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Posets, Regular CW Complexes and Bruhat Order

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1. INTRODUCTION

In this paper some questions are discussed concerning the topological interpretation of posets (partially ordered sets). In particular, the class of posets which are the combinatorial counterparts of regular CW complexes is singled out. Posets of this kind arise in some purely combinatorial contexts, and our main purpose is to show how the present considerations can lead to additional insight into such examples.

It is common practice to associate with a poset P the simplicial complex $\Delta(P)$ of its finite chains. In this manner each poset determines a topological space via the realization of $\Delta(P)$. In some cases, however, $\Delta(P)$ has the nature of a subdivision of a more intrinsic cell structure, which is in general not simplicial. Indeed, several combinatorial examples suggest the desirability of being able to directly interpret the elements of P as topological cells and the order relation as the incidence relation of a cell complex. This is of course possible only for a restricted class of posets, and the natural requirement that the topology of the cell complex should be unambiguously determined by the poset restricts the class even further.

In the next few sections the precise conditions for the preceding program are discussed. The most suitable class of cell complexes for combinatorial purposes seems to be the 'regular CW complexes', and we propose to call the corresponding posets 'CW posets'. Such posets are defined by the fact that all lower intervals are spherical. A certain notion of shellability for regular CW complexes is discussed and it is shown by way of the related posets that most of the important facts from simplicial shellability theory generalize. A number of combinatorial criteria for the topological interpretation of posets are then developed, for instance the following: a finite poset determines a shellable regular CW decomposition of a topological sphere if and only if it is thin and dual lexicographically shellable.

In the last two sections we mention some applications. One of the main results of [4] concerning the Bruhat ordering of a Coxeter group is sharpened and some related results derived. Specifically, it is shown that every Coxeter group (W, S) determines a topological space with a regular cell decomposition such that the Bruhat ordering of W is isomorphic to the incidence ordering of the cells. The associated space is a sphere when W is finite and is contractible when W is infinite. Finally, evidence is gathered in support of a positive answer to a question of Lindström [15] concerning convex polytopes.

2. CW POSETS

The following poset terminology will be considered familiar: chain, length of a chain, covering, least element $\hat{0}$, greatest element $\hat{1}$, interval, pure poset, shellable poset, order complex and Möbius function. The definitions can be found in [1], [3] or [5]. Whenever a topological statement is made about a poset P we are tacitly referring to its order complex $\Delta(P)$.

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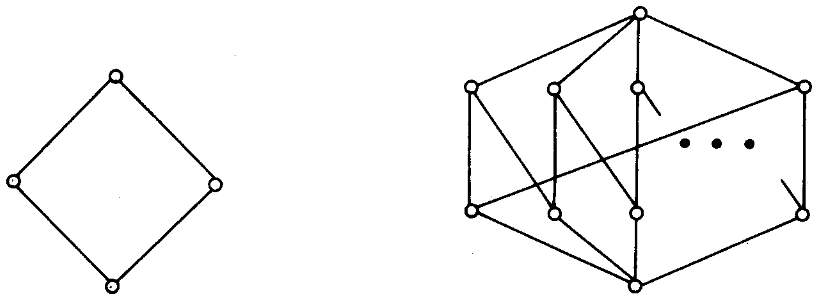
DEFINITION 2.1. A poset P is said to be a *CW poset* if
(a) P has a least element $\hat{0}$,
(b) P is nontrivial, i.e., has more than one element,
(c) for all $x \in P - \{\hat{0}\}$ the open interval $(\hat{0}, x)$ is homeomorphic to a sphere.

One immediate consequence of the definition is that a CW poset satisfies the Jordan–Dedekind chain condition. This is because every triangulation of a sphere is pure. In particular, each $x \in P$ has a well-defined *rank* $r(x)$, such that all unrefinable chains from $\hat{0}$ to x have length $r(x)$ and $(\hat{0}, x)$ is homeomorphic to the $(r(x)-2)$-sphere. The closed intervals of length ≤ 3 in a CW poset are of a prescribed combinatorial type, as shown in Figure 1. About the structure of intervals in general the following can be said:



(a) (b)

FIGURE 1. (a) length 2, (b) length 3.

an open interval (x, y), $x < y$, in a CW poset P is a homology $[r(y)-r(x)-2]$-sphere, and if $r(x)=1$ even a homotopy $[r(y)-3]$-sphere. As a consequence the Möbius function μ of P takes the form $\mu(x, y) = (-1)^{r(y)-r(x)}$, for all $x \leq y$. Since (x, y) is a link in the triangulation of the sphere $(\hat{0}, y)$, these statements follow from the well-known fact that all proper links in a triangulated manifold are homology spheres. The $r(x)=1$ case corresponds to that of the link of a vertex, which is a homotopy sphere, cf. [17, Prop. 3.4.3].

The result of Danaraj and Klee [9, Prop. 1.2] that a shellable simplicial pseudomanifold triangulates a sphere leads to the following purely combinatorial criterion for a poset to be CW.

