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Remarks on Nonlinear Contraction and Comparison Principle in Abstract Cones

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INTRODUCTION

The contraction mapping principle and the Schauder principle can both be viewed as a comparison of maps. For the former one has a condition of the type

$$\rho(Tx, Ty) \leqslant \psi\rho(x, y) \tag{1.1}$$

and for the latter one has a condition of the type

$$\gamma(T(S)) \leqslant \psi(\gamma(S)), \tag{1.2}$$

where ρ is the metric and γ is the Kuratowski measure of noncompactness. If ψ is a linear map $\psi x = kx$ from the nonnegative real R^+ into itself then the map T satisfying (1.1) is said to be k-contractive and the map satisfying (1.2) is said to be k-set contractive. It is also usually assumed that $k < 1$ in which case the adjective "strict" is used to describe the contractive property.

Instead of taking ψ to be a linear map on the cone $R^+ \to R^+$, ψ can be chosen as a nonlinear map from a cone of a Banach space into itself [1, 4]. This innovation provides for greater flexibility in the choice of ψ and it also has the advantage of stronger convergence properties and more accurate estimates. The comparison map ψ is positive (in the sense that it takes values in a cone), monotone (nondecreasing) and has a unique fixed point which is the zero element of the cone. For a regular cone (such as cones in L_p, $1 \leqslant p < \infty$) ψ need only satisfy the weak continuity condition: upper semicontinuous from above (or from the right). However, in the case of a normal cone which is not regular (such as $C[0, 1]$) it is assumed in [1, 4] that ψ is completely continuous. The complete continuity condition which is also used by Krasnoselskii [7, p. 127] may be replaced by a weaker compactness-type condition in terms of measure of noncompactness along with upper semicontinuity from above. We also manage to avoid strict contractive conditions.

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The paper is organized as follows. In Section 2 we state definitions regarding the theory of cones and some propositions which are used as lemmas or to amplify results proved later on. In Section 3 we present some results dealing with maximal fixed points of monotone maps. As a consequence we obtain a generalized Bellman–Gronwall–Reid inequality. In Section 4 we present a generalization of the contraction mapping principle.

For applications see [1–4]. Also see [5] for modifications using minimal solutions in place of maximal solutions.

2. Cones

Let E be a real Banach space. A set $k \subset E$ is called a *cone* if: (i) k is closed; (ii) if $u, v \in k$, then $\alpha u + \beta v \in k$ for all $\alpha, \beta \geqslant 0$; (iii) of each pair of vectors u, $-v$ at least one does not belong to k, provided $u \neq \theta$, where θ is the zero of the space E. We say that $u \geqslant v$ if and only if $u - v \in k$. A cone is called *normal* if a $\delta > 0$ exists such that $\| e_1 + e_2 \| > \delta$ for $e_1, e_2 \in k$ and $\| e_1 \| = \| e_2 \| = 1$. The norm in E is called *semimonotone* if for arbitrary $x, y \in k$ it follows from $x \leqslant y$ that $\| x \| \leqslant N \| y \|$, where the constant N does not depend on x and y.

Proposition 2.1 [7]. *A necessary and sufficient condition for the cone K to be normal is that the norm be semimonotone.*

Proposition 2.2. *A decreasing sequence $u_0 \geqslant u_1 \geqslant \cdots$ in a space with a normal cone is convergent if it has a convergent subsequence.*

Proof. Let $u_{n_k} \to u_\infty$, as $n \to \infty$. Then for $m \geqslant m_k$, $u_m - u_\infty \leqslant u_{n_k} - u_\infty$. By Proposition 2.1, $\| u_m - u_\infty \| \leqslant N \| u_{n_k} - u_\infty \| \to 0$ as $n_k \to \infty$. Thus u_n converges to u_∞.

A cone is said to be *regular* if every decreasing sequence $u_0 \geqslant u_1 \geqslant \cdots$ which is bounded from below; i.e., there is a vector v such that $u_n \geqslant v$, $n = 0, 1, \ldots$, is convergent.

The *conical segment* $\langle x_0, u_0 \rangle$ is the subset of E of vectors x satisfying $x_0 \leqslant x \leqslant u_0$.

