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# Codes over rings of size $p^2$ and lattices over imaginary quadratic fields

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ABSTRACT

Let $\ell > 0$ be a square-free integer congruent to 3 mod 4 and $\mathcal{O}_K$ the ring of integers of the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-\ell})$. Codes $C$ over rings $\mathcal{O}_K/p\mathcal{O}_K$ determine lattices $\Lambda_\ell(C)$ over $K$. If $p \nmid \ell$ then the ring $\mathcal{R} := \mathcal{O}_K/p\mathcal{O}_K$ is isomorphic to $\mathbb{F}_{p^2}$ or $\mathbb{F}_p \times \mathbb{F}_p$. Given a code $C$ over $\mathcal{R}$, theta functions on the corresponding lattices are defined. These theta series $\theta_{\Lambda_\ell(C)}(q)$ can be written in terms of the complete weight enumerators of $C$. We show that for any two $\ell < \ell'$ the first $\frac{\ell+1}{4}$ terms of their corresponding theta functions are the same. Moreover, we conjecture that for $\ell > \frac{p(n+1)(n+2)}{2}$ there is a unique symmetric weight enumerator corresponding to a given theta function. We verify the conjecture for primes $p < 7$, $\ell \leqslant 59$, and small $n$.



## 1. Introduction

Let $\ell > 0$ be a square-free integer congruent to 3 modulo 4, $K = \mathbb{Q}(\sqrt{-\ell})$ be the imaginary quadratic field, and $\mathcal{O}_K$ its ring of integers. Codes, Hermitian lattices, and their theta-functions over rings $\mathcal{R} := \mathcal{O}_K/p\mathcal{O}_K$, for small primes $p$, have been studied by many authors, see [1,4,5], among others. In [1], explicit descriptions of theta functions and MacWilliams identities are given for $p = 2, 3$. In [6] we explored codes $C$ defined over $\mathcal{R}$ for $p > 2$. For any $\ell$ one can construct a lattice $\Lambda_\ell(C)$ via Construction A and define theta functions based on the structure of the ring $\mathcal{R}$. Such constructions suggested some relations between the complete weight enumerator of the code and the theta function of the corresponding lattice. In this paper we give complete proofs of some of the theorems

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in [6]. Furthermore, we study the weight enumerators of such codes in terms of the theta functions of the corresponding lattices. This paper is organized as follows.

In Section 2 we give a brief overview of the basic definitions for codes and lattices and define theta functions over $\mathbb{F}_p$. We define the theta series $\theta_{\Lambda_{a,b}}(q)$ for all cosets in $p\mathcal{O}_K$ and determine relations among such theta series. Two such theta series $\theta_{\Lambda_{a,b}}(q)$ and $\theta_{\Lambda_{m,n}}(q)$ are the same when $(m,n)$ is congruent modulo $p$ to one of the ordered pairs $(a,b), (-a-b,b), (-a,-b), (a+b,-b)$. This implies that we have at most $\frac{(p+1)^2}{4}$ theta series, and when $\ell > 12p^2+1$ we have exactly $\frac{(p+1)^2}{4}$ theta series. In Section 3 we define theta functions on the lattice defined over $\mathcal{R} := \mathcal{O}_K/p\mathcal{O}_K$. We prove in [6] that such a theta series is equal to the evaluation of the complete weight enumerator of the code on the theta series of cosets of $p\mathcal{O}_K$. We also define the symmetric weight enumerator of a code and show that such a theta series is equal to the symmetric weight enumerator evaluated on the theta series of certain cosets of $p\mathcal{O}_K$.

In Section 4, we address a special case of a general problem of the construction of lattices: the injectivity of Construction A. For codes defined over an alphabet of size four (regarded as a quotient of the ring of integers of an imaginary quadratic field), the problem is solved completely in [7]. We expect that similar results as for $p=2$ hold also for odd primes. However, we are not able to get explicit bounds for $p>2$. In Section 5 we display some computational results for $p=3$. Such results confirm our results of Section 4. We compute the theta series for $p=3$, $n=3,4,5$, and $\ell \leqslant 59$. We conjecture that for $\ell > \frac{p(n+1)(n+2)}{2}$ for each given theta series there exists at most one symmetric weight enumerator polynomial corresponding to this theta series.

