lambda_A$ and $e \in \lambda_A \setminus \lambda$, where $e = (1, 1, 1, \ldots)$ and $e^{(n)}$ is a sequence whose only non-zero term is a 1 in nth place for each $n \in \mathbb{N}$. That is to say that the sequence spaces λ_A and λ overlap but neither contains the other. The approach constructing a new sequence space by means of the matrix domain of a particular limitation method has recently been employed by Wang [1], Ng and Lee [2], Malkowsky [3], Altay and Başar [4–9], Malkowsky and Savaş [10], Başarır [11], Aydın and Başar [12–16], Başar et al. [17], Şengönül and Başar [18], Altay [19], Polat and Başar [20] and, Malkowsky et al. [21]. In Table 1; Δ, Δ^2 and Δ^m are the transpose of the matrices $\Delta^{(1)}$, $\Delta^{(2)}$ and $\Delta^{(m)}$, respectively, and $c_0(u, p)$ and $c(u, p)$ are the spaces consisting of the sequences $x = (x_k)$ such that $ux = (u_k x_k)$ in the spaces $c_0(p)$ and $c(p)$ for $u \in \mathcal{U}$, respectively, and studied by Başarır [11]. Finally, the new technique for deducing certain topological properties, for example *AB*-, *KB*-, *AD*-properties, solidity and monotonicity etc., and determining the β- and γ-duals of the domain of a triangle matrix in a sequence space is given by Altay and Başar [8].

Let $r, s \in \mathbb{R} \setminus \{0\}$ and define the generalized difference matrix $B(r, s) = \{b_{nk}(r, s)\}$ by

$$b_{nk}(r, s) := \begin{cases} r, & (k = n), \\ s, & (k = n-1), \\ 0, & (0 \le k < n-1 \text{ or } k > n), \end{cases}$$

for all $k, n \in \mathbb{N}$. We should record here that the matrix $B(r, s)$ can be reduced to the difference matrix $\Delta^{(1)}$ in case $r = 1, s = -1$. So, the results related to the matrix domain of the matrix $B(r, s)$ are more general and more comprehensive

Table 1
The domains of some triangle matrices in certain sequence spaces.

λ	A	λ_A	Refer to:
c	N_q	c_{N_q}	[1]
$\ell_p,\ (1\le p\le\infty)$	C	$X_p,\ X_\infty$	[2]
$X_p,\ (1\le p\le\infty)$	Δ^m	$C_p(\Delta^m),\ C_\infty(\Delta^m)$	[22]
$c_0,\ c,\ \ell_\infty$	R^q	$(\overline{N},q)_0,\ (\overline{N},q),\ (\overline{N},q)_\infty$	[3]
$c_0,\ c,\ \ell_\infty$	$\Delta^{(1)}$	$c_0(\Delta),\ c(\Delta),\ \ell_\infty(\Delta)$	[23]
$c_0,\ c,\ \ell_\infty$	Δ^2	$c_0(\Delta^2),\ c(\Delta^2),\ \ell_\infty(\Delta^2)$	[24]
$c_0,\ c,\ \ell_\infty$	$u\Delta^2$	$c_0(u;\Delta^2),\ c(u;\Delta^2),\ \ell_\infty(u;\Delta^2)$	[25]
$c_0,\ c,\ \ell_\infty$	Δ^2	$c_0(\Delta^2),\ c(\Delta^2),\ \ell_\infty(\Delta^2)$	[24]
$c_0,\ c,\ \ell_p$	$G(u,v)$	$Z(u,v;c_0),\ Z(u,v;c),\ Z(u,v;\ell_p)$	[10]
$c_0,\ c$	C	$\widetilde{c}_0,\ \widetilde{c}$	[18]
$c_0,\ c$	E^r	$e_0^r,\ e_c^r$	[4]
$c_0,\ c$	$G(u,v)$	$(c_0)_{G(u,v)},\ c_{G(u,v)}$	[26]
$c_0,\ c$	A_1^r	$a_0^r,\ a_c^r$	[12]
$\ell_p,\ (1\le p\le\infty)$	A_1^r	$a_p^r,\ a_\infty^r$	[15]
$\ell_p,\ (1\le p\le\infty)$	E^r	$e_p^r,\ e_\infty^r$	[27,28]
$a_0^r,\ a_c^r$	$\Delta^{(1)}$	$a_0^r(\Delta),\ a_c^r(\Delta)$	[14]
$\ell_p,\ (1\le p<\infty)$	$G(u,v)$	ℓ_A^p	[29]
$\ell_p,\ (1\le p<\infty)$	$\Delta^{(1)}$	bv_p	[30,31]
$\ell_p,\ (0<p<1)$	$\Delta^{(1)}$	bv_p	[9]
$c_0,\ c,\ \ell_\infty$	Δ^m	$c_0(\Delta^m),\ c(\Delta^m),\ \ell_\infty(\Delta^m)$	[32,33]
$\ell_p,\ (1\le p<\infty)$	$\Delta^{(m)}$	$\ell_p(\Delta^{(m)})$	[19]
$c_0,\ c,\ \ell_\infty$	$\Delta^{(m)}$	$c_0(\Delta^{(m)}),\ c(\Delta^{(m)}),\ \ell_\infty(\Delta^{(m)})$	[34]
$e_0^r,\ e_c^r$	$\Delta^{(m)}$	$e_0^r(\Delta^{(m)}),\ e_c^r(\Delta^{(m)})$	[20]
$w_0^p,\ w^p,\ w_\infty^p$	Δ	$w_0^p(\Delta),\ w^p(\Delta),\ w_\infty^p(\Delta)$	[35]
$w_0^p,\ w^p,\ w_\infty^p$	T	$w_0^p(T),\ w^p(T),\ w_\infty^p(T)$	[36]
$\ell_\infty(p)$	S	$bs(p)$	[37,38]
$\ell(p)$	A_u^r	$a^r(u,p)$	[16]
$\ell(p)$	$B(r,s)$	$\widehat{\ell(p)}$	[39]
$\ell(p)$	S	$\overline{\ell(p)}$	[40]
$c_0(p),\ c(p),\ \ell_\infty(p)$	Δ	$\Delta c_0(p),\ \Delta c(p),\ \Delta\ell_\infty(p)$	[41]
$c_0(p),\ c(p),\ \ell_\infty(p)$	$u\Delta$	$c_0(u,\Delta,p),\ c(u,\Delta,p),\ \ell_\infty(u,\Delta,p)$	[42]
$c_0(p),\ c(p),\ \ell_\infty(p)$	$u\Delta^2$	$c_0(u,\Delta^2,p),\ c(u,\Delta^2,p),\ \ell_\infty(u,\Delta^2,p)$	[43]
$c_0(p),\ c(p),\ \ell_\infty(p)$	$G(u,v)$	$c_0(u,v;p),\ c(u,v;p),\ \ell_\infty(u,v;p)$	[6]
$\ell(p)$	$G(u,v)$	$\ell(u,v;p)$	[7]
$\ell(p),\ \ell_\infty(p)$	A^z	$bv(z,p),\ bv_\infty(z,p)$	[17]
$c_0(u,p),\ c(u,p)$	A_1^r	$a_0^r(u,p),\ a_c^r(u,p)$	[13]
$\ell(p)$	R^t	$r^t(p)$	[44]
$c_0(p),\ c(p),\ \ell_\infty(p)$	R^t	$r_0^t(p),\ r_c^t(p),\ r_\infty^t(p)$	[5]
$c_0(p),\ c(p),\ \ell_\infty(p)$	Δ^m	$\Delta^m c_0(p),\ \Delta^m c(p),\ \Delta^m\ell_\infty(p)$	[45]
$c_0(p),\ c(p),\ \ell_\infty(p)$	$u\Delta^{(m)}$	$\Delta_u^{(m)}c_0(p),\ \Delta_u^{(m)}c(p),\ \Delta_u^{(m)}\ell_\infty(p)$	[21]

