Available online at www.sciencedirect.com
SCIENCE DIRECT
ACADEMIC PRESS
Journal of Combinatorial Theory, Series A 104 (2003) 29–48
Journal of Combinatorial Theory Series A
http://www.elsevier.com/locate/jcta

The toric Hilbert scheme of a rank two lattice is smooth and irreducible☆

Diane Maclagan[a] and Rekha R. Thomas[b]

[a] Department of Mathematics, Stanford University, Stanford, CA 94305, USA
[b] Department of Mathematics, University of Washington, Box 354350, Seattle, WA 98195, USA

Received 20 October 2002

Abstract

The toric Hilbert scheme of a lattice $\mathscr{L} \subseteq \mathbb{Z}^n$ is the multigraded Hilbert scheme parameterizing all homogeneous ideals I in $S = \mathbf{k}[x_1, \ldots, x_n]$ such that the Hilbert function of the quotient S/I has value one for every g in the grading monoid $G^+ = \mathbb{N}^n/\mathscr{L}$. In this paper we show that if $\mathscr{L}$ is two-dimensional, then the toric Hilbert scheme of $\mathscr{L}$ is smooth and irreducible. This result is false for lattices of dimension three and higher as the toric Hilbert scheme of a rank three lattice can be reducible.

Keywords: toric Hilbert scheme; Connectedness; Codimension two; Triangulations

1. Introduction

The main result of this paper is the following structure theorem.

Theorem 1.1. *Let $\mathscr{L}$ be a two-dimensional lattice contained in $\mathbb{Z}^n$. Then the toric Hilbert scheme $H_{\mathscr{L}}$ is smooth and irreducible.*

Consider a sublattice $\mathscr{L} \subseteq \mathbb{Z}^n$ of dimension (rank) r and the abelian group $G = \mathbb{Z}^n/\mathscr{L}$. Let $S := \mathbf{k}[x_1, \ldots, x_n]$ be the polynomial ring in n variables over an arbitrary infinite field $\mathbf{k}$. We grade S by setting $\deg(x_i) = e_i + \mathscr{L}$ for $i = 1, \ldots, n$

☆ Research partially supported by an AWM Mentoring Travel Grant and NSF Grant DMS-0100141.
E-mail addresses: maclagan@math.stanford.edu (D. Maclagan), thomas@math.washington.edu (R.R. Thomas).

0097-3165/03/$ - see front matter
doi:10.1016/S0097-3165(03)00117-1

where $e_1, \dots, e_n$ are the unit vectors of $\mathbb{Z}^n$. The set of possible degrees under this grading is $G^+ := \mathbb{N}^n/\mathscr{L}$.

Definition 1.2. A homogeneous ideal $I \subseteq S$ is *$\mathscr{L}$-graded* if the value of its Hilbert function $\dim_{\mathbf{k}}((S/I)_g) = 1$ for all $g \in G^+$.

The notion of $\mathscr{L}$-graded ideals extends the notion of A-graded ideals, first introduced in [1] and further developed in [17,18, Chapter 10]. In the A-graded situation, $\mathscr{L} = \ker_{\mathbb{Z}}(A) := \{u \in \mathbb{Z}^n : Au = 0\}$ where A is an integer matrix, and an ideal $I \subseteq S$ is *A-graded* if and only if it is $\ker_{\mathbb{Z}}(A)$-graded in the sense of Definition 1.2. The *toric Hilbert scheme H_A* [15,17] parameterizes all A-graded ideals for a given A. Haiman and Sturmfels [7] have introduced *multigraded Hilbert schemes* which provide a uniform setting for many known Hilbert schemes, including H_A. The multigraded Hilbert scheme is a quasi-projective scheme which parameterizes all ideals in a polynomial ring that are homogeneous with respect to grading by a fixed abelian group and whose quotients have a fixed Hilbert function. In the special case where this Hilbert function takes value one for all elements in G^+, Haiman and Sturmfels call the resulting multigraded Hilbert scheme a toric Hilbert scheme [7, Section 5].

Definition 1.3. The *toric Hilbert scheme* of the lattice $\mathscr{L}$, denoted by $H_{\mathscr{L}}$, is the multigraded Hilbert scheme that parameterizes all $\mathscr{L}$-graded ideals in S.

Two special cases of Theorem 1.1 are in the literature. When the two-dimensional lattice $\mathscr{L}$ equals $\ker_{\mathbb{Z}}(A)$ (for an integer matrix A of corank two), Theorem 1.1 was proved by Gasharov and Peeva [6] and Peeva and Stillman [15]. In this case, $G = \mathbb{Z}^n/\mathscr{L}$ is a free abelian group. At the other extreme, when $\mathscr{L}$ is a two-dimensional lattice in $\mathbb{Z}^2$ then G is a finite abelian group, and the lattice gives an embedding of G into $GL(2)$. In this case the toric Hilbert scheme is Nakamura's G-Hilbert scheme. This can be seen by comparing Reid's functorial description [4, Section 4.1] with that of the toric Hilbert scheme given in [7]. The fact that the G-Hilbert scheme is smooth and irreducible for abelian subgroups of $GL(2)$ is due to Kidoh [9].

Although the G-Hilbert scheme is smooth and irreducible for abelian subgroups of $SL(3)$ [3,4] there is no hope for a further common generalization, as [18, Theorem 10.4] shows that the toric Hilbert scheme of a rank three lattice can be reducible.

Theorem 1.1 firstly is a common generalization of the above results to all lattice ideals of codimension two. The two special cases lie at the two extreme ends of the spectrum for which Theorem 1.1 holds. Further, the proofs in this paper are much more combinatorial and directly exploit the geometry and combinatorics of lattice ideals of codimension two. One byproduct is simpler proofs of the two special cases. We also establish new results about the structure of codimension two lattice ideals, including some (e.g. Corollary 3.21) which were known for toric ideals of codimension two.

For an r-dimensional lattice $\mathscr{L} \subseteq \mathbb{Z}^n$ and a vector $u \in \mathscr{L}$, we write $u = u^+ - u^-$ where $u^+, u^- \in \mathbb{N}^n$ are defined by setting $(u^+)_i = u_i$ if $u_i > 0$, and $(u^+)_i = 0$ otherwise, and $u^- = (-u)^+$. If $u \in \mathbb{N}^n$ we write x^u for the monomial $\prod_{i=1}^n x_i^{u_i}$. The *lattice ideal* of $\mathscr{L}$ is the $(n-r)$-dimensional binomial ideal

$$I_{\mathscr{L}} = \langle x^{u^+} - x^{u^-} : u = u^+ - u^- \in \mathscr{L} \rangle \subseteq S.$$

If $\mathscr{L} = \ker_{\mathbb{Z}}(A)$ for some $A \in \mathbb{Z}^{d \times n}$ of corank $n - d = r$, then $\mathscr{L}$ is saturated and $I_{\mathscr{L}}$ is the toric ideal of A, denoted by I_A. The ideal $I_{\mathscr{L}}$ is a distinguished point on the toric Hilbert scheme $H_{\mathscr{L}}$. It lies on an irreducible component of $H_{\mathscr{L}}$ called the *coherent component* [7, p. 30]. The algebraic torus $(\mathbf{k}^*)^n$ acts on $\mathscr{L}$-graded ideals by scaling variables. This action translates into an action on $H_{\mathscr{L}}$, and the *coherent component* of $H_{\mathscr{L}}$ is the closure of the $(\mathbf{k}^*)^n$-orbit of $I_{\mathscr{L}}$. An ideal $J \in H_{\mathscr{L}}$ lies on the coherent component, and is thus called *coherent*, if there is some weight vector $w \in \mathbb{Z}^n$ and a $\lambda \in (\mathbf{k}^*)^n$ such that $J = \lambda \, \mathrm{in}_w(I_{\mathscr{L}})$, where $\mathrm{in}_w(I_{\mathscr{L}})$ is the *initial ideal* of $I_{\mathscr{L}}$ with respect to w. Note that $I_{\mathscr{L}} = \mathrm{in}_0(I_{\mathscr{L}})$ and hence is coherent. For an arbitrary lattice $\mathscr{L}$, $H_{\mathscr{L}}$ could have other components. Theorem 1.1 asserts that when $\mathscr{L}$ is two dimensional, the coherent component is the unique component of $H_{\mathscr{L}}$ and that it is smooth.

In Section 2 we establish some general results about $\mathscr{L}$-graded ideals for lattices $\mathscr{L}$ of arbitrary dimension, while Sections 3 and 4 focus on two-dimensional lattices. Our proof of Theorem 1.1 is in two parts. In Section 3 we show that all the monomial ideals on $H_{\mathscr{L}}$ are coherent (Theorem 3.1). Hence all the fixed points of $H_{\mathscr{L}}$ under the action of $(\mathbf{k}^*)^n$ lie on the coherent component. Gröbner basis theory implies that every irreducible component of $H_{\mathscr{L}}$ contains a monomial ideal. Thus Theorem 3.1 implies that every irreducible component of $H_{\mathscr{L}}$ intersects the coherent component at a monomial ideal and so $H_{\mathscr{L}}$ is connected. In Section 4 we show that the ideals parameterized by the torus fixed curves between two monomial ideals in $H_{\mathscr{L}}$ are also coherent (Theorem 4.4) which lets us prove that the Zariski tangent space at each monomial $\mathscr{L}$-graded ideal is two-dimensional (Lemma 4.5). As the coherent component is itself two-dimensional, it is therefore smooth. Since $H_{\mathscr{L}}$ is both connected and smooth, it is irreducible.

2. General lattice lemmas

Before we restrict to the case where $\dim(\mathscr{L})$, the dimension of $\mathscr{L}$, equals two, we establish some basic results for monomial $\mathscr{L}$-graded ideals when $\dim(\mathscr{L}) = r \leqslant n$.

Definition 2.1. A binomial $x^u - x^v \in I_{\mathscr{L}}$ is a *Graver binomial* if there is no other binomial $x^{u'} - x^{v'} \in I_{\mathscr{L}}$ such that $x^{u'}$ divides x^u and $x^{v'}$ divides x^v. The set of Graver binomials of $I_{\mathscr{L}}$ is called the *Graver basis* of $I_{\mathscr{L}}$ and is denoted by $Gr_{\mathscr{L}}$.

Example 2.2. For the two-dimensional lattice $\mathscr{L} \subset \mathbb{Z}^4$ generated by $(2,0,-2,-2)$ and $(0,1,1,0)$, we have $Gr_{\mathscr{L}} = \{x_2x_3 - 1, x_3^2x_4^2 - x_1^2, x_1^2x_2^2 - x_4^2, x_1^2x_2 - x_3x_4^2\}$. Note that the first two binomials come from the two generators of the lattice.