PROPOSITION 2.2. *Let P be a nontrivial poset such that*
(a) *P has a least element* $\hat{0}$,
(b) *every interval* $[x, y]$ *of length two has cardinality four* [*as in Figure* 1(a)],
(c) *every interval* $[\hat{0}, x]$, $x \in P$, *is finite and shellable.*
Then P is a CW poset.

Let us consider some combinatorial examples. It seems practical to classify these according to the type of their spherical lower intervals.

2.3. BOOLEAN TYPE. Let us call a nontrivial poset P with $\hat{0}$ a *poset of Boolean type* if every lower interval $[\hat{0}, x]$, $x \in P$, is isomorphic to a finite Boolean algebra. Clearly, such a poset is CW. Important examples are the face posets of simplicial complexes. It is not hard to prove that a poset of Boolean type is the face poset of a simplicial complex if and only if it is a meet-semilattice.

A more general class of posets of Boolean type is obtained as follows. Let K be a simplicial complex and suppose that $\sim$ is an equivalence relation on K such that

(a) $\sigma \sim \tau$, $\sigma \neq \tau$, implies that σ and τ are of the same dimension and there is no simplex in K having both σ and τ as faces,

(b) if $\sigma \sim \tau$ and $\sigma' \subseteq \sigma$, then there exists $\tau' \in K$ such that $\tau' \sim \sigma'$ and $\tau' \subseteq \tau$.

Partially order the equivalence classes $K/\sim$ by $C_1 \leq C_2$ if and only if C_1 has a member which is a subset of a member of C_2. Then $K/\sim$ is a poset of Boolean type. Conversely, it can be shown that every poset of Boolean type arises in this fashion.

Posets of Boolean type were considered by Garsia and Stanton [13] in connection with a study of orbit complexes which arise when subgroups of a finite Coxeter group act on the Coxeter complex. In fact, the situation described in the preceding paragraph arises whenever a group acts on a simplicial complex in such a way that a simplex is invariant if and only if all of its vertices are fixed. Then the equivalence classes are the orbits, and these form a poset of Boolean type.

2.4. Polyhedral Type. Let us call a nontrivial poset P with $\hat{0}$ *a poset of polyhedral type* if every lower interval $[\hat{0}, x]$, $x \in P$, is isomorphic to the face lattice $\mathcal{F}(\mathcal{C})$ of some convex polytope $\mathcal{C}$. Such a poset is CW, because the proper part of $\mathcal{F}(\mathcal{C})$ triangulates a sphere, namely the boundary of $\mathcal{C}$. The most important examples are the face posets of polyhedral complexes. See Grünbaum [14] for definitions and details concerning polytopes and polyhedral complexes. The 'abstract d-complexes' of M. A. Perles [14, p. 206] are also posets of polyhedral type.

We have the following inclusions among some classes of posets of polyhedral type:
$\{P|P$ isomorphic to an order ideal in the face lattice of a convex polytope$\}$
$\subseteq \{P|P$ isomorphic to a face poset of a finite polyhedral complex$\}$
$\subseteq \{P|P$ a finite meet-semilattice of polyhedral type$\}$.
The second inclusion is strict, as the examples of Perles on pp. 206–207 of [14] show. It is unknown to us whether the first inclusion is strict. The question is motivated by the fact that the corresponding classes of posets of Boolean type all coincide.

By straightforward generalization of the construction for simplicial complexes described above one can define a class of equivalence relations on polyhedral complexes such that the quotients are posets of polyhedral type. It can also be shown that every poset of polyhedral type arises from a polyhedral complex in such a way.

2.5. Bruhat Type. Let us call a nontrivial poset P with $\hat{0}$ a poset of *Bruhat type* if every lower interval $[\hat{0}, x]$, $x \in P$, is isomorphic to an interval in the Bruhat ordering of a Coxeter group. Such intervals are spherical [4, Theorem 5.4], so these posets are CW posets. Clearly, every nontrivial interval or order ideal in the Bruhat ordering of a Coxeter group is a poset of Bruhat type.

Let A be a finite or infinite nonempty alphabet. A word in A^*, i.e., a finite string of letters from A, is said to be *injective* if no letter occurs more than once, and *normal* if no two consecutive letters are equal. A word w is a *subword* of w' if by erasing some letters in w' and closing the gaps one can obtain w. The posets I_A and N_A of all injective and all normal words, respectively, ordered by the subword relation, were introduced and studied by Farmer [11]. I_A is clearly a poset of Boolean type, and it was shown in [5, Section 6] that N_A is a poset of Bruhat type.

Having encountered these examples, let us mention some simple ways of deriving new CW posets from the given ones.