than the corresponding consequences of the matrix domain of $\Delta^{(1)}$, and include them. For the literature concerning with the domain λ_A of the infinite matrix A in the sequence space λ, the following table may be useful:

The main purpose of the present paper is to introduce the sequence space $\lambda_{B(r,s)}$, and to determine the β- and γ-duals of the space, where λ denotes the any one of the spaces ℓ_∞, c, c_0 or ℓ_p. Furthermore, the Schauder bases for the spaces $\widehat{c}, \widehat{c_0}$ and $\widehat{\ell_p}$ are given, and some topological properties of the spaces $\widehat{c_0}$, $\widehat{\ell_1}$ and $\widehat{\ell_p}$ are examined. Finally, some classes of matrix mappings on the space $\lambda_{B(r,s)}$ are characterized.

The paper is organized as follows: In Section 2, we summarize the studies on the difference sequence spaces. In Section 3, we introduce the domain $\lambda_{B(r,s)}$ of the generalized difference matrix $B(r,s)$ in the sequence space λ with $\lambda \in \{\ell_\infty, c, c_0, \ell_p\}$ and determine the β-, and γ-duals of $\lambda_{B(r,s)}$. After proving the fact that under which conditions the inclusion $\lambda \subset \lambda_{B(r,s)}$ and the equality $\lambda = \lambda_{B(r,s)}$ hold, we give the Schauder basis of the spaces $(c_0)_{B(r,s)}$, $c_{B(r,s)}$ and $(\ell_p)_{B(r,s)}$. Finally, we investigate some topological properties of the spaces $(c_0)_{B(r,s)}$, $(\ell_1)_{B(r,s)}$ and $(\ell_p)_{B(r,s)}$ with $p > 1$. In Section 4 we state and prove a general theorem characterizing the matrix transformations from the domain of a triangle matrix to any sequence space. As an application of this basic theorem, we make a table which gives the necessary and sufficient conditions of the matrix transformations from $\lambda_{B(r,s)}$ to μ, where $\lambda \in \{\ell_\infty, c, c_0, \ell_p\}$ and $\mu \in \{\ell_\infty, c, c_0, \ell_1\}$. In the final section of the paper, we mention the significance of working on the space f of almost convergent sequences and note further suggestions.

2. Difference sequence spaces

In this section, we give some knowledge about the literature concerning with the spaces of difference sequences.

Let λ denote any one of the classical sequence spaces ℓ_∞, c or c_0. Then, $\lambda(\Delta)$ consisting of the sequences $x = (x_k)$ such that $\Delta x = (x_k - x_{k+1}) \in \lambda$ is called as the *difference sequence spaces* which were introduced by Kızmaz [23]. Kızmaz [23] proved that $\lambda(\Delta)$ is a Banach space with the norm

$$\|x\|_{\Delta} = |x_1| + \|\Delta x\|_{\infty}; \quad x = (x_k) \in \lambda(\Delta)$$

and the inclusion relation $\lambda \subset \lambda(\Delta)$ strictly holds. He also determined the α-, β- and γ-duals of the difference spaces and characterized the classes $(\lambda(\Delta) : \mu)$ and $(\mu : \lambda(\Delta))$ of infinite matrices, where $\lambda, \mu \in \{\ell_\infty, c\}$. Following Kızmaz [23], Sarıgöl [46] extended the difference spaces $\lambda(\Delta)$ to the spaces $\lambda(\Delta_r)$ defined by

$$\lambda(\Delta_r) := \left\{x = (x_k) \in \omega : \Delta_r x = \left\{k^r (x_k - x_{k+1})\right\} \in \lambda \text{ for } r < 1\right\}$$

and computed the α-, β-, γ-duals of the space $\lambda(\Delta_r)$, where $\lambda \in \{\ell_\infty, c, c_0\}$. It is easily seen that $\lambda(\Delta_r) \subset \lambda(\Delta)$, if $0 < r < 1$ and $\lambda(\Delta) \subset \lambda(\Delta_r)$, if $r < 0$.

