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Note

A note on 3-choosability of planar graphs without certain cycles

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Abstract

Steinberg asked whether every planar graph without 4 and 5 cycles is 3-colorable. Borodin, and independently Sanders and Zhao, showed that every planar graph without any cycle of length between 4 and 9 is 3-colorable. We improve this result by showing that every planar graph without any cycle of length 4, 5, 6, or 9 is 3-choosable.

Keywords: Planar graph; Cycle; Choosability

1. Introduction

All graphs considered here are finite, undirected and simple. Let $G=(V(G),E(G))$ be a graph. For a vertex v, $d_G(v)$ and $N_G(v)$ (also $d(v)$ and $N(v)$ for short) denote its degree and the set of its neighbors in G, respectively. $\delta(G)$ denotes the minimum degree of G. If $S\subset V(G)$, then $G-S$ is the subgraph obtained from G by deleting the vertices in S and all the edges incident with some vertices in S, and $G[S]$ is the subgraph of G induced by the subset S. A *k-coloring* of G is a mapping φ from $V(G)$ to a k-element set such that $\varphi(x)\neq\varphi(y)$ for any adjacent vertices x and y. The graph is *k-colorable* if it has a k-coloring.

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A *list-assignment* L to the vertices of G is the assignment of a set $L(v)$ of colors to every vertex v of G. If G has a proper coloring φ such that $\varphi(v) \in L(v)$ for all vertices v, then we say that G is *L-colorable* or φ is an *L-coloring* of G. We say that G is *k-list colorable* or *k-choosable* if it is L-colorable for every list-assignment L satisfying $|L(v)| = k$ for all vertices v.

Let G be a plane graph. $F(G)$ denotes the set of faces of G. For a face $f \in F(G)$, $\lambda(f)$ denotes the number of edges on the boundary of f, where each cut edge is counted twice. A face f is called *simple* if its boundary is a cycle. Let us denote the set of vertices on the boundary of f by $V(f)$. If a vertex v satisfies $v \in V(f)$, then we say that v and f are *incident*; two faces f_1 and f_2 are *adjacent* if they have some common edges. We often call a vertex $v \in V(G)$ a *k-vertex* if $d(v) = k$, and similarly call a face $f \in F(G)$ a *k-face* if $\lambda(f) = k$.

Various results are known on the 3-colorability and 3-choosability of classes of planar graphs. Alon and Tarsi [2] proved that every planar bipartite graph is 3-choosable. Thomassen [9] modified the classical theorem of Grötzsch [6] saying that every planar graph without 3 cycles is 3-colorable by showing that every planar graph without 3 and 4 cycles is 3-choosable. In the mid-1970s, Steinberg (see [8]) asked whether every planar graph without 4 and 5 cycles is 3-colorable. Essentially no progress was made on this question for fifteen years, leading Erdös, in 1990, to suggest the following relaxation by asking if there is an integer $k \geqslant 5$ such that if G is a planar graph without any cycles of length between 4 and k, then is G 3-colorable? Abbott and Zhou [1] proved that Erdös' relaxation holds for $k = 11$ and later Borodin [4] established it when $k = 10$. The best-known result for $k = 9$ is due to Borodin [3] and independently to Sanders and Zhao [7]. Borondin's proof is based on the following structural property of planar graphs.

Theorem 1.1 (*Borodin [3]*). *Let G be a plane graph without any cycles of length between 4 and 9. If $\delta(G) \geqslant 3$, then G contains a 10-face incident with ten 3-vertices and adjacent to five 3-faces.*

In fact, note that one can obtain the following stronger result from Theorem 1.1.