## 2. Preliminaries

Let $\ell > 0$ be a square free integer and $K = \mathbb{Q}(\sqrt{-\ell})$ be the imaginary quadratic field with discriminant $d_K$. Recall that $d_K = -\ell$ if $\ell \equiv 3 \bmod 4$, and $d_K = -4\ell$ otherwise. Let $\mathcal{O}_K$ be the ring of integers of $K$. A lattice $\Lambda$ over $K$ is an $\mathcal{O}_K$-submodule of $K^n$ of full rank. The Hermitian dual is defined by

$$\Lambda^* = \left\{x \in K^n \;\middle|\; x \cdot \bar{y} \in \mathcal{O}_K, \text{ for all } y \in \Lambda\right\}, \tag{1}$$

where $x \cdot y := \sum_{i=1}^n x_i y_i$ and $\bar{y}$ denotes component-wise complex conjugation. In the case that $\Lambda$ is a free $\mathcal{O}_K$-module, for every $\mathcal{O}_K$ basis $\{v_1, v_2, \dots, v_n\}$ we can associate a Gram matrix $G(\Lambda)$ given by $G(\Lambda) = (v_i . v_j)_{i,j=1}^n$ and the determinant $\det \Lambda := \det(G)$ defined up to squares of units in $\mathcal{O}_K$. If $\Lambda = \Lambda^*$ then $\Lambda$ is Hermitian self-dual (or unimodular) and integral if and only if $\Lambda \subset \Lambda^*$. An integral lattice has the property $\Lambda \subset \Lambda^* \subset \frac{1}{\det \Lambda}\Lambda$. An integral lattice is called even if $x \cdot x \equiv 0 \bmod 2$ for all $x \in \Lambda$, and otherwise it is odd. An odd unimodular lattice is called a Type 1 lattice and even unimodular lattice is called a Type 2 lattice.

The theta series of a lattice $\Lambda$ in $K^n$ is given by

$$\theta_\Lambda(\tau) = \sum_{z \in \Lambda} e^{\pi i \tau z \cdot \bar{z}},$$

where $\tau \in H = \{z \in \mathbb{C}\colon\ Im(z) > 0\}$. Usually we let $q = e^{\pi i \tau}$. Then, $\theta_\Lambda(q) = \sum_{z \in \Lambda} q^{z \cdot \bar{z}}$. The one-dimensional theta series (or Jacobi's theta series) and its shadow are given by

$$\theta_3(q) = \sum_{n \in \mathbb{Z}} q^{n^2}, \qquad \theta_2(q) = \sum_{n \in \mathbb{Z}} q^{(n+1/2)^2} = \sum_{n \in \mathbb{Z}+\frac{1}{2}} q^{n^2}.$$

Let $\ell \equiv 3 \bmod 4$ and $d$ be a positive number such that $\ell = 4d-1$. Then, $-\ell \equiv 1 \bmod 4$. This implies that the ring of integers is $\mathcal{O}_K = \mathbb{Z}[\omega_\ell]$, where $\omega_\ell = \frac{-1+\sqrt{-\ell}}{2}$ and $\omega_\ell^2 + \omega_\ell + d = 0$. The principal norm form of $K$ is given by

$$Q_d(x,y) = |x - y\omega_\ell|^2 = x^2 + xy + dy^2. \tag{2}$$

The structure of $\mathcal{O}_K/p\mathcal{O}_K$ depends on the value of $\ell$ modulo $p$. For $(\frac{a}{p})$ the Legendre symbol,

$$\mathcal{O}_K/p\mathcal{O}_K = \begin{cases} \mathbb{F}_p \times \mathbb{F}_p & \text{if } (\frac{-\ell}{p}) = 1, \\ \mathbb{F}_{p^2} & \text{if } (\frac{-\ell}{p}) = -1, \\ \mathbb{F}_p + u\mathbb{F}_p \text{ with } u^2 = 0 & \text{if } p \mid \ell. \end{cases} \tag{3}$$

In this paper we will focus on the cases when $p \nmid \ell$.