In the same year, Ahmad and Mursaleen [41] extended these spaces to $\lambda(p, \Delta)$ and studied related problems. Malkowsky [47] determined the Köthe–Toeplitz duals of the sets $\ell_\infty(p, \Delta)$ and $c_0(p, \Delta)$, and give new proofs of the characterization of the matrix transformations considered in [41]. In 1993, Choudhary and Mishra [48] studied some properties of the sequence space $c_0(\Delta_r)$, for $r \geq 1$. The same year, Mishra [49] gave a characterization of BK-spaces which contain subspace isomorphic to $sc_0(\Delta)$ in terms of matrix maps and sufficient condition for a matrix map from $s\ell_\infty(\Delta)$ into a BK-space to be a compact operator. He showed that any matrix from $s\ell_\infty(\Delta)$ into a BK-space which does not contain any subspace isomorphic to $s\ell_\infty(\Delta)$ is compact, where

$$s\lambda(\Delta) = \{x = (x_k) \in \omega : (\Delta x_k) \in \lambda,\ x_1 = 0 \text{ for } \lambda = \ell_\infty \text{ or } c_0\}.$$

In 1996, Mursaleen et al. [50] defined and studied the sequence space

$$\ell_\infty(p, \Delta_r) = \{x = (x_k) \in \omega : \Delta_r x \in \ell_\infty(p)\}, \quad (r > 0).$$

Gnanaseelan and Srivastava [51] defined and studied the spaces $\lambda(u, \Delta)$ for a sequence $u = (u_k)$ of non-complex numbers such that

(i) $\frac{|u_k|}{|u_{k+1}|} = 1 + O(1/k)$ for each $k \in \mathbb{N}_1 = \{1, 2, 3, \ldots\}$.
(ii) $k^{-1}|u_k| \sum_{i=0}^{k} |u_i|^{-1} = O(1)$.
(iii) $\left(k\left|u_k^{-1}\right|\right)$ is a sequence of positive numbers increasing monotonically to infinity.

The same year, Malkowsky [52] defined the spaces $\lambda(u, \Delta)$ for an arbitrary fixed sequence $u = (u_k)$ without any restrictions on u. He proved that the sequence spaces $\lambda(u, \Delta)$ are BK-spaces with the norm defined by

$$\|x\| = \sup_{k \in \mathbb{N}} |u_{k-1}(x_{k-1} - x_k)| \quad \text{with } u_0 = x_0 = 1.$$

Later, Gaur and Mursaleen [53] extended the space $S_r(\Delta)$ to the space $S_r(p, \Delta)$, where

$$S_r(p, \Delta) = \left\{x = (x_k) \in \omega : \left(k^r |\Delta x_k|\right) \in c_0(p)\right\}, \quad (r \geq 1)$$

and characterized the matrix classes $(S_r(p, \Delta) : \ell_\infty)$ and $(S_r(p, \Delta) : \ell_1)$. Malkowsky et al. [54], and independently, Asma and Çolak [42] extended the space $\lambda(u, \Delta)$ to the space $\lambda(p, u, \Delta)$ and gave Köthe–Toeplitz duals of this spaces, for $\lambda = \ell_\infty, c$ or c_0. Recently Malkowsky and Mursaleen [55] characterized the matrix classes $(\Delta\lambda : \mu)$ and $(\Delta\lambda : \Delta\mu)$ for $\lambda = c_0(p), c(p), \ell_\infty(p)$ and $\mu = c_0(q), c(q), \ell_\infty(q)$.

Recently, the difference spaces bv_p consisting of the sequences $x = (x_k)$ such that $(x_k - x_{k-1}) \in \ell_p$ have been studied in the case $0 < p < 1$ by Altay and Başar [9], and in the case $1 \leq p \leq \infty$ by Başar and Altay [30], and Çolak et al. [31].

3. Some new sequence spaces derived by the domain of the matrix $B(r, s)$

In this section, we define the sequence spaces $\widehat{\ell_\infty}, \widehat{c}, \widehat{c_0}$ and $\widehat{\ell_p}$, and determine the β- and γ-duals of the spaces.

