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Convexity in Oriented Matroids

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We generalize to oriented matroids classical notions of Convexity Theory: faces of convex polytopes, convex hull, etc., and prove some basic properties. We relate the number of acyclic orientations of an orientable matroid to an evaluation of its Tutte polynomial.

The structure of oriented matroids (oriented combinatorial geometries) [1] retains many properties of vector spaces over ordered fields. In the present paper we consider classical notions such as those of faces of convex polytopes, convex hull, etc., and show that usual definitions can be extended to finite oriented matroids in a natural way. We prove some basic properties: lattice structure with chain property of the set of faces of an acyclic oriented matroid, lower bounds for the numbers of vertices and facets and characterization of extremal cases, analogues of the Krein–Milman Theorem. In Section 3 we relate the number of acyclic orientations of an orientable matroid M to the evaluation $t(M; 2, 0)$ of its Tutte polynomial.

The present paper is a sequel to [1]. Its content constituted originally Sections 6, 7 and 8 of "Matroïdes orientables" (preprint, April 1974) [8]. Basic notions on oriented matroids are given in [1]. For completeness we recall the main definitions [1, Theorems 2.1 and 2.2]:

All considered matroids are on finite sets.

A *signed set* X is a set $\underline{X}$ partitioned into two distinguished subsets: the set X^+ of *positive* elements and the set X^- of *negative* elements. The *opposite* $-X$ of X is defined by $(-X)^+ = X^-$ and $(-X)^- = X^+$.

An oriented matroid M on a (finite) set E is defined by its collection $\mathscr{O}$ of *signed circuits*: $\mathscr{O}$ is a set of signed subsets of E satisfying

(O1) $X \in \mathscr{O}$ implies $X \neq \varnothing$ and $-X \in \mathscr{O}$; $X_1, X_2 \in \mathscr{O}$ and $\underline{X}_1 \subseteq \underline{X}_2$ imply $X_1 = X_2$ or $X_1 = -X_2$.

(O2) (*elimination property*) for all $X_1, X_2 \in \mathscr{O}$, $x \in X_1^+ \cap X_2^-$ and $y \in X_1^+ - X_2^-$ there exists $X_3 \in \mathscr{O}$ such that $y \in X_3$, $X_3^+ \subseteq (X_1^+ \cup X_2^+) - \{x\}$ and $X_3^- \subseteq (X_1^- \cup X_2^-) - \{x\}$.

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Clearly by forgetting the orientation we obtain a (non-oriented) matroid $\underline{M}$. The circuits of the orthogonal matroid $\underline{M}^{\perp}$ (i.e., the cocircuits of $\underline{M}$) can be oriented in a unique way such that the collection $\mathcal{O}^{\perp}$ of signed cocircuits of M satisfies the *orthogonality property*: for all $X \in \mathcal{O}$ and $Y \in \mathcal{O}^{\perp}$ such that $|X \cap Y| = 2$, both $(X^+ \cap Y^+) \cup (X^- \cap Y^-)$ and $(X^+ \cap Y^-) \cup (X^- \cap Y^+)$ are non-empty. Then $\mathcal{O}^{\perp}$ satisfies (O1) and (O2), and defines an oriented matroid $M^{\perp}$, the *orthogonal* of M. The orthogonality property holds for all $X \in \mathcal{O}$ and $Y \in \mathcal{O}^{\perp}$ such that $X \cap Y \neq \varnothing$. We have $(M^{\perp})^{\perp} = M$.

1. Faces in Acyclic Oriented Matroids

Let $\mathbb{R}$ be the real field and E be a finite subset of $\mathbb{R}^d$. A non-empty subset X of E is an *affine dependency* of E over $\mathbb{R}$ if there is a non-zero mapping $\lambda: X \to \mathbb{R}$ such that $\sum_{x \in X} \lambda(x) = 0$ and $\sum_{x \in X} \lambda(x) \cdot x = 0$. The affine dependencies minimal with respect to inclusion constitute the circuits of a matroid M on E. The ordering of $\mathbb{R}$ induces a canonical orientation of M: observe that if X is a circuit of M then a mapping λ such that $\sum_{x \in X} \lambda(x) = 0$ and $\sum_{x \in X} \lambda(x) \cdot x = 0$ has non-zero values and is unique up to multiplication by a non-zero number. The minimal affine dependencies X of E over $\mathbb{R}$ signed by $X^+ = \{x \in X: \lambda(x) > 0\}$ and $X^- = \{x \in X: \lambda(x) < 0\}$ constituted the signed circuits of an oriented matroid [1, Ex. 3.5]. We denote this oriented matroid, the *oriented matroid of affine dependencies* of E over $\mathbb{R}$, by $\mathbf{Aff}_{\mathbb{R}}(E)$. Note that $\mathbf{Aff}_{\mathbb{R}}(E)$ contains no positive circuits (signed circuits X with $X^- = \varnothing$); we say that $\mathbf{Aff}_{\mathbb{R}}(E)$ is *acyclic*.