Proposition 2.6. *Let P and Q be CW posets [of type X]. Then*

(a) *the direct product $P \times Q$ is a CW poset [of type X],*

(b) *every nontrivial order ideal in P is a CW poset [of type X],*
(c) *if P is of type X and $y \in P$ then $\{p \in P \mid p \geq y\}$ is a CW poset of type X.*

PROOF. By a result of Walker [19, Theorem 4.3] the open interval $((\hat{0}, \hat{0}), (x, y))$ in $P \times Q$ is homeomorphic to the suspension of the join of $(\hat{0}, x)$ and $(\hat{0}, y)$, hence if these intervals are spherical then so is $((\hat{0}, \hat{0}), (x, y))$. Also, one can show that if $[\hat{0}, x]$ and $[\hat{0}, y]$ are intervals of Boolean, polyhedral or Bruhat isomorphism type, then so is their direct product $[\hat{0}, x] \times [\hat{0}, y] \cong [(\hat{0}, \hat{0}), (x, y)]$. This settles part (a). Parts (b) and (c) are straightforward. Notice that (c) does not necessarily apply to arbitrary CW posets.

3. REGULAR CW COMPLEXES

The notion of a CW complex will be considered familiar. Detailed treatments can be found in e.g. Lundell and Weingram [16], Maunder [17] or Spanier [18]. Without stating a complete definition, let us simply remind the reader that a CW complex $\mathscr{K}$ is a Hausdorff topological space $X_{\mathscr{K}}$ together with a certain kind of cellular decomposition. In particular, $X_{\mathscr{K}}$ is the disjoint union of open cells e_α, each nonsingleton cell homeomorphic to an open ball $\{x \in \mathbb{R}^n \mid \|x\| < 1\}$ under the restriction of a characteristic map $\varphi_\alpha : E^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\} \to X_{\mathscr{K}}$. Singleton cells are similarly considered to have characteristic maps from $E^0 = \{\text{point}\}$. Thus, each cell has a naturally defined dimension and one can speak of the n-skeleton $\mathscr{K}^n$ with space $X^n_{\mathscr{K}} = \bigcup_{\dim(e_\alpha) \leq n} e_\alpha$. The sets $\varphi_\alpha(E^n)$ are called closed cells, and it is required that $\varphi_\alpha(\dot{E}^n) \subseteq X^{n-1}_{\mathscr{K}}$. A CW complex is said to be *regular* if all closed cells are homeomorphic to closed balls E^n (cf. [16, p. 78]). Finally, recall that there is a useful inductive characterization of CW complexes in terms of successively attaching cells to the skeleta dimension by dimension (cf. [16, p. 47]).

Given a CW complex $\mathscr{K}$ we define its *face poset* $\mathscr{F}(\mathscr{K})$ to be the set of closed cells ordered by containment and augmented by a bottom element $\hat{0}$. Figure 2(a) shows a regular CW decomposition of S^2 (the 2-sphere) and Figure 2(b) the corresponding face poset. In general the order complex $\Delta(\mathscr{F}(\mathscr{K}) - \{\hat{0}\})$ does not reveal the topology of $X_{\mathscr{K}}$. E.g., the ball E^2 and the real projective plane both have CW decompositions whose face poset is the 4-element chain. However, if $\mathscr{K}$ is a regular CW complex then $\Delta(\mathscr{F}(\mathscr{K}) - \{\hat{0}\}) \cong X_{\mathscr{K}}$ (cf. the proof of Theorem 1.7, [16, p. 80]). Thus the class of regular CW complexes is combinatorially tractable in the sense that the incidence relations of cells determine the topology.

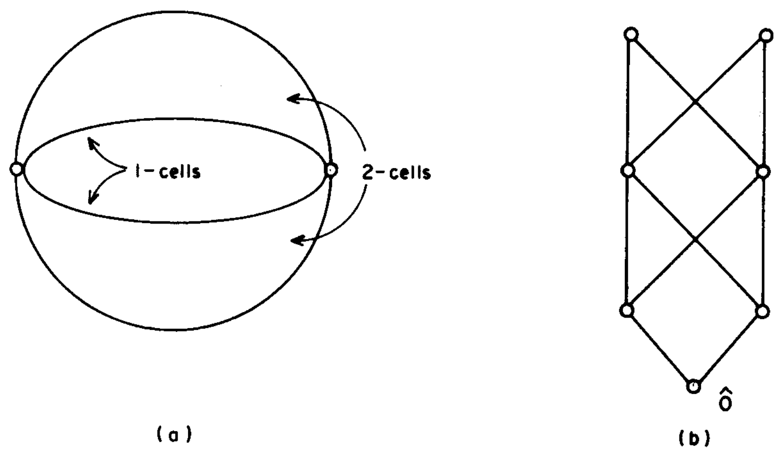


FIGURE 2.

PROPOSITION 3.1. *A poset P is a CW poset if and only if it is isomorphic to the face poset of a regular CW complex.*

PROOF. Let $\bar{e}_\alpha$ be a closed n-dimensional cell in a regular CW complex $\mathcal{K}$. Then its boundary $\dot{e}_\alpha$ is homeomorphic to S^{n-1}. Also, the interval $(\hat{0}, \bar{e}_\alpha)$ in $\mathcal{F}(\mathcal{K})$ consists of cells which form a regular CW decomposition of $\dot{e}_\alpha$ (cf. [16, p. 82]). Hence, $\Delta((\hat{0}, \bar{e}_\alpha)) \cong S^{n-1}$.