We introduce the sequence spaces $\widehat{\ell_\infty}, \widehat{c}, \widehat{c_0}$ and $\widehat{\ell_p}$ as the set of all sequences whose $B(r, s)$-transforms are in the spaces ℓ_∞, c, c_0 and ℓ_p, respectively, that is

$$\widehat{\ell_\infty} := \left\{x = (x_k) \in \omega : \sup_{k \in \mathbb{N}} |sx_{k-1} + rx_k| < \infty\right\},$$

$$\widehat{c} := \left\{x = (x_k) \in \omega : \exists l \in \mathbb{C} \ni \lim_{k \to \infty} |sx_{k-1} + rx_k - l| = 0\right\},$$

$$\widehat{c_0} := \left\{x = (x_k) \in \omega : \lim_{k \to \infty} |sx_{k-1} + rx_k| = 0\right\},$$

$$\widehat{\ell_p} := \left\{x = (x_k) \in \omega : \sum_k |sx_{k-1} + rx_k|^p < \infty\right\}.$$

With the notation of (1.2), we can redefine the spaces $\widehat{\ell_\infty}$, $\widehat{c}$, $\widehat{c_0}$ and $\widehat{\ell_p}$ by

$$\widehat{\ell_\infty} := \{\ell_\infty\}_{B(r,s)}, \qquad \widehat{c} := c_{B(r,s)}, \qquad \widehat{c_0} := \{c_0\}_{B(r,s)}, \qquad \widehat{\ell_p} := \{\ell_p\}_{B(r,s)}.$$

Define the sequence $y = (y_k)$ by the $B(r, s)$-transform of a sequence $x = (x_k)$, i.e.

$$y_k := sx_{k-1} + rx_k, \quad (k \in \mathbb{N}). \tag{3.1}$$

Since the spaces λ and $\lambda_{B(r,s)}$ are linearly isomorphic one can easily observe that $x = (x_k) \in \lambda_{B(r,s)}$ if and only if $y = (y_k) \in \lambda$, where the sequences $x = (x_k)$ and $y = (y_k)$ are connected with the relation (3.1).

Prior to quoting the lemmas which are needed for deriving some consequences given in Corollary 3.4 below, we give an inclusion theorem related with these new spaces.

Theorem 3.1. *Let $\lambda \in \{\ell_\infty, c, c_0, \ell_p\}$ and $B = B(r, s)$. Then,*

(i) $\lambda = \lambda_B$, *if* $|s/r| < 1$.
(ii) $\lambda \subset \lambda_B$ *is strict, if* $|s/r| \geq 1$.

Proof. Let $\lambda \in \{\ell_\infty, c, c_0, \ell_1\}$ and $B = B(r, s)$. Since the matrix B satisfies the conditions;

$$\sup_{n\in\mathbb{N}} \sum_k |b_{nk}| = |r| + |s|,$$
$$\lim_{n\to\infty} b_{nk} = 0,$$
$$\lim_{n\to\infty} \sum_k b_{nk} = r + s$$

and

$$\sup_{k\in\mathbb{N}} \sum_n |b_{nk}| = |r| + |s|,$$

$B \in (\lambda : \lambda)$. For any sequence $x \in \lambda$, $Bx \in \lambda$ hence $x \in \lambda_B$. This shows that $\lambda \subset \lambda_B$.

(i) Let $|s/r| < 1$. Since the inverse matrix $B^{-1} = (b_{nk}^{-1})$ of the matrix B also satisfies the conditions;

$$\sup_{n\in\mathbb{N}} \sum_k \left|b_{nk}^{-1}\right| = \left|\frac{1}{r}\right| \sum_k \left|\frac{s}{r}\right|^k < \infty,$$
$$\lim_{n\to\infty} b_{nk}^{-1} = \lim_{n\to\infty} \frac{1}{r}\left(\frac{-s}{r}\right)^n = 0,$$
$$\lim_{n\to\infty} \sum_k b_{nk}^{-1} = \lim_{n\to\infty} \sum_{k=0}^{n} \left(\frac{-s}{r}\right)^k \quad \text{exists}$$

and

$$\sup_{n\in\mathbb{N}} \sum_k \left|b_{nk}^{-1}\right| = \left|\frac{1}{r}\right| \sum_k \left|\frac{s}{r}\right|^k < \infty,$$

$B^{-1} \in (\lambda : \lambda)$. Therefore, if $x \in \lambda_B$ then $y = Bx \in \lambda$ and $x = B^{-1}y \in \lambda$. Thus the opposite inclusion $\lambda_B \subset \lambda$ is just proved. This completes the proof of the part (i).

(ii) Let us consider the sequences $u^1 := \{(-s/r)^n/r\}$, $u^2 := (n/r)$, $u^3 := \{(-1)^n(n + 1)\}$ and $u^4 := \{[1 + (-1)^n]/2\}$.

If $|s/r| > 1$, then $Bu^1 = e^{(0)} = (1, 0, 0, \ldots) \in \lambda$. Hence $u^1 \in \lambda_B \setminus \lambda$.

Suppose that $|s/r| = 1$.

(a) If $\lambda = c_0, \ell_p$, then $u^1 \in \lambda_B \setminus \lambda$.

(b) Let $\lambda = \ell_\infty, c$. If $s = -r$, then $Bu^2 = e \in \lambda$. Hence $u^2 \in \lambda_B \setminus \lambda$. If $s = r$, then $Bu^3 = \{r(-1)^n\} \in \ell_\infty$, $Bu^4 = (r, r, r, \ldots) \in c$. Hence $u^3 \in (\ell_\infty)_B \setminus \ell_\infty$ and $u^4 \in c_B \setminus c$.

This step completes the proof. □

The set $S(\lambda, \mu)$ defined by

$$S(\lambda, \mu) := \{z = (z_k) \in \omega : xz = (x_k z_k) \in \mu \text{ for all } x = (x_k) \in \lambda\} \tag{3.2}$$

is called the *multiplier space* of the spaces λ and μ. One can easily observe for a sequence space ν with $\lambda \supset \nu \supset \mu$ that the inclusions

$$S(\lambda, \mu) \subset S(\nu, \mu) \quad \text{and} \quad S(\lambda, \mu) \subset S(\lambda, \nu)$$

hold. With the notation of (3.2), the α-, β- *and* γ-*duals* of a sequence space λ, which are respectively denoted by λ^α, λ^β and λ^γ, are defined by

$$\lambda^\alpha := S(\lambda, \ell_1), \qquad \lambda^\beta := S(\lambda, cs) \quad \text{and} \quad \lambda^\gamma := S(\lambda, bs).$$

Table 2
The characterization of the class $(\lambda_1 : \lambda_2)$ with $\lambda_1 \in \{\ell_\infty, c, c_0, \ell_p, \ell_1\}$ and $\lambda_2 \in \{\ell_\infty, c\}$.