The Graver basis of $I_{\mathscr{L}}$ is finite and [7, Proposition 5.2] shows that it gives rise to a finite set of determinantal equations that cut out $H_{\mathscr{L}}$. For $u \in \mathbb{Z}^n$, let $\mathrm{supp}(u) = \{i : u_i \neq 0\}$ be the support of u and for a monomial x^u, we define its support $\mathrm{supp}(x^u) := \mathrm{supp}(u)$. If $x^u - x^v \in Gr_{\mathscr{L}}$, then x^u and x^v have disjoint supports. We repeatedly use the following lemma from [12,15].

Lemma 2.3 (Maclagan and Thomas [12, Lemma 2.4]; Peeva and Stillman [15, Lemma 2.2]). *If I is an $\mathscr{L}$-graded ideal with $x^u - cx^v$ ($c \in \mathbf{k}$, possibly zero) in some reduced Gröbner basis for I, then $x^u - x^v$ is a Graver binomial. If $c = 0$, then we here assume that x^v is a monomial not in I of the same $\mathscr{L}$-degree as x^u.*

Although in [12] it is assumed that $\mathscr{L} = \ker_{\mathbb{Z}}(A)$ and $\mathscr{L} \cap \mathbb{N}^n = \{0\}$, the proof there is valid for general lattices. The same is true for other results quoted later from [12].

Fix a matrix $B \in \mathbb{Z}^{n \times r}$ whose columns form a $\mathbb{Z}$-basis for $\mathscr{L}$. This implies that $\mathscr{L} = \{Bz : z \in \mathbb{Z}^r\}$ and the map

$$\phi : \mathbb{Z}^r \to \mathscr{L} \quad \text{given by } z \mapsto Bz \tag{1}$$

is bijective. Let b_i denote the ith row of B. The vector configuration $\mathscr{B} = \{b_1, \ldots, b_n\} \subset \mathbb{R}^r$ is called the *Gale diagram* of $\mathscr{L}$. For a subset $\tau \subseteq [n] := \{1, \ldots, n\}$ let B_τ be the submatrix of B whose rows are indexed by τ and $\mathscr{B}_\tau := \{b_i : i \in \tau\}$. Write $\mathrm{pos}(\mathscr{B}_\tau)$ for the cone $\{xB_\tau : x \geqslant 0, x \in \mathbb{R}^{|\tau|}\} \subseteq \mathbb{R}^r$. If $\mathrm{pos}(\mathscr{B}) = \mathbb{R}^r$, $\mathscr{B}$ is said to be *totally cyclic*. If $r = 2$ and $\mathscr{B}$ is totally cyclic, we set $b_{n+1} := b_1$.

Example 2.4. Let $\mathscr{L}$ be the lattice from Example 2.2. Then the Gale diagram of $\mathscr{L}$ is $\mathscr{B} = \{(2,0), (0,1), (-2,1), (-2,0)\}$. This is illustrated in Fig. 1.

Remark 2.5. We may assume that no $b_i = 0$, and that for $i \neq j$, $b_i \neq mb_j$ for any $m \in \mathbb{Z}_+$. If some $b_i = 0$ then $l_i = 0$ for all $l \in \mathscr{L}$ and x_i would not appear in any generator of any $\mathscr{L}$-graded ideal. This means that there would be a bijection between the set of $\mathscr{L}$-graded ideals in S and the set of $\mathscr{L}'$-graded ideals in $\mathbf{k}[x_1, \ldots, \widehat{x_i}, \ldots, x_n]$, where $\mathscr{L}'$ is the projection of $\mathscr{L}$ which removes the ith coordinate. Similarly if $b_i = mb_j$ for $m \in \mathbb{Z}_+$, the map which takes every occurrence of x_j to $x_i^m x_j$ would give a bijection between $\mathscr{L}'$-graded ideals and $\mathscr{L}$-graded ideals. These bijections would give rise to an isomorphism between $H_{\mathscr{L}}$ and $H_{\mathscr{L}'}$, and so it suffices to prove Theorem 1.1 for the smaller lattice $\mathscr{L}'$.

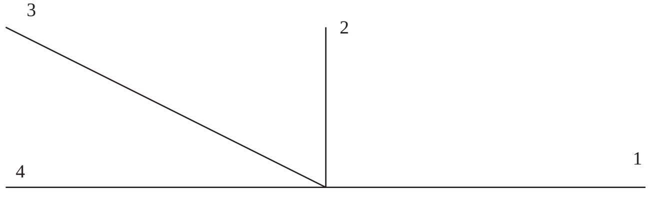

Fig. 1. The configuration $\mathscr{B}$ from Example 2.4.

We say that $\mathrm{pos}(\mathcal{B}_\tau)$, or τ, is a q-simplex if $|\tau| = q$ and $\mathrm{pos}(\mathcal{B}_\tau)$ is q-dimensional. A *triangulation* T of $\mathcal{B}$ is a collection of r-simplices such that (i) for $\tau, \tau' \in T$, $\mathrm{pos}(\mathcal{B}_{\tau \cap \tau'}) = \mathrm{pos}(\mathcal{B}_\tau) \cap \mathrm{pos}(\mathcal{B}_{\tau'})$, and (ii) $\mathrm{pos}(\mathcal{B}) = \bigcup_{\tau \in T} \mathrm{pos}(\mathcal{B}_\tau)$. To be completely accurate, we must complete T to a simplicial complex on $[n]$ by adding to T all subsets of $\tau \in T$. Depending on the context, a simplex $\tau \in T$ will be either the set $\tau \subseteq [n]$ or the cone $\mathrm{pos}(\mathcal{B}_\tau) \subseteq \mathrm{pos}(\mathcal{B})$.

Recall that all monomial prime ideals in S are of the form $P_\sigma := \langle x_j : j \notin \sigma \rangle$ for a set $\sigma \subseteq [n]$. We denote by $\Delta(I)$ the simplicial complex whose Stanley-Reisner ideal is $\mathrm{rad}(I)$. The following lemma is a mild extension of a special case of [18, Theorem 10.10].

Lemma 2.6. *Let I be a monomial $\mathcal{L}$-graded ideal, for a lattice $\mathcal{L}$ with* $\dim(\mathcal{L}) = r < n$. *Let* $\mathrm{rad}(I)$ *be its radical and* $\mathrm{rad}(I) = \bigcap_{\sigma \in \Delta(I)} P_\sigma$ *be the unique prime decomposition of* $\mathrm{rad}(I)$. *Fix $A = [a_1, \dots, a_n] \in \mathbb{Z}^{(n-r)\times n}$ such that $AB = 0$ and rank$(A) = n - r$, and let $\mathcal{A} = \{a_1, \dots, a_n\}$. Then $\Delta(I)$ is a triangulation of $\mathcal{A}$.*

Proof. We first note that since $I_{\mathcal{L}}$ is $(n-r)$-dimensional, the same is true for I, so if $\sigma \in \Delta(I)$ we have $|\sigma| \leqslant n - r$. We will show that $\Delta(I)$ is a triangulation of $\mathcal{A}$, by showing firstly that the interiors of any two simplices in $\Delta(I)$ do not intersect in this embedding, and secondly that each point in $\mathrm{pos}(\mathcal{A})$ is covered by one of these simplices. This will also show that $|\sigma| = n - r$ for all $\sigma \in \Delta(I)$.

Suppose that σ and τ are two simplices contained in simplices in $\Delta(I)$ such that their relative interiors intersect. This means that there is some vector $c \in \mathrm{pos}(\mathcal{A})$ with $c = \sum_{i \in \sigma} \lambda_i a_i = \sum_{j \in \tau} \mu_j a_j$, where we may take λ_i and μ_j to be positive integers. Then $\lambda - \mu \in \ker_{\mathbb{Z}}(A)$, and so there is some multiple $t(\lambda - \mu) \in \mathcal{L}$. Let $t(\lambda - \mu) = u - v$, where $\mathrm{supp}(u) \cap \mathrm{supp}(v) = \emptyset$. Then since $\mathrm{supp}(u) \subseteq \sigma$, we know that $x^u \notin I$. Similarly, $\mathrm{supp}(v) \subseteq \tau$, so $x^v \notin I$. But this means there are two standard monomials of I of the same $\mathcal{L}$-degree, contradicting the fact that I is an $\mathcal{L}$-graded ideal. This shows that c does not exist, and so the relative interiors of different simplices in $\Delta(I)$ do not intersect. Note that we made no assumption on the dimension of σ and τ, so in particular they need not be r-dimensional.

We now show that each point $c \in \{u : u = \sum_i n_i a_i,\ n_i \in \mathbb{Z}_+\}$ is covered by $\mathrm{pos}(\mathcal{A}_\sigma)$ for some $\sigma \in \Delta(I)$. Grade the polynomial ring S by setting $\deg(x_i) = a_i$. The $\mathcal{L}$-grading of S refines this grading. It suffices to show that there is some monomial x^u of A-degree tc for some $t > 0$ with $x^u \notin \mathrm{rad}(I)$. This will imply that $\mathrm{supp}(u) \subseteq \sigma$ for some $\sigma \in \Delta(I)$, and so the point c will be contained in a simplex of $\Delta(I)$.

Consider the $\mathbf{k}$-algebra $U = \bigoplus_{t \geqslant 0} S_{tc}$, where S_{tc} is the coarsely-graded part of S of degree tc. We claim that U is a finitely generated algebra, generated by a finite number of monomials. To see this consider the sequence of ideals $P_t = \langle x^u : Au = tc \rangle$. By Maclagan [11, Theorem 1.1] only a finite number of the P_t are not contained in other ideals of this form. Let $x^{u_1}, \dots, x^{u_s}$ be the monomial generators of these inclusion-maximal P_t. If x^u is any other monomial of degree tc for some $t > 0$, then x^u is divisible by one of the x^{u_i}. Since x^{u-u_i} also has degree a multiple of c, we see that in fact we can write x^u as a product of some (possibly repeated) of the x^{u_i}, and so

$x^{u_1}, \ldots, x^{u_s}$ generate U as an algebra. Let $J = \bigoplus_{t \geq 0} I_{tc}$. If all of the x^{u_i} lay in the radical of I, then there would be an N for which $x^{Nu_i} \in I$ for all i and hence $x^{Nu_i} \in J$ for all i. But this would mean that U/J was a finite-dimensional algebra, which contradicts the fact that for all $t > 0$ there is a standard monomial for I of degree tc. Indeed, since the $\mathscr{L}$-grading refines the A-grading, there may well be more than one standard monomial of each degree tc. From this contradiction we can conclude that there is a generator x^{u_j} of U with $x^{u_j} \notin \mathrm{rad}(I)$, which in turn implies that the simplices of $\Delta(I)$ cover $\mathrm{pos}(\mathscr{A})$. $\square$

Example 2.7. For the lattice of Example 2.2, we have

$$B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ -2 & 1 \\ -2 & 0 \end{pmatrix}, \text{ so we can take } A = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & -1 & 1 & -1 \end{pmatrix}.$$

Note that $\ker_{\mathbb{Z}}(A)$ is the *saturation* of $\mathscr{L}$; the lattice generated by the vectors $(1,0,-1,-1)$ and $(0,1,1,0)$. With this choice of A we have $\mathscr{A} = \{(1,0),(-1,-1),(1,1),(0,-1)\}$.