Conversely, suppose that P is a CW poset. We will inductively construct a regular CW complex $\mathcal{K}$ such that $\mathcal{F}(\mathcal{K}) \cong P$. Let $P_k = \{x \in P \mid r(x) = k\}$ and suppose a regular CW complex $\mathcal{K}_{n-1}$ has been constructed so that $\mathcal{F}(\mathcal{K}_{n-1}) \cong \bigcup_{k \leq n} P_k$. Under this isomorphism every $x \in P_{n+1}$ corresponds via the open interval $(\hat{0}, x)$ to a regular CW subcomplex $\mathcal{K}_x \subseteq \mathcal{K}_{n-1}$ such that $\Delta(\mathcal{F}(\mathcal{K}_x) - \{\hat{0}\}) \cong S^{n-1}$. Hence, we can attach an n-cell to $\mathcal{K}_{n-1}$ for each $x \in P_{n+1}$ by a map $S^{n-1} \cong X_{\mathcal{K}_x} \hookrightarrow X_{\mathcal{K}_{n-1}}$. This way we obtain a CW complex $\mathcal{K}_n$ such that $\mathcal{F}(\mathcal{K}_n) \cong \bigcup_{k \leq n+1} P_k$. Furthermore, $\mathcal{K}_n$ is regular since the attaching maps are homeomorphisms onto their images. The construction leads to a regular CW complex $\mathcal{K} = \bigcup_n \mathcal{K}_n$ for which $\mathcal{F}(\mathcal{K}) \cong P$.

Notice that the complex $\mathcal{K}$ constructed in the preceding proof is uniquely determined by P not only up to cellular incidence structure but also topologically, since $X_{\mathcal{K}} \cong \Delta(P - \{\hat{0}\})$. The construction can be elaborated in the following way. Define a *semiregular map* $f: \mathcal{K} \to \mathcal{K}'$ between CW complexes to be a continuous map $f: X_{\mathcal{K}} \to X_{\mathcal{K}'}$ having the property that for each open cell $e_\alpha \in \mathcal{K}$ there is an open cell $e_\beta \in \mathcal{K}'$ such that $f(e_\alpha) \subseteq e_\beta$ and $\dim(e_\alpha) \geq \dim(e_\beta)$. Such maps are cellular, and regular cellular maps are semiregular. Notice that e_β is uniquely determined by e_α. Call a map $\varphi: P \to Q$ between CW posets *admissible* if it is order-preserving and $1 \leq r \circ \varphi(x) \leq r(x)$ for all $x \in P - \{\hat{0}\}$. By extending the preceding proof it can be shown that up to a certain equivalence relation we have a 'functor' $\mathcal{K}(\cdot)$ from the category of CW posets and admissible maps to the category of regular CW complexes and semiregular maps. On the other hand, the face poset functor $\mathcal{F}(\cdot)$ connects these categories in the opposite direction. Clearly, $\mathcal{F} \circ \mathcal{K}(P) \cong P$ for every CW poset P and $\mathcal{K} \circ \mathcal{F}(\mathcal{K}') \cong \mathcal{K}'$ for every regular CW complex $\mathcal{K}'$.

Having established the equivalence between CW posets and regular CW complexes one might ask what it means in terms of a correspondence between various poset constructions and topological constructions. The following can be said in connection with Proposition 2.6. Let $\mathcal{K}$ and $\mathcal{K}'$ be regular CW complexes. Then the nontrivial order ideals of $\mathcal{F}(\mathcal{K})$ are in one-to-one correspondence with the subcomplexes of $\mathcal{K}$, and the direct product $\mathcal{F}(\mathcal{K}) \times \mathcal{F}(\mathcal{K}')$ is the face poset of a regular CW decomposition of the join of spaces $X_{\mathcal{K}} * X_{\mathcal{K}'}$. Part (iii) of Proposition 2.6 shows that in regular CW complexes whose face posets are of Boolean, polyhedral or Bruhat type the combinatorial notion of 'link' has a cellular counterpart.

4. SHELLINGS

The idea of a shelling of a simplicial complex can be generalized to cell complexes in several different ways (cf. e.g. [9, Section 1]). In this section we will define what seems to be a useful notion of shellability for regular CW complexes. It has the advantage that the pleasant topological properties of simplicial shellability (cf. e.g. [2, Section 1]) remain intact.

For convenience, let the term *d-CW-complex* mean a finite regular CW complex such that every closed cell is a face of a d-dimensional cell. Hence, $\mathcal{K}$ is a d-CW-complex if and only if $\mathcal{F}(\mathcal{K})$ is a finite pure CW poset of length $d+1$. If σ is a d-cell (meaning, a d-dimensional closed cell) of a regular CW complex $\mathcal{K}$, then let $\partial\sigma$ denote the $(d-1)$-CW-complex consisting of all proper faces of σ. Thus, $\partial\sigma$ is a subcomplex of $\mathcal{K}$ affording a cell decomposition of the topological boundary $\dot{\sigma}$.

