Ain Shams Engineering Journal (2014) 5, 999–1004

Ain Shams University
Ain Shams Engineering Journal
www.elsevier.com/locate/asej
www.sciencedirect.com

ENGINEERING PHYSICS AND MATHEMATICS

Analytical solutions of time-fractional models for homogeneous Gardner equation and non-homogeneous differential equations

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Received 22 January 2014; revised 2 March 2014; accepted 28 March 2014
Available online 24 May 2014

KEYWORDS

Non-homogeneous equation;
Gardner equation;
Buck-master equation;
Fractional derivative;
q-Homotopy Analysis Method

Abstract In this paper, we obtain analytical solutions of homogeneous time-fractional Gardner equation and non-homogeneous time-fractional models (including Buck-master equation) using q-Homotopy Analysis Method (q-HAM). Our work displays the elegant nature of the application of q-HAM not only to solve homogeneous non-linear fractional differential equations but also to solve the non-homogeneous fractional differential equations. The presence of the auxiliary parameter h helps in an effective way to obtain better approximation comparable to exact solutions. The fraction-factor in this method gives it an edge over other existing analytical methods for non-linear differential equations. Comparisons are made upon the existence of exact solutions to these models. The analysis shows that our analytical solutions converge very rapidly to the exact solutions.

1. Introduction

The frequently used analytical methods to solve non-linear differential equations have different restrictions and discretization of variables are involved in numerical techniques which leads to rounding off errors see [19].

The Gardner equation (combined KdV–mKdV or eKdV equation) is a useful model for the description of internal solitary waves in shallow water while the buck-master's equation is used in thin viscous fluid sheet flows and have been widely studied by the various methods see [2,21].

Generally, for the past three decades, fractional calculus has been considered with great importance due to its various applications in physics, fluid flow, chemical physics, control theory of dynamical systems, electrical networks, and so on. The quest of getting accurate methods for solving resulted non-linear models involving fractional order is of utmost concern of many researchers in this field today.

Various analytical methods have been put to use successfully to obtain solutions of classical Gardner equations and Buck-Master equations such as the method of planar dynamical systems approach, exp-function method, bilinear method and extended homo-clinic test approach, fractional variational iteration method (FVIM) and generalized double reduction theorem see [1,3–7,13,16–18]. Recently, a modified HAM called q-Homotopy Analysis Method was introduced in [8], see also [10–12]. It was proven that the presence of fraction

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Peer review under responsibility of Ain Shams University.

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http://dx.doi.org/10.1016/j.asej.2014.03.014

factor in this method enables a fast convergence better than the usual HAM which then makes is more reliable.

To the best of our knowledge, no attempt has been made regarding analytical solutions of time-fractional homogeneous Gardner equation and time fractional non-homogeneous Buck-Master equation using q-Homotopy Analysis Method. In this paper, we consider these equations subject to some appropriate initial conditions. Comparison analysis of our results is carried out with exact solutions when they exist. The numerical results of the problems are presented graphically, obtained using Mathematica 9 and MATLAB R2012b.

2. Preliminaries

This section is devoted to some definitions and some known results. Caputo's fractional derivative is adopted in this work.

Definition 2.1. The Riemann–Liouville's (RL) fractional integral operator of order $\alpha \geqslant 0$, of a function $f \in L^1(a,b)$ is given as

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau)d\tau, \quad t>0, \quad \alpha>0, \tag{1}$$

where Γ is the Gamma function and $I^0 f(t) = f(t)$.

Definition 2.2. The fractional derivative in the Caputo's sense is defined as [20],

$$\mathcal{D}^{\alpha}f(t) = I^{n-\alpha}D^n f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}f^{(n)}(\tau)d\tau, \tag{2}$$

where $n-1 < \alpha \leqslant n, n \in \mathbb{N}, t > 0$.

