

An Optimal Family of Directed, Bounded-Degree Broadcast Networks

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Abstract

Increasingly, the design of efficient computer networks and multi-processor configurations are considered important applications of computer science. There are some constraints in network design which are usually created by economic and physical limitations. One constraint is the bounded degree, which is the limited number of connections between one node to others. Another possible constraint is a bound on the time that a message can afford to take during a “broadcast”. We present, for the first time, a set of largest-known directed networks satisfied specified bounds on node degree and broadcast time. We also presents a family of optimal $(\Delta, \Delta + 1)$ broadcast digraphs. That is, digraphs with a proven maximum number of nodes, having maximum degree Δ and broadcast time at most $\Delta + 1$.

Key words: broadcast time, Cayley digraphs, bounded-degree networks

1 Introduction

Some fundamental design problems related with the topology of networks have been widely studied. One facet of those design problems is the study of network constructions under the constraint of a bound on the maximum node

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degree imposed by economic and physical limitations. A lot of research has been done recently on the design of networks with the largest number of nodes (order) satisfying this constraint on the number of direct neighbor connections (see [1,2,4,6,9,12]).

An important feature characterizing the “quality” of an interconnection network for parallel computing is the ability to effectively disseminate the information among its processors. One of the main problems of information dissemination is broadcasting, which is the process of sending a message originating at one node of a network to all other nodes. There exists two kinds of network connection models: point-to-point model and multi-cast model. The minimum time of broadcasting in the interconnection network for those two different models may not be the same. With these two connection models, two different basic design sub-problems occur satisfying the constraint of bounded node degree [3]:

1. *The Degree/Diameter Problem.*

The *degree/diameter problem* is the design problem for finding a network with the largest possible order satisfying the bounds on node degree and diameter. The network connection model is the multi-cast connection model. That is, each node can communicate with all of its neighbor nodes in one time step. The diameter is the maximum time delay for broadcasting a message throughout the whole network under this model.

2. *The Degree/Broadcast-Time Problem.*

The *degree/broadcast-time problem* asks for the construction of a largest possible network satisfying the bounds on node degree and maximum broadcast time. The network connection model is the point-to-point connection model. That is, each node can communicate with just one of its neighbor nodes at one time step. Here, the broadcast time is the maximum time for disseminate a message throughout the whole network under that model.

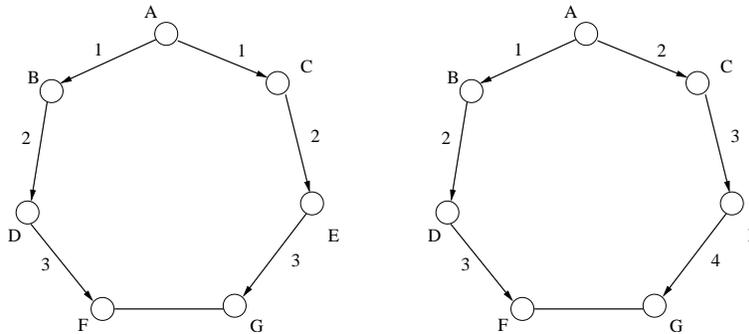


Fig. 1. A comparison between (a) the diameter and (b) the broadcast time.

We illustrate the difference between the diameter and broadcast time in Fig. 1 with originating node A . Good survey papers about the general broad-

casting problem are given in [13,14].

Generally a network's (minimum) broadcast time is larger than its diameter since there are more constraints on message routing. Also computing the broadcast time of the network is harder than computing the diameter when the topology of the network becomes more complicated.

The original broadcast design problem was introduced by Farley in [8]. It is slightly different from what we discuss in this paper. This is the problem of finding graphs for a given order with the least number of edges that one can broadcast from each vertex in minimum time. The minimum time for broadcasting in a network of order n is $\lceil \log_2 n \rceil$, because the number of vertices receiving the message at most doubles at each time step on the broadcasting schedule. Recent results for Farley's minimum broadcast problem is presented in [7] and [10]. A bounded-degree version of Farley's model is given in [15].

For our degree/broadcast-time problem, we constrain both the degree Δ and broadcast time T while maximizing the order of the network instead of fixing the order and minimizing the number of edges. Specially, for this paper, we will focus on the directed broadcast network, as defined in the next section.