From	ℓ_∞	c	c_0	ℓ_p	ℓ_1
To					
ℓ_∞	**1.**	**1.**	**1.**	**2.**	**3.**
c	**4.**	**5.**	**6.**	**7.**	**8.**

Lemma 3.2 ([56, p. 52, Exercise 2.5(i)]). *Let λ, μ be the sequence spaces and $\xi \in \{\alpha, \beta, \gamma\}$. If $\lambda \subset \mu$, then $\mu^\xi \subset \lambda^\xi$.*

We read the following useful results from Stieglitz and Tietz [57]:

$$\sup_{n\in\mathbb{N}} \sum_k |a_{nk}|^q < \infty. \tag{3.3}$$

$$\sup_{k,n\in\mathbb{N}} |a_{nk}| < \infty. \tag{3.4}$$

$$\lim_{n\to\infty} a_{nk} = \alpha_k, \quad (k \in \mathbb{N}). \tag{3.5}$$

$$\lim_{n\to\infty} \sum_k |a_{nk}| = \sum_k |\alpha_k|. \tag{3.6}$$

$$\lim_{n\to\infty} \sum_k a_{nk} = \alpha. \tag{3.7}$$

Lemma 3.3. *The necessary and sufficient conditions for $A \in (\lambda : \mu)$ when $\lambda \in \{\ell_\infty, c, c_0, \ell_1, \ell_p\}$ and $\mu \in \{\ell_\infty, c\}$ can be read from Table 2: where*

1. (3.3) *with* $q = 1$.	**2.** (3.3).
3. (3.4).	**4.** (3.5) *and* (3.6).
5. (3.3) *with* $q = 1$, (3.5) *and* (3.7).	**6.** (3.3) *with* $q = 1$ *and* (3.5).
7. (3.3) *and* (3.5).	**8.** (3.4) *and* (3.5).

Basic Lemma ([8, Theorem 3.1]). *Let $C = (c_{nk})$ be defined via a sequence $a = (a_k) \in \omega$ and the inverse matrix $V = (v_{nk})$ of the triangle matrix $U = (u_{nk})$ by*

$$c_{nk} := \begin{cases} \sum_{j=k}^{n} a_j v_{jk}, & (0 \le k \le n), \\ 0, & (k > n), \end{cases}$$

for all $k, n \in \mathbb{N}$. Then,

$$\{\lambda_U\}^\gamma := \{a = (a_k) \in \omega : C \in (\lambda : \ell_\infty)\}$$

and

$$\{\lambda_U\}^\beta := \{a = (a_k) \in \omega : C \in (\lambda : c)\}.$$

Combining Lemma 3.3 with Basic Lemma, we have:

Corollary 3.4. *Define the sets $d_1(r,s)$, $d_2(r,s)$, $d_3(r,s)$, $d_4(r,s)$ and $d_5(r,s)$ by*

$$d_1(r,s) := \left\{ a = (a_k) \in \omega : \sup_{n\in\mathbb{N}} \sum_{k=0}^{n} \left| \frac{1}{r} \sum_{j=k}^{n} \left(\frac{-s}{r}\right)^{j-k} a_j \right|^q < \infty \right\},$$

$$d_2(r,s) := \left\{ a = (a_k) \in \omega : \lim_{n\to\infty} \sum_{j=k}^{n} \left(\frac{-s}{r}\right)^{j-k} a_j \text{ exists} \right\},$$

$$d_3(r,s) := \left\{ a = (a_k) \in \omega : \lim_{n\to\infty} \sum_{k=0}^{n} \left| \frac{1}{r} \sum_{j=k}^{n} \left(\frac{-s}{r}\right)^{j-k} a_j \right| = \sum_{k=0}^{\infty} \left| \lim_{n\to\infty} \frac{1}{r} \sum_{j=k}^{n} \left(\frac{-s}{r}\right)^{j-k} a_j \right| \right\},$$

$$d_4(r,s) := \left\{ a = (a_k) \in \omega : \lim_{n\to\infty} \sum_{k=0}^{n} \left[\frac{1 - \left(\frac{-s}{r}\right)^{k+1}}{1 + \frac{s}{r}} \right] a_k \text{ exists} \right\},$$

and

$$d_5(r,s) := \left\{ a = (a_k) \in \omega : \sup_{k,n\in\mathbb{N}} \left| \sum_{j=k}^{n} \left(\frac{-s}{r}\right)^{j-k} a_j \right| < \infty \right\}.$$

Then,

(i) $\{\widehat{\ell_\infty}\}^\gamma := \widehat{c}^\gamma := \{\widehat{c_0}\}^\gamma := d_1(r,s)$ *with* $q = 1$.
(ii) $\{\widehat{\ell_p}\}^\gamma := d_1(r,s)$.
(iii) $\{\widehat{\ell_1}\}^\gamma := d_5(r,s)$.
(iv) $\{\widehat{\ell_\infty}\}^\beta := d_2(r,s) \cap d_3(r,s)$.
(v) $\widehat{c}^\beta := d_1(r,s) \cap d_2(r,s) \cap d_4(r,s)$ *with* $q = 1$.
(vi) $\{\widehat{c_0}\}^\beta := d_1(r,s) \cap d_2(r,s)$ *with* $q = 1$.
(vii) $\{\widehat{\ell_p}\}^\beta := d_1(r,s) \cap d_2(r,s)$.
(viii) $\{\widehat{\ell_1}\}^\beta := d_2(r,s) \cap d_5(r,s)$.