Suppose E is of affine dimension d: the affine closure $[E]$ of E in $\mathbb{R}^d$ is the whole space. Consider a hyperplane H of $\mathbf{Aff}_{\mathbb{R}}(E)$: $[H]$ is an affine hyperplane of $\mathbb{R}_d$. Let S_1, S_2 be the two open half-spaces of $\mathbb{R}^d$ determined by $[H]$. In $\mathbf{Aff}_{\mathbb{R}}(E)$ $E - H$ is the support of two signed cocircuits Y and its opposite $-Y$. We have $Y^+ = S_1 \cap E$, $Y^- = S_2 \cap E$. This example suggests the following definitions:

Let M be an acyclic oriented matroid on a (finite) set E. An *open half-space* of M is a subset of E of the form Y^+, where Y is a signed cocircuit. A *facet* of M is a hyperplane H such that $E - H$ supports a positive cocircuit. A *face* of M is any intersection of facets. In particular an *extreme point* (or *vertex*) of M is a face of rank 1. (These definitions still make sense when M is not acyclic. However, this assumption is necessary for certain desirable properties to hold.)

The following generalization of Minty's 3-Painting Lemma to oriented matroids is a fundamental tool in the sequel:

The 3-Painting Lemma [1, Theorem 2.2]. *Let M be an oriented matroid on a set E. Given any 3-partition (3-painting) of E into 3 classes B (black),*

G (green), R (red) and $e \in B$, then either there exists a black and red signed circuit X such that $e \in B \cap X \subseteq X^+$, or there exists a black and green signed cocircuit Y such that $e \in B \cap Y \subseteq Y^+$.

By the 3-Painting Lemma for $G = R = \varnothing$, in an oriented matroid an element belongs to a positive circuit or to a positive cocircuit, but not to both. We say that an oriented matroid is *totally cyclic* if every element belongs to some positive circuit.

Thus an oriented matroid M is acyclic if and only if its orthogonal $M^\perp$ is totally cyclic. In consequence, by orthogonality, every property of acyclic oriented matroids is equivalent to a property of totally cyclic oriented matroids. Both points of view have applications: natural examples of acyclic oriented matroids are provided by affine dependencies in vector spaces over ordered fields, of totally cyclic oriented matroids by elementary circuits of strongly connected directed graphs. The present paper is mainly in terms of acyclic oriented matroids. An example of the second point of view is given below in this section.

THEOREM 1.1. *Let M be an acyclic oriented matroid. The set of faces of M ordered by inclusion is a lattice with the Jordan–Dedekind chain property.*

LEMMA 1.1.1. *Let M be an acyclic oriented matroid on a set E and E' be a subset of E. Given any face F of M, $F \cap E'$ is a face of $M(E')$.*[1]

Proof. F is an intersection of facets of M, i.e., $E - F$ is a union of positive cocircuits of M. Then $E' - F$ is an union of positive cocircuits of $M(E')^{(1)}$, i.e., $F \cap E'$ is a face of $M(E')$.

LEMMA 1.1.2. *Let M be an oriented matroid, $X_1, X_2, \dots, X_k$ be positive circuits and X be a signed circuit. Then for any $e \in X - A$, where $A = \bigcup_{i=1}^{i=k} X_i$, there is a signed circuit Z of M such that $e \in Z \subseteq X \cup A$, $Z^+ \subseteq X^+ \cup A$ and $Z^- \subseteq X^- - A$ (i.e., Z is positive on A and has the sign of X outside A).*

Proof. Let Z be a signed circuit of M such that $e \in Z \subseteq X \cup A$ and Z has the sign of X outside A (X has these properties). Suppose Z chosen such that $|Z^- \cap A|$ is minimal. We show that $Z^- \cap A = \varnothing$. Let $x \in Z^- \cap X_1$. By the elimination property there is a signed circuit Z' of M such that $e \in Z' \subseteq (Z \cup X_1) - \{x\}$, $Z'^+ \subseteq Z^+ \cup X_1^+$ and $Z'^- \subseteq Z^- \cup X_1^-$. We have

[1] Let M be an oriented matroid on a set E and A be a subset of E. We denote by $M - A$, or alternatively by $M(E - A)$), resp. M/A, the oriented matroid obtained from M by deleting, resp. contracting, A. We have $(M - A)^\perp = M^\perp/A$. Given a signed circuit X of M and $x \in X - A$ there is a signed circuit X' of M/A such that $x \in X'$, $X'^+ \subseteq X^+ - A$ and $X'^- \subseteq X^- - A$ [1, Proposition 4.4].

$e \in Z' \subseteq X \cup A$, Z' has the sign of X outside A and $Z'^{-} \cap A \subseteq Z^{-} \cap A - \{x\}$, contradicting the definition of Z.

LEMMA 1.1.3. *Let M be an acyclic oriented matroid and F be a face of M. Any face of $M(F)$ is a face of M.*

Proof. Let F' be a face of $M(F)$. For all $x \in F - F'$ there is a positive cocircuit Y' of $M(F)$ such that $x \in Y' \subseteq F - F'$. Let Y be a signed cocircuit of M such that $Y' = Y \cap F$. Then $Y \cap F$ is positive; on the other hand, F being a face of M, $E - F$ is an union of positive cocircuits of M. By Lemma 1.1.2 there is a positive cocircuit Z of M such that $x \in Z \subseteq Y \cup (E - F)$. Hence $E - F'$ is an union of positive cocircuits of M, i.e., F' is a face of M.