DEFINITION 4.1. An ordering $\sigma_1, \sigma_2, \ldots, \sigma_t$ of the d-cells of a d-CW-complex $\mathcal{K}$ is said to be a *shelling* if $d=0$ or if $d>0$ and

(a) $\partial\sigma_1$ has a shelling,

(b) $\partial\sigma_j \cap (\bigcup_{i=1}^{j-1} \partial\sigma_i)$ is a $(d-1)$-CW-complex, $j=2,3,\ldots,t$,

(c) $\partial\sigma_j$ has a shelling in which the $(d-1)$-cells of $\partial\sigma_j \cap (\bigcup_{i=1}^{j-1} \partial\sigma_i)$ come first, $j=2,3,\ldots,t$.

$\mathcal{K}$ is said to be *shellable* if it admits a shelling.

For simplicial complexes this definition is equivalent with the usual concept of shellability. For polyhedral complexes it coincides with the definition in [5, Section 4]. In a Boolean CW complex (i.e., a complex whose face poset is of Boolean type) $\partial\sigma_j$ is always the boundary complex of a d-simplex, for which every ordering of the $(d-1)$-cells is a shelling, so conditions (a) and (c) are vacuously satisfied. Hence, the definition of a shelling of a Boolean d-CW-complex has a nonrecursive formulation using just condition (b), and this formulation can be seen to be equivalent with the definition used by Garsia and Stanton [13].

From now on we will assume familiarity with the concept of lexicographic shellability of posets, more precisely with the notion of CL-shellability as defined in [3]–[5]. For any complex $\mathcal{K}$ let $\hat{\mathcal{F}}(\mathcal{K})$ denote the face poset of $\mathcal{K}$ augmented by a new greatest element $\hat{1}$. The proof of Corollary 4.4 in [5] carries over to the present context and yields the following.

PROPOSITION 4.2. *A d-CW-complex $\mathcal{K}$ is shellable if and only if its face poset $\hat{\mathcal{F}}(\mathcal{K})$ is dual CL-shellable.*

It is possible using this fundamental correspondence to easily transfer topological information from the simplicial case to the case of regular CW complexes.

PROPOSITION 4.3. *Let $\mathcal{K}$ be a shellable d-CW-complex. Then*

(a) *$X_{\mathcal{K}}$ has the homotopy type of a wedge of d-spheres (or equivalently, $X_{\mathcal{K}}$ is $(d-1)$-connected),*

(b) *if every $(d-1)$-cell is a face of exactly two d-cells then $X_{\mathcal{K}}$ is homeomorphic to the sphere S^d.*

(c) *if every $(d-1)$-cell is a face of at most two d-cells and some $(d-1)$-cell is a face of only one d-cell then $X_{\mathcal{K}}$ is homeomorphic to the ball E^d.*

PROOF. Use the fact that $X_{\mathcal{K}} \cong \Delta(\mathcal{F}(\mathcal{K})-\{\hat{0}\})$, [16, p. 80], and that the (dual) CL-shellability of $\hat{\mathcal{F}}(\mathcal{K})$ implies the shellability of $\Delta(\mathcal{F}(\mathcal{K})-\{\hat{0}\})$, [4, Theorem 3.3]. Now, (a) is a well-known property of shellable simplicial d-complexes, [1, Appendix] or [2, Section 1]. Also, under the conditions (b) and (c) the complex $\Delta(\mathcal{F}(\mathcal{K})-\{\hat{0}\})$ becomes a shellable pseudomanifold or a shellable pseudomanifold with boundary, so the conclusions follow from the result of Danaraj and Klee [9, Prop. 1.2].

The preceding parts (b) and (c) show that the intersections $\partial\sigma_j \cap (\bigcup_{i=1}^{j-1} \partial\sigma_i)$ in Definition 4.1 are always topological $(d-1)$-balls or $(d-1)$-spheres. Furthermore, the number of times that the latter case occurs equals the number of d-spheres in the wedge of part (a).

PROPOSITION 4.4. *Let $\mathcal{K}$ be a shellable d-CW-complex. Then*

(a) *the barycentric subdivision of $\mathcal{K}$ is shellable,*

(b) *every skeleton $\mathcal{K}^n$, $n=0,1,\ldots,d$, is shellable.*

PROOF. The barycentric subdivision equals $\Delta(\mathscr{F}(\mathscr{K})-\{\hat{0}\})$ as in the preceding proof. The face poset $\hat{\mathscr{F}}(\mathscr{K}^n)$ is obtained by deleting the rank-levels for $r=n+2, n+3, \ldots, d+1$ from $\hat{\mathscr{F}}(\mathscr{K})$, so part (b) follows from Proposition 4.2 and the fact that rank-selection preserves CL-shellability of posets, [5, Theorem 8.1].