A sequence space λ with a linear topology is called a *K-space* provided each of the maps $p_i : \lambda \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$. A *K*-space λ is called an *FK-space* provided λ is a complete linear metric space. An *FK*-space whose topology is normable is called a *BK-space*. If a normed sequence space λ contains a sequence (b_n) with the property that for every $x \in \lambda$ there is a unique sequence of scalars (α_n) such that

$$\lim_{n\to\infty} \|x - (\alpha_0 b_0 + \alpha_1 b_1 + \cdots + \alpha_n b_n)\| = 0$$

then (b_n) is called a *Schauder basis* (or briefly *basis*) for λ. The series $\sum \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n), and written as $x = \sum \alpha_k b_k$.

Since, it is known that the matrix domain λ_A of a normed sequence space λ has a basis if and only if λ has a basis whenever $A = (a_{nk})$ is a triangle (cf. [58, Remark 2.4]), we have:

Corollary 3.5. *Define the sequences* $z = (z_n)$ *and* $b^{(k)}(r,s) = \left\{ b_n^{(k)}(r,s) \right\}_{n\in\mathbb{N}}$ *for every fixed* $k \in \mathbb{N}$ *by*

$$z_n := \frac{1}{r} \sum_{k=0}^{n} \left(\frac{-s}{r}\right)^k \quad \text{and} \quad b_n^{(k)}(r,s) := \begin{cases} 0, & (n < k), \\ \dfrac{1}{r}\left(\dfrac{-s}{r}\right)^n, & (n \geq k). \end{cases}$$

Then,

(a) *The sequence* $\{b^{(k)}(r,s)\}_{k\in\mathbb{N}}$ *is a basis for the spaces* $\widehat{c_0}$ *and* $\widehat{\ell_p}$, *and any* x *in* $\widehat{c_0}$ *or in* $\widehat{\ell_p}$ *has a unique representation of the form*

$$x := \sum_k \alpha_k(r) b^{(k)}(r,s),$$

where $\alpha_k(r) := \{B(r,s)x\}_k$ *for all* $k \in \mathbb{N}$.

(b) *The set* $\left\{ z, b^{(k)}(r,s) \right\}$ *is a basis for the space* $\widehat{c}$, *and any* x *in* $\widehat{c}$ *has a unique representation of the form*

$$x := lz + \sum_k [\alpha_k(r) - l] b^{(k)}(r,s),$$

where $l := \lim_{k\to\infty} \{B(r,s)x\}_k$.

By $\lambda\mu$, we mean the set

$$\lambda\mu := \{ z = (z_k) \in \omega : z_k = x_k y_k \ \forall k \in \mathbb{N},\ x = (x_k) \in \lambda,\ y = (y_k) \in \mu \}$$

for the sequence spaces λ and μ.

Given a *BK*-space $\lambda \supset \phi$, we denote the *nth section of a sequence* $x = (x_k) \in \lambda$ by $x^{[n]} := \sum_{k=0}^{n} x_k e^{(k)}$, and we say that x has the property

$$AK \text{ if } \lim_{n\to\infty} \|x - x^{[n]}\|_\lambda = 0 \quad \text{(abschnittskonvergenz)},$$

$$AB \text{ if } \sup_{n\in\mathbb{N}} \|x^{[n]}\|_\lambda < \infty \quad \text{(abschnittsbeschränktheit)},$$

$$AD \text{ if } x \in \overline{\phi} (\text{ closure of } \phi \subset \lambda) \quad \text{(abschnittsdichte)},$$

$$KB \text{ if the set } \{x_k e^{(k)}\} \text{ is bounded in } \lambda \quad \text{(koordinatenweise beschränkt)}.$$

If one of these properties holds for every $x \in \lambda$ then we say that the space λ has that property, (cf. [59]). It is trivial that *AK* implies *AD* and *AK* iff *AB* and *AD*. For example, c_0 and ℓ_p are *AK*-spaces and, c and ℓ_∞ are not *AD*-spaces.

The sequence space λ is said to be *solid* if and only if

$$\tilde{\lambda} := \{(u_k) \in \omega : \exists (x_k) \in \lambda \text{ such that } |u_k| \leq |x_k| \text{ for all } k \in \mathbb{N}\} \subset \lambda.$$

For a sequence J of $\mathbb{N}$ and a sequence space λ, we define λ_J by

$$\lambda_J := \left\{x = (x_i) : \text{ there is a } y = (y_i) \in \lambda \text{ with } x_i = y_{n_i}, \forall n_i \in J\right\}$$

and call λ_J the J-stepspace or J-sectional subspace of λ. If $x_J \in \lambda_J$, then the canonical preimage of x_J is the sequence $\bar{x}_J$ which agrees with x_J on the indices in J and is zero elsewhere. Then, λ is called *monotone* provided λ contains the canonical preimages of all its stepspaces.