A map ψ from a subset of E into E is said to be *monotone* if $\psi u \geqslant \psi v$ when $u \geqslant v$.

If (A, ρ) is a bounded metric space, we define $\gamma(A)$, the *measure of noncompactness* of A, to be $\inf\{d > 0 \mid A$ can be covered by a finite number of sets of diameter less than or equal to $d\}$.

Proposition 2.3 (Kuratowski [8]). *Let (X, ρ) be a complete metric space and let $A_1 \supset A_2 \supset \cdots$ be a decreasing sequence of nonempty, closed subsets of X. Assume* *that $\gamma(A_n)$ converges to zero. Then if we write $A_\infty = \bigcap_{n\geq 1} A_n$, A_∞ is a nonempty compact set and A_n approaches A_∞ in the Hausdorff metric.*

With regard to Kuratowski's theorem we say that a map ψ on a complete metric space (A_0, ρ) into itself is *quasi-compact* if the sequence of measure of noncompactness $\gamma(A_n)$ of the closed sets $A_{n+1} = \mathrm{cl}(\psi(A_n))$, $n \geq 0$, approaches zero.

A mapping ψ on a partially ordered set into itself is said to be *upper semicontinuous from above* if whenever $u_0 \geq u_1 \geq \cdots$ and $\psi u_0 \geq \psi u_1 \geq \cdots$ are both monotonic, convergent sequences and $w = \lim u_n$ is in the domain of ψ, then $\psi w \geq \lim \psi u_n$.

PROPOSITION 2.4. *Let ψ be monotone and upper semicontinuous from above and suppose that the sequence of iterates $u_n = \psi^n u_0$, of a vector u_0, is decreasing and convergent to a vector u_∞ which is in the domain of ψ. Then u_∞ is a fixed point of ψ, i.e., $\psi u_\infty = u_\infty$.*

Proof. Clearly $\psi u_n = u_{n+1}$ is also decreasing and convergent to u_∞. From $u_n \geq u_\infty$ and the monotone property we deduce that $\psi u_n \geq \psi u_\infty$ and hence $u_\infty \geq \psi u_\infty$. The reverse inequality follows from the upper-semicontinuous-from-above property.

PROPOSITION 2.5. *Let f be monotone and upper semicontinuous from above from an interval $[0, a]$ of real numbers into itself such that $f(x) = x$ if and only if $x = 0$. Let ψ be a map from a complete metric space (A_0, ρ) into itself such that*

$$\gamma(\psi A) \leq f(\gamma(A)) \tag{2.1}$$

for any subset A of A_0. Then ψ is quasi-compact.

Proof. Let $A_{n+1} = \mathrm{cl}(\psi A_n)$, $n \geq 0$. Put $r_n = \gamma(A_n)$, $n \geq 0$. One then has from condition (2.1) that $r_{n+1} \leq f(r_n)$, $n \geq 0$. From the monotone property of f, it follows that $r_n \leq t_n$, $n \geq 0$ where sequence t_n is defined by $t_0 = r_0$, $t_{n+1} = f(t_n)$. By the monotone property of f, t_n is a decreasing sequence. Let $t_\infty = \lim t_n$. From Proposition 2.4, t_∞ is a fixed point f and hence $t_\infty = 0$. Clearly, r_n converges to zero.

Remark. The map ψ in Proposition 2.5 is called α-set contractive, $\alpha > 0$, if it is continuous and $\gamma(\psi(A)) \leq \alpha\gamma(A)$ for any bounded subset A. If ψ is α-set contractive with $\alpha < 1$ then ψ is quasi-compact since we may take $f(x) = \alpha x$ in this case.