The converse to part (a) is false, as an example due to Walker [20] shows. The example is a 3-CW-complex $\mathscr{W}$ obtained as a regular cell decomposition of the cube I^3. The complex $\mathscr{W}$ has 8 0-cells, 16 1-cells, 12 2-cells and 3 3-cells. It is strongly unshellable in the sense that the removal of any 3-cell leaves a subcomplex which is not simply connected. Yet the barycentric subdivision of $\mathscr{W}$ is shellable.

The time has come to sum up those parts of the discussion in the last few sections which will be particularly useful for our original purpose, i.e., the topological interpretation of posets. A poset P will be called *thin* if all its intervals of length two have cardinality four [as in Figure 1(a)]. As before, $\hat{P}$ denotes $P\cup\{\hat{1}\}$ for a new greatest element $\hat{1}$.

PROPOSITION 4.5. *Let P be a nontrivial, finite, pure poset of length $d+1$ with least element $\hat{0}$. Then the following implications hold*:

(a) *P shellable and thin $\Rightarrow P\cong\mathscr{F}(\mathscr{K})$, $\mathscr{K}$ a d-CW-complex,*

(b) *$\hat{P}$ dual CL-shellable and P thin $\Leftrightarrow P\cong\mathscr{F}(\mathscr{K})$, $\mathscr{K}$ a shellable d-CW-complex,*

(c) *$\hat{P}$ shellable and thin $\Rightarrow P\cong\mathscr{F}(\mathscr{K})$, $\mathscr{K}$ a d-CW-complex homeomorphic to the d-sphere,*

(d) *$\hat{P}$ dual CL-shellable and thin $\Leftrightarrow P\cong\mathscr{F}(\mathscr{K})$, $\mathscr{K}$ a shellable d-CW-complex homeomorphic to the d-sphere.*

PROOF. Every interval in a shellable poset is shellable [1, Proposition 4.2], so by Proposition 2.2 P is a CW poset. The corresponding d-CW-complex $\mathscr{K}$ is then constructed as in Proposition 3.1. This settles part (a). The hypothesis of part (c) means that $\Delta(P-\{\hat{0}\})$ is a shellable pseudomanifold, and by [9, Proposition 1.2] it then triangulates the d-sphere. Therefore, part (c) follows via the homeomorphism $X_{\mathscr{K}}\cong\Delta(P-\{\hat{0}\})$. Finally, parts (b) and (d) are obtained by combining the preceding arguments with Proposition 4.2.

A few more remarks about the topological nature of the realization of shellable posets can be made. Recall that by a *p.l. sphere* and *p.l. ball* ('p.l.' = piecewise linear) is meant a Euclidean polyhedron (i.e., finite union of simplices in $\mathbb{R}^n$) admitting a subdivision isomorphic to some subdivision of the boundary of a simplex or of a simplex, respectively. When a d-CW-complex $\mathscr{K}$ is erected upon a thin shellable poset P as in Proposition 4.5 one can arrange that all the constituent cells are in fact p.l. balls and in parts (c) and (d) that the whole complex is a p.l. sphere. This follows from the construction together with the well-known facts that the barycentric subdivision of $\mathscr{K}$ embeds linearly into $\mathbb{R}^{2d+1}$ (cf. [18, Theorem 3.2.9]) and that shellable simplicial spheres and balls are p.l. spheres and p.l. balls (cf. [9, Proposition 1.2] or [2, Theorem 1.5]). In particular, every shellable d-CW-complex is piecewise linear, in the sense that it can be isomorphically 'realized' as a polyhedron in $\mathbb{R}^n$ with a regular cell decomposition into p.l. balls. Similarly, the complexes in parts (b) and (c) of Proposition 4.3 have isomorphic 'realizations' as p.l. spheres and p.l. balls.

5. BRUHAT ORDER

Here we will assume familiarity with the notion of a Coxeter group and the Bruhat partial ordering of the group elements. See [4, Section 2] or [7] for the basic facts and standard notation.

THEOREM 5.1. *Let (W, S) be a Coxeter group and $w, w' \in W$, $w < w'$, $d = l(w') - l(w) - 2 \geq 0$. Then the interval $[w, w']$ is isomorphic to the face poset $\hat{\mathcal{F}}(\mathcal{K})$ of a shellable regular CW decomposition $\mathcal{K}$ of the sphere S^d.*

PROOF. By [4, Theorem 4.2] the poset $[w, w']$ is dual CL-shellable and thin, so Proposition 4.5 (d) is applicable.

The preceding result sharpens Theorem 5.4(i) of [4], which follows from the above by subdivision. In the case of a finite Coxeter group even more can be said. Let w_0 as usual denote the element of maximal length.