Some good results have been presented by using group-theoretic methods for designing large "efficient" networks. For the undirected case, a table of largest-known degree/diameter graphs is given in [5] using Cayley graphs. For the directed case, Faber and Moore in [9] study families of digraphs on permutations and give a table of largest-known vertex symmetric digraphs for the degree/diameter problem. Other more-recent results of finding large digraphs of small diameter are presented in [1] and [12]. A table of the largest-known (undirected) broadcast graphs of bounded degree may be found in the paper [6]. There is no such (published) result for the directed degree/broadcast-time problem until now.

In this paper we establish the first table of largest-known (Δ, T) broadcast digraphs. We will give some upper and lower bounds on the directed degree/broadcast-time problem in Section 3.1. Many of the lower bounds were obtained by searching through random Cayley digraphs based on the semi-product group of cyclic groups³, as explained in Section 3.2. The most important result of this paper is the presentation of an easily constructed infinite family of optimal $(\Delta, \Delta + 1)$ broadcast digraphs, which is presented in Section 4.

2 Some Formal Definitions

We are ready to give some standard definitions and notations.

Definition 2.1 A **digraph** $G = (V, E)$ consists of two finite sets V and E . The elements of V are called the **vertices** (or **nodes**), and the elements of E are called **edges** (or **arcs**). Each edge $(u, v) \in E$, $u \neq v$, is an ordered pair of

³ This construction technique was first applied in [4] and later utilized in [5,12,6].

vertices $u \in V$ and $v \in V$. If $(u, v) \in E$ implies $(v, u) \in E$ then we call G an (undirected) **graph**.

The **degree** of a vertex u , denoted by $\deg(u)$, is the number of edges $(u, v) \in E$, where $v \in V$. The **diameter** of a graph is the maximum of the shortest distances $d(u, v)$ between each pair of u and v of V .

Definition 2.2 A (Δ, D) **graph** is a graph $G = (V, E)$ satisfying: (1) $\deg(v) \leq \Delta$ for all vertices $v \in V$ and (2) diameter of G is less than or equal to D . A (Δ, D) **digraph** is defined similarly; in this case, each in-degree and out-degree must be bounded by Δ . A (Δ, D) graph (digraph) is **optimal** if it has the maximum order possible for a (Δ, D) graph (digraph).

Definition 2.3 A **broadcast protocol (scheme)** for a vertex v (called the originator) for a graph $G = (V, E)$ may be presented as a sequence $V_0 = \{v\}, E_1, V_1, E_2, V_2, \dots, E_t, V_t = V$ such that each $V_i \subseteq V$, each $E_i \subseteq E$, and for $1 \leq i \leq t$,

- (i) Each edge in E_i has exactly one vertex in V_{i-1} .
- (ii) No two edges in E_i share a common vertex.
- (iii) $V_i = V_{i-1} \cup \{w \mid (u, w) \in E_i\}$.

The (minimum) **broadcast time** $B(G, v)$ for a graph $G = (V, E)$ and originator v is the size of the minimum length broadcast protocol, where the **length** is defined to be t , as given in Definition 2.3. The **broadcast time** of a graph $B(G)$ is the maximum broadcast time over all $B(G, v)$, $v \in V$.

Definition 2.4 A (Δ, T) **broadcast graph** is a graph $G = (V, E)$ satisfying: (1) $\deg(v) \leq \Delta$ for all vertices $v \in V$ and (2) broadcast time $B(G) \leq T$. A (Δ, T) **broadcast digraph** is defined similarly; in this case, all in-degree and out-degree must be bounded by Δ . A (Δ, T) graph (digraph) is **optimal** if it has the maximum order possible for any (Δ, T) broadcast graph (digraph).

Finally we define a standard algebraic method for producing nice vertex-symmetric graphs to act as a model for our communication networks.

Definition 2.5 Given a finite group A and a set S of generators for A the **Cayley digraph** $G = (V, E)$, denoted by $\langle A, S \rangle$, is constructed as follows:

- (i) The elements of the group A are the vertices V of the digraph G .
- (ii) An edge (a, b) is in E if and only if $ag = b$ for some generator g in S .

If we also requires $S = S \cup S^{-1}$ then G is a **Cayley graph**.

Table 1
Some (Δ, T) broadcast digraph upper bounds.