3. Fixed Points in Spaces with Cones

THEOREM 3.1. *Let A be a closed bounded subset of a Banach space which is partially ordered with respect to a normal cone. Let ψ be a monotone, quasi-compact, upper-semicontinuous-from-above map from A into itself. Let*

$$U = \{u \in A \mid \psi u \leqslant u\}, \tag{3.1}$$

$$L = \{u \in A \mid \psi u \geqslant u\}, \tag{3.2}$$

$$F = \{u \in A \mid \psi u = u\}. \tag{3.3}$$

Then (i) *U, L, F are invariant under ψ.* (ii) *Let ψ_n denote the restriction of ψ^n to U, $n \geqslant 0$. Then the sequence ψ_n is decreasing, i.e., $\psi_n u \geqslant \psi_{n+1} u$ for $u \in U$, and pointwise convergent to a map ϕ, i.e., $\psi_n u \to \phi u$ for $u \in U$.* (iii) *The range of ϕ is F which is precompact.* (iv) *The map ϕ is monotone.* (v) *If $v \in L$ and $v \leqslant u \in U$, then $v \leqslant \phi u \leqslant u$.*

Proof. Statement (i) is obvious from the monotone property of ψ which also implies that $\psi_n u \geqslant \psi_{n+1} u$ when $u \in U$. Let $u \in A$, then by Proposition 2.3, with $A_n = \mathrm{cl}(\psi^n(A))$, and from the quasi-compact property, there is a compact set C and sequence $c_n \in C$ such that $\| \psi^n u - c_n \| \to 0$ as $n \to \infty$. Since the sequence c_n has a convergent subsequence, so does the sequence $u_n = \psi^n u$. If $u \in U$, we deduce from Proposition 2.2 that u_n is convergent. This establishes statement (ii). The fact that $\phi u \in F$ follows from Proposition 2.4. Now if $u \in F$ then $\phi u = u$ so that ϕ maps U into F. Also $\mathrm{cl}(F)$ is precompact because $F \subset C$. This completes the proof of statement (iii). Statement (iv) follows because each of the maps ψ_n are monotone. For statement (v), note that $v \leqslant u$ implies $v \leqslant \psi^n v \leqslant \psi^n u \to \phi u$. Thus $v \leqslant \phi u$. This completes the proof.

THEOREM 3.2. *Let ψ be a monotone, upper-semicontinuous map from a conical segment $\langle \theta, u_0 \rangle$ into itself and let the following condition be satisfied.*

> *Condition* (H): *either ψ is quasi-compact and the cone is normal or the cone is regular* (*or both*).

Then the sequence of iterates $\psi^n u_0$ is decreasing and convergent to fixed point w of ψ. Moreover, $v \leqslant \psi u$ implies $v \leqslant w$. In particular, w is the maximal fixed point of ψ in the segment.

Proof. If ψ is quasi-compact and the cone is normal then the result is a corollary of Theorem 3.1. If the cone is regular the result follows from [1, Theorem 3.1].

The following result is a generalization of the Bellman–Gronwall–Reid inequality.

COROLLARY 3.1. *Let the hypothesis of Theorem* 3.1 *be satisfied and let p be a mapping of a set X into the segment $\langle \theta, u_0 \rangle$. Suppose T is a mapping of X into itself such that $pTa \leqslant \psi pa$, $a \in X$. Then if b is a fixed point of T, $pb \leqslant w$ where w is the maximal fixed point of ψ.*

Proof. Set $u = pb$. Then $u = pTb \leqslant \psi pb = \psi u$. By Theorem 3.2, $u \leqslant w$.

4. CONTRACTION MAPPING PRINCIPLE

Let X be a set and let ρ be a mapping from $X \times X$ into a cone k of a Banach space. The map ρ is said to be a *k-metric* on X if it satisfies the properties:

$$\rho(x, y) = \rho(y, x), \qquad \rho(x, y) = \theta \quad \text{iff} \quad x = y,$$
$$\rho(x, y) \leqslant \rho(x, z) + \rho(z, y).$$

A sequence x_n in the k-metric space (X, ρ) is said to be *Cauchy* if $\lim_{m \geqslant n \to \infty} \rho(x_n, x_m) \to \theta$. The sequence x_n is said to be *convergent* if there is a $y \in X$ such that $\lim_{n \to \infty} \rho(x_n, y) = \theta$. A k-metric space is *complete* if every Cauchy sequence is convergent. A convergent sequence x_n is said to be k-convergent to y if there is a sequence $u_n \to \theta$ in k such that $\rho(x_n, y) \leqslant u_n$.