There are four monomial $\mathscr{L}$-graded ideals. We list them in the table below, each with its radical and the prime decomposition of the radical.

Monomial $\mathscr{L}$-graded ideal	radical	Prime decomposition of radical
$\langle x_2x_3, x_1^2 \rangle$	$\langle x_2x_3, x_1 \rangle$	$\langle x_1, x_2 \rangle \cap \langle x_1, x_3 \rangle$
$\langle x_2x_3, x_1^2x_2, x_3^2x_4^2 \rangle$	$\langle x_1x_2, x_2x_3, x_3x_4 \rangle$	$\langle x_1, x_3 \rangle \cap \langle x_2, x_3 \rangle \cap \langle x_2, x_4 \rangle$
$\langle x_2x_3, x_1^2x_2^2, x_3x_4^2 \rangle$	$\langle x_1x_2, x_2x_3, x_3x_4 \rangle$	$\langle x_1, x_3 \rangle \cap \langle x_2, x_3 \rangle \cap \langle x_2, x_4 \rangle$
$\langle x_2x_3, x_4^2 \rangle$	$\langle x_2x_3, x_4 \rangle$	$\langle x_2, x_4 \rangle \cap \langle x_3, x_4 \rangle$

Note that the second and third ideals have the same radical. The three radicals correspond to the triangulations of $\mathrm{pos}(\mathscr{A})$ shown in Fig. 2 in order from left to right.

Remark 2.8. The proof of Lemma 2.6 shows that each $\sigma \in \Delta(I)$ has cardinality $n - r$. Since the ideals P_σ, $\sigma \in \Delta(I)$ are precisely the minimal primes of I, this shows that I is

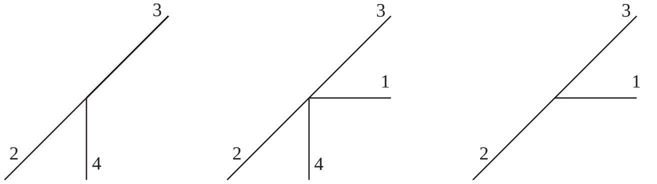

Fig. 2. The triangulations of $\mathscr{A}$ corresponding to $\mathscr{L}$-graded ideals.

an equidimensional ideal of dimension $n - r$. This is also true if $r = n$, as then I is a zero-dimensional ideal.

Let P_σ be a minimal prime of I, so $|\sigma| = n - r$. We can localize I at P_σ to get an ideal I_σ. We identify I_σ with the projection $\pi_\sigma(I)$ where π_σ is the map:

$$\pi_\sigma : S \to S_\sigma := \mathbf{k}[x_i : i \notin \sigma],$$

$$x_j \mapsto \begin{cases} x_j & \text{if } j \notin \sigma, \\ 1 & \text{if } j \in \sigma. \end{cases}$$

The ideal I_σ is also the image under π_σ of the P_σ-primary component of I from an irredundant primary decomposition of I. Similarly, let $\widehat{\pi}_\sigma$ be the map that projects $u \in \mathbb{Z}^n$ to the $|\overline{\sigma}|$-vector obtained by restricting u to its coordinates indexed by $\overline{\sigma}$, where $\overline{\sigma} = [n] \backslash \sigma$.

Definition 2.9. A homogeneous ideal $J \subseteq S$ is *weakly $\mathscr{L}$-graded* if its Hilbert function

$$\dim_{\mathbf{k}}((S/J)_g) \leqslant 1 \quad \text{for all } g \in G^+.$$

We recall [12, Lemma 2.6] that a monomial ideal I is weakly $\mathscr{L}$-graded if and only if for each $x^u - x^v \in Gr_{\mathscr{L}}$, at least one of x^u or x^v lies in I.

Lemma 2.10. *Let P_σ be a minimal prime of a monomial $\mathscr{L}$-graded ideal I.*

(1) *For each $l \in \mathscr{L}$ there is some $i \in \overline{\sigma}$ for which $l_i \neq 0$.*

(2) *The localized ideal $I_\sigma \subset S_\sigma$ is weakly $\mathscr{L}_\sigma$-graded, where $\mathscr{L}_\sigma = \widehat{\pi}_\sigma(\mathscr{L}) \subseteq \mathbb{Z}^{|\overline{\sigma}|}$. This implies that I_σ is an artinian monomial ideal.*

(3) *If $I = \mathrm{in}_w(I_{\mathscr{L}})$, where $w_i = 0$ for $i \in \sigma$, then $I_\sigma = \mathrm{in}_{\widehat{\pi}_\sigma(w)}(I_{\mathscr{L}_\sigma})$. In particular, I_σ is $\mathscr{L}_\sigma$-graded and coherent.*

Proof. (1) Since P_σ is a minimal prime of I, if $x^u \notin P_\sigma$ then $x^u \notin I$. If $l \in \mathscr{L}$, then x^{l^+} and x^{l^-} have the same $\mathscr{L}$-degree, so at most one of them can be a standard monomial of I. This means that at least one of x^{l^+} and x^{l^-} must lie in P_σ, and so there is some $i \in \overline{\sigma}$ with $l_i \neq 0$.

(2) Let x^u and x^v be two monomials in S_σ with $u - v \in \mathscr{L}_\sigma$. Then there exist u', v' with $\mathrm{supp}(u'), \mathrm{supp}(v') \subseteq \sigma$ and $(u + u') - (v + v') \in \mathscr{L}$. But this means that either $x^{u+u'} \in I$ or $x^{v+v'} \in I$, and thus one of x^u or x^v must lie in I_σ. This shows that I_σ has at most one standard monomial of each $\mathscr{L}_\sigma$-degree.

(3) By the previous part we know that I_σ is a weakly $\mathscr{L}_\sigma$-graded ideal, so it suffices to show that it is contained in the $\mathscr{L}_\sigma$-graded ideal $\mathrm{in}_{\widehat{\pi}_\sigma(w)}(I_{\mathscr{L}_\sigma})$. Let x^u be a generator of I_σ. Then there is a $u' \in \mathbb{N}^n$ with $\mathrm{supp}(u') \subseteq \sigma$ and $x^{u+u'} \in I$. Let x^v be the standard monomial of I in the same degree as $x^{u+u'}$. Then $w(u + u' - v) > 0$, so $\widehat{\pi}_\sigma(w)(u - \widehat{\pi}_\sigma(v)) > 0$, and so $x^u \in \mathrm{in}_{\widehat{\pi}_\sigma(w)}(I_{\mathscr{L}_\sigma})$. $\square$

Remark 2.11. A stronger result than part two of Lemma 2.10 is true, as I_σ is in fact $\mathscr{L}_\sigma$-graded. The proof is longer, however, so we prove it in the next section only in the case where $\mathscr{L}$ is two dimensional.

3. Monomial ideals

In the rest of this paper we assume that $\dim(\mathscr{L}) = 2$. In this section we show that all monomial ideals in the toric Hilbert scheme $H_{\mathscr{L}}$ are coherent.

Theorem 3.1. *Let I be a monomial $\mathscr{L}$-graded ideal where $\mathscr{L}$ is a two-dimensional sublattice of $\mathbb{Z}^n$. Then $I = \text{in}_w(I_{\mathscr{L}})$ for some $w \in \mathbb{Z}^n$.*

The proof of Theorem 3.1 is established in several steps. We first show that the localization of a monomial $\mathscr{L}$-graded ideal I at a minimal prime P_σ is coherent in the sense that it is an initial ideal of $I_{\mathscr{L}_\sigma}$. One of these coherent localizations is special in the sense that there is no other monomial $\mathscr{L}$-graded ideal with the same localization. This localization determines I. The coherence of this special localization implies that it is the localization of a monomial initial ideal of $I_{\mathscr{L}}$. Thus I is this initial ideal.

We first recall the result of Lee [10] that all triangulations of n vectors in $\mathbb{R}^{n-2}$ are *regular*. Recall [18, Chapter 8] that a triangulation Δ of $\{p_1, \dots, p_n\} \subseteq \mathbb{R}^{n-2}$ is regular if there exists a cost-vector $w \in \mathbb{R}^n$ such that $\sigma \in \Delta$ if and only if there exists an $x \in \mathbb{Z}^{n-2}$ for which $p_i x = w_i$ for $i \in \sigma$, and $p_i x < w_i$ for $i \notin \sigma$. If such a w exists we denote Δ as Δ_w. Applying this definition to triangulations of $\mathscr{A}$, we get that $\Delta = \Delta_w$ if and only if $\sigma \in \Delta$ exactly when $wB = (w - xA)B \in \text{pos}(\mathscr{B}_{\overline{\sigma}})$. (Recall that $AB = 0$.) Thus for a $w \in \mathbb{Z}^n$ such that $wB \in \text{pos}(\mathscr{B})$, the maximal simplices of the regular triangulation Δ_w of $\mathscr{A}$ are the $(n-2)$-simplices $\sigma \subset [n]$ such that $wB \in \text{pos}(\mathscr{B}_{\overline{\sigma}})$. This means that there is a bijection between the regular triangulations of $\mathscr{A}$ and the *chambers* of $\mathscr{B}$.

Definition 3.2. Given a collection $\mathscr{P}$ of vectors in $\mathbb{R}^n$, the *chamber complex* $\Sigma(\mathscr{P})$ of $\mathscr{P}$ is the polyhedral fan obtained by intersecting all the simplices in $\mathscr{P}$. If $n = 2$, then the chamber complex is the collection of cones formed by taking the positive hull of adjacent vectors in $\mathscr{P}$. We identify chambers in the chamber complex with the collection of maximal simplices (of $\mathscr{P}$) which contain them. If $\mathscr{L}$ is a lattice with generating matrix B, we denote by $\Sigma(\mathscr{L})$ the chamber complex $\Sigma(\mathscr{B})$.

Example 3.3. For $\mathscr{B} = \{(2,0), (0,1), (-2,1), (-2,0)\}$, the chamber complex is the collection of three cones shown in Fig. 1.

By Lee's result when $\mathscr{L}$ is two dimensional, Lemma 2.6 reduces to:

Lemma 3.4. *Let I be a monomial $\mathscr{L}$-graded ideal where $\dim(\mathscr{L}) = 2$. Then the collection $\{\overline{\sigma} : \sigma \in \Delta(I)\} = \{\overline{\sigma} : P_\sigma$ is a minimal prime of $I\}$ is a chamber of $\mathscr{B}$.*

Definition 3.5. If the monomial $\mathscr{L}$-graded ideal I maps to the chamber $\text{pos}(b_i, b_j)$ then the $(n-2)$-simplex $\sigma = [n]\backslash\{i,j\}$ is called the *special simplex* of I, and the localization $I_\sigma \subset S_\sigma = \mathbf{k}[x_i, x_j]$, the *special localization* of I. If $n = 2$ we set $I_\sigma = I$.