$\Delta \backslash T$	2	3	4	5	6	7	8	9	10
2	4	7	12	20	33	54	88	143	232
3	4	8	15	28	52	96	177	326	600
4	4	8	16	31	60	116	224	432	833
5	4	8	16	32	63	124	244	480	944
6	4	8	16	32	64	127	252	500	992
7	4	8	16	32	64	128	255	508	1012
8	4	8	16	32	64	128	256	511	1020
9	4	8	16	32	64	128	256	512	1023
10	4	8	16	32	64	128	256	512	1024

3 Efficient Directed Broadcast Network Constructions

In this section, we present the largest-known broadcast directed networks satisfying the bounds on maximum vertex degree and broadcast time.

The main purpose of the degree/broadcast-time problem is to provide constructions of the largest possible (Δ, T) broadcast graphs (digraphs). In this section we calculate some upper bounds on these orders so that we know how close to optimal are the currently-known large broadcast digraphs (which provide our lower bounds).

3.1 Upper Bounds and Lower Bounds

A simple recurrence relation is available on the upper bound of the maximum order of broadcast directed networks. Let $f(\Delta, T)$ be the branch-out upper bound of the maximum order with out-degree Δ and broadcast time T of a rooted directed tree.

$$f(\Delta, 0) = 1$$

$$f(\Delta, T) = \sum_{i=1}^{\min(\Delta, T)} f(\Delta, T - i) + 1$$

From the above recurrence relation, we get Table 1 of some (Δ, T) broadcast digraph upper bounds.

For a given (Δ, T) broadcast digraph, a lower bound for the order of other larger (Δ', T') broadcast digraphs can be obtained from the following theorem. The proof of this theorem is similar to that of the theorem for the undirected networks (see [6]).

Table 2
The largest-known (Δ, T) broadcast digraphs.

D\T	2	3	4	5	6	7	8	9	10
2	4	7	12*	20*	27*	42*	64*	84*	126*
3		8	15	28*	48*	80*	110*	220*	328*
4			16	31	56*	96*	165*	300*	506*
5				32	63	116	210	390	686
6					64	127	234	440	840
7						128	255	486	952
8							256	511	1000
9								512	1023
10									1024

Theorem 3.1 *If $B(\Delta, T)$ denotes the order of the largest (Δ, T) broadcast digraph, then $B(\Delta + 1, T + 1) \geq 2 \cdot B(\Delta, T)$.*

Similarly, we have a more general result as follows:

Corollary 3.2 $B(\Delta + n, T + n) \geq 2^n \cdot B(\Delta, T)$, $n \geq 2$.

Table 2 shows the current largest-known broadcast digraphs. For the reader's convenience the bold entries in Table 2 show where the upper bounds have been achieved. The asterisks in the table denote where our random search algorithm has been used. The other plain entries show the results that have been achieved in [6] for the undirected case, which implicitly give lower bounds for the directed case.

3.2 Cayley Digraph Construction

In this section, we will describe our main Cayley digraph construction technique for finding large (Δ, T) broadcast digraphs. The digraphs, which correspond to the starred entries in Table 2, were created by using semi-direct products of cyclic groups⁴. The other bold (optimal) entries of the table are discussed later in Section 4.

As explained in [5], when given two cyclic groups Z_m and Z_n , a semi-direct product group $G = Z_m \times_{\sigma} Z_n$ is formed by defining an appropriate homomorphism $\sigma : Z_m \rightarrow \text{Aut}(Z_n)$. We define a mapping $\sigma^l(k) = (r^c)^k = r^{ck}$ where r belongs to $\text{Aut}(Z_n)$ and c is chosen so that $r^{cm} = 1$. The multiplication

⁴ The reader may see the digraph specifications in Appendix A of [17].