Proof of Theorem 1.1. It follows immediately from the definition that the intersection of two faces of M is also a face. Hence the faces of M ordered by inclusion constitute a lattice. The Jordan–Dedekind property is a straightforward consequence of Lemmas 1.1.1 and 1.1.3: Let F, F' be two faces of M such that $F' \subseteq F$. By Lemma 1.1.1 F' is a face of $M(F)$. There is a facet F_1 of $M(F)$ such that $F' \subseteq F_1$. Now F_1 is a face of M by Lemma 1.1.3. We have $F' \subseteq F_1 \subset F$ and $r_M(F_1) = r_M(F) - 1$, where $r_M(F)$ is the rank of F in M. Hence by induction: any maximal chain between two faces F, F' such that $F' \subseteq F$ has length $r_M(F) - r_M(F')$.

Remark 1.1. The same proof shows, more precisely, that given any maximal chain $F_0 \subset F_1 \subset \cdots \subset F_k$ of faces of M, then $k = r(M)$ and there exist $r(M)$ facets of M, $H_1, H_2, \ldots, H_{r(M)}$, such that $F_{r(M)-i} = H_1 \cap H_2 \cap \cdots \cap H_i$ for $i = 0, 1, \ldots, r(M)$.

PROPOSITION 1.2. *In an acyclic oriented matroid any non-empty open half-space contains an extreme point.*

Let F be a face of an acyclic oriented matroid M on a set E. By definition $E - F$ is a union of positive cocircuits of M; equivalently $E - F$ is an union of positive cocircuits of M/F, i.e., M/F is acyclic. A face of M is thus a closed subset of E such that M/F is acyclic.

Proof of Proposition 1.2. Let M be an acyclic oriented matroid on a set E and Y be a cocircuit of M such that $Y^+ \neq \varnothing$. Let $Y^+ = P_1 + P_2 + \cdots + P_k$ be the partition of Y^+ into points of M. Suppose that M/P_i is not acyclic for $i = 1, 2, \ldots, k$.

M/P_i being not acyclic, contains a positive circuit X'_i. Let X_i be a signed circuit of M such that $X'_i = X_i - P_i$. We have $X_i^- \subseteq P_i$; on the other hand $|X_i \cap P_i| \leqslant 1$. Hence, since M is acyclic, $X_i^- = \{e_i\}$ $e_i \in P_i$.

We prove that for $j = 0, 1, \ldots, k-1$ there exist signed circuits Z_i, $i = 1, 2, \ldots, k-j$ such that $Z_i^- = \{e_i\}$ and $Z_i \cap Y^+ \subseteq \{e_1, e_2, \ldots, e_{k-j}\}$.

The assertion is true for $j=0$. Suppose it has been proved for $j<k-1$. Let i $0 \leqslant i \leqslant k-j-1$. There is a signed circuit Z_i such that $Z_i^- = \{e_i\}$ and $Z_i \cap Y^+ \subseteq \{e_1, e_2, ..., e_{k-j}\}$. If $e_{k-j} \notin Z_i$ we set $Z_i' = Z_i$. Suppose $e_{k-j} \in Z_i$. We have $Z_i^- = \{e_i\}$, $Z_{k-j}^- = \{e_{k-j}\}$; hence $Z_i \neq -Z_{k-j}$, otherwise $\{e_i, e_{k-j}\}$ would be a circuit of M, contradicting the fact that e_i and e_{k-j} are in 2 different points of M. By the elimination property there is a signed circuit Z_i' of M contained in $(Z_i \cup Z_{k-j}) - \{e_{k-j}\}$ such that $Z_i'^+ \subseteq Z_i^+ \cup Z_{k-j}^+$ and $Z_i'^- \subseteq Z_i^- \cup Z_{k-j}^-$. We have $Z_i'^- \subseteq \{e_i\}$ hence $Z_i'^- = \{e_i\}$ since M is acyclic. Thus the assertion holds also for $j+1$.

By induction we obtain for $j=k-1$ that there exists a signed circuit Z of M such that $Z^- = \{e_1\}$ and $Z \cap Y^+ = \{e_1\}$, but this contradicts the orthogonality property.

Let M be a matroid on a set E. We say that $r(M)$ points $P_1, P_2, ..., P_{r(M)}$ of M constitute a *point-basis* of M if there is a basis $\{e_1, e_2, ..., e_{r(M)}\}$ such that $e_i \in P_i$ for $i = 1, 2, ..., r(M)$.

THEOREM 1.3. *An acyclic oriented matroid M of rank $r(M)$ contains at least $r(M)$ extreme points and $r(M)$ facets. Furthermore there is a point-basis of M constituted of $r(M)$ extreme points.*

Proof. The proof is by induction on $r(M)$. By Proposition 1.2 M has at least one extreme point P. M/P is acyclic, hence by induction, contains at least $r(M/P) = r(M) - 1$ facets: $E - P$ contains at least $r(M) - 1$ positive cocircuits of M/P, i.e., at least $r(M) - 1$ positive cocircuits of M. Since P is contained in at least one positive cocircuit of M, M contains at least $r(M)$ positive cocircuits, i.e., at least $r(M)$ facets.