Lemma 2.1. *Let* $t \in (a,b]$. *Then*

$$\left[I_a^{\alpha}(t-a)^{\beta}\right](t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}(t-a)^{\beta+\alpha}, \qquad \alpha \geqslant 0, \quad \beta > 0. \tag{3}$$

Definition 2.3. The Mittag-Leffler function for two parameters is defined as,

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, \qquad \alpha,\beta,z \in \mathcal{C}, \quad Re(\alpha) \geqslant 0 \tag{4}$$

3. q-Homotopy Analysis Method (q-HAM)

Differential equation of the form

$$N\left[\mathcal{D}_t^{\alpha}u(x,t)\right] - f(x,t) = 0 \tag{5}$$

is considered, where N is a nonlinear operator, $\mathcal{D}_t^{\alpha}$ denotes the Caputo fractional derivative, (x,t) are independent variables, f is a known function and u is an unknown function. To generalize the original Homotopy method, Liao [9] construct what is generally known as the zeroth-order deformation equation

$$\begin{aligned}(1-nq)L(\phi(x,t;q) - u_0(x,t)) \\ = qhH(x,t)\left(N[\mathcal{D}_t^{\alpha}\phi(x,t;q)] - f(x,t)\right),\end{aligned} \tag{6}$$

where $n \geqslant 1, q \in \left[0, \frac{1}{n}\right]$ denotes the so-called embedded parameter, L ia an auxiliary linear operator, $h \neq 0$ is an auxiliary parameter, $H(x,t)$ is a non-zero auxiliary function.

It is clearly seen that when $q = 0$ and $q = \frac{1}{n}$, Eq. (1) becomes

$$\phi(x,t;0) = u_0(x,t) \quad \text{and} \quad \phi\left(x,t;\frac{1}{n}\right) = u(x,t) \tag{7}$$

respectively. So, as q increases from 0 to $\frac{1}{n}$, the solution $\phi(x,t;q)$ varies from the initial guess $u_0(x,t)$ to the solution $u(x,t)$.

If $u_0(x,t), L, h, H(x,t)$ are chosen appropriately, solution $\phi(x,t;q)$ of Eq. (1) exists for $q \in \left[0, \frac{1}{n}\right]$.

Expansion of $\phi(x,t;q)$ in Taylor series gives

$$\phi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty}u_m(x,t)q^m. \tag{8}$$

where

$$u_m(x,t) = \frac{1}{m!}\left.\frac{\partial^m\phi(x,t;q)}{\partial q^m}\right|_{q=0}. \tag{9}$$

Assume that the auxiliary linear operator L, the initial guess u_0, the auxiliary parameter h and $H(x,t)$ are properly chosen such that the series (8) converges at $q = \frac{1}{n}$, then we have

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty}u_m(x,t)\left(\frac{1}{n}\right)^m. \tag{10}$$

Let the vector u_n be define as follows:

$$\vec{u}_n = \{u_0(x,t), u_1(x,t), \ldots, u_n(x,t)\}. \tag{11}$$

Differentiating Eq. (6) m-times with respect to the (embedding) parameter q, then evaluating at $q = 0$ and finally dividing them by $m!$, we have what is known as the mth-order deformation equation (Liao [14,15]) as

$$L\left[u_m(x,t) - \chi_m^* u_{m-1}(x,t)\right] = hH(x,t)\mathcal{R}_m(\vec{u}_{m-1}). \tag{12}$$

with initial conditions

$$u_m^{(k)}(x,0) = 0, \quad k = 0,1,2,\ldots,m-1. \tag{13}$$

where

$$\mathcal{R}_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!}\left.\frac{\partial^{m-1}\left(N[\mathcal{D}_t^{\alpha}\phi(x,t;q)] - f(x,t)\right)}{\partial q^{m-1}}\right|_{q=0} \tag{14}$$

and

$$\chi_m^* = \begin{cases} 0 & m \leqslant 1 \\ n & \text{otherwise,} \end{cases} \tag{15}$$

Remark 1. It should be emphasized that $u_m(x,t)$ for $m \geqslant 1$, is governed by the linear operator (12) with the linear boundary conditions that come from the original problem. The existence of the factor $\left(\frac{1}{n}\right)^m$ gives more chances for better convergence, faster than the solution obtained by the standard Homotopy method. Off course, when $n = 1$, we are in the case of the standard Homotopy method.