### 2.1. Theta functions over $\mathbb{F}_p$

Let $q = e^{\pi i \tau}$. For integers $a$ and $b$ and a prime $p$, let $\Lambda_{a,b}$ denote the coset $a - b\omega_\ell + p\mathcal{O}_K$. The theta series associated to this coset is

$$\theta_{\Lambda_{a,b}}(q) = \sum_{m,n\in\mathbb{Z}} q^{|a+mp-(b+np)\omega_\ell|^2} = \sum_{m,n\in\mathbb{Z}} q^{Q_d(mp+a,np+b)} = \sum_{m,n\in\mathbb{Z}} q^{p^2 Q_d(m+a/p,n+b/p)}. \tag{4}$$

For a prime $p$ and an integer $j$, consider the one-dimensional theta series

$$\theta_{p,j}(q) := \sum_{n\in\mathbb{Z}} q^{(n+j/2p)^2}. \tag{5}$$

Note that $\theta_{p,j}(q) = \theta_{p,k}(q)$ if and only if $j \equiv \pm k \bmod 2p$.

**Lemma 1.** *One can write $\theta_{\Lambda_{a,b}}(q)$ in terms of one-dimensional theta series defined above in Eq.* (5)*. In particular,*

$$\theta_{\Lambda_{a,b}}(q) = \theta_{p,b}\big(q^{p^2\ell}\big)\theta_{p,2a+b}\big(q^{p^2}\big) + \theta_{p,b+p}\big(q^{p^2\ell}\big)\theta_{p,2a+b+p}\big(q^{p^2}\big). \tag{6}$$

**Proof.** We use the fact that $Q_d(m,n) = m^2 + mn + dn^2 = (m + \frac{n}{2})^2 + \frac{\ell n^2}{4}$.

$$\begin{aligned}
\theta_{\Lambda_{a,b}}(q) &= \sum_{m,n\in\mathbb{Z}} q^{Q_d(mp+a,np+b)} \\
&= \sum_{m,n\in\mathbb{Z}} q^{(mp+a+\frac{np+b}{2})^2 + \frac{\ell(np+b)^2}{4}} \\
&= \sum_{n\in\mathbb{Z}} q^{\ell\frac{(np+b)^2}{4}} \sum_{m\in\mathbb{Z}} q^{(mp+a+\frac{np+b}{2})^2} \\
&= \sum_{n\in\mathbb{Z}} q^{\ell p^2(\frac{n}{2}+\frac{b}{2p})^2} \sum_{m\in\mathbb{Z}} q^{p^2(m+\frac{2a}{2p}+\frac{n}{2}+\frac{b}{2p})^2} \\
&= \sum_{n \text{ even}} q^{\ell p^2(\frac{n}{2}+\frac{b}{2p})^2} \sum_{m\in\mathbb{Z}} q^{p^2(m+\frac{n}{2}+\frac{2a+b}{2p})^2} + \sum_{n \text{ odd}} q^{\ell p^2(\frac{n}{2}+\frac{b}{2p})^2} \sum_{m\in\mathbb{Z}} q^{p^2(m+\frac{n}{2}+\frac{2a+b}{2p})^2} \\
&= \theta_{p,b}\big(q^{p^2\ell}\big)\theta_{p,2a+b}\big(q^{p^2}\big) + \theta_{p,b+p}\big(q^{p^2\ell}\big)\theta_{p,2a+p+b}\big(q^{p^2}\big).
\end{aligned}$$

This completes the proof. $\square$

It would be interesting to determine what happens to the distribution of points on these cosets as $\ell$ increases. In other words, is there any relation among $\theta_{\Lambda_{a,b}}(q)$ as $\ell$ increases?

**Lemma 2.** *For any integers $a, b, m, n$, if the ordered pair $(m, n)$ is congruent modulo $p$ to one of $(a, b)$, $(-a-b, b)$, $(-a, -b)$, $(a+b, -b)$, then $\theta_{\Lambda_{m,n}}(q) = \theta_{\Lambda_{a,b}}(q)$.*

**Proof.** We aim to find sufficient conditions on $a, b, m, n$ so that $\theta_{\Lambda_{m,n}}(q) = \theta_{\Lambda_{a,b}}(q)$. By Lemma 1,

$$\theta_{\Lambda_{m,n}}(q) = \theta_{p,n}\big(q^{p^2\ell}\big)\theta_{p,2m+n}\big(q^{p^2}\big) + \theta_{p,n+p}\big(q^{p^2\ell}\big)\theta_{p,2m+n+p}\big(q^{p^2}\big)$$

and

$$\theta_{\Lambda_{a,b}}(q) = \theta_{p,b}\big(q^{p^2\ell}\big)\theta_{p,2a+b}\big(q^{p^2}\big) + \theta_{p,b+p}\big(q^{p^2\ell}\big)\theta_{p,2a+b+p}\big(q^{p^2}\big).$$