By induction we know that M/P contains at least $r(M) - 1$ extreme points $P_1', P_2', ..., P_{r(M)-1}'$, and we may suppose that they constitute a point-basis of M/P. For $i = 1, 2, ..., r(M) - 1$, $P \cup P_i'$ is a face of rank 2 of M. It is easily seen that an acyclic oriented matroid has at least 2 extreme points (let M be an acyclic oriented matroid of rank 2: if M is not the direct sum of two points, M contains a 3-elements signed circuit $xy\bar{z}$, the two open half-spaces defined by z are both non-empty; hence M contains at least two extreme points by Proposition 1.2). By Lemma 1.1.1 P is an extreme point of $M(P \cup P_i')$. Let P_i be another extreme point of $M(P \cup P_i')$. Then P_i is an extreme point of M by Lemma 1.1.3. Now P and $P_1, P_2, ..., P_{r(M)-1}$ are extreme points of M constituting a point-basis.

Remark 1.3. We have in fact proved that any extreme point of M is contained in a point-basis constituted of extreme points.

Proposition 1.2 and Theorem 1.3 are well-known properties of $\mathbb{R}^d$. We consider the case of graphs:

Let G be a directed graph with edge-set E. The signed sets of edges of elementary circuits of G constitute the signed circuits of an oriented matroid

$\mathbb{C}(G)$. The signed sets of edges with one end-vertex in A and the other in B for some partition $A+B$ of the vertex-set of G such that $c(G[A]) + c(G[B]) = c(G) + 1$, where $c(G[A])$ denotes the number of connected components of the subgraph $G[A]$ of G induced by A, constitute the signed circuits of an oriented matroid $\mathbb{B}(G)$. The oriented matroids $\mathbb{B}(G)$ and $\mathbb{C}(G)$ are orthogonal [1, Ex. 3.3]. $\mathbb{B}(G)$ acyclic is equivalent to $(\mathbb{B}(G))^{\perp} = \mathbb{C}(G)$ totally cyclic. Now $\mathbb{C}(G)$ is totally cyclic if and only if every connected component of G is strongly connected.

An extreme point of $\mathbb{B}(G)$ is a set P of edges such that $\mathbb{B}(G)/P$ is acyclic: equivalently $(\mathbb{B}(G)/P)^{\perp} = \mathbb{C}(G) - P = \mathbb{C}(G - P)$ is totally cyclic, i.e., every connected component of $G - P$ is strongly connected. By Proposition 1.2 applied to $\mathbb{B}(G)$, every elementary circuit γ of a strongly connected directed graph G contains a non-empty set P of edges consistently directed on γ, such that every connected component of $G - P$ is strongly connected. This statement is exactly Lemma 1 of [7].

F is a facet of $\mathbb{B}(G)$ if and only if $E - F$ supports a positive cocircuit of $\mathbb{B}(G)$, or equivalently a positive circuit of $\mathbb{C}(G)$, i.e., a directed circuit of G. By Theorem 1.3, a strongly connected directed graph G contains at least $r(\mathbb{B}(G)) = e(G) - v(G) + 1$ directed elementary circuits, where $v(G)$ is the number of vertices of G and $e(G)$ its number of edges. This statement is a well-known elementary result of Graph Theory.

Our next proposition characterizes the case of equality in Theorem 1.3. The particular case of graphs was considered by Chaty in [4].

PROPOSITION 1.4. *Let M be an acyclic oriented matroid. The following properties are equivalent*:

(i) *M has exactly $r(M)$ extreme points.*

(ii) *M has exactly $r(M)$ facets.*

(iii) *There is a point-basis of M constituted of $r(M)$ extreme points such that the $r(M)$ associated hyperplanes are facets of M.*

LEMMA 1.4.1. *Let M be an acyclic oriented matroid. M has exactly $r(M)$ facets if and only if the set of elements of M contained in exactly one positive cocircuit contains a basis.*

Proof. Let E be the set of elements of M and A be the set of elements contained in exactly one positive cocircuit. Suppose A is contained in a hyperplane of M. By Proposition 1.2 $E - A$ contains an extreme point P. By Theorem 1.3 M/P has at least $r(M/P) = r(M) - 1$ facets, hence $E - P$ contains at least $r(M) - 1$ positive cocircuits of M. Since at least two positive cocircuits meet P by definition of A, M contains at least $r(M) + 1$ positive cocircuits.

Conversely suppose A contains a basis B of M. Let $Y_1, Y_2, \ldots, Y_k$ be the positive cocircuits of M. We have $B \cap Y_i \neq \varnothing$ for $i = 1, 2, \ldots, k$. On the other hand $B \cap Y_i \cap Y_j = \varnothing$ for $i \neq j$ by definition of A. Since $k \geqslant r(M)$ by Theorem 1.3, we have $|B \cap Y_i| = 1$, for $i = 1, 2, \ldots, k$ and $k = r(M)$.

LEMMA 1.4.2. *Let M be an oriented matroid. If there is a basis such that the $r(M)$ associated cocircuits are positive, then M has exactly $r(M)$ positive cocircuits.*

Proof. Let B be a basis of M such that the $r(M)$ associated cocircuits are positive and Y be a signed cocircuit of M such that $|Y \cap B| \geqslant 2$. Suppose $Y \cap B$ is positive and Y is chosen such that $|Y \cap B|$ is minimal with these properties. Let $x \in Y \cap B$. By hypothesis the cocircuit Z such that $Z \cap B = \{x\}$ is positive. By the elimination property applied to Y, $-Z$ and x there is a cocircuit Y' contained in $(Y \cup Z) - \{x\}$ such that $Y'^{+} \subseteq Y^{+} \cup Z^{-}$ and $Y'^{-} \subseteq Y^{-} \cup Z^{+}$. Now $Y' - B$ contains both positive and negative elements, hence $|Y' \cap B| \geqslant 2$, and $Y' \cap B$ is positive, contradicting the definition of Y since $|Y' \cap B| < |Y \cap B|$.