4. The time-fractional Garner equation

We consider the time fractional homogeneous time-fractional Garner equation. Let

$$D_t^{\alpha}u + 6(u - \varepsilon^2u^2)u_x + u_{xxx} = 0, \qquad 0 \leqslant x \leqslant 1, \quad t > 0, \quad 0 < \alpha \leqslant 1 \tag{16}$$

subjected to the initial condition

$$u(x,0)=\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{x}{2}\right). \tag{17}$$

The exact solution to this problem, when $\varepsilon=1$ and $\alpha=1$, is

$$u(x,t)=\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{x-t}{2}\right). \tag{18}$$

4.1. Application of q-HAM

In order to use q-HAM to solve the problem considered in (16), we choose the linear operator

$$L[\phi(x,t;q)]=\mathcal{D}_t^{\alpha}\phi(x,t;q) \tag{19}$$

with property that $L[c_1]=0, c_1$ is constant.

We use initial approximation $u_0(x,t)=\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{x}{2}\right)$. We can then define the non-linear operator as

$$N[\phi(x,t;q)]=\mathcal{D}_t^{\alpha}\phi(x,t;q)+6\phi(x,t;q)\phi_x(x,t;q)-6\varepsilon^2(\phi(x,t;q))^2\phi_x(x,t;q)+\phi_{xxx}(x,t;q). \tag{20}$$

We construct the zeroth order deformation equation

$$(1-nq)L[\phi(x,t;q)-u_0(x,t)]=qhH(x,t)N\big[\mathcal{D}_t^{\alpha}\phi(x,t;q)\big]. \tag{21}$$

We choose $H(x,t)=1$ to obtain the mth-order deformation equation to be

$$L\big[u_m(x,t)-\chi_m^* u_{m-1}(x,t)\big]=h\mathcal{R}_m(\vec{u}_{m-1}), \tag{22}$$

with initial condition for $m\geqslant 1, u_m(x,0)=0, \chi_m^*$ is as defined in (15) and

$$\mathcal{R}_m(\vec{u}_{m-1})=\mathcal{D}_t^{\alpha}u_{m-1}+6\sum_{k=0}^{m-1}u_k(u_{m-1-k})_x-6\varepsilon^2\sum_{k=0}^{m-1}\left(\sum_{i=0}^{k}u_i u_{k-i}\right)u_{(m-1-k)x}+u_{(m-1)xxx}. \tag{23}$$

So, the solution to Eq. (16) for $m\geqslant 1$ becomes

$$u_m(x,t)=\chi_m^* u_{m-1}+hI_t^{\alpha}[\mathcal{R}_m(\vec{u}_{m-1})]. \tag{24}$$

We therefore obtain components of the solution using q-HAM successively as follows

$$\begin{aligned}u_1(x,t)&=\chi_1^* u_0+hI^{\alpha}\big[\mathcal{D}_t^{\alpha}u_0+6u_0(u_0)_x-6\epsilon^2u_0^2(u_0)_x+(u_0)_{xxx}\big]\\&=\frac{1}{8}hsech^4\left(\frac{x}{2}\right)\left[1+(4-3\varepsilon^2)\cosh(x)-3(-1+\varepsilon^2)\sinh(x)\right]\\&\quad\times\frac{t^{\alpha}}{\Gamma(1+\alpha)}\end{aligned} \tag{25}$$