Corollary 1.2. *Every planar graph without i-cycles for each $i \in \{4, \dots, 9\}$ is 3- choosable.*

Proof. Suppose the plane graph G is a counterexample of minimum order; then $\delta(G) \geqslant 3$. Since G has no i-cycles for all $i = 4, 5, 6, 7, 8, 9$, then by Theorem 1.1, G has a 10-face f incident with ten 3-vertices. Setting $G' = G - V(f)$, let L be a list-assignment of G in which $|L(v)| = 3$ for each vertex $v \in V(G)$. By the minimality of G, G' admits a list coloring φ' with list L restricted to G'. For $v \in V(f)$, let $L'(v) = L(v) \backslash \{\varphi'(u) | u \in V(G) \cap N(u)\}$, then $|L'(v)| \geqslant 2$. Since $G[V(f)]$ is isomorphic to C_{10}, the cycle of length 10, and by the well-known fact that every even cycle is 2-choosable, $G[V(f)]$ admits a list coloring with L'. Thus, we obtain a list coloring of G, a contradiction. □

In this note, we will show that the lack of cycles C_4, C_5, C_6 and C_9 is sufficient for a plane graph having the same sub-structure as confirmed in Theorem 1.1.

Theorem 1.3. *Let G be a plane graph without any cycles of length in $\{4, 5, 6, 9\}$. If $\delta(G) \geqslant 3$, then G contains a 10-face incident with ten 3-vertices and adjacent to five 3-faces.*

By the similar proof as in Theorem 1.2, we have the following result.

Corollary 1.4. *Let G be a plane graph without any cycles of length in* $\{4, 5, 6, 9\}$. *Then G is* 3-*choosable*.

2. Proof of Theorem 1.3

Suppose that G is a counterexample of minimum order. Then, G is connected and $\delta(G) \geqslant 3$. We define a weight function w on $V(G) \cup F(G)$ by letting $w(v) = d(v) - 6$ for $v \in V(G)$ and $w(f) = 2\lambda(f) - 6$ for $f \in F(G)$. Applying Euler's formula $|V(G)| - |E(G)| + |F(G)| = 2$ and the handshaking lemmas for vertices and faces for a plane graph, we have $\sum_{x \in V(G) \cup F(G)} w(x) = \sum_{v \in V(G)} (d(v) - 6) + \sum_{f \in F(G)} (2\lambda(f) - 6) = -12$.

First, we assume that G is 2-connected, which implies that every face of G is simple. We shall discharge the weight of every non-triangular face f to its incident vertices v with $d(v) \leqslant 5$. Let $z(f, v)$ be the amount of weight that is transferred from f to v according to the following rules.

R1. $d(v) = 3$: $z(f, v) = \frac{3}{2}$ *if* v *is incident with a triangle; otherwise*, $z(f, v) = 1$.
R2. $d(v) = 4$: $z(f, v) = 1$ *if* v *is incident with either two triangles or one triangle not adjacent to f; otherwise*, $z(f, v) = \frac{1}{2}$.
R3. $d(v) = 5$: $z(f, v) = \frac{1}{3}$.

Let $w'(x)$ be the new weight of $x \in V(G) \cup E(G)$ after the discharging process is finished. Since we discharge weight from one element to another, the total weight is kept fixed during the discharging. Thus, $\sum_{x \in V(G) \cup F(G)} w'(x) = -12$. On the other hand, we shall obtain a contradiction by verifying $w'(x) \geqslant 0$ for every $x \in V(G) \cup F(G)$. Note that $w'(v) = w(v) = d(v) - 6 \geqslant 0$ if $d(v) \geqslant 6$ and $w'(f) = w(f) = 0$ for a 3-face f. Thus, it remains to verify that the new weights are also nonnegative for the vertices $v \in V(G)$ with $d(v) \leqslant 5$ and for the faces f with $\lambda(f) \notin \{3, 4, 5, 6, 9\}$.

Let $T(v)$ be the set of 3-faces incident with a vertex v. Since G contains no C_4, G has no two adjacent 3-faces, and thus $|T(v)| \leqslant d(v)/2$. If $d(v) = 3$, then we have $w(v) = -3$ and $|T(v)| \leqslant 1$. Hence, by R1 $w'(v) \geqslant w(v) + \frac{3}{2} \times 2 = 0$ if $|T(v)| = 1$; otherwise, $w'(v) \geqslant w(v) + 1 \times 3 = 0$. If $d(v) = 4$, then $w(v) = -2$ and $|T(v)| \leqslant 2$. By R2, $w'(v) \geqslant w(v) + 1 \times 2 = 0$ if $|T(v)| = 2$; $w'(v) \geqslant w(v) + \frac{1}{2} \times 2 + 1 \geqslant 0$ if $|T(v)| = 1$, and $w'(v) \geqslant w(v) + \frac{1}{2} \times 4 = 0$ if $|T(v)| = 0$. If $d(v) = 5$, then $w(v) = -1$ and $|T(v)| \leqslant 2$. Hence, by R3, $w'(v) \geqslant w(v) + \frac{1}{3} \times 3 = 0$.