Lemma 3.6 ([8, *Theorem 2.1 and Lemma 4.1*]). *Let λ, μ be the BK-spaces and $C_\mu^U = (c_{nk})$ be defined via the sequence $\alpha = (\alpha_k) \in \mu$ and the triangle matrix $U = (u_{nk})$ by*

$$c_{nk} := \sum_{j=k}^{n} \alpha_j u_{nj} v_{jk}$$

for all $k, n \in \mathbb{N}$. Then, the domain of the matrix U in the sequence space λ has the property

(i) *KB if and only if $C_{\ell_1}^U \in (\lambda : \lambda)$.*
(ii) *AB if and only if $C_{bv_0}^U \in (\lambda : \lambda)$.*
(iii) *Monotone if and only if $C_{m_0}^U \in (\lambda : \lambda)$.*
(iv) *Solid if and only if $C_{\ell_\infty}^U \in (\lambda : \lambda)$.*

From Lemma 3.6, we have

Corollary 3.7. *If $|s/r| = 1$, then $\widehat{\ell}_1$ has the KB- and AB-properties.*

Lemma 3.8 ([8, *Theorem 2.2*]). *Let λ be a BK-space which has AK-property, U be a triangle matrix and $\lambda_U \supset \phi$. Then, the sequence space λ_U has the AD-property if and only if the fact $tU = \theta$ for $t \in \lambda^\beta$ implies the fact $t = \theta$.*

Since c_0 and ℓ_p have the AK-property, we can employ Lemma 3.8 for the matrix $U = B(r, s)$. Then, we have:

Corollary 3.9. *$\widehat{c}_0$ and $\widehat{\ell}_p$ $(p > 1)$ have the AD-property if and only if $|s/r| \leq 1$.*

4. Some matrix transformations related to the sequence spaces $\widehat{\ell}_\infty$, $\widehat{c}$, $\widehat{c}_0$ and $\widehat{\ell}_1$

In the present section, we characterize some classes of infinite matrices related with new sequence spaces.

Theorem 4.1. *Let λ be an FK-space, U be a triangle, V be its inverse and μ be arbitrary subset of ω. Then we have $A = (a_{nk}) \in (\lambda_U : \mu)$ if and only if*

$$C^{(n)} = \left(c_{mk}^{(n)}\right) \in (\lambda : c) \quad \text{for all } n \in \mathbb{N} \tag{4.1}$$

and

$$C = (c_{nk}) \in (\lambda : \mu), \tag{4.2}$$

where $c_{mk}^{(n)} := \begin{cases} \sum_{j=k}^{m} a_{nj} v_{jk}, & (0 \leq k \leq m), \\ 0, & (k > m) \end{cases}$ *and* $c_{nk} := \sum_{j=k}^{\infty} a_{nj} v_{jk}$ *for all $k, m, n \in \mathbb{N}$.*

Proof. Let $A = (a_{nk}) \in (\lambda_U : \mu)$ and take $x \in \lambda_U$. Then, we obtain the equality

$$\begin{aligned} \sum_{k=0}^{m} a_{nk} x_k &= \sum_{k=0}^{m} a_{nk} \left(\sum_{j=0}^{k} v_{kj} y_j \right) \\ &= \sum_{k=0}^{m} \left(\sum_{j=k}^{m} a_{nj} v_{jk} \right) y_k = \sum_{k=0}^{m} c_{nk}^{(n)} y_k \end{aligned} \tag{4.3}$$

for all $m, n \in \mathbb{N}$. Since Ax exists, $C^{(n)}$ must belong to the class $(\lambda : c)$. Letting $m \to \infty$ in the equality (4.3) we have $Ax = Cy$. Since $Ax \in \mu$, then $Cy \in \mu$, i.e. $C \in (\lambda : \mu)$.

Table 3
The characterization of the class $(\widehat{\lambda} : \mu)$ with $\lambda \in \{\ell_\infty, c, c_0, \ell_p, \ell_1\}$ and $\mu \in \{\ell_\infty, c, c_0, \ell_1\}$.

From	$\widehat{\ell_\infty}$	$\widehat{c}$	$\widehat{c_0}$	$\widehat{\ell_p}$	$\widehat{\ell_1}$
To					
ℓ_∞	**1.**	**2.**	**3.**	**4.**	**5.**
c	**6.**	**7.**	**8.**	**9.**	**10.**
c_0	**11.**	**12.**	**13.**	**14.**	**15.**
ℓ_1	**16.**	**17.**	**18.**	**19.**	**20.**

Conversely, let (4.1), (4.2) hold and take $x \in \lambda_U$. Then, we have $(c_{nk})_{k\in\mathbb{N}} \in \lambda^\beta$ which gives together with (4.1) that $(a_{nk})_{k\in\mathbb{N}} \in \lambda_U^\beta$ for all $n \in \mathbb{N}$. Hence Ax exists. Therefore, we obtain from the equality (4.3) as $m \to \infty$ that $Ax = Cy$ and this shows that $A \in (\lambda_U : \mu)$. $\square$