In particular, if we have

$$\theta_{p,n}\big(q^{p^2\ell}\big) = \theta_{p,b}\big(q^{p^2\ell}\big), \tag{7}$$

$$\theta_{p,2m+n}\big(q^{p^2}\big) = \theta_{p,2a+b}\big(q^{p^2}\big), \tag{8}$$

$$\theta_{p,n+p}\big(q^{p^2\ell}\big) = \theta_{p,b+p}\big(q^{p^2\ell}\big), \tag{9}$$

$$\theta_{p,2m+n+p}\big(q^{p^2}\big) = \theta_{p,2a+b+p}\big(q^{p^2}\big) \tag{10}$$

(that is, equating the first terms, equating the second terms, etc.) then we will have $\theta_{\Lambda_{m,n}}(q) = \theta_{\Lambda_{a,b}}(q)$.

Similarly, if we change the order of the terms in $\theta_{\Lambda_{a,b}}(q)$ to obtain

$$\theta_{\Lambda_{a,b}}(q) = \theta_{p,b+p}\big(q^{p^2\ell}\big)\theta_{p,2a+b+p}\big(q^{p^2}\big) + \theta_{p,b}\big(q^{p^2\ell}\big)\theta_{p,2a+b}\big(q^{p^2}\big),$$

we will have $\theta_{\Lambda_{m,n}}(q) = \theta_{\Lambda_{a,b}}(q)$ if

$$\theta_{p,n}\big(q^{p^2\ell}\big) = \theta_{p,b+p}\big(q^{p^2\ell}\big), \tag{11}$$

$$\theta_{p,2m+n}\big(q^{p^2}\big) = \theta_{p,2a+b+p}\big(q^{p^2}\big), \tag{12}$$

$$\theta_{p,n+p}\big(q^{p^2\ell}\big) = \theta_{p,b}\big(q^{p^2\ell}\big), \tag{13}$$

$$\theta_{p,2m+n+p}\big(q^{p^2}\big) = \theta_{p,2a+b}\big(q^{p^2}\big). \tag{14}$$

Eqs. (7)–(10) are satisfied if

$$\theta_{p,n}(q) = \theta_{p,b}(q) \quad \text{and} \quad \theta_{p,2m+n}(q) = \theta_{p,2a+b}(q). \tag{15}$$

Eqs. (11)–(14) are satisfied if

$$\theta_{p,n}(q) = \theta_{p,b+p}(q) \quad \text{and} \quad \theta_{p,2m+n}(q) = \theta_{p,2a+b+p}(q). \tag{16}$$

That is, if Eq. (15) or (16) holds, then $\theta_{\Lambda_{m,n}}(q) = \theta_{\Lambda_{a,b}}(q)$.

From Eq. (15), we have four subcases corresponding to $n \equiv \pm b \bmod 2p$ and $2m + n \equiv \pm(2a + b) \bmod 2p$. If $n \equiv b \bmod 2p$, one finds that $m \equiv a \bmod p$ or $m \equiv -a - b \bmod p$. If $n \equiv -b \bmod 2p$, one finds that $m \equiv a + b \bmod p$ or $m \equiv -a \bmod p$.

From Eq. (16), we have four subcases as well, corresponding to $n \equiv \pm(b + p) \bmod 2p$ and $2m + n \equiv \pm(2a + b + p) \bmod 2p$. If $n \equiv b + p \bmod 2p$, then either $m \equiv a \bmod p$ or $m \equiv -a - b \bmod p$. And if

$n \equiv -b - p \bmod 2p$, then either $m \equiv a + b \bmod p$ or $m \equiv -a \bmod p$. Therefore, if $n \equiv b \bmod p$, then $m \equiv a \bmod p$ or $m \equiv -a - b \bmod p$. If $n \equiv -b \bmod p$, then $m \equiv a + b \bmod p$ or $m \equiv -a \bmod p$. $\square$