Proof of Proposition 1.4. Property (i) implies (ii): Suppose M has exactly $r(M)$ extreme points. We show that an extreme point is contained in exactly one positive cocircuit of M. Suppose on the contrary that $P \cap F = P \cap F' = \varnothing$ for an extreme point P and facets F, F' of M. By Theorem 1.3 and Lemma 1.1.3 F contains $r(M) - 1$ extreme points of M constituting a point-basis of $M(F)$, and similarly for F'. If $F \neq F'$ then $F \cap F'$ contains at least $r(M)$ extreme points of M, and this implies that M has at least $r(M) + 1$ extreme points. On the other hand, by Proposition 1.2 given any facet F there is an extreme point P such that $P \cap F = \varnothing$. There is thus a 1–1 correspondence between extreme points and facets of M.

Property (ii) implies (i): By Remark 1.1 an extreme point of M is an intersection of $r(M) - 1$ facets of M. If M has exactly $r(M)$ facets, M can have at most $r(M)$ extreme points, hence exactly $r(M)$ by Theorem 1.3.

Property (i) implies (iii) by Lemma 1.4.1 and the second part of its proof. Conversely (iii) implies (i) by Lemma 1.4.2.

PROPOSITION 1.5. *Let M be an acyclic oriented matroid without isthmuses on a set E. Suppose M has exactly $r(M)$ facets. Let B be a basis of M such that the associated hyperplanes are the $r(M)$ facets of M. Then the oriented matroid M' obtained from $M^{\perp}$ by reversing signs on B is acyclic and has exactly $r(M') = |E| - r(M)$ facets. The $|E| - r(M)$ facets of M' are the hyperplanes associated with the basis $E - B$ of M.*

Proposition 1.5 is an easy corollary of Lemma 1.4.2, and also of Theorem 2.2. We mention the following characterization of extreme points:

PROPOSITION 1.6. *Let M be an acyclic oriented matroid. A point P of M is an extreme point if and only if there is a signed cocircuit of M such that $Y^+ = P$.*

Proof. If there is a signed cocircuit Y such that $Y^+ = P$, then P is an extreme point by Proposition 1.2. Conversely let P be an extreme point of M and E be the set of elements. There is a cocircuit Y such that $P \subseteq Y^+$ (since M being acyclic has no loops). Now by definition $E - P$ is an union of positive cocircuits: by Lemma 1.1.2 there is a signed cocircuit Z of M such that $Z^+ = P$.

In this section we have generalized to oriented matroids some properties of faces of convex polytopes. Actually for any such property, see, for example, [6], the question arises whether or not it generalizes to oriented matroids. At the present time we have no example of an acyclic oriented matroid whose lattice of faces is not the lattice of faces of a convex polytope. We ask:

Problem. Is there an acyclic oriented matroid M such that the lattice of faces of M is not isomorphic to the lattice of faces of some convex polytope in an Euclidean space $\mathbb{R}^d$?

2. CONVEXITY IN ORIENTED MATROIDS

Let M be an oriented matroid matroid on a set E. We define the *convex hull in M* of a subset A of E as $\mathrm{Conv}_M(A) = A \cup \{x \in E - A$: there is a signed circuit X of M such that $X^- = \{x\}$ and $X^+ \subseteq A\}$

This definition reduces to the usual one when M arises from affine dependences in $\mathbb{R}^d$. In the case of a directed graph G, the convex hull in $\mathbb{C}(G)$ of a set A of edges is the transitive closure of A in G.

Clearly $\mathrm{Conv}_M(A)$ is contained in the closure $\bar{A}^M$ of A in M. The mapping $A \rightsquigarrow \mathrm{Conv}_M(A)$ is a closure: It follows immediately from the definition that $A \subseteq \mathrm{Conv}_M(A)$ and $\mathrm{Conv}_M(A) \subseteq \mathrm{Conv}_M(B)$ if $A \subseteq B$. The idempotence property $\mathrm{Conv}_M(\mathrm{Conv}_M(A)) = \mathrm{Conv}_M(A)$ follows from Proposition 2.1: We call *closed half-space* of M any subset of E of the form $(E - Y) \cup Y^+$ for a signed cocircuit Y of M.

PROPOSITION 2.1. *Let M be an oriented matroid on a set E. For any $A \subseteq E$, $\mathrm{Conv}_M(A)$ is equal to the intersection of the closed half-spaces of M containing A.*

Note that by definition the intersection of the closed half-spaces containing A is equal to the set of elements $x \in E$ such that for all signed cocircuits Y of M $x \in Y^+$ implies $Y^+ \cap A \neq \varnothing$.