$$\begin{aligned}u_2(x,t)&=\chi_2^* u_1+hI^{\alpha}\\&\quad\times\big[\mathcal{D}_t^{\alpha}u_1+6u_0(u_1)_x+6u_1(u_0)_x-6\varepsilon^2u_0^2(u_1)_x-12\varepsilon^2u_0u_1(u_0)_x+(u_1)_{xxx}\big]\\&=\frac{h(n+h)}{8}sech^7\left(\frac{x}{2}\right)\cosh^3\left(\frac{x}{2}\right)\left[1-(3\varepsilon^2-4)\cosh(x)-3(\varepsilon^2-1)\sinh(x)\right]\frac{t^{\alpha}}{\Gamma(1+\alpha)}\\&\quad+\frac{6h^2sech^7\left(\frac{x}{2}\right)}{64}\left[4(\varepsilon^2-1)\cosh\left(\frac{x}{2}\right)+(22-37\varepsilon^2+15\varepsilon^4)\cosh\left(\frac{3x}{2}\right)\right]\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\\&\quad-\frac{6h^2sech^7\left(\frac{x}{2}\right)}{64}\left[4\cosh\left(\frac{5x}{2}\right)+3\varepsilon^4\cosh\left(\frac{5x}{2}\right)-7\varepsilon^2\cosh\left(\frac{5x}{2}\right)\right]\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\\&\quad-\frac{h^2sech^7\left(\frac{x}{2}\right)}{64}\left[(206-204\varepsilon^2)\sinh\left(\frac{x}{2}\right)+(222\varepsilon^2-129-90\varepsilon^4)\sinh\left(\frac{3x}{2}\right)\right]\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\\&\quad+\frac{h^2sech^7\left(\frac{x}{2}\right)}{64}\left[(42\varepsilon^2-25-18\varepsilon^4)\sinh\left(\frac{5x}{2}\right)\right]\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}.\end{aligned} \tag{26}$$

In the same way, $u_m(x,t)$ for $m=3,4,5,\ldots$ can be obtained using Mathematica 9.

Then the series solution expression by q-HAM can be written in the form

$$u(x,t;n;h)\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{x}{2}\right)+\sum_{i=1}^{\infty}u_i(x,t;n;h)\left(\frac{1}{n}\right)^i. \tag{27}$$

Eq. (27) is an appropriate solution to the problem (16) in terms of convergence parameter h and n.

5. The time-fractional non-homogeneous differential equations

We consider the following time-fractional non-homogeneous differential equations of the form

$$IHP_1\begin{cases}\frac{\partial^{\beta}u}{\partial t^{\beta}}+x\frac{\partial u}{\partial x}+\frac{\partial^2 u}{\partial x^2}=2t^{\beta}+2x^2+2, & x\geqslant 0,\quad t\geqslant 0,\quad 0<\beta\leqslant 1,\\ u(x,0)=x^2.\end{cases} \tag{28}$$

The exact solution to this problem is

$$u(x,t)=x^2+\frac{2\Gamma(1+\beta)}{\Gamma(1+2\beta)}t^{2\beta}. \tag{29}$$

Also,

$$IHP_2\begin{cases}\mathcal{D}_t^{\beta}u-(u^4)_{xx}-(u^3)_x=-12x^2e^{4t}-3x^2e^{3t}+xe^t, & 0\leqslant x\leqslant 1,\quad t\geqslant 0,\quad 0<\beta\leqslant 1,\\ u(x,0)=x.\end{cases} \tag{30}$$

The exact solution to the problem (30) when $\beta=1$ is $u=xe^t$.

5.1. Application of q-HAM

5.1.1. Problem $\boldsymbol{IHP_1}$

We follow the same procedure as in first case using the initial approximation to be $u_0(x,t)=x^2$.

We construct the zeroth order deformation equation

$$(1-nq)L[\phi(x,t;q)-u_0(x,t)]=qhH(x,t)\big(N\big[\mathcal{D}_t^{\beta}\phi(x,t;q)\big]-2t^{\beta}-2x^2-2\big). \tag{31}$$

We choose $H(x,t)=1$ to obtain the mth-order deformation equation to be

$$L\big[u_m(x,t)-\chi_m^* u_{m-1}(x,t)\big]=h\mathcal{R}_m(\vec{u}_{m-1}), \tag{32}$$

with initial condition for $m\geqslant 1, u_m(x,0)=0, \chi_m^*$ is as defined in (15) and

$$\mathcal{R}_m(\vec{u}_{m-1})=\mathcal{D}_t^{\beta}u_{m-1}+xu_{(m-1)x}+u_{(m-1)xx}-2t^{\beta}-2x^2-2. \tag{33}$$

So, the solution to Eq. (28) for $m\geqslant 1$ becomes

$$u_m(x,t)=\chi_m^* u_{m-1}+hI_t^{\beta}[\mathcal{R}_m(\vec{u}_{m-1})]. \tag{34}$$