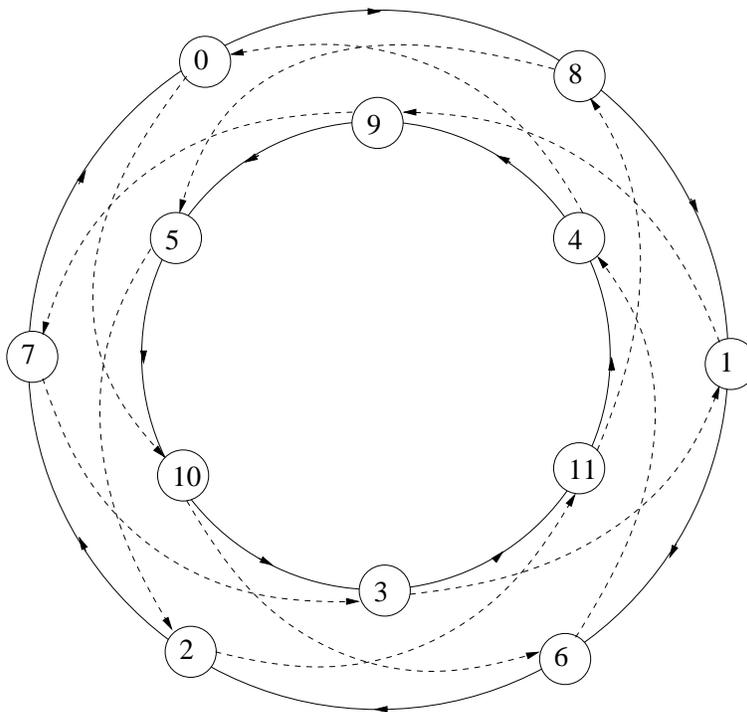


Fig. 2. An optimal $(2, 4)$ broadcast digraph.

table of the semi-direct product group G is defined by

$$(a_0, a_1) *_{\sigma} (b_0, b_1) = (a_0 + b_0 \bmod m, (a_1 + \sigma'(a_0) \cdot b_1) \bmod n).$$

For $a \in Z_m$ and $b \in Z_n$, $(\sigma(a))b = \sigma'(a) \cdot b$ is a suitable homomorphism. Note that $(0, 0)$ is the group identity for the semi-direct product group.

In Fig. 2, we give an example of the above construction. The digraph is a largest possible $(2, 4)$ broadcast digraph with 12 vertices based on the group $Z_4 \times_{\sigma} Z_3$. For $a \in Z_4$ and $b \in Z_3$, a vertex of the digraph whose corresponding group element is (a, b) and is labeled $4a + b$. The normal edge presents the use of the first generator $(2, 2)$ while the dashed edges represent the second generator $(3, 1)$. Another example is the optimal $(2, 5)$ broadcast digraph, given in Fig. 3, based on the group $Z_4 \times_{\sigma} Z_5$.

4 Optimal Families of (Δ, T) Broadcast Digraphs

We will now continue to describe the broadcast digraphs corresponding to the entries listed in Table 2. We can omit the entries below the diagonal (i.e., $\Delta > T$) since those entries follow from the $\Delta = T$ cases.

We now focus on two theorems that yield optimal broadcast digraphs (the main diagonal and off diagonal of Table 2). The first theorem, follows from the undirected (Δ, T) broadcast network problem (see, e.g., [6]). The second theorem is analogous to the (undirected) dihedral broadcast family of [4], where the broadcast time T is one greater than the maximum degree Δ .

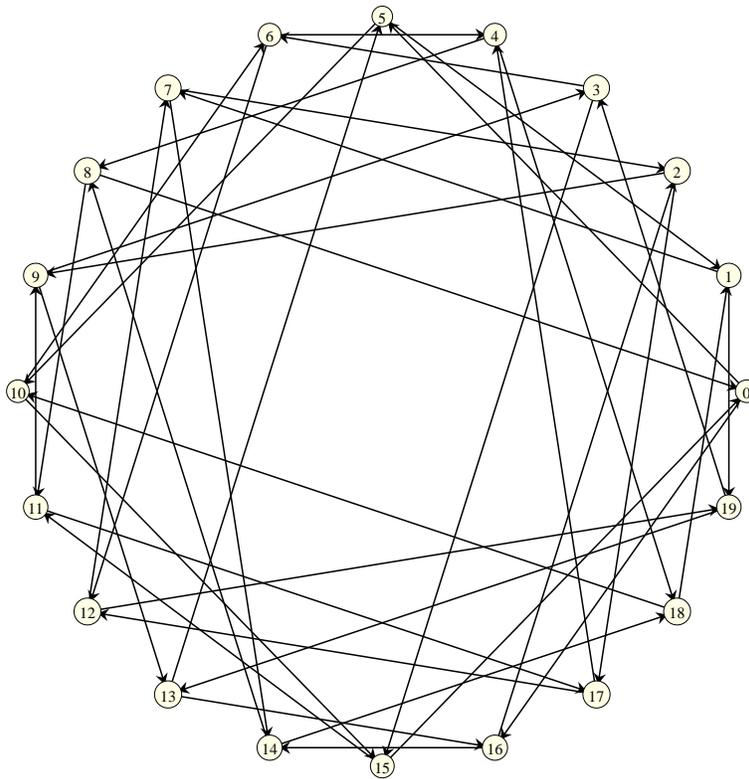


Fig. 3. An optimal $(2, 5)$ broadcast digraph.