THEOREM 4.1. *Let (X, ρ) be a complete k-metric space. Let ψ be a monotone, upper-semicontinuous-from-above map from the conical segment $\langle \theta, u_0 \rangle$ into itself such that condition* (H) *is satisfied and such that θ is the unique fixed point of ψ. Let T be a map from X into itself such that*

$$\rho(Tx, Ty) \leqslant u_0, \qquad x, y \in X, \tag{4.1}$$
$$\rho(Tx, Ty) \leqslant \psi\rho(x, y) \quad \textit{when} \quad \rho(x, y) \leqslant u_0. \tag{4.2}$$

Then for arbitrary $x_0 \in X$, the sequence of iterates $x_n = T^n x_0$ k-converges to a fixed point y of T and y is the unique fixed point of T.

Proof. For any pair of integers $m \geqslant n \geqslant 1$ we have $\rho(x_n, x_m) = \rho(Tx_{n-1}, Tx_{m-1}) \leqslant u_0$. Hence, $\rho(x_{n+1}, x_{m+1}) \leqslant \psi\rho(x_n, x_m)$. Repeating this argument we find that $\rho(x_{n+1}, x_{m+1}) \leqslant \psi^m \rho(Tx_0, Tx_m)$. Since $\rho(Tx_0, Tx_m) \leqslant u_0$ and since ψ is monotone, $\rho(x_{n+1}, x_{m+1}) \leqslant \psi^m u_0$. But by Theorem 3.2 and since θ is the maximal fixed point of ψ, $\psi^m u_0$ decreases to θ. Thus x_n is a Cauchy sequence. Let $y = \lim x_n$. Then by letting $m \to \infty$ in the above inequality we have $\rho(x_{n+1}, y) \leqslant \psi^n u_0$. Thus the sequence x_n k-converges to y. Since $\rho(x_{n+1}, y) \leqslant u_0$, $\rho(Tx_{n+1}, Ty) \leqslant \psi\rho(x_{n+1}, y) \leqslant \psi^{n+1} u_0 \to \theta$. Therefore $\rho(x_{n+2}, Ty) \to \theta$ so that $y = Ty$, i.e., y is a fixed point of T. Suppose also $z = Tz$ is also a fixed point. Then $\rho(y, z) = \rho(Ty, Tz) \leqslant u_0$. Hence $\rho(y, z) = \rho(Ty, Tz) \leqslant \psi\rho(y, z)$. By Theorem 3.2, $\rho(y, z) \leqslant \theta$. This means that $\rho(y, z) = \theta$ or $y = z$. The proof is now complete.

The above result weakens assumptions made by the authors in [1] regarding the complete continuity of ψ and the strict inequality $\psi u_0 \leqslant u_0$. Still it assumes too much. The following theorem represents an economization of Theorem 4.1.

Recall that a map T is *closed* if whenever $x_n \in$ Domain (T), $x_n \to u$, $Tx_n \to v$ then $u \in$ Domain (T) *and* $Tu = v$.

THEOREM 4.2. *Let (X, ρ) be a k-metric space and let ψ be a monotone map from a segment $\langle \theta, u_0 \rangle$ into itself such that*

$$\lim \psi^n u_0 \to \theta \tag{4.3}$$

Suppose T is a closed map from a subset D of X into X such that

$$\rho(Tx, Ty) \leqslant \psi\rho(x, y), \qquad x, y \in D, \quad \rho(x, y) \leqslant u_0 . \tag{4.4}$$

Suppose further that $x \in \mathscr{D}$ and $x_n \in T^n x \in D$, $n = 1, 2, \ldots$, and that

$$\rho(x_n , x_0) \leqslant u_0 . \tag{4.5}$$

Then x_n k-converges to a fixed point of T.

Proof. As in the proof of Theorem 4.1, $\rho(x_n , x_m) \leqslant \psi^n \rho(Tx_0 , Tx_n) \leqslant \psi^{n+1}\rho(x_0 , x_n) \leqslant \psi^{n+1}u_0$ for $m > n > 0$. Whence by (4.3), x_n is Cauchy. Let $x_n \to w$. Then $Tx_n = x_{n+1} \to w$. Since T is closed, $Tw = w$. Moreover from $\rho(x_n , w) \leqslant \psi^{n+1}u_0$ we conclude that x_n k-converges to w.

See [9] for further conditions under which an iterative process converges to a fixed point.

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