We therefore obtain components of the solution using q-HAM successively as follows

$$\begin{aligned}u_1(x,t)&=\chi_1^* u_0+hI^{\beta}\big[\mathcal{D}_t^{\beta}u_0+xu_{0x}+u_{0xx}-2t^{\beta}-2x^2-2\big]\\&=-\frac{2h\Gamma(1+\beta)}{\Gamma(1+2\beta)}t^{2\beta}\end{aligned} \tag{35}$$

$$\begin{aligned}u_2(x,t)&=\chi_2^* u_1+hI^{\beta}\big[\mathcal{D}_t^{\beta}u_1+xu_{1x}+u_{1xx}-2t^{\beta}-2x^2-2\big]\\&=-\frac{2h(n+h+1)\Gamma(1+\beta)}{\Gamma(1+2\beta)}t^{2\beta}-\frac{2h(x^2+1)}{\Gamma(1+\beta)}t^{\beta}.\end{aligned} \tag{36}$$

In the same way, $u_m(x,t)$ for $m=3,4,5,\ldots$ can be obtained using Mathematica 9.

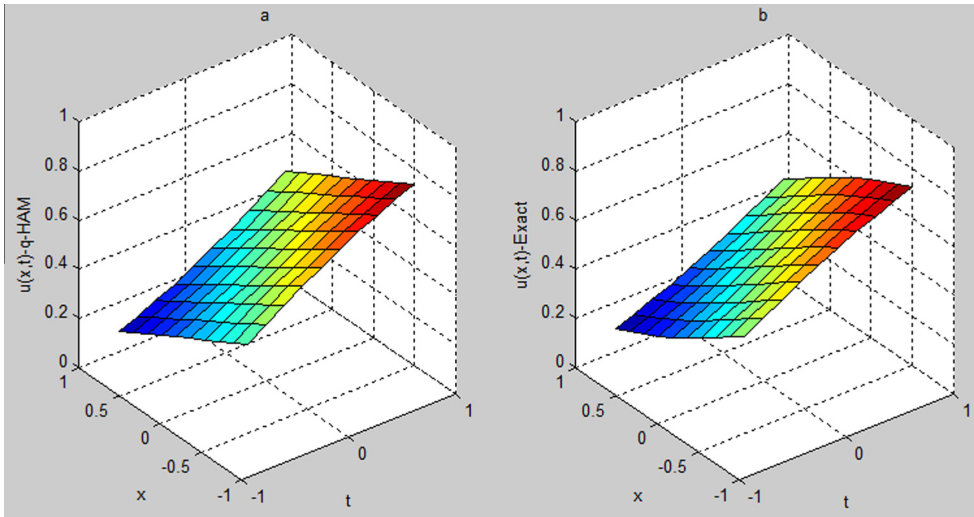

Figure 1 (a) q-HAM solution plot and (b) exact solution plot where $\alpha = 1, h = -0.0465$, and $n = 1$.

Then the series solution expression by q-HAM can be written in the form

$$u(x,t;n;h) = x^2 + \sum_{i=1}^{\infty} u_i(x,t;n;h)\left(\frac{1}{n}\right)^i$$

Eq. (43) is an appropriate solution to the problem (28) in terms of convergence parameter h and n.

Remark 2. Using the first two terms of the q-HAM series in (43), when $n = 1$, we choose appropriate $h = -1$, to get

$$u(x,t) = x^2 + \frac{2\Gamma(1+\beta)}{\Gamma(1+2\beta)} t^{2\beta}. \tag{37}$$

Hence, we obtain exact solution to the non-homogeneous differential Eq. (28) given by just two terms of the series.

*5.1.2. Problem **IHP$_2$**: time-fractional Buck-Master's equation*

We follow the same procedure as in first case using the initial approximation to be $u_0(x,t) = x$.