The Klein 4-group generated by matrices

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

acts on $(Z/pZ)^2$. The orbits form equivalence classes on $\mathbb{Z}^2$. This equivalence is given by

$$(a,b) \sim (m,n) \quad \text{if } (m,n) \equiv (a,b),\ (-a-b,b),\ (-a,-b),\ \text{or } (a+b,-b) \bmod p.$$

By Lemma 2, if $(a,b) \sim (m,n)$, then

$$\theta_{\Lambda_{a,b}}(q) = \theta_{\Lambda_{m,n}}(q).$$

Then we have the following result:

**Corollary 3.** *For any odd prime $p$, the set $\{\theta_{\Lambda_{a,b}}(q)\colon a,b \in \mathbb{Z}\}$ contains at most $\frac{(p+1)^2}{4}$ elements.*

**Proof.** We will prove this by showing that there are $\frac{(p+1)^2}{4}$ equivalence classes under the relation $\sim$. This will imply that there are at most $\frac{(p+1)^2}{4}$ theta functions. Note that $(a,b) \sim (a+mp, b+np)$ for any $m,n \in \mathbb{Z}$. Thus, it is enough to consider only $a,b \in \{0,\dots,p-1\}$.

Consider the equivalence class of $(a,b)$, which is

$$\big\{(a,b), (-a-b,b), (-a,-b), (a+b,-b)\big\}.$$

This set contains either 1, 2, or 4 elements. (If two elements are equal, then the two remaining elements are also equal.) If $b = 0$, the set contains $(a,0)$ and $(-a,0)$, which are equal if $a = 0$ and non-equal if $a \neq 0$ (using the fact that $p$ is odd). Thus, if $b = 0$, there is one equivalence class corresponding to $a = 0$ and there are $\frac{p-1}{2}$ equivalence classes containing elements with $a \neq 0$.

If $b \neq 0$, then $b \not\equiv -b \bmod p$, so $(a,b)$ and $(-a,-b)$ are distinct mod $p$. This means there are either 2 or 4 elements in the equivalence class of $(a,b)$. Further, $(a,b)$ and $(-a-b,b)$ are congruent mod $p$ if and only if $(-a,-b)$ and $(a+b,-b)$ are congruent mod $p$ if and only if $2a \equiv -b \bmod p$. Thus, if $2a \equiv -b \bmod p$, the equivalence class of $(a,b)$ has 2 elements. There are $p-1$ pairs $(a,b)$ with $b \neq 0$ and $2a \equiv -b \bmod p$, which gives $\frac{p-1}{2}$ equivalence classes. There are $(p-1)^2$ remaining pairs $(a,b)$ for which $b \neq 0$ and $2a \not\equiv -b \bmod p$. The equivalence classes for these pairs contain 4 elements, leading to $\frac{(p-1)^2}{4}$ equivalence classes. Summed up, we have $1 + \frac{p-1}{2} + \frac{p-1}{2} + \frac{(p-1)^2}{4} = \frac{(p+1)^2}{4}$ equivalence classes, meaning there are at most $\frac{(p+1)^2}{4}$ theta functions. $\square$

The next result determines in what cases we have exactly $\frac{(p+1)^2}{4}$ theta functions.

**Theorem 4.** *For any odd prime $p$ and any $d > 3p^2$, the set $\{\theta_{\Lambda_{a,b}}(q)\colon a,b \in \mathbb{Z}\}$ spans a $\frac{(p+1)^2}{4}$ dimension vector space in $\mathbb{Z}[\![q]\!]$. Hence, Lemma 2 is an "if and only if" statement for large enough $d$.*