Now let f be a face with $\lambda(f) \notin \{3, 4, 5, 6, 9\}$. Let a, b, and c be, respectively, the number of 3, 4 and 5 vertices incident with f. Clearly, $a + b + c \leqslant \lambda(f)$. If $\lambda(f) \geqslant 12$, then $w'(f) \geqslant w(f) - \frac{3}{2}a - b - \frac{1}{3}c \geqslant 2\lambda - 6 - \frac{3}{2}a - b - c \geqslant 2\lambda - 6 - \frac{3}{2}a - (\lambda - a) = \lambda - 6 - \frac{1}{2}a \geqslant (\lambda/2) - 6 \geqslant 0$. If $\lambda(f) = 8$, f cannot be adjacent to a 3-face since G has no C_9. Hence $z(f, v) \leqslant 1$ for any $v \in V(f)$ by the discharging rule, and thus $w'(f) \geqslant 2 \times 8 - 6 - 8 > 0$. For the same reason, if $\lambda(f) = 7$, f is adjacent to atmost one 3-face. It follows that f is incident with atmost two vertices receiving $\frac{3}{2}$ each, so that $w'(f) \geqslant w(f) - (\frac{3}{2} \times 2 + 5 \times 1) = 0$. If $\lambda(f) = 11$, there are atmost 10 vertices of $V(f)$ that receive $\frac{3}{2}$ from f. So $w'(f) \geqslant w(f) - (\frac{3}{2} \times 10 + 1) = 0$.

Finally, let $\lambda(f) = 10$. By assumption, f cannot be incident with ten vertices receiving $\frac{3}{2}$ each. If f is incident with 9 vertices receiving $\frac{3}{2}$ each, then theremaining vertex can receive atmost $\frac{1}{2}$ from f. Thus, $w'(f) \geqslant w(f) - (\frac{3}{2} \times 9 + \frac{1}{2}) = 0$. Otherwise, there are atmost 8 vertices in $V(f)$ that receive $\frac{3}{2}$ from f, so that $w'(f) \geqslant w(f) - (\frac{3}{2} \times 8 + 1 \times 2) = 0$.

This gives a contradiction, and proves our theorem for 2-connected graphs. Suppose now that G is not 2-connected, and let B be a block containing a unique cut vertex v_0 of G. Thus, $d_B(v_0) \geqslant 2$ and $d_B(v) \geqslant 3$ for every $v \in V(B)\backslash\{v_0\}$. We define the weight function w on $V(B) \cup F(B)$ by $w(v) = d_B(v) - 6$ for $v \in V(B)$ and $w(f) = 2\lambda_B(f) - 6$ for $f \in F(B)$. Since $w(v_0) = d_B(v_0) - 6 \geqslant -4$, we have $\sum\{w(x)|x \in (V(B) \cup F(B))\backslash\{v_0\}\} = -12 - w(v_0) \leqslant -8$. Let w' denote the new weight after discharging the weights by the same rules as in the preceding proof, except that v_0 receives nothing from its incident faces. We have $\sum\{w'(x)|x \in (V(B) \cup F(B))\backslash\{v_0\}\} \leqslant -8$ since the total weight is kept fixed as we have seen before. However, by the similar argument as for the 2-connected case, we obtain $w'(x) \geqslant 0$ for all $x \in (V(B) \cup F(B))\backslash\{v_0\}$. It follows that $\sum\{w'(x)|x \in (V(B) \cup F(B))\backslash\{v_0\}\} \geqslant 0$, a contradiction. This completes the proof of Theorem 1.3.

Remark. Borodin et al. [5] currently move a further step closer to solving Steinberg's problem by showing that a planar graph *G* is 3-colorable if *G* contains no cycles of length in {4, 5, 6, 7}. In [10], we also proved that a planar graph *G* is 3-choosable if *G* contains no cycles of length in {4, 5, 7, 9}.

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