Theorem 4.1 *The directed hypercube Q_Δ is an optimal (Δ, Δ) broadcast digraph.*

The following construction is the main result of this paper.

Theorem 4.2 *The Cayley digraph from the cyclic group Z_{2^n-1} with generators $\{g_1, g_2, \dots, g_{n-1}\}$ where $g_1 = 1, g_2 = 3, g_3 = 7, \dots$, and $g_{n-1} = 2^{n-1} - 1$ forms an optimal $(n-1, n)$ broadcast digraph, denoted by \mathcal{A}_{n-1} .*

Proof. Before proving the general version of this theorem, we illustrate it by building a simple $(2, 3)$ broadcast digraph \mathcal{A}_2 in Fig. 4. The bold edges in Fig. 4 denote the routing scheme from node 0. For this graph, the generators are 1 and 3. Let $V_i = \{v \mid \text{vertex } v \text{ has received the message by time } i\}$. We start the broadcasting at the identity $V_0 = \{0\}$. Using 1 as the first generator yields $V_1 = V_0 \cup \{1\} = \{0, 1\}$. Then, using 1 as the generator for node 1 and using 3 as another generator for node 0 so that

$$V_2 = V_1 \cup \{1 + 1, 0 + 3\} = \{0, 1\} \cup \{2, 3\} = \{0, 1, 2, 3\}.$$

Finally, using 3 as the third generator, the final broadcast time yields $V_3 = V_2 \cup \{0 + 3, 1 + 3, 2 + 3, 3 + 3\} = \{0, 1, 2, 3, 4, 5, 6\} = Z_7$.

This shows that the broadcast time for \mathcal{A}_2 is at most 3, which is the best possible.

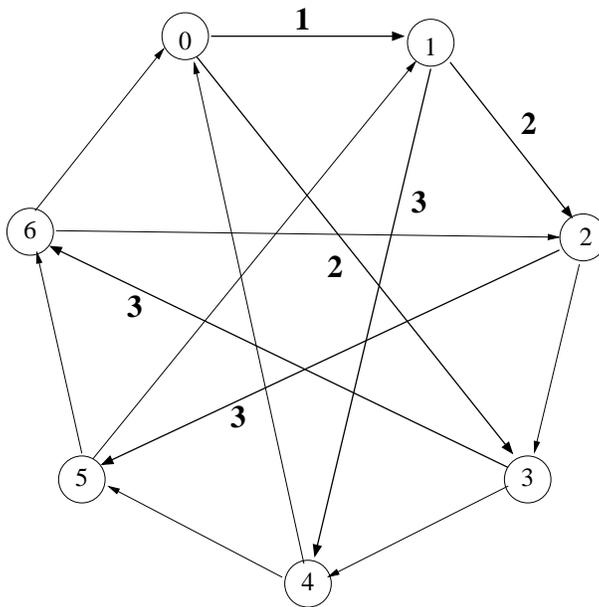


Fig. 4. An optimal $(2, 3)$ broadcast digraph, \mathcal{A}_2 .

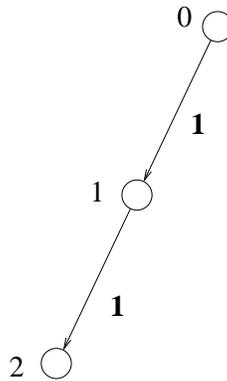


Fig. 5. A spanning broadcast tree of time 2 in the $(1, 2)$ broadcast digraph \mathcal{A}_1 .

We now give the complete proof of this theorem. First denote

$f_i =$ “a spanning broadcast tree of time $i + 1$ in \mathcal{A}_i ”.

Proof by induction.