Proof of Proposition 2.1. Let $x \in \mathrm{Conv}_M(A)$ and Y be a signed cocircuit of M such that $x \in Y^+$. By definition there is a signed circuit X of M such that $X^- = \{x\}$ and $X^+ \subseteq A$. We have $X^- \cap Y^+ = X \cap Y^+ = \{x\}$, hence $X^+ \cap Y^+ \neq \varnothing$ by orthogonality, and thus $Y_+ \cap A \neq \varnothing$.

Conversely let $x \in E - A$ be such that $Y^+ \cap A \neq \varnothing$ for all signed cocircuits Y of M such that $x \in Y^+$. Apply the 3-Painting Lemma in ${}_{\bar{x}}M$ (definition in Section 3) with $B = A \cup \{x\}$. $G = E - (A \cup \{x\})$, $R = \varnothing$ and $e = x$: there is no signed cocircuit Y such that $x \in Y^-$ and $A \cap Y \subseteq Y^+$, hence there is a signed circuit X such that $X^- = \{x\}$ and $X^+ \subseteq A$, i.e., $x \in \mathrm{Conv}_M(A)$.

We say that a subset A of E is *convex in* M if $A = \mathrm{Conv}_M(A)$. The following subsets of E are convex in M: a flat of M, an open half-space, a closed half-space, the intersection of two convex sets.

Theorem 2.2 is a version of Krein–Milman Theorem:

THEOREM 2.2. *Let M be an acyclic oriented matroid on a set E and A be a subset of E. Let A' be the union of the extreme points of $M(A)$. We have*

$$\mathrm{Conv}_M(A') = \mathrm{Conv}_M(A)$$

Proof. We have $\mathrm{Conv}_M(A') \subseteq \mathrm{Conv}_M(A)$. Conversely consider $x \in \mathrm{Conv}_M(A)$. Let Y be a signed cocircuit of M such that $x \in Y^+$. We have $Y^+ \cap A \neq \varnothing$ by Proposition 2.1. There is a signed cocircuit Y' of $M(A)$ such that $\varnothing \neq Y'^+ \subseteq Y^+ \cap A$. By Proposition 1.2 Y'^+ contains an extreme point P' of $M(A)$, hence $Y^+ \cap A' \neq \varnothing$. Therefore $x \in \mathrm{Conv}_M(A')$.

PROPOSITION 2.3. *Let M be an acyclic oriented matroid on a set E and A be a subset of E.*

(i) *Let F be a face of $M(\mathrm{Conv}_M(A))$: then $F = \mathrm{Conv}_M(F \cap A)$ and $F \cap A$ is a face of $M(A)$.*

(ii) *Let F be a face of $M(A)$: then $\mathrm{Conv}_M(F) = \bar{F}^M \cap \mathrm{Conv}_M(A)$ is a face of $M(\mathrm{Conv}_M(A))$.*

Proof. Without loss of generality we may suppose that $\mathrm{Conv}_M(A) = E$ and $1 \leqslant r_M(F) \leqslant r(M) - 1$.

(i) Since F is a flat we have $\mathrm{Conv}_M(F \cap A) \subseteq F$. To show the equality consider $x \in F$ and a signed cocircuit Y of M such that $x \in Y^+$. F being a face of M, by definition, $E - F$ is an union of positive cocircuits of M. By Lemma 1.1.2 there is a signed cocircuit Z such that $x \in Z^+ \subseteq Y^+ \cap F$. Now $x \in \mathrm{Conv}_M(A) = E$ implies $Z^+ \cap A \neq \varnothing$, hence $Y^+ \cap F \cap A \neq \varnothing$. Therefore $x \in \mathrm{Conv}_M(F \cap A)$.

$F \cap A$ is a face of $M(A)$ by Lemma 1.1.1.

(ii) F being a face of $M(A)$, by definition, $A - F$ is an union of positive cocircuits $Y'_1, Y'_2, \ldots, Y'_k$ of $M(A)$. Let Y_1 be a signed cocircuit of M such that $Y'_1 = Y_1 \cap A$. We have $Y_1 \cap A \subseteq Y_1^+$ and $\mathrm{Conv}_M(A) = E$, hence Y_1 is positive by Proposition 2.1. On the other hand $Y_1 \subseteq E - F$ implies

$Y_1 \subseteq E - \bar{F}^M$. Hence $A - F \subseteq Y_1 \cup Y_2 \cup \cdots \cup Y_k \subseteq E - \bar{F}^M$. In fact we have $Y_1 \cup Y_2 \cup \cdots \cup Y_k = E - \bar{F}_M$. Otherwise consider $x \in (E - \bar{F}) - (\bigcup Y_i)$. There is a signed cocircuit Y of M contained in $E - \bar{F}$ such that $x \in Y$. By Lemma 1.1.2 there is a signed cocircuit Z of M such that $x \in Z^+ \subseteq Y^+ - (\bigcup Y_i)$. Now $x \in \mathrm{Conv}_M(A) = E$, hence $Z^+ \cap A \neq \varnothing$, a contradiction, since $Y \subseteq E - \bar{F}$ and $A - F \subseteq \bigcup Y_i$ imply $(Y^+ - (\bigcup Y_i)) \cap A = \varnothing$. Thus $\bar{F}^M$ is a face of M.