THEOREM 5.2. *Let (W, S) be a finite Coxeter group, and let $d = l(w_0) - 2 \geq 0$. Then there exists a regular CW decomposition $\mathcal{K}$ of S^d such that $\hat{\mathcal{F}}(\mathcal{K})$ and W are isomorphic as posets. Furthermore,*

(a) *every ordering of the d-cells of $\mathcal{K}$ is a shelling,*
(b) *if $\sigma_1, \sigma_2, \ldots, \sigma_t$ is any one such ordering, then each σ_j has a face $\mathcal{R}(\sigma_j)$ such that $\mathcal{F}(\mathcal{K}) = \biguplus_{j=1}^{t} [\mathcal{R}(\sigma_j), \sigma_j]$ (disjoint union) and $\mathcal{R}(\sigma_i) \subseteq \sigma_j$ implies $i \leq j$, for all $1 \leq i, j \leq t$.*

PROOF. The d-cells of $\mathcal{K}$ correspond to the elements sw_0, $s \in S$, of W. Assume that the desired ordering of the d-cells corresponds to the ordering $s_1 w_0, s_2 w_0, \ldots, s_t w_0$. The way that a shelling order for $\mathcal{K}$ is derived from a dual CL-labeling of W, described in the proofs of Theorems 3.2 and 4.3 of [5], shows that it suffices to find a dual CL-labeling of W such that $1 \leq p < q \leq t$ implies that the label of $w_0 \to s_p w_0$ is less than the label of $w_0 \to s_q w_0$. The way that a dual CL-labeling of W is derived from a reduced expression for w_0, described in [4, Section 4], shows that this in turn is equivalent with finding a reduced expression $w_0 = r_1 r_2 \cdots r_{d+2}$, $r_i \in S$, such that $1 \leq p < q \leq t$ implies that $\varphi(p) < \varphi(q)$, where φ is the 'position' function uniquely defined by $s_i = r_1 r_2 \cdots r_{\varphi(i)} r_{\varphi(i)-1} \cdots r_1$. Such a reduced expression can be obtained as follows. For $j = 1, 2, \ldots, t$, let w_j be the top element of the parabolic subgroup $W_{\{s_1, s_2, \ldots, s_j\}}$. Thus, $s_1 = w_1 < w_2 < \cdots < w_t = w_0$. Put $r_1 = s_1 = w_1$, and continue by induction: given a reduced expression $w_i = r_1 r_2 \cdots r_{\varphi(i)}$ one can find $r_{\varphi(i)+1}, r_{\varphi(i)+2}, \ldots, r_{\varphi(i+1)} \in S$ such that $w_{i+1} = r_1 r_2 \cdots r_{\varphi(i)} \cdots r_{\varphi(i+1)}$ is reduced (this defines the same function φ as before). It is easily checked that the resulting reduced expression $w_0 = r_1 r_2 \cdots r_{\varphi(t)}$ has the desired property, hence part (a) is proved.

For part (b), transfer the statement from $\mathcal{F}(\mathcal{K})$ to $W - \{w_0\}$ via the given isomorphism and apply the translation $w \mapsto ww_0$ which is an anti-isomorphism of $W - \{w_0\}$ onto $W - \{e\}$ (cf. [7, p. 43]). The statement then becomes equivalent to the following: For every ordering $s_1, s_2, \ldots, s_t$ of S elements $\rho(s_j) \in W - \{e\}$ can be found so that $W - \{e\} = \biguplus_{j=1}^{t} [s_j, \rho(s_j)]$ and $s_j \leq \rho(s_i)$ implies $i \leq j$. This is accomplished by letting $\rho(s_j)$ be the top element of the parabolic subgroup $W_{\{s_j, s_{j+1}, \ldots, s_t\}}$.

We remark that for Boolean CW complexes the property (b) of the preceding Theorem is equivalent with shellability (cf. [2, Prop. 1.2]). However, for general d-CW-complexes property (b) neither implies nor is implied by shellability, and its presence in these Bruhat order cell complexes seems to be a singular occurrence.

THEOREM 5.3. *Let (W, S) be an infinite Coxeter group. Then there exists an infinite-dimensional regular CW complex $\mathcal{K}$ such that $\mathcal{F}(\mathcal{K})$ and W are isomorphic as posets. Furthermore,*

(a) *$\mathcal{K}$ is contractible,*
(b) *every skeleton $\mathcal{K}^n$ is $(n-1)$-connected, $n = 0, 1, 2, \ldots$, and if $|S| < \infty$ every skeleton $\mathcal{K}^n$ is shellable.*

PROOF. The existence of $\mathscr{K}$ is clear since W is a CW poset of infinite length (cf. 2.5 and 3.1 above). W is a directed poset [5, Lemma 6.4], hence $\Delta(W-\{e\})$ is contractible [12, Proposition 13]. Since $X_{\mathscr{K}} \cong \Delta(W-\{e\})$, part (a) follows. If $|S|<\infty$ then $\hat{\mathscr{F}}(\mathscr{K}^n)$ is dual CL-shellable [5, Theorem 6.5], hence by Proposition 4.5(b) $\mathscr{K}^n$ is a shellable n-CW-complex, and therefore in particular $(n-1)$-connected. If S is infinite the $(n-1)$-connectedness of $\mathscr{K}^n$ can be deduced from the finite cardinality case e.g. by cellular approximation.