(i) When $i=1$, f_i is given by Fig. 5.

In the figure, the message is originated from vertex 0. When the time is 1, there exists an arc from vertex 0 to vertex 1 by using generators 1. When the time is 2, vertex 1 sends the message to vertex 2 by using generator 1 again. The bold numbers in Fig. 5 denote the generator being used. As we know, f_1 is a spanning broadcast tree of time 2 in \mathcal{A}_1 . Obviously, $|f_1|=3$, which equals the upper bound of the order of an optimal $(1, 2)$ broadcast digraph.

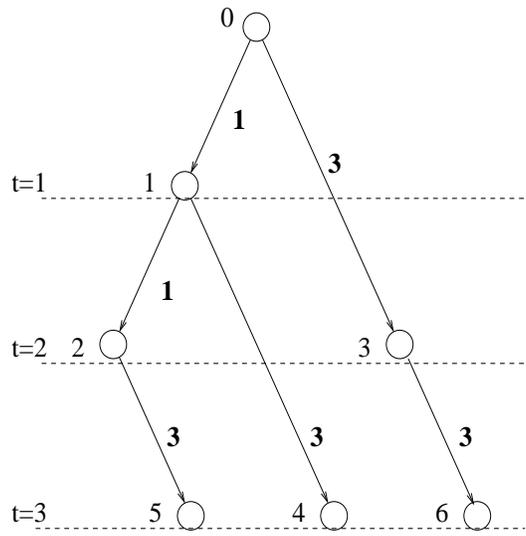


Fig. 6. A spanning broadcast tree of time 3 in the $(2, 3)$ broadcast digraph \mathcal{A}_2 .

(ii) When $i=2$, We show f_2 in Fig. 6.

When the time is 1, the routing scheme is the same as that in f_1 . There is a slight difference at time 2. We add one more arc from vertex 0 to vertex 3 by using a new generator 3. When the time is 3, three arcs are used from vertices 1, 2 and 3 to vertices 4, 5 and 6, respectively. Here, we use generator $g_2=3$ again. Then $|f_2|=4+3=7$. That is $|\mathcal{A}_2|=7$. So the order of \mathcal{A}_2 is equal to the upper bound for an optimal $(2,3)$ broadcast digraph.

(iii) When $i > 1$, we do the following.

In Fig. 7, f_i is the spanning broadcast tree of time $i + 1$ in \mathcal{A}_i while f_{i+1} is a new spanning broadcast tree of time $i + 2$ in \mathcal{A}_{i+1} . We construct f_{i+1} from f_i . The routing scheme is designed by adding one more arc from vertex 0 to vertex $2^{i+1} - 1$ by using the new generator $2^{i+1} - 1$ when the time is $i + 1$. Then, when time is $i + 2$, there are arcs from each vertex from f_i (except 0) to the new $2^{i+1} - 1$ vertices of f'_i by using generator $2^{i+1} - 1$. So we have

$$\begin{aligned} f'_i &= \{1 + 2^{i+1} - 1, 2 + 2^{i+1} - 1, \dots, (2^{i+1} - 1) + (2^{i+1} - 1)\} \\ &= \{2^{i+1}, 2^{i+1} + 1, \dots, 2^{i+2} - 2\} \end{aligned}$$

Then,

$$|f_{i+1}| = 2|f'_i| + 1 = 2(2^{i+1} - 1) + 1 = 2^{i+2} - 1$$

After time $i + 2$, all vertices in \mathcal{A}_{i+1} have received the message. Because the order of \mathcal{A}_{i+1} equals to $2^{i+2} - 1$ is the upper bound and $i + 1$ is the degree of \mathcal{A}_{i+1} , each \mathcal{A}_Δ is optimal $(\Delta, \Delta+1)$ broadcast digraph. \square

We now give another example of an optimal broadcast digraph \mathcal{A}_Δ . Fig. 8 shows an optimal $(3,4)$ broadcast digraph \mathcal{A}_3 with 15 vertices. In that digraph, the three generators are $\{1, 3, 7\}$ and the broadcast time is 4.