We have $\mathrm{Conv}_M(F) \subseteq \bar{F}^M$. To show the equality consider $x \in \bar{F}^M$ and a signed cocircuit Y such that $x \in Y^+$. By Lemma 1.1.2 there is a signed cocircuit Z of M such that $x \in Z^+ \subseteq Y^+ \cap \bar{F}^M$. Again $x \in \mathrm{Conv}_M(A) = E$ implies $Z^+ \cap A \neq \varnothing$, hence $Y^+ \cap F \neq \varnothing$ since $\bar{F}^M \cap A = F$. Therefore $x \in \mathrm{Conv}_M(F)$.

By Proposition 2.3 the lattices of faces of $M(A)$ and $M(\mathrm{Conv}_M(A))$ in an acyclic oriented matroid M are isomorphic. It results from Theorem 2.2 and Proposition 2.3 that the study of lattices of faces in oriented matroids can be restricted to acyclic oriented matroids such that all points are extreme points.

A further step in the study of convexity would be separation theorems: analogues of the Hahn–Banach Theorem, etc. This study requires the notion of extension of an oriented matroid, since it may happen that the convex hulls of two sets A, B are disjoint in an acyclic oriented matroid M but have a non-empty intersection in some extension N of M. On the other hand, in matroids, separation by hyperplanes does not cover all cases: examples can be constructed where $\mathrm{Conv}_N(A) \cap \mathrm{Conv}_N(B) = \varnothing$ in all extensions N of M but no hyperplane of M separates A, B. A different definition of separation has to be considered: We say that A, B are *separable* if there is no signed circuit X of M such that $X^+ \subseteq A$ and $X^- \subseteq B$. We have studied 1-extensions of oriented matroids in [9].

3. The Number of Acyclic Orientations of an Orientable Matroid

Stanley has shown that the number of acyclic orientations of an undirected graph G is equal to $\chi(G; -1)$, where $\chi(G)$ denotes the chromatic polynomial of G [10]. This result generalizes as follows to oriented matroids:

THEOREM 3.1. *Let M be an oriented matroid on a set E.*

The number of subsets A of E such that ${}_{\bar{A}}M$ is acyclic is equal to $t(M; 2, 0)$, where $t(M)$ denotes the Tutte polynomial of M.

Theorem 3.1 contains the extension of Stanley's theorem to unimodular submodules of $\mathbb{Z}^n$ given by Brylawski and Lucas in [2].

We recall that ${}_{\bar{A}}M$ denotes the oriented matroid obtained from M by reversing signs on A [1, Sect. 2]: the signed circuits of ${}_{\bar{A}}M$ are all the signed sets ${}_{\bar{A}}X$ defined by $({}_{\bar{A}}X)^+ = (X^+ - A) \cup (X^- \cap A)$ and $({}_{\bar{A}}X)^- = (X^- - A) \cup (X^+ \cap A)$ for X a signed circuit of M.

The Tutte polynomial of M [5] is the polynomial

$$t(M; \zeta, \eta) = \sum_{X \subseteq E} (\zeta - 1)^{r(M) - rM(X)} (\eta - 1)^{|X| - rM(X)}.$$

$t(M)$ is the unique solution of the inductive relations:

(1) if $e \in E$ is neither an isthmus nor a loop of M

$$t(M; \zeta, \eta) = t(M - e; \zeta, \eta) + t(M/e; \zeta, \eta);$$

(2) if $e \in E$ is an isthmus of M

$$t(M; \zeta, \eta) = \zeta t(M - e; \zeta, \eta);$$

(3) if $e \in E$ is a loop of M

$$t(M; \zeta, \eta) = \eta t(M - e; \zeta, \eta);$$

(4) $t(\phi; \zeta, \eta) = 1$.

Let G be a graph. We have (notations of Section 1):

$$\chi(G; q) = (-1)^{v(G)} (-q)^{c(G)} t(C(G); 1 - q, 0).$$

LEMMA 3.1.1. *Let M an oriented matroid on a set E and $e \in$ be an element of E.*

(i) *If both M and ${}_{\bar{e}}M$ are acyclic, then both $M - e$ and M/e are acyclic.*

(ii) *If M is acyclic and ${}_{\bar{e}}M$ is not acyclic, then $M - e$ is acyclic and M/e is not acyclic.*

(iii) *If e is not a loop of M and both M and ${}_{\bar{e}}M$ are not acyclic, then both $M - e$ and M/e are not acyclic.*

Proof. (i) M acyclic implies clearly $M - e$ acyclic. Suppose M/e contains a positive circuit X'. Let X be a signed circuit of M such that $X - \{e\} = X'$. We have $X^- \subseteq \{e\}$, hence ${}_{\bar{e}}X$ is a positive circuit of ${}_{\bar{e}}M$ a contradiction.

(ii) $M - e$ is clearly acyclic. Since ${}_{\bar{e}}M$ is not acyclic, M contains a signed circuit X with $X^- \subseteq \{e\}$. M being acyclic necessarily $X^- = \{e\}$ and $X^+ \neq \varnothing$. Then $X - e$ is a positive circuit of M/e.

(iii) Let X_1 be a positive circuit of M. Since e is not a loop of M, X_1 is not reduced to e, hence $X_1 - \{e\}$ non-empty contains a positive circuit of M/e.