Note that the special simplex determines the corresponding triangulation of $\mathscr{A}$ [5, Corollary 5.9]. This is because since b_i and b_j are adjacent in the Gale diagram $\mathscr{B}$, $\text{pos}(b_i, b_j)$ does not contain any other chamber of $\mathscr{B}$.

Example 3.6. Let I be the $\mathscr{L}$-graded ideal $\langle x_2x_3, x_1^2x_2, x_3^2x_4^2 \rangle$ from Example 2.2. We saw in Example 2.7 that I corresponds to the chamber $\text{pos}(b_2, b_3)$, so the special simplex is $\sigma = \{1, 4\}$. Note that the corresponding triangulation in Fig. 2 is the only triangulation of $\mathscr{A}$ containing the simplex $\text{pos}(a_1, a_4)$.

From now on, we fix a monomial $\mathscr{L}$-graded ideal I and let $\sigma = [n]\backslash\{i,j\}$ always be its special simplex. We now show that I_σ is coherent by showing that it is an initial ideal of $I_{\mathscr{L}_\sigma}$. To prove this it suffices to show that there exists some $\widetilde{w} \in \mathbb{Z}^2$ such that $\widetilde{w} \cdot \widehat{\pi}_\sigma(u - v) > 0$ for all $x^u - x^v \in Gr_{\mathscr{L}}$ such that $\pi_\sigma(x^u) \in I_\sigma$ and $\pi_\sigma(x^v) \notin I_\sigma$. That this suffices follows from Lemma 2.3 and the fact that $Gr_{\mathscr{L}_\sigma} \subseteq \{\pi_\sigma(x^u - x^v) : x^u - x^v \in Gr_{\mathscr{L}}\}$.

Proposition 3.7. (1) *There exists a cost-vector $w \in \mathbb{Z}^n$ such that $w(u - v) > 0$ whenever $x^u - x^v \in Gr_{\mathscr{L}}$ with $\pi_\sigma(x^v) \notin I_\sigma$. We can choose w so that $w_i = 0$ for $i \in \sigma$. This implies that $I_\sigma = \text{in}_{\widehat{\pi_\sigma(w)}}(I_{\mathscr{L}_\sigma})$ and is therefore coherent.*

(2) *The special simplex σ of I is also the special simplex of the initial ideal $\text{in}_w(I_{\mathscr{L}})$ of $I_{\mathscr{L}}$, and $(\text{in}_w(I_{\mathscr{L}}))_\sigma = I_\sigma$.*

Proof. (1) Let x_i^a and x_j^b be minimal generators of I_σ which exist by Lemma 2.10(2), and let $w \in \mathbb{N}^n$ be the cost-vector with $w_i = b, w_j = a$, and $w_k = 0$ for $k \neq i, j$. Suppose $x^u - x^v \in Gr_{\mathscr{L}}$, with $\pi_\sigma(x^v) \notin I_\sigma$. Since x^u and x^v have disjoint support, if $v_i \neq 0$, we must have $u_i = 0$. But then we must have $u_j \geqslant b$, since $\pi_\sigma(x^u) \in I_\sigma$. Since $\pi_\sigma(x^v) \notin I_\sigma$, we must have $v_i < a$. But now $w(u - v) = -w_iv_i + w_ju_j = -bv_i + au_j > -ba + ab = 0$. Similarly, if $v_j \neq 0$ we must have $v_j < b$ and $u_i \geqslant a$, so $w(u - v) > 0$. Finally, if $v_i = v_j = 0$, then $w(u - v) = bu_i + au_j$. If $u_i = u_j = 0$ then that would mean that $u - v = l \in \mathscr{L}$ and $l_i = l_j = 0$ which contradicts Lemma 2.10(1). So we conclude that $w \cdot (u - v) > 0$ as required.

(2) It suffices to show that σ is the special simplex for $\text{in}_w(I_{\mathscr{L}})$, as then Lemma 2.10(3) and part (1) of this proposition together imply that $(\text{in}_w(I_{\mathscr{L}}))_\sigma = \text{in}_{\widehat{\pi_\sigma(w)}}(I_{\mathscr{L}_\sigma}) = I_\sigma$. Let x^v be a monomial with $v_i = v_j = 0$. Then for all x^u such that $x^u - x^v \in I_{\mathscr{L}}$, either $u_i > 0$ or $u_j > 0$ by Lemma 2.10(1). This implies that $\text{in}_w(x^u - x^v) = x^u$ and hence $x^v \notin \text{in}_w(I_{\mathscr{L}})$. This shows that $\text{in}_w(I_{\mathscr{L}}) \subseteq P_\sigma$. Since $\text{in}_w(I_{\mathscr{L}})$ is $(n-2)$-dimensional, P_σ must be a minimal prime of $\text{in}_w(I_{\mathscr{L}})$ and hence σ appears in $\Delta(\text{in}_w(I_{\mathscr{L}}))$. However this implies that $\Delta(I) = \Delta(\text{in}_w(I_{\mathscr{L}}))$ since σ appears in only one triangulation of $\mathscr{A}$. Hence σ is the special simplex of $\text{in}_w(I_{\mathscr{L}})$. $\square$

Notice that the proof of Proposition 3.7(1) works for any localization I_τ of I at a minimal prime P_τ and hence all these localizations of I are coherent. Proposition 3.7 proves that Theorem 3.1 holds when $\mathscr{L} \subseteq \mathbb{Z}^2$ since in this case, $\mathscr{B} = \{b_1, b_2\}$ and $I = I_\emptyset = I_\sigma$.

Example 3.8. Continuing Example 3.6, $I_\sigma = \langle x_2, x_3^2 \rangle$, so from the proof of Proposition 3.7 we see that $w = (0, 2, 1, 0)$ satisfies $w(u - v) > 0$ whenever $x^u - x^v \in Gr_{\mathscr{L}}$ with $\pi(x^v) \notin I_\sigma$. Every Graver binomial satisfies this condition on $\pi(x^v)$, and it is easy to check that they all also satisfy the condition on $w(u - v)$.

In the rest of this section we fix w to be the vector constructed in the proof of Proposition 3.7 for the special localization I_σ. The final step in the proof of Theorem 3.1 is to show that $I = \mathrm{in}_w(I_{\mathscr{L}})$. This requires understanding the *Gröbner fan* of $I_{\mathscr{L}}$ [2,13]. This is a fan in $\mathbb{R}^n$ whose cells (which are open polyhedral cones) are in bijection with the initial ideals of $I_{\mathscr{L}}$. The open cone indexing an initial ideal J of $I_{\mathscr{L}}$ is the set of all $p \in \mathbb{Z}^n$ such that $J = \mathrm{in}_p(I_{\mathscr{L}})$. The closure of this cone is called the *Gröbner cone* of J. Full dimensional Gröbner cones index the monomial initial ideals of $I_{\mathscr{L}}$.

The Gröbner fan of $I_{\mathscr{L}}$ can be drawn in $\mathbb{R}^2$ as follows. By Lemma 2.3, the normal vector to a facet (*wall*) of a full dimensional Gröbner cone is an $l \in \mathscr{L}$ such that $x^{l^+} - x^{l^-} \in Gr_{\mathscr{L}}$. Using the injective map ϕ from (1) we can represent l by $\phi^{-1}(l) \in \mathbb{R}^2$ and the wall with normal l by the ray in $\mathbb{R}^2$ with normal $\phi^{-1}(l)$. We will always mean this two-dimensional fan when we refer to the Gröbner fan of $I_{\mathscr{L}}$. If a wall of the fan (ray in $\mathbb{R}^2$) is $\mathrm{pos}(g)$ for some $g \in \mathbb{R}^2$, then we always mean the primitive clockwise normal vector $g^\perp$ to g by the normal to this wall. We now recall that the Gröbner fan of $I_{\mathscr{L}}$ is a refinement of $\Sigma(\mathscr{L})$. Let $\mathbb{Z}\mathscr{B}$ be the lattice in $\mathbb{Z}^2$ generated by the elements of $\mathscr{B}$. Recall that $\mathbb{Z}\mathscr{B} = \mathbb{Z}^2$ if and only if $\mathscr{L}$ is saturated or equivalently, if $I_{\mathscr{L}}$ is a toric ideal.

Definition 3.9. Let K be a two-dimensional pointed rational polyhedral cone in $\mathbb{R}^2$ and H be the Hilbert basis of $K \cap \mathbb{Z}\mathscr{B}$ (i.e., H is a minimal generating set for the semigroup $K \cap \mathbb{Z}\mathscr{B}$). We call the fan obtained by subdividing K by drawing in the rays $\mathbb{R}_{\geqslant 0} \cdot h$ for each $h \in H$ the *Hilbert refinement* of K.

By Lemma 3.3.3 in [16], the Gröbner fan of $I_{\mathscr{L}}$ is supported on $\mathrm{pos}(\mathscr{B})$. Also, for any $\{r, s\} \subseteq [n]$ and $\tau = [n] \backslash \{r, s\}$ such that $B_{\{r,s\}}$ is non-singular, the Gröbner fan of $I_{\mathscr{L}_\tau}$ is supported on $\mathrm{pos}(b_r, b_s)$, and Example 3.3.4 in [16] shows that the Gröbner fan of $I_{\mathscr{L}_\tau}$ is the Hilbert refinement of $\mathrm{pos}(b_r, b_s)$. Theorem 3.3.8 in [16] implies the following.

Lemma 3.10 (Saito et al. [16]). *The Gröbner fan of $I_{\mathscr{L}}$ is the fan obtained by taking the Hilbert refinement of each full dimensional cone (chamber) in the chamber complex $\Sigma(\mathscr{L})$.*

A two-dimensional rational cone K is *unimodular* if the primitive integer generators of the two extreme rays of the cone form a basis for the lattice $\mathbb{Z}\mathscr{B}$.

The next two results rely on the geometry of the Hilbert basis of a two-dimensional rational polyhedral cone K. It is known that the Hilbert basis elements of K are precisely the lattice points that lie on the bounded faces of the polyhedron $K' = \mathrm{conv}\{z \in K \cap \mathbb{Z}\mathscr{B} : z \neq 0\}$. See [14, Proposition 1.19] for instance. For Corollaries 3.11 and 3.13, assume that the Hilbert basis elements of a chamber $\mathrm{pos}(b_r, b_s)$ in $\Sigma(\mathscr{L})$ are $g_0 = b_r, g_1, \ldots, g_t, b_s = g_{t+1}$ in the order they occur on the boundary of K' consisting of its bounded faces.