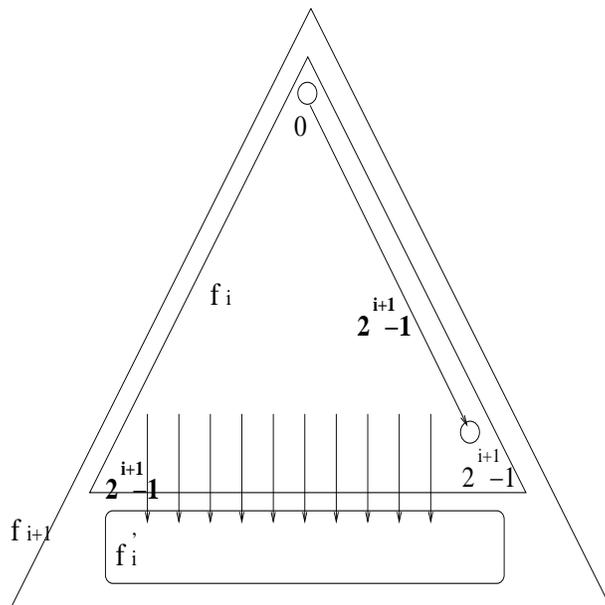


Fig. 7. A spanning broadcast tree of time $i + 2$ in the $(i + 1, i + 2)$ broadcast digraph \mathcal{A}_{i+1} .

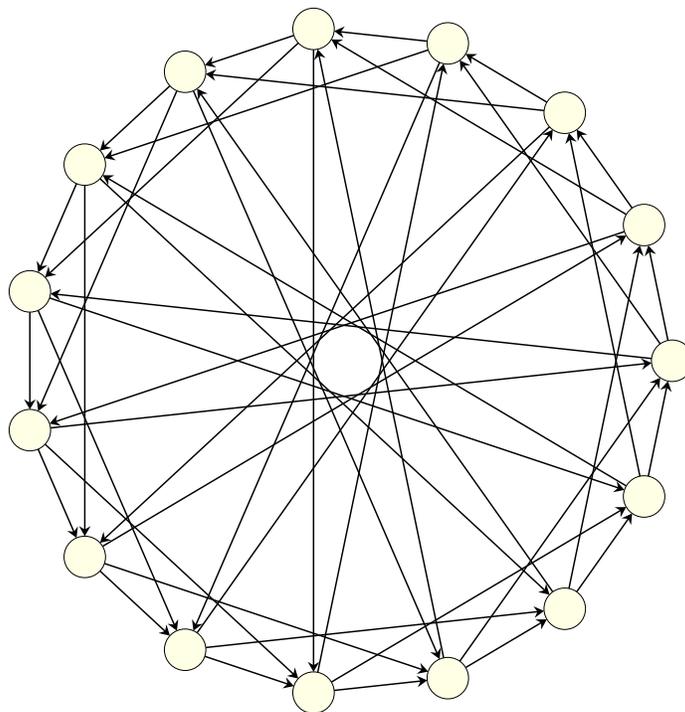


Fig. 8. An optimal $(3, 4)$ broadcast digraph, \mathcal{A}_3 .

5 Conclusion

In this paper we have studied a natural network design problem of finding the largest directed broadcast networks, with constraints of bounded degree Δ and maximum broadcast time T . In doing so, we have provided tables of upper and lower bounds for $2 \leq \Delta, T \leq 10$. We utilized Cayley digraphs (based on semi-direct product of cyclic groups) as the model for trying to increase the envelope on the lower bounds. From the theoretical front, we have proved a new family of optimal $(\Delta, \Delta + 1)$ broadcast digraphs based on Cayley digraphs from the cyclic groups $Z_{2^n - 1}$ with generators $\{1, 3, 7, \dots, 2^{n-1} - 1\}$.

An additional benefit of all of the network examples given in this paper is the fact that they are vertex symmetric. That is, the broadcast time $B(G)$ of a digraph G is equal to the minimum broadcast time $B(G, v)$ from any originator v . There is a lot of work still needed to be done in this area, such as the following:

- (i) Are there better upper bounds than the simple recurrence function we used in Section 3.1?
- (ii) To compute the exact broadcast time, unlike the diameter, is often impossible to calculate as the order and degree increases. Better algorithms, such as those presented in [16] and [11], need to surface.
- (iii) Are there other simple families of optimal (Δ, T) broadcast digraphs?
- (iv) What other types of network constructions can yield good bounded-degree broadcast digraphs? For example, can the graph compounding techniques (e.g. [7]) be of use here?

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