We construct the zeroth order deformation equation

$$\begin{aligned}&(1-nq)L[\phi(x,t;q) - u_0(x,t)]\\ &\quad = qhH(x,t)\left(N\left[\mathcal{D}_t^{\beta}\phi(x,t;q)\right] + 12x^2e^{4t} + 3x^2e^{3t} - xe^t\right).\end{aligned} \tag{38}$$

We choose $H(x,t) = 1$ to obtain the mth-order deformation equation to be

$$L\left[u_m(x,t) - \chi_m^* u_{m-1}(x,t)\right] = h\mathcal{R}_m(\vec{u}_{m-1}), \tag{39}$$

with initial condition for $m \geqslant 1, u_m(x,0) = 0, \chi_m^*$ is as defined in (15) and

$$\mathcal{R}_m(\vec{u}_{m-1}) = \mathcal{D}_t^{\beta}u_{m-1} - \left(u_{m-1}^4\right)_{xx} - \left(u_{m-1}^3\right)_x + 12x^2e^{4t} + 3x^2e^{3t} - xe^t \tag{40}$$

So, the solution to Eq. (28) for $m \geqslant 1$ becomes

$$u_m(x,t) = \chi_m^* u_{m-1} + hI_t^{\beta}[\mathcal{R}_m(\vec{u}_{m-1})]. \tag{41}$$

We therefore obtain components of the solution using q-HAM successively as follows

$$\begin{aligned}u_1(x,t) &= \chi_1^* u_0 + hI^{\beta}\left[\mathcal{D}_t^{\beta}u_0 - \left(u_0^4\right)_{xx} - \left(u_0^3\right)_x + 12x^2e^{4t} + 3x^2e^{3t} - xe^t\right]\\ &= -\frac{15hx^2t^{\beta}}{\Gamma(1+\beta)} + 12hx^2t^{\beta-1}E_{1,\beta}(4t) + 3x^2ht^{\beta-1}E_{1,\beta}(3t) + xht^{\beta-1}E_{1,\beta}(t).\end{aligned} \tag{42}$$

In the same way, $u_m(x,t)$ for $m = 3,4,5,\ldots$ can be obtained using Mathematica 9.

Then the series solution expression by q-HAM can be written in the form

$$u(x,t;n;h) = x^2 + \sum_{i=1}^{\infty} u_i(x,t;n;h)\left(\frac{1}{n}\right)^i \tag{43}$$

Eq. (43) is an appropriate solution to the problem (30) in terms of convergence parameter h and n.

5.2. Numerical results

We present the numerical results obtained to demonstrate the effectiveness of the modified Homotopy analysis method (q-HAM) presented in this paper. The figures below show the q-HAM and exact solutions of the time-fractional equations considered for different values of n, α and h.

Remark 3. Fig. 1(a) displays the solution plot of the time-fractional Gardner's equation obtained by the q-HAM while Fig. 1(b) displays the exact solutions for the same equation when $\alpha = 1$. It should be noted that only three terms of the q-HAM series solution is used for the plot. The results match comparatively with results of other analytical methods.

Remark 4. Fig. 2(a and b) display the solution plot of the time-fractional non-homogeneous problem in (28) for different values of β. Here, the q-HAM solution and the exact coincide.