Corollary 3.11. *Each full dimensional Gröbner cone of $I_{\mathscr{L}}$ is unimodular.*

Proof. If g_k and g_{k+1} are adjacent Hilbert basis elements, then there is no element of the lattice $\mathbb{Z}\mathscr{B}$ in the convex hull of 0, g_k, and g_{k+1} other than the three vertices. Indeed, if a lattice point g existed, then the fact that the g_i are vertices of a convex polyhedron means that g cannot be written as an integral combination of any of the g_i, which would mean that the Hilbert basis was not complete. Now consider the triangle with vertices g_k, g_{k+1} and $g_k + g_{k+1}$. If there was a lattice point v in the interior of this triangle, then $v = \lambda g_k + \mu g_{k+1}$ where $0 < \lambda, \mu < 1$ and $\lambda + \mu > 1$. This implies that $g_k + g_{k+1} - v = g_k(1-\lambda) + g_{k+1}(1-\mu)$ lies in the interior of the convex hull of 0, g_k, and g_{k+1} since $1-\lambda, 1-\mu > 0$ and their sum $1-\lambda+1-\mu = 2-(\lambda+\mu) < 1$. Since this contradicts the earlier observation, we can conclude that there are no lattice points in the interior of the parallelogram spanned by g_k and g_{k+1} which is hence a fundamental domain of $\mathbb{Z}\mathscr{B}$. Therefore, $\mathrm{pos}(g_k, g_{k+1})$ is unimodular. $\square$

Example 3.12. The Gröbner fan for the lattice of Example 2.2 is shown in Fig. 3. Note that of the three cones in the chamber complex, only the middle one contains an extra Hilbert basis element, so we get four Gröbner cones.

Corollary 3.13. *Let $\mathrm{pos}(b_r, b_s)$ be a chamber of $\mathscr{B}$ and $g_0 = b_r, g_1, g_2, \ldots, g_t, b_s = g_{t+1}$ be the elements in the Hilbert basis of $\mathrm{pos}(b_r, b_s)$ in clockwise order. Then $b_s \cdot g_k^{\perp} < b_s \cdot g_{k+1}^{\perp}$ for all $k = 1, \ldots, t$. Similarly, $b_r \cdot g_k^{\perp} > b_r \cdot g_{k+1}^{\perp}$ for all $k = 1, \ldots, t$.*

Proof. Since rotating the cone does not affect the statement, for the first assertion, we may assume without loss of generality that $b_r = (0, y)$, $y \in \mathbb{Z}_+$ and $b_s = (p, q) \in \mathbb{Z}^2$

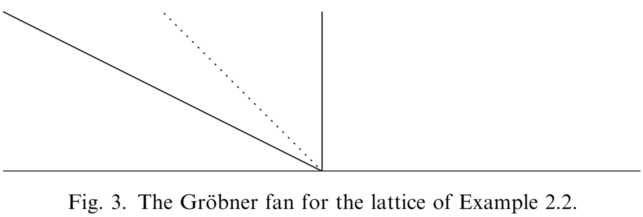

Fig. 3. The Gröbner fan for the lattice of Example 2.2.

with $p>0$. Let $m_0, m_1, \ldots, m_t$ be the slopes of the line segments $[g_0, g_1], [g_1, g_2], \ldots, [g_t, g_{t+1}]$. Since the g_i are vertices of a polyhedron, it now follows that $m_0 \leqslant m_1 \leqslant \cdots \leqslant m_t < q/p$. Let $g_k = (a,b)$ and $g_{k+1} = (c,d)$. Then $g_k^{\perp} = (b,-a)$ and $g_{k+1}^{\perp} = (d,-c)$. Since $m_k = (b-d)/(a-c) < q/p$, we get that $pb - pd < qa - qc$ which implies that $b_s \cdot g_k^{\perp} = pb - qa < pd - qc = b_s \cdot g_{k+1}^{\perp}$. The assertion for b_r is proved similarly. $\square$

We now begin the arguments to prove that $I = \mathrm{in}_w(I_{\mathscr{L}})$. Recall that $\sigma = [n]\backslash\{i,j\}$ is the special simplex of I. Let $\mathrm{int}(C)$ denote the interior of a cone C.

Lemma 3.14. (1) *If* $-b_k \in \mathrm{int}(\mathrm{pos}(b_i, b_j))$, *then* $k \in \tau$ *for all* $\tau \in \Delta(I)$.

(2) *If* $b_k \cdot \phi^{-1}(l) = 0$ *for* $l \in \mathscr{L}$, $l_k = 0$.

Proof. (1) Recall that $k \notin \tau \in \Delta(I)$ if and only if there exists some q such that $\mathrm{pos}(b_q, b_k) \supseteq \mathrm{pos}(b_i, b_j)$. However, if this was the case, then $-b_k \in \mathrm{int}(\mathrm{pos}(b_i, b_j))$ would mean $-b_k \in \mathrm{int}(\mathrm{pos}(b_k, b_q))$ which is not possible.

(2) Let $z = \phi^{-1}(l)$. Recall that $z \cdot b_i = l_i$ for all i, and so if $z \cdot b_k = 0$ then $l_k = 0$. $\square$

Let $C^* = \mathrm{pos}(g_1, g_2)$ be the Gröbner cone of $\mathrm{in}_w(I_{\mathscr{L}})$ and $\phi(g_1^{\perp}) = l_1, \phi(g_2^{\perp}) = l_2 \in \mathscr{L}$. Then C^* is also the Gröbner cone of $(\mathrm{in}_w(I_{\mathscr{L}}))_\sigma = \mathrm{in}_{\widehat{\pi(w)}}(I_{\mathscr{L}_\sigma})$ and hence lies in $\mathrm{pos}(b_i, b_j)$ which is the support of the Gröbner fan of $I_{\mathscr{L}_\sigma}$. We focus on the cones $\mathrm{pos}(b_i, b_j)$ and $C^* = \mathrm{pos}(g_1, g_2)$ in the rest of this section. Assume they are as in Fig. 4(a), where it could be that $b_i = g_1$ or $b_j = g_2$.

Lemma 3.15. *Suppose* $g \in \mathrm{pos}(g_1, g_2)$ *such that* $g = \lambda g_1 + \mu g_2 \in \mathbb{Z}^2$ *with* $\lambda, \mu \in \mathbb{Z}_+$ *and* $\phi(g^{\perp}) = l \in \mathscr{L}$. *Then* $x^{l_1^+}$ *divides* x^{l^+} *and* $x^{l_2^-}$ *divides* x^{l^-}.

Proof. We refer to Fig. 4(b) for this proof. If there is some index k for which $(l_1)_k$ and $(l_2)_k$ have opposite signs, then there must be some vector $l \in \mathscr{L} \cap \mathrm{int}(\mathrm{pos}(l_1, l_2))$ with $l_k = 0$. By Lemma 3.14(2), $\phi^{-1}(l)b_k = 0$ which implies that $\pm b_k \in \mathrm{int}(C^*)$. Since $b_k \notin \mathrm{int}(C^*)$ for any k, this can only happen if $-b_k \in \mathrm{int}(C^*)$.

Let $U \subseteq [n]$ be the set of indices k for which $-b_k \in \mathrm{int}(C^*)$, where U may be empty. Then l, l_1, l_2 are sign compatible in all slots except those indexed by U. If $k \in U$ then $g_1^{\perp}(-b_k) > 0$, and so $(l_1)_k < 0$. This means that $\mathrm{supp}(l_1^+) \cap U = \emptyset$. Similarly for $k \in U$, $g_2^{\perp}(-b_k) < 0$, so $(l_2)_k > 0$, and thus $\mathrm{supp}(l_2^-) \cap U = \emptyset$. Therefore, looking at $l^+ - l^- = \lambda(l_1^+ - l_1^-) + \mu(l_2^+ - l_2^-) = (\lambda l_1^+ + \mu l_2^+) - (\lambda l_1^- + \mu l_2^-)$, we see that $l_1^+ \leqslant l^+$ and $l_2^- \leqslant l^-$. $\square$

Definition 3.16. Let $M \subseteq S$ be a weakly $\mathscr{L}$-graded *monomial* ideal. The *forced ideal* of M is the ideal generated by all monomials x^u for which $x^u - x^v \in Gr_{\mathscr{L}}$ and $x^v \notin M$.

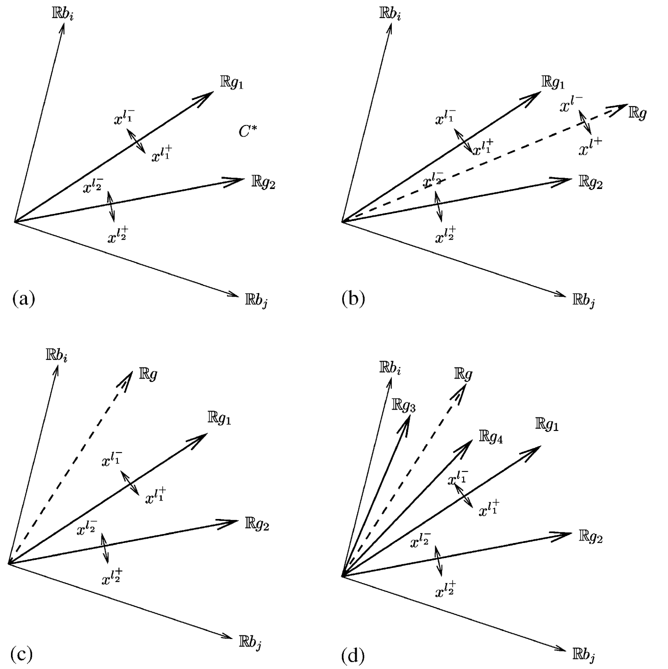

Fig. 4. The cones $\mathrm{pos}(b_i, b_j)$ and $C^* = \mathrm{pos}(g_1, g_2)$.

Remark 3.17. If M is a weakly $\mathscr{L}$-graded monomial ideal, then note that the forced ideal of M is contained in M as well as in all $\mathscr{L}$-graded monomial ideals contained in M. In particular, if the forced ideal of M is $\mathscr{L}$-graded then it is the unique $\mathscr{L}$-graded ideal contained in M since one monomial $\mathscr{L}$-graded ideal cannot be contained in another.

In Proposition 3.19 we will prove that I is the forced ideal of I_σ. We now collect together some facts needed for this proof.

Lemma 3.18. (1) *The special localization $I_\sigma \subset \mathbf{k}[x_i, x_j] \subset S$, of the monomial $\mathscr{L}$-graded ideal I, is weakly $\mathscr{L}$-graded when considered as an ideal in S.*

(2) *The monomials $x^{\widehat{\pi_\sigma(l_1^+)}}, x^{\widehat{\pi_\sigma(l_2^-)}}$ are minimal generators of $I_\sigma \subset \mathbf{k}[x_i, x_j]$ and the monomials $x^{\widehat{\pi_\sigma(l_1^-)}}, x^{\widehat{\pi_\sigma(l_2^+)}}$ are not in $I_\sigma \subset \mathbf{k}[x_i, x_j]$. Hence $x^{l_1^+}, x^{l_2^-}$ lie in the forced ideal of $I_\sigma \subset S$.*

(3) *If $g \in \mathrm{int}(\mathrm{pos}(b_i, b_j))$ and $\phi(g^\perp) = l \in Gr_{\mathscr{L}}$ then $x_i | x^{l^-}$ and $x_j | x^{l^+}$. If $g \notin \mathrm{pos}(b_i, b_j)$ and $\mathrm{pos}(b_i, b_j)$ lies on the same side of $\mathbb{R}g$ as $g^\perp$, then $x^{\widehat{\pi_\sigma(l^-)}} = 1$ which implies that $x^{l^+} \in I_\sigma \subset S$.*

Proof. (1) If there exists $x^u, x^v \in S$ of the same $\mathscr{L}$-degree such that $x^u, x^v \notin I_\sigma$ then $x^u, x^v \notin I \subseteq I_\sigma \subseteq S$. This cannot happen as I is $\mathscr{L}$-graded.