**Proof.** We prove this by calculating the minimal exponent appearing in the power series of $\theta_{\Lambda_{a,b}}(q)$ for any $a,b \in \mathbb{Z}$. We will find that there are $\frac{(p+1)^2}{4}$ different such minimal exponents, indicating that there is no linear relationship between the $\frac{(p+1)^2}{4}$ corresponding theta series. From Corollary 3, there are at most $\frac{(p+1)^2}{4}$ such series, so we can then conclude that there are exactly $\frac{(p+1)^2}{4}$ such series. Let $a, b \in \mathbb{Z}$ with $0 \leqslant a < p$ and $0 \leqslant b < p$. Expanding $\theta_{\Lambda_{a,b}}(q)$, one finds that

$$\begin{aligned}
\theta_{\Lambda_{a,b}}(q) &= \theta_{p,b}\big(q^{p^2\ell}\big)\theta_{p,2a+b}\big(q^{p^2}\big) + \theta_{p,b+p}\big(q^{p^2\ell}\big)\theta_{p,2a+b+p}\big(q^{p^2}\big) \\
&= \sum_{n\in\mathbb{Z}} q^{p^2\ell(n+b/2p)^2} \sum_{m\in\mathbb{Z}} q^{p^2(m+a/p+b/2p)^2} \\
&\quad + \sum_{n\in\mathbb{Z}} q^{p^2\ell(n+1/2+b/2p)^2} \sum_{m\in\mathbb{Z}} q^{p^2(m+1/2+a/p+b/2p)^2} \\
&= \sum_{n\in\mathbb{Z}} q^{(\ell/4)(2pn+b)^2} \sum_{m\in\mathbb{Z}} q^{(1/4)(2pm+2a+b)^2} \\
&\quad + \sum_{n\in\mathbb{Z}} q^{(\ell/4)(2pn+p+b)^2} \sum_{m\in\mathbb{Z}} q^{(1/4)(2pm+p+2a+b)^2}.
\end{aligned}$$

Using the fact that $0 \leqslant b < p$, the term with the smallest exponent in the first summation is $q^{(\ell/4)b^2}$ and the term with the smallest exponent in the second summation is either $q^{(1/4)(2a+b)^2}$ or $q^{(1/4)(2a+b-2p)^2}$ (depending on how big $2a + b$ is). Thus, the term with minimal exponent in the product of the first two summations is either

$$q^{(\ell/4)b^2} \cdot q^{(1/4)(2a+b)^2} \quad \text{or} \quad q^{(\ell/4)b^2} \cdot q^{(1/4)(2a+b-2p)^2}.$$

Using the fact that $\ell = 4d - 1$, this term is either

$$q^{a^2+ab+db^2} \quad \text{or} \quad q^{(a-p)^2+(a-p)b+db^2}.$$

Working analogously with the product of the second pair of summations, one finds the term with smallest exponent there is either

$$q^{a^2+a(b-p)+d(b-p)^2} \quad \text{or} \quad q^{(a-p)^2+(a-p)(b-p)+d(b-p)^2}.$$

Thus, in the theta series $\theta_{\Lambda_{a,b}}(q)$, the smallest power of $q$ is the minimum of

$$\begin{aligned}
a^2 + ab + db^2, &\qquad (a-p)^2 + (a-p)b + db^2, \\
a^2 + a(b-p) + d(b-p)^2, &\qquad (a-p)^2 + (a-p)(b-p) + d(b-p)^2.
\end{aligned}$$

Let $min(\theta_{\Lambda_{a,b}}(q))$ denote this minimal exponent. Suppose that $min(\theta_{\Lambda_{a,b}}(q)) = min(\theta_{\Lambda_{m,n}}(q))$ for some integers $a, b, m, n \in \{0, 1, \dots, p-1\}$ and some value of $d$. Then, $min(\theta_{\Lambda_{a,b}}(q)) = u^2 + uv + dv^2$, where $u = a$ or $u = a - p$ and $v = b$ or $v = b - p$. Similarly, $min(\theta_{\Lambda_{m,n}}(q)) = x^2 + xy + dy^2$ where $x = m$ or $x = m - p$ and $y = n$ or $y = n - p$. Note that we have $|u|, |v|, |x|, |y| \leqslant p$. We have two cases to consider, either $v^2 \neq y^2$ or $v^2 = y^2$.

If $v^2 \neq y^2$, then, solving for $d$, we find that $d = \frac{u^2+uv-x^2-xy}{y^2-v^2}$. Thus,

$$|d| \leqslant \frac{|u^2 - x^2| + |uv| + |-xy|}{|y^2 - v^2|} \leqslant \frac{p^2 + p^2 + p^2}{|y^2 - v^2|} \leqslant 3p^2.$$

If $v^2 = y^2$, then given that $u^2 + uv + dv^2 = x^2 + xy + dy^2$, we find that $y = \pm v$. If $y = v$, then we find $u^2 + uv = x^2 + xv$, so $u^2 - x^2 + uv - xv = 0$, so $(u - x)(u + x + v) = 0$. Thus, $x = u$ or $x = -u - v$. Similarly, if $y = -v$, then $u^2 + uv = x^2 - xv$, which implies that $x = -u$ or $x = u + v$.