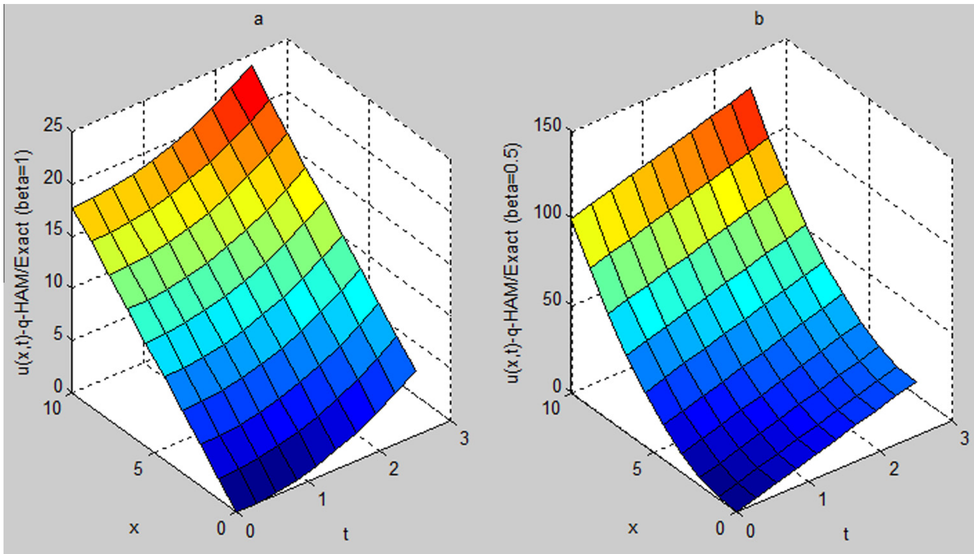

Figure 2 (a) q-HAM/Exact solution plot for $h = -1, n = 1$ and $\beta = 1$ and (b) q-HAM/Exact solution plot for $h = -1, n = 1$ and $\alpha = 0.5$.

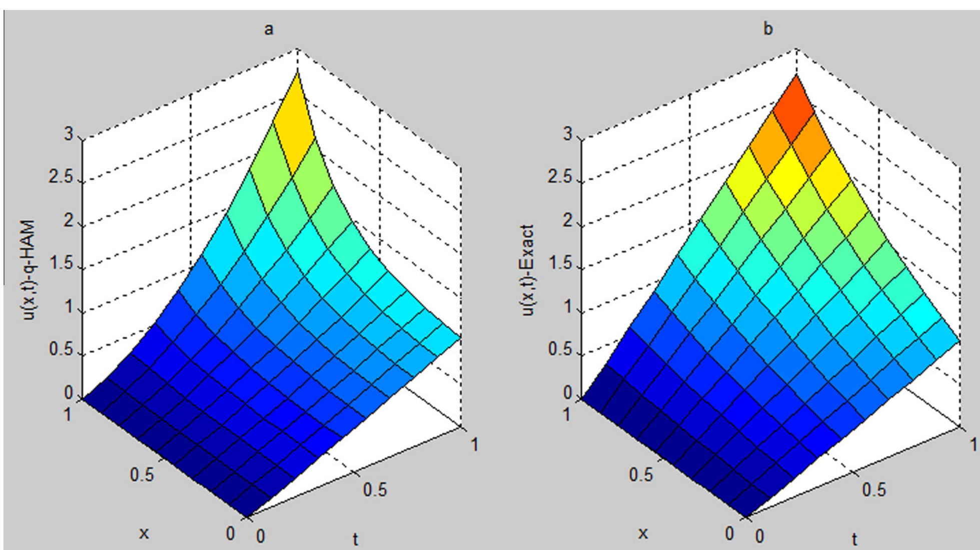

Figure 3 (a) q-HAM solution plot and (b) exact solution plot where $\beta = 1, h = -0.0025$, and $n = 1$.

Remark 5. Fig. 3(a) displays the solution plot of the time-fractional Buck-Master's equation obtained by the q-HAM while Fig. 3(b) displays the exact solutions for the same equation when $\beta = 1$. It should be noted that only two terms of the q-HAM series solution is used for the plot. The results match comparatively with results of other analytical methods.

6. Conclusion

The major achievement of this paper is the demonstration of the successful application of the q-HAM to obtain analytical solutions of the time-fractional homogeneous Gardner's equation and time-fractional non-homogeneous differential equations (including Buck-Master's equation). Our results confirm that the method is really effective for handling solutions of a class of non-linear partial differential equations of fractional order system both homogeneous and non-homogeneous. The comparison made with the exact solutions and other analytical methods, enables us to see clearly the accuracy of q-HAM in the sense that just three and two terms of the series solutions are needed in the case of the homogeneous Gardner equation and the non-homogeneous Buck-Master equation respectively unlike other methods. We are able to obtain exact solution in the case of the problem (28). This method is a potential analytical method for further works on strongly non-linear fractional differential equations both homogeneous and non-homogeneous cases.

Authors' contributions

All authors contributed equally and significantly. All authors read and approved the final manuscript.

Acknowledgement

The financial support received from King Fahd University of Petroleum and Minerals (KFUPM) is gratefully acknowledged.

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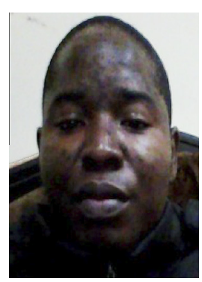

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