(2) Recall that C^* is the Gröbner cone of $\mathrm{in}_{\widehat{\pi_\sigma(w)}}(I_{\mathscr{L}_\sigma})$ with the binomials $x^{\widehat{\pi_\sigma(l_1^+)}} - x^{\widehat{\pi_\sigma(l_1^-)}}$ and $x^{\widehat{\pi_\sigma(l_2^-)}} - x^{\widehat{\pi_\sigma(l_2^+)}}$ defining the facets $\mathrm{pos}(g_1)$ and $\mathrm{pos}(g_2)$ of C^*. Further, the initial terms in both binomials are the positive terms $x^{\widehat{\pi_\sigma(l_1^+)}}$ and $x^{\widehat{\pi_\sigma(l_2^-)}}$ which are minimal generators of $\mathrm{in}_{\widehat{\pi_\sigma(w)}}(I_{\mathscr{L}_\sigma})$. It is known that $\mathrm{in}_{g_1}(I_{\mathscr{L}_\sigma}) = \langle x^{\widehat{\pi_\sigma(l_1^+)}} - x^{\widehat{\pi_\sigma(l_1^-)}} \rangle + \langle x^p : x^p \text{ is a minimal generator of } \mathrm{in}_{\widehat{\pi_\sigma(w)}}(I_{\mathscr{L}_\sigma}),\ x^p \neq x^{\widehat{\pi_\sigma(l_1^+)}} \rangle$. Hence if $x^{\widehat{\pi_\sigma(l_1^-)}}$ is also in $\mathrm{in}_{\widehat{\pi_\sigma(w)}}(I_{\mathscr{L}_\sigma})$, then we will get that $\mathrm{in}_{g_1}(I_{\mathscr{L}_\sigma}) \subseteq \mathrm{in}_{\widehat{\pi_\sigma(w)}}(I_{\mathscr{L}_\sigma})$ which is impossible as an initial ideal of an ideal cannot be contained in another initial ideal of the same ideal. Therefore, $x^{\widehat{\pi_\sigma(l_1^-)}}$ and similarly, $x^{\widehat{\pi_\sigma(l_2^+)}}$ are not in $I_\sigma = \mathrm{in}_{\widehat{\pi_\sigma(w)}}(I_{\mathscr{L}_\sigma}) \subset \mathbf{k}[x_i, x_j]$. This implies that $x^{l_1^-}, x^{l_2^+}$ are not in $I_\sigma \subset S$ and hence $x^{l_1^+}, x^{l_2^-}$ lie in the forced ideal of $I_\sigma \subset S$.

(3) Since $l_i = b_i \cdot g^\perp = 0$ if and only if g and b_i are dependent, it follows that l_i changes sign as g is rotated from one side of b_i to the other. Similarly l_j changes sign as g is rotated from one side of b_j to the other. In particular, if $g \notin \mathrm{pos}(b_i, b_j)$, then l_i and l_j must have the same sign, while if $g \in \mathrm{int}(\mathrm{pos}(b_i, b_j))$, then l_i and l_j must have opposite signs. Further, if g lies on the clockwise side of $\mathrm{pos}(b_i, b_j)$ as in the claim, then $\pi_\sigma(x^{l^+})$ lies in all initial ideals corresponding to Gröbner cones in $\mathrm{pos}(b_i, b_j)$ which shows that $l_i^+, l_j^+ > 0$, as $\pi_\sigma(x^{l^-}) = 1$ which does not lie in any initial ideal. Finally, in the case that $g \in \mathrm{pos}(b_i, b_j)$, the fact that $\pi_\sigma(x^{l^+})$ lies in all initial ideals of $I_{\mathscr{L}_\sigma}$ corresponding to Gröbner cones that lie between g and b_j while $\pi_\sigma(x^{l^-})$ lies in all initial ideals of $I_{\mathscr{L}_\sigma}$ corresponding to Gröbner cones that lie between g and b_i means that $l_i < 0$ and $l_j > 0$. $\square$

Proposition 3.19. *The monomial $\mathscr{L}$-graded ideal I is the forced ideal of $I_\sigma \subset S$.*

Proof. Let J be the forced ideal of I_σ. Then $J \subseteq I \subseteq I_\sigma$. To show that $J = I$, it suffices to show that J is weakly $\mathscr{L}$-graded. This is true if for every $x^u - x^v \in Gr_{\mathscr{L}}$ either $x^u \in J$ or $x^v \in J$. By the definition of J, if one of x^u or x^v does not lie in I_σ then the other is in J, so we may assume that $x^u, x^v \in I_\sigma$. Let $g^\perp = \phi^{-1}(u - v)$ and $g \in \mathbb{Z}^2$ be the primitive vector such that $g^\perp$ is the clockwise normal to the ray $\mathrm{pos}(g)$. Since $x^u, x^v \in I_\sigma$, Lemma 3.18(3) lets us assume $g \in \mathrm{pos}(b_i, b_j)$ (where we use $v - u$ instead of $u - v$ if necessary).

Since $C^* = \mathrm{pos}(g_1, g_2)$ is unimodular (Corollary 3.11), we can write $u - v = \lambda l_1 + \mu l_2$ for $\lambda, \mu \in \mathbb{Z}$. If $\lambda\mu = 0$, then since $x^u - x^v \in Gr_{\mathscr{L}}$ either $u - v = l_1$ or $u - v = l_2$. In either case, x^u and x^v do not both belong to I_σ by Lemma 3.18(2) which contradicts our assumption. So $\lambda\mu \neq 0$. The proof now breaks into two cases, depending on the sign of $\lambda\mu$.

Case 1: $\lambda\mu > 0$. From our assumption that $g \in \mathrm{pos}(b_i, b_j)$ it follows that $\lambda, \mu > 0$. Then $g \in \mathrm{pos}(g_1, g_2)$ and hence by Lemma 3.15, $x^{l_1^+} | x^u$ and $x^{l_2^-} | x^v$ which implies that $x^u, x^v \in J$ by Lemma 3.18(2).

Case 2: $\lambda\mu < 0$. Then $g \in \mathrm{int}(\mathrm{pos}(b_i, b_j)) \backslash C^*$. We may assume that C^* lies on the same side of $\mathbb{R}g$ as $g^\perp$ (see Figs. 4(c) and (d)) and consider two subcases.

Subcase 2.1: Suppose g is in the Hilbert basis of $\mathrm{pos}(b_i, b_j)$. Then by Lemma 3.18(3), $v_i = -(b_i g^\perp) > 0$. Also, $(l_2^-)_i = -(b_i g_2^\perp)$. Then Corollary 3.13 implies that $b_i g^\perp > b_i g_2^\perp$ which implies that $v_i < (l_2^-)_i$. Hence $x^{\widehat{\pi_\sigma}(v)} = x_i^{v_i} \notin I_\sigma$ since $x_i^{(l_2^-)_i}$ is a minimal generator of I_σ. This means $x^v \notin I_\sigma$ which is a contradiction.

Subcase 2.2: Suppose g is not in the Hilbert basis of $\mathrm{pos}(b_i, b_j)$. Since g lies in $\mathrm{int}(\mathrm{pos}(b_i, b_j)) \backslash C^*$, it lies in the interior of some Gröbner cone of $I_{\mathscr{L}}$ contained in $\mathrm{pos}(b_i, b_j)$ different from C^*. Suppose this Gröbner cone is $\mathrm{pos}(g_3, g_4)$ with $\phi(g_3^\perp) = l_3 \in Gr_{\mathscr{L}}$ and $\phi(g_4^\perp) = l_4 \in Gr_{\mathscr{L}}$ as in Fig. 4(d). By Lemma 3.15, $x^{l_3^+} | x^u$. Since g_3 lies in the Hilbert basis of $\mathrm{pos}(b_i, b_j)$, the arguments in the previous subcase show that $x^{l_3^-} \notin I_\sigma$ which implies that $x^{l_3^+} \in J$ and hence $x^u \in J$. $\square$

Example 3.20. In Example 3.6 we saw that for $I = \langle x_2x_3, x_1^2x_2, x_3^2x_4^2 \rangle$ we have $\sigma = \{1, 4\}$, and $I_\sigma = \langle x_2, x_3^2 \rangle$. Then in the notation of the preceding proofs, $b_i = b_3$, $b_j = b_2$, $g_1 = (-1, 1)$, and $g_2 = b_2$. We now check that the forced ideal of I_σ is weakly $\mathscr{L}$-graded, by checking that each binomial in the Graver basis has one of its monomials lying in I_σ. The Graver binomial $x_2x_3 - 1$ is covered by part 3 of Lemma 3.18, as $1 \notin I_\sigma$. The binomial $x_3^2x_4^2 - x_1^2$ comes from $\phi(b_2^\perp)$, while $x_1^2x_2 - x_3x_4^2$ comes from $\phi(g_1^\perp)$, so they are both dealt with by part 2 of Lemma 3.18. Note that x_1^2 and $x_3x_4^2$ do not lie in I_σ. Finally, $x_1^2x_2^2 - x_4^2$ comes from $\phi(b_3^\perp)$, so is covered by subcase 2.1. of Proposition 3.19. Note that $x_4^2 \notin I_\sigma$.

The intersection of the minimal primary components of a monomial ideal M is called $Top(M)$ in [16]. Theorem 3.3.6 in [16] and Theorem 4.4 in [8] then follow from the following corollary of Proposition 3.19.

Corollary 3.21. *If* $\dim(\mathscr{L}) = 2$ *and* I *and* I' *are two distinct monomial* $\mathscr{L}$*-graded ideals, then* $\mathrm{Top}(I) \neq \mathrm{Top}(I')$.

Proof. If $I \neq I'$, then Proposition 3.19 implies that $I_\sigma \neq I'_{\sigma'}$, where σ and σ' are the special simplices of the two ideals. Since the minimal primary components are uniquely determined from $\mathrm{Top}(I)$, this implies that $\mathrm{Top}(I) \neq \mathrm{Top}(I')$.

Propositions 3.7 and 3.19 combine to prove Theorem 3.1.