Suppose $M - e$ is acyclic. Then necessarily $e \in X_1$. Similarly ${}_{\bar{e}}M$ contains a positive circuit ${}_{\bar{e}}X_2$ and we have $X_2^- = \{e\}$. Now by the elemination property there is a positive circuit of M contained in $(X_1 \cup X_2) - \{e\}$.

Proof of Theorem 3.1. Let $f(M)$ denote the number of subsets A of E such that ${}_{\bar{A}}M$ is acyclic.

Clearly the inductive relations (2), (3), (4) are satisfied for $\zeta = 2$ and $\eta = 0$. We have to show that if $e \in E$ is neither an isthmus nor a loop of M then $f(M) = f(M - e) + f(M/e)$.

For $A \subseteq E$ set $f(M; A) = 0$ if ${}_{\bar{A}}M$ is not acyclic, $f(M; A) = 1$ if ${}_{\bar{A}}M$ is acyclic. We have

$$f(M) = \sum_{A \subseteq E} f(M; A)$$

It follows immediately from Lemma 3.1.1 that for $e \in E$ not a loop of M and $A \subseteq E - \{e\}$ we have

$$f(M; A) + f({}_{\bar{e}}M; A) = f(M - e; A) + f(M/e; A)$$

Now $f({}_{\bar{e}}M; A) = f(M; A \cup \{e\})$. Summing up for all $A \subseteq E - \{e\}$ we get $f(M) = f(M - e) + f(M/e)$ as required.

As pointed out in [1] the collection of orientations of an oriented matroid M on a set E is partitioned into classes by operations of sign reversal on subsets of E. Clearly each class contains exactly $2^{|E| - c(M)}$ different orientations, where $c(M)$ denotes the number of connected components of M; Theorem 3.1 implies that each class contains the same number of acyclic orientations of M, namely $2^{-c(M)} t(M; 2, 0)$. The problem of determining the exact number of classes seems difficult. We recall that this number is 1 when M is a binary oriented matroid [1, Proposition 6.2].

COROLLARY 3.2. *Let M be an oriented matroid on a set E. The number of subsets A of E such that ${}_{\bar{A}}M$ is totally cyclic is equal to $t(M; 0, 2)$.*

Proof. Apply Theorem 3.1 to $M^\perp$ and use the relation

$$t(M^\perp; \zeta, \eta) = t(M; \eta, \zeta).$$

In particular, the number of strongly connected orientations of a connected graph G without loops is equal to $t(\mathbb{C}(G); 0, 2)$.

COROLLARY 3.3. *Let M be an oriented matroid without loops on a set E. There exists a subset A of E such that ${}_{\bar{A}}M$ is acyclic.*

Proof. Clearly if M has no loops then $M - e$ has no loops for any $e \in E$. It follows immediately by induction that $t(M; 2, 0) \geqslant 2$.

Corollary 3.3 has an easy direct proof using Lemma 1.1.2.

In the case of an oriented matroid M on a set E arising from a vector space over an ordered field, it follows from a result of Camion [3, Chap. III, Theorem 3], that there always exists a subset A of E and a basis B of M such that ${}_{\bar{A}}M$ is acyclic and all the signed cocircuits of M associated with B are positive, equivalently, by Proposition 1.4, ${}_{\bar{A}}M$ has exactly $r(M)$ facets. We conjecture that this property holds for any oriented matroid.

Note added in proof. We mention some recent related papers: J. Folkman and J. Lawrence, Oriented matroids, *J. Combinatorial Theory B* **25** (1978), 199–236; R. G. Bland, A combinatorial abstraction of linear programming, *J. Combinatorial Theory B* **23** (1977), 33–57; R. G. Bland and M. Las Vergnas, Minty colorings and orientations of matroids, *Ann. N.Y. Acad. Sci.* **319** (1979), 86–92; R. Cordovil, M. Las Vergnas, and A. Mandel, Euler's relation, Möbius functions and matroid identities, to appear.

A. Mandel has announced a negative answer to the problem of extending the Hahn–Banach theorem raised in Section 2, by exhibiting an example of oriented matroid with two separable subsets A, B such that $\mathrm{Conv}(A) \cap \mathrm{Conv}(B) = \varnothing$ in every extension of M (private communication).

Theorem 3.1 contains as special cases, in dual form, Zaslavsky's theorems on the numbers of regions determined by arrangements of hyperplanes in real Euclidean and projective spaces [T. Zaslavsky, Facing up to arrangements: Face-count formulas for partitions of spaces of spaces by hyperplanes, *Mem. Amer. Math. Soc.*, No. 154 (1975)]. The precise relationship between these theorems is given in M. Las Vergnas, Sur les activités des orientations d'une géométrie combinatoire, *in* "Actes Colloque Mathématiques Discrètes (Bruxelles 1978)," Vol. 20, pp. 293–300, Cahiers du Centre d'Etudes de Recherche Opérationnelle, Brussels, 1978. Theorem 3.1 is generalized to morphisms of matroids in M. Las Vergnas, Acyclic and totally cyclic orientations of combinatorial geometries, *Discrete Math.* **20** (1977), 51–61. See also the preceding reference, and M. Las Vergnas, On the Tutte polynomial of a morphism of matroids, *in* "Proceedings of the Joint Canada–France Combinatorial Colloquium (Montréal 1979)," Annals of Discrete Mathematics, to appear.

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