Using the facts that $u \equiv a \bmod p$, $v \equiv b \bmod p$, $x \equiv m \bmod p$, and $y \equiv n \bmod p$, we find that $(m,n)$ is congruent modulo $p$ to one of the ordered pairs $(a,b)$, $(-a-b,b)$, $(-a,-b)$, $(a+b,-b)$. Hence, if $d > 3p^2$, then $\theta_{\Lambda_{a,b}}(q) = \theta_{\Lambda_{m,n}}(q)$ if and only if $(a,b) \sim (m,n)$. By the above corollary, there are precisely $\frac{(p+1)^2}{4}$ equivalence classes. Hence, there are precisely $\frac{(p+1)^2}{4}$ theta functions. Furthermore, since these theta functions all have different leading exponents, they are linearly independent. This completes the proof. $\square$

**Remark 5.** The bound for $d$ given in Theorem 4 is not sharp. For instance, using a computer algebra package, one finds that for $d=2$, there are $\frac{(p+1)^2}{4}$ equivalence classes for all primes $p \leqslant 19$.

## 3. Theta functions of codes over $\mathcal{O}_K/p\mathcal{O}_K$

Let $p \nmid \ell$ and $\mathcal{R} := \mathcal{O}_K/p\mathcal{O}_K = \{a + b\omega\colon a, b \in \mathbb{F}_p,\ \omega^2+\omega+d=0\}$. We have the map

$$\rho_{\ell,p} : \mathcal{O}_K \to \mathcal{O}_K/p\mathcal{O}_k =: \mathcal{R}.$$

A linear code $C$ of length $n$ over $\mathcal{R}$ is an $\mathcal{R}$-submodule of $\mathcal{R}^n$. The dual is defined as $C^\perp = \{u \in \mathcal{R}^n\colon u \cdot \bar{v} = 0 \text{ for all } v \in C\}$. If $C = C^\perp$ then $C$ is self-dual. We define

$$\Lambda_\ell(C) := \big\{u = (u_1, \ldots, u_n) \in \mathcal{O}_K^n\colon \big(\rho_{\ell,p}(u_1), \ldots, \rho_{\ell,p}(u_n)\big) \in C\big\}.$$

In other words, $\Lambda_\ell(C)$ consists of all vectors in $\mathcal{O}_K^n$ in the inverse image of $C$, taken componentwise by $\rho_{\ell,p}$. This method of lattice construction is known as Construction A.

For notation, let $r_{a+pb+1} = a - b\omega$, so $\mathcal{R} = \{r_1, \ldots, r_{p^2}\}$. For a codeword $u = (u_1, \ldots, u_n) \in \mathcal{R}^n$ and $r_i \in \mathcal{R}$, we define the counting function $n_i(u) := \#\{i\colon u_i = r_i\}$. The complete weight enumerator of the $\mathcal{R}$ code $C$ is the polynomial

$$cwe_C(z_1, z_2, \ldots, z_{p^2}) = \sum_{u \in C} z_1^{n_1(u)} z_2^{n_2(u)} \ldots z_{p^2}^{n_{p^2}(u)}. \tag{17}$$

We can use this polynomial to find the theta function of the lattice $\Lambda_\ell(C)$. For a proof of the following result see [6].

**Lemma 6.** *Let $C$ be a code defined over $\mathcal{R}$ and $cwe_C$ its complete weight enumerator as above. Then,*

$$\theta_{\Lambda_\ell(C)}(q) = cwe_C\big(\theta_{\Lambda_{0,0}}(q), \theta_{\Lambda_{1,0}}(q), \ldots, \theta_{\Lambda_{p-1,p-1}}(q)\big).$$

**Remark 7.** The connection between complete weight enumerators of self-dual codes over $\mathbb{F}_p$ and Siegel theta series of unimodular lattices is well known. Construction A associates to any length $n$ code $C = C^\perp$ an $n$-dimensional unimodular lattice; see