Proof of Theorem 3.1. Proposition 3.19 says that I is generated by monomials x^u for which there is some $x^v \notin I_\sigma$ with $x^u - x^v \in Gr_{\mathscr{L}}$. Proposition 3.7 says that there is a cost-vector w for which $w(u - v) > 0$ for all such Graver binomials $x^u - x^v$. This implies that $I \subseteq \mathrm{in}_w(I_{\mathscr{L}})$. Since we cannot have a proper inclusion of monomial $\mathscr{L}$-graded ideals, we conclude that $I = \mathrm{in}_w(I_{\mathscr{L}})$. $\square$

4. The general case

In this section we finish proving Theorem 3.2. We first recall from [12] the notion of a *flip*.

Definition 4.1. Let M_1 be a monomial $\mathscr{L}$-graded ideal with minimal generator x^u. Let x^v be the standard monomial for M_1 of the same degree as x^u. The *wall ideal W* associated to M_1 and x^u is the ideal obtained by replacing the generator x^u of M_1 by the binomial $x^u - x^v$. We say the binomial $x^u - x^v$ is *flippable* if $\mathrm{in}_{\prec}(W) = M_1$ for any term order with $x^v \prec x^u$. If there is some other term order $\prec'$ with $x^u \prec' x^v$ then we set M_2 to be the $\mathscr{L}$-graded monomial ideal $\mathrm{in}_{\prec'}(W)$. We say that M_2 and M_1 differ by a flip over the *true flip* $x^u - x^v$. If there is no such term order $\prec'$ we say that $x^u - x^v$ is a *fake flip*.

Remark 4.2. (1) In [12] the assumption that $\ker(A) \cap \mathbb{N}^n = \{0\}$ (the *positively-graded* assumption) avoided the possibility of fake flips.

(2) For a true flip, we recall Definition 2.7 and Lemma 2.9 of [12] which says that $M_2 = \langle x^\alpha \in M_1 : \alpha \neq u \text{ and } \exists x^\beta \notin M_1 \text{ with } x^\alpha - x^\beta \in Gr_{\mathscr{L}} \rangle + \langle x^v \rangle$.

Example 4.3. Let $\mathscr{L}$ be the lattice of Example 2.2. For the $\mathscr{L}$-graded ideal $\langle x_2x_3, x_1^2 \rangle$ the binomial $x_2x_3 - 1$ is a fake flip, and the binomial $x_1^2 - x_3^2x_4^2$ is a true flip. Flipping over the true flip, we get the other initial ideal of $\langle x_1^2 - x_3^2x_4^2, x_2x_3 \rangle$, which is $\langle x_2x_3, x_1^2x_2, x_3^2x_4^2 \rangle$. This has two true flips: $x_3^2x_4^2 - x_1^2$ and $x_1^2x_2 - x_3x_4^2$. Flipping over the second true flip gives the ideal $\langle x_2x_3, x_1^2x_2^2, x_3x_4^2 \rangle$, which again has two true flips: $x_3x_4^2 - x_1^2x_2$ and $x_1^2x_2^2 - x_4^2$. Flipping over $x_1^2x_2^2 - x_4^2$ we get the last $\mathscr{L}$-graded ideal $\langle x_2x_3, x_4^2 \rangle$. For this ideal $x_4^2 - x_1^2x_2^2$ is a true flip, and $x_2x_3 - 1$ is a fake flip.

We first prove that every wall ideal is coherent. If x^a is a monomial, we write $x^{\mathrm{supp}(a)}$ for the monomial $\prod_{i \in \mathrm{supp}(a)} x_i$.

Lemma 4.4. *Let $\mathscr{L}$ be a two-dimensional lattice contained in $\mathbb{Z}^n$, and let I and J be two monomial initials ideals of $I_{\mathscr{L}}$ which differ by a flip. Then the wall ideal W of I and J is coherent.*

Proof. Let the flip be over the binomial $x^u - x^v$, with $x^u \in I \backslash J$, and $x^v \in J \backslash I$. Choose cost-vectors w_0 and w_1 for which $\mathrm{in}_{w_0}(I_{\mathscr{L}}) = I$, and $\mathrm{in}_{w_1}(I_{\mathscr{L}}) = J$. Let w' be the composite cost-vector $(w_0(u-v))w_1 - (w_1(u-v))w_0$. This is chosen so that $w'(u-v) = 0$. Pick a monomial generator x^α of W. If the standard monomial of the same degree as x^α is the same monomial x^c for both I and J, then $w_i(\alpha - c) > 0$ for $i = 0, 1$. This means that $w'(\alpha - c) > 0$, and so $x^\alpha \in \mathrm{in}_{w'}(I_{\mathscr{L}})$. If this is the case for all such minimal monomial generators x^α, then $W \subseteq \mathrm{in}_{w'}(I_{\mathscr{L}})$. As we cannot have a proper inclusion of $\mathscr{L}$-graded ideals, we conclude that in this case we have $W = \mathrm{in}_{w'}(I_{\mathscr{L}})$.

Suppose on the contrary that there is some minimal generator x^α of W for which the standard monomials for I and J are different monomials x^c and x^d, respectively. The rest of the proof deals with this case. Let $x^e - x^f$ be a Graver binomial with x^e dividing x^c and x^f dividing x^d. Then we must have $x^e \in J \backslash I$ and $x^f \in I \backslash J$. The second observation in Remark 4.2 now implies that $e = v$ and $f = u$. So we can write $c = kv + \gamma$, and $d = ku + \gamma$, where $\gamma = \gamma_u + \gamma_v + \gamma'$ with $\text{supp}(\gamma_u) \subseteq \text{supp}(u)$, $\text{supp}(\gamma_v) \subseteq \text{supp}(v)$, and $\text{supp}(\gamma') \cap (\text{supp}(u) \cup \text{supp}(v)) = \emptyset$. Also note that by Lemma 2.3 $x^\alpha - x^c$ and $x^\alpha - x^d$ are both Graver basis elements, so $\text{supp}(\alpha) \cap \text{supp}(c) = \text{supp}(\alpha) \cap \text{supp}(d) = \emptyset$.

We next claim that $x^{u+v+\gamma} \notin \text{rad}(W)$. If $x^{u+v+\gamma} \in \text{rad}(W)$, there is some $l \in \mathbb{N}$ for which $x^{l(u+v+\gamma)} \in W$. We may assume that $l \geqslant k$. Choose a monomial x^g with $\text{supp}(g) \subseteq \text{supp}(u)$ so that $x^{\gamma_u + g} = x^{pu}$ for some integer p. Since $x^u - x^v \in W$, and $x^{l(u+v+\gamma)+lg} \in W$, it follows that $x^{l((2+p)v+\gamma_v+\gamma')} \in W$, and thus $x^{l((2+p)v+\gamma_v+\gamma')} \in I$. Note that $\text{supp}(l((2+p)v + \gamma_v + \gamma')) = \text{supp}(v) \cup \text{supp}(\gamma')$.

Let x^s be a monomial not lying in $\text{rad}(I)$. Since $x^\alpha \in W$, and thus $x^\alpha \in I$, we cannot have $\text{supp}(\alpha) \subseteq \text{supp}(s)$. Similarly, we cannot have $\text{supp}(u) \subseteq \text{supp}(s)$ or $\text{supp}(v) \cup \text{supp}(\gamma') \subseteq \text{supp}(s)$. Since these three sets are pairwise disjoint, we conclude that $|\text{supp}(s)| \leqslant n - 3$. But since s was arbitrary this contradicts the observation of Remark 2.8 that I is a $(n-2)$-dimensional ideal. From this we conclude that $x^{u+v+\gamma} \notin \text{rad}(W)$.

Since $\alpha - c$ and $\alpha - d$ are not linearly dependent, they must be a basis for the real span of $\mathscr{L}$, so every vector $l \in \mathscr{L}$ can be written as a (possibly non-integral) linear combination of $\alpha - c$ and $\alpha - d$.

Let w be the cost-vector with $w_i = 1$ if $i \in \text{supp}(\alpha)$, and $w_i = 0$ otherwise. We will show that $W = \text{in}_w(I_{\mathscr{L}})$. Let F be the generating set for $\text{in}_w(I_{\mathscr{L}})$ obtained from the corresponding reduced Gröbner basis for $I_{\mathscr{L}}$. If $f \in F$ then f is either a binomial or a monomial. If f is a binomial, then f comes from a lattice vector $l = \lambda(\alpha - c) + \mu(\alpha - d)$ with $wl = 0$. This means that $\lambda = -\mu$, and so $l = ku - kv$. Since f must be a Graver binomial, we know that $k = 1$, and thus $f = x^u - x^v \in W$. Now consider the case that f is a monomial x^β, and let x^δ be a standard monomial of $\text{in}_w(I_{\mathscr{L}})$ with $x^\beta - x^\delta \in Gr_{\mathscr{L}}$. Since $\beta - \delta \in \mathscr{L}$, it equals $\lambda(\alpha - c) + \mu(\alpha - d)$ for some constants λ and μ, so $\beta - \delta = (\lambda + \mu)\alpha - k\lambda v - k\mu u - (\lambda + \mu)\gamma$. Because x^β is the leading term of $x^\beta - x^\delta$ with respect to the cost-vector w, we know that $\lambda + \mu > 0$. It follows that $\text{supp}(\delta) \subseteq \text{supp}(u) \cup \text{supp}(v) \cup \text{supp}(\gamma)$, so $x^\delta \notin W$. But this means that there is some constant ρ for which $x^\beta - \rho x^\delta \in W$, and so $x^{N\beta} - \rho^N x^{N\delta} \in W$ for all $N \in \mathbb{N}$. We can choose N sufficiently large so that $N(\lambda + \mu) > 1$, so x^α divides x^β. But x^α was chosen to be in W, so if $\rho \neq 0$ this implies that $x^{N\delta} \in W$, and so $x^\delta \in \text{rad}(W)$. From this contradiction we conclude that $\rho = 0$, and so $x^\beta \in W$. This shows that every minimal generator of the $\mathscr{L}$-graded ideal $\text{in}_w(I_{\mathscr{L}})$ lies in W, and so since W is itself $\mathscr{L}$-graded, we must have $W = \text{in}_w(I_{\mathscr{L}})$. $\square$

Corollary 4.5. *Let $\mathscr{L}$ be a two-dimensional lattice. Then every monomial $\mathscr{L}$-graded ideal has exactly two flips.*

Proof. Let I be a monomial $\mathscr{L}$-graded ideal. By Theorem 3.1 we know that I is an initial ideal of $I_{\mathscr{L}}$, and thus corresponds to a cone in the Gröbner fan of $I_{\mathscr{L}}$. Now Lemma 4.4 says that any true flip of I connects I to an initial ideal J that corresponds to an adjacent Gröbner cone. There are thus exactly two true flips of I, unless I corresponds to a cone which is adjacent to the boundary of the Gröbner fan of $I_{\mathscr{L}}$, in which case I has only one true flip.

It thus suffices to show that I has no fake flips unless it corresponds to a cone adjacent to the boundary of the Gröbner fan, in which case I has exactly one fake flip.