Let G be a complex reductive algebraic group and B a Borel subgroup. Then the homogeneous space G/B has a cell decomposition $G/B = \biguplus_{w\in W} C_w$ induced by the Bruhat decomposition of G. The open cells C_w are indexed by the Weyl group W of G, which is a finite Coxeter group, and $\bar{C}_w \subseteq \bar{C}_{w'}$ if and only if $w \leq w'$ in Bruhat order. See [6, Section 3] for all this. Thus, the Bruhat ordering of a finite Weyl group is always the face poset of a cellular decomposition of a complex projective variety by a general algebraic-geometric construction. This decomposition has cells only in even dimensions, hence is not regular. On the other hand, the synthetic combinatorial construction discussed in this paper has shown that the Bruhat ordering of any Coxeter group (or interval in such) is the face poset of a regular CW complex. It would be of considerable interest to know which (CW) posets can be reasonably interpreted as face posets of cellular decompositions of complex algebraic varieties, and whether there is a synthetic construction for doing so. In particular, can 'synthetic Schubert varieties' be naturally associated with the (lower) Bruhat intervals of any Coxeter group?

6. Convex Polytopes

For a finite poset P, let $\mathrm{Int}(P)$ denote the poset of all intervals $[x, y]$, $x \leq y$, ordered by containment. By convention the empty interval $\varnothing$ is also included in $\mathrm{Int}(P)$. In [15] B. Lindström asked whether given a convex polytope $\mathscr{C}$ with face lattice $\mathscr{F}(\mathscr{C})$ one can always find a convex polytope $\mathscr{D}$ such that $\mathscr{F}(\mathscr{D}) \cong \mathrm{Int}[\mathscr{F}(\mathscr{C})]$? We will answer the topological content of the question in the affirmative. However, the step from a topological realization to a convex realization of course remains open. For convenience we will work with the equivalent dual formulation: given $\mathscr{C}$ find $\mathscr{D}$ such that $\mathscr{F}(\mathscr{D}) \cong \mathrm{Int}[\mathscr{F}(\mathscr{C})]^*$, where the asterisk denotes dual poset (reverse ordering).

THEOREM 6.1. *Given a d-dimensional convex polytope $\mathscr{C}$ one can find a shellable regular CW decomposition $\mathscr{K}$ of the d-sphere such that $\hat{\mathscr{F}}(\mathscr{K}) \cong \mathrm{Int}[\mathscr{F}(\mathscr{C})]^*$ and all d-cells of $\mathscr{K}$ are combinatorially equivalent to (i.e., have the facial incidence structure of) convex d-polytopes.*

PROOF. $\mathscr{F}(\mathscr{C})$ is a CL-shellable poset [5, Theorem 4.5] of length $d+1$. Hence $\mathrm{Int}[\mathscr{F}(\mathscr{C})]$ is CL-shellable (by [1, Theorem 4.6] and [5, Theorem 8.5]) of length $d+2$. Thus, $\mathrm{Int}[\mathscr{F}(\mathscr{C})]^*$ is isomorphic to the face poset $\hat{\mathscr{F}}(\mathscr{K})$ of a shellable regular CW decomposition $\mathscr{K}$ of S^d, by Proposition 4.5(d). It remains to show that $\mathscr{F}(\mathscr{K})$ is a poset of polyhedral type. Via the given isomorphism this is equivalent to showing that all maximal intervals $[[x, x], [\hat{0}, \hat{1}]]$ in $\mathrm{Int}[\mathscr{F}(\mathscr{C})]-\{\varnothing\}$ are of polyhedral type. But $[[x, x], [\hat{0}, \hat{1}]] \cong [\hat{0}, x]^* \times [x, \hat{1}]$, so what we need follows from the following general fact: Given a d_1-polytope $\mathscr{C}_1$ and a d_2-polytope $\mathscr{C}_2$ there exists a (d_1+d_2+1)-polytope $\mathscr{C}_3$ such that $\mathscr{F}(\mathscr{C}_3) \cong \mathscr{F}(\mathscr{C}_1) \times \mathscr{F}(\mathscr{C}_2)$. There seems to be no reference in the literature for this, so we sketch the construction. Let V_1 and V_2 be a pair of orthogonal subspaces of dimensions d_1 and d_2 respectively in the hyperplane $x_1 = 0$ of Euclidean space $\mathbb{R}^{d_1+d_2+1}$. Translate V_2 by a nonzero vector parallel with $(1, 0, 0, \ldots, 0)$ to obtain an affine flat V_2' disjoint from V_1. Now place $\mathscr{C}_1$ in V_1 and $\mathscr{C}_2$ in V_2' and let $\mathscr{C}_3$ be the convex hull of $\mathscr{C}_1 \cup \mathscr{C}_2$.

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