Now, we list the following conditions:

$$\sup_{m\in\mathbb{N}} \sum_{k=0}^{m} \left| \frac{1}{r} \sum_{j=k}^{m} \left(\frac{-s}{r}\right)^{j-k} a_{nj} \right|^q < \infty \tag{4.4}$$

$$\lim_{m\to\infty} \frac{1}{r} \sum_{j=k}^{m} \left(\frac{-s}{r}\right)^{j-k} a_{nj} = c_{nk} \tag{4.5}$$

$$\lim_{m\to\infty} \sum_{k=0}^{m} \left| \frac{1}{r} \sum_{j=k}^{m} \left(\frac{-s}{r}\right)^{j-k} a_{nj} \right| = \sum_k |c_{nk}| \quad \text{for each } n \in \mathbb{N} \tag{4.6}$$

$$\lim_{m\to\infty} \sum_{k=0}^{m} \frac{1}{r} \left[\frac{1 - \left(\frac{-s}{r}\right)^{k+1}}{1 + \frac{s}{r}} \right] a_{nk} = \alpha_n \quad \text{for each } n \in \mathbb{N} \tag{4.7}$$

$$\sup_{k,m\in\mathbb{N}} \left| \frac{1}{r} \sum_{j=k}^{m} \left(\frac{-s}{r}\right)^{j-k} a_{nj} \right| < \infty \tag{4.8}$$

$$\sup_{n\in\mathbb{N}} \sum_k |c_{nk}|^q < \infty \tag{4.9}$$

$$\lim_{n\to\infty} c_{nk} = \beta_k \tag{4.10}$$

$$\lim_{n\to\infty} \sum_k |c_{nk}| = \sum_k |\beta_k| \tag{4.11}$$

$$\lim_{n\to\infty} \sum_k c_{nk} = \beta \tag{4.12}$$

$$\sup_{k,n\in\mathbb{N}} |c_{nk}| < \infty \tag{4.13}$$

$$\sup_{k\in\mathbb{N}} \sum_n |c_{nk}| < \infty \tag{4.14}$$

$$\lim_{n\to\infty} \sum_k c_{nk} = 0 \tag{4.15}$$

$$\sup_{N,K\in\mathcal{F}} \left| \sum_{n\in N} \sum_{k\in K} c_{nk} \right| < \infty \tag{4.16}$$

$$\sup_{N\in\mathcal{F}} \sum_k \left| \sum_{n\in N} c_{nk} \right|^q < \infty, \tag{4.17}$$

where $\mathcal{F}$ denotes the collection of all finite subsets of $\mathbb{N}$.

We have from Theorem 4.1:

Corollary 4.2. *The necessary and sufficient conditions for $A \in (\lambda : \mu)$ when $\lambda \in \{\widehat{\ell_\infty}, \widehat{c}, \widehat{c_0}, \widehat{\ell_p}, \widehat{\ell_1}\}$ and $\mu \in \{\ell_\infty, c, c_0, \ell_1\}$ can be read from the following Table 3: where*

1.	(4.5), (4.6) and (4.9) *with* $q = 1$.
2.	(4.5), (4.7) and (4.4), (4.9) *with* $q = 1$.
3.	(4.5) *and* (4.4), (4.9) *with* $q = 1$.
4.	(4.4), (4.5) and (4.9).
5.	(4.5), (4.8) and (4.13).
6.	(4.5), (4.6), (4.10) and (4.11).
7.	(4.5), (4.7), (4.10), (4.12) and (4.4) ,(4.9) *with* $q = 1$.
8.	(4.5), (4.10) and (4.4), (4.9) *with* $q = 1$.
9.	(4.4), (4.5), (4.9) and (4.10).
10.	(4.5), (4.8), (4.10) and (4.13).
11.	(4.5), (4.6) and (4.15).
12.	(4.5), (4.7), (4.10) *with* $\beta_k = 0$ *and* (4.12) *with* $\beta = 0$, *and* (4.4), (4.9) *with* $q = 1$.
13.	(4.5), (4.10) *with* $\beta_k = 0$ *and* (4.4), (4.9) *with* $q = 1$.
14.	(4.4), (4.5), (4.9) and (4.10) *with* $\beta_k = 0$.
15.	(4.5), (4.8), (4.10) *with* $\beta_k = 0$ *and* (4.13).
16.	(4.5), (4.6) and (4.16).
17.	(4.4) *with* $q = 1$, (4.5), (4.7) *and* (4.16).
18.	(4.4) *with* $q = 1$, (4.5) *and* (4.16).
19.	(4.4), (4.5) and (4.17).
20.	(4.5), (4.8) and (4.14).

Now, we may present our final lemma given by Başar and Altay [30, Lemma 5.3] which is useful for obtaining the characterization of some new matrix classes from Corollary 4.2.

Lemma 4.3. *Let λ, μ be any two sequence spaces, A be an infinite matrix and U a triangle matrix. Then, $A \in (\lambda : \mu_U)$ if and only if $UA \in (\lambda : \mu)$.*

We should finally note that, if a_{nk} is replaced by $ra_{nk} + sa_{n-1,k}$ for all $k, n \in \mathbb{N}$ in Corollary 4.2, then one can derive the characterization of the class $(\widehat{\lambda} : \widehat{\mu})$ from Lemma 4.3 with $U = B(r, s)$.

5. Conclusion

Although the concept of almost convergence was defined by Lorentz [60], in 1948, neither the algebraic structure nor the topological structure of the space f was studied, until now. So, working the domain of generalized difference matrix $B(r, s)$ in the space f and deriving the β- and γ-duals of the space $\widehat{f}$ are meaningful which are filling up a gap in the existing literature as well as the domain of generalized difference matrix $B(r, s)$ in the classical sequence spaces ℓ_∞, c, c_0 and ℓ_p, with $1 \le p < \infty$. So, we should note from now on that the domain of generalized difference matrix $B(r, s)$ in the space f will be examined in the next paper. Since we employ a different technique for determining the dual spaces than Kızmaz [23] and the other authors following him, we are able to determine β- and γ-duals of the generalized difference spaces $\widehat{\ell_\infty}$, $\widehat{c}$, $\widehat{c_0}$ and $\widehat{\ell_p}$. It is natural that the investigation of the existence of the Schauder basis both for the space f and for the space derived as the domain of an infinite matrix in the space f will lead us to the significant topological results concerning with these spaces.

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