We first recall that if $x^u - x^v$ is a flip of I then $x^u - x^v \in Gr_{\mathscr{L}}$. This means that x^u and x^v have disjoint supports, and so if $\mathrm{supp}(v) \neq \emptyset$ there is some term order $\prec$ with $x^u \prec x^v$. Thus the only way for $x^u - x^v$ to be a fake flip is to have $x^v = 1$.

Suppose $x^u - 1$ is a fake flip for I. If $x^{u'}$ were another minimal generator of I with $\mathrm{supp}(u) \cap \mathrm{supp}(u') \neq \emptyset$, then the S-pair $S(x^u - 1, x^{u'})$ would divide $x^{u'}$, contradicting the flippability of $x^u - 1$. On the other hand, if $x^{u'}$ is a minimal generator of I with $\mathrm{supp}(u) \cap \mathrm{supp}(u') = \emptyset$, then $S(x^u - 1, x^{u'}) = x^{u'} \in I$. So we see that a necessary and sufficient condition for a minimal generator x^u of I of the same $\mathscr{L}$-degree as 1 to give rise to a fake flip $x^u - 1$ is for $\mathrm{supp}(u) \cap \mathrm{supp}(u') = \emptyset$ for all other minimal generators $x^{u'}$ of I.

We note that for any vector $x^u - 1 \in I_{\mathscr{L}}$ (such as a fake flip) there is a vector $v \in \mathbb{Z}^2$ for which $u = \phi(v)$, and $b_i v \geqslant 0$ for all i. Note that we can have $b_i v = 0$ for at most two values of i, and if $b_i v = b_j v = 0$, then b_i is a negative multiple of b_j.

Let the Gröbner cone of I, $\mathrm{pos}(g_1, g_2)$, be contained in the secondary cone $\mathrm{pos}(b_i, b_j)$. Let $g_k^{\perp}$ be the clockwise normal to g_k for $k = 1, 2$, and let $l_k = \phi(g_k^{\perp}) \in \mathscr{L}$. We assume that g_1 and g_2 are in clockwise order.

We first consider the case where I is a monomial $\mathscr{L}$-graded ideal with two true flips. Then $x^{l_1^+}, x^{l_2^-} \in I$. Note that $(l_1)_j = g_1^{\perp} b_j > 0$, and $(l_2)_i = g_2^{\perp} b_i < 0$. Hence $i, j \notin \mathrm{supp}(u)$. This means that if $x^u - 1$ is a fake flip with $u = \phi(v)$, then $v b_j = v b_i = 0$. Since $\mathrm{pos}(b_i, b_j)$ is a pointed cone this is not possible for $v \neq 0$, so we conclude that I has no fake flips.

We now consider the case where I is a monomial $\mathscr{L}$-graded ideal with at most one true flip, so I corresponds to a cone on the boundary of the Gröbner fan of $I_{\mathscr{L}}$. Without loss of generality we may assume that this boundary is the counter-clockwise outer wall b_i. We first note that the fact that the Gröbner fan has a boundary implies that it is pointed. If the fan is not pointed, then for every vector $x \in \mathbb{Z}^2$ there is some b_k with $x \cdot b_k < 0$, and so $\mathscr{L} \cap \mathbb{N}^n = \{0\}$. This means that the ideal $I_{\mathscr{L}}$ is positively graded, which in turn means that the Gröbner region of the ideal is all of $\mathbb{R}^n$, so the fan does not have a boundary. The fact that the Gröbner fan is pointed means that $b_i^{\perp} b_j \geqslant 0$ for all i, and so if $u = \phi(b_i^{\perp})$, then $u \geqslant 0$, and we will show that $x^u - 1$ is a fake flip.

We know that the Gröbner cone for I is $\mathrm{pos}(b_i, g_2)$ for some vector g_2, and $x^{l_2^-} \in I$. Note that $l_2^- = a e_i$ for some $a \in \mathbb{N}$. We now show that this x_i^a is the only generator of I divisible by x_i. Let $l' \in \mathscr{L}$, and write $l' = \phi(g^{\perp})$ for a vector $g^{\perp} \in \mathbb{Z}^2$. Write g for the primitive vector in $\mathbb{Z}^2$ whose clockwise normal is $g^{\perp}$. By Corollary 3.11 we know that $\text{pos}(b_i, g_2)$ is a unimodular cone, so $g = \lambda b_i + \mu g_2$ with $\lambda, \mu \in \mathbb{Z}$. Since $\phi(b_i^{\perp})_i = 0$, we know that $l'_i = \mu\phi(g_2^{\perp})_i = -\mu a$. Since $|\mu a| > a$ for $|\mu| > 1$, we see that $x^{l'^-}$ is in I but not a minimal generator, so l' does not give rise to a minimal generator of I unless $l' = l_1$ or l_2. So the two generators of I are $x^{l_1^+} = x^u$ and $x^{l_2^-} = x_i^a$. Since $u_i = b_i^{\perp} b_i = 0$, it follows from the above characterization of fake flips that $x^u - 1$ is a fake flip. The only other possible flip come from x_i^a. If I has any true flips, this must come from x_i^a, giving I a total of two flips. Otherwise I is the only monomial $\mathscr{L}$-graded ideal, and g_2 must be the other boundary of the Gröbner fan, so x_i^a gives rise to a second fake flip. $\square$

We recall the following theorem which is Corollary 5.2 of [15].

Theorem 4.6. *Let I be a monomial $\mathscr{L}$-graded ideal. Then the number of flips of I is equal to* $\dim_{\mathbf{k}}(\text{Hom}_S(I, S/I))_0$.

We note that in [15] the assumption was made that $\mathscr{L}$ was saturated and positively graded. The proof goes through word for word in the case of general lattices.

We also recall the following theorem which is Proposition 1.6 of [7] applied to $H_{\mathscr{L}}$.

Theorem 4.7. *The Zariski tangent space to the toric Hilbert scheme $H_{\mathscr{L}}$ at an ideal I is canonically isomorphic to* $(\text{Hom}_S(I, S/I))_0$.

We now complete the proof of Theorem 1.1.

Proof of Theorem 1.1. There is an action of the torus $(\mathbf{k}^*)^n$ on $H_{\mathscr{L}}$ given by scaling the variables occurring in an ideal I. Since the singular locus of $H_{\mathscr{L}}$ must be fixed under this torus action, and the monomial $\mathscr{L}$-graded ideals are the torus-fixed points, to show that the scheme is smooth we need only show that it is smooth at each monomial ideal. Now Corollary 4.5 and Theorem 4.6 imply that for a monomial $\mathscr{L}$-graded ideal we have $\dim_{\mathbf{k}}\text{Hom}(I, S/I) = 2$, which Theorem 4.7 says is the dimension of the tangent space of $H_{\mathscr{L}}$ at I.

Pick any two linearly independent vectors $b_i, b_j \in \mathscr{B}$. We now define a map λ which takes an element of $(\mathbf{k}^*)^2$ to a vector in $(\mathbf{k}^*)^n$ that we use to scale $I_{\mathscr{L}}$. Let $\lambda(a,b) \in (\mathbf{k}^*)^n$ have $(\lambda(a,b))_l = 1$ for $l \neq i, j$, $(\lambda(a,b))_i = a$, and $(\lambda(a,b))_j = b$. There exist $l, l' \in \mathscr{L}$ with $l_i = 0, l_j \neq 0$ and $l'_i \neq 0, l'_j = 0$, so considering the action of $\lambda(a,b)$ on the binomials $x^{l^+} - x^{l^-}$ and $x^{l'^+} - x^{l'^-}$ we see that the map of $(\mathbf{k}^*)^2$ to the underlying reduced scheme of $H_{\mathscr{L}}$ given by $(a,b) \mapsto \lambda(a,b) I_{\mathscr{L}}$ is injective, so the coherent component of $H_{\mathscr{L}}$ is at least two dimensional. Since we showed above that the dimension of the tangent space to $H_{\mathscr{L}}$ at each monomial ideal is two, this shows that $H_{\mathscr{L}}$ is smooth at every monomial ideal, and thus everywhere. Every irreducible component must contain a monomial ideal, so Theorem 3.1 says that every irreducible component must intersect the coherent component. Since we just showed that $H_{\mathscr{L}}$ is smooth, we conclude that it is also irreducible. $\square$

Acknowledgments

We wish to thank Alastair Craw for details and references on G-Hilb.

References

[1] V.I. Arnold, A-graded algebras and continued fractions, Commun. Pure Appl. Math. 42 (1989) 993–1000.
[2] D. Bayer, I. Morrison, Gröbner bases and geometric invariant theory I, J. Symbolic Comput. 6 (1998) 209–217.
[3] T. Bridgeland, A. King, M. Reid, The McKay correspondence as an equivalence of derived categories, J. Amer. Math. Soc. 14 (3) (2001) 535–554 (electronic).
[4] A. Craw, M. Reid, How to calculate A-Hilb $\mathbb{C}^3$, Séminaires et Congrès 6 (2002) 129–154 math.AG/9909085.
[5] J. Deloera, S. Hoşten, F. Santos, B. Sturmfels, The polytope of all triangulations of a point configuration, Documenta Mathematica 1 (1996) 103–119.
[6] V. Gasharov, I. Peeva, Deformations of codimension 2 toric varieties, Compositio Mathematica 123 (2000) 225–241.
[7] M. Haiman, B. Sturmfels, Multigraded Hilbert schemes, math.AG/0201271, J. Algebraic Geometry, to appear.
[8] S. Hoşten, R.R. Thomas, Standard pairs and group relaxations in integer programming, J. Pure Appl. Algebra 139 (1999) 133–157.
[9] R. Kidoh, Hilbert schemes and cyclic quotient singularities, Hokkaido Math. J. 30 (2001) 91–103.
[10] C. Lee, Regular triangulations of convex polytopes, in: P. Gritzmann, B. Sturmfels (eds.), Applied Geometry and Discrete Mathematics - The Victor Klee Festschrift, Vol. 4, Dimacs Series, AMS, Providence, RI, 1991, pp. 443–456.
[11] D. Maclagan, Antichains of monomial ideals are finite, Proc. AMS 129 (2001) 1609–1615.
[12] D. Maclagan, R. Thomas, Combinatorics of the toric Hilbert scheme, Discrete Comput. Geom. 27 (2002) 249–264.
[13] T. Mora, L. Robbiano, The Gröbner fan of an ideal, J. Symbolic Comput. 6 (1988) 183–208.
[14] T. Oda, Convex Bodies and Algebraic Geometry, Springer, Berlin, 1985.
[15] I. Peeva, M. Stillman, Toric Hilbert schemes, Duke Math. J. 111 (3) (2002) 419–449.
[16] M. Saito, B. Sturmfels, N. Takayama, Gröbner Deformations of Hypergeometric Differential Equations, Vol. 6, Algorithms and Computation in Mathematics, Springer, New York, 1999.
[17] B. Sturmfels, The geometry of A-graded algebras, 1994, math.AG/9410032.
[18] B. Sturmfels, Gröbner Bases and Convex Polytopes, American Mathematical Society, Providence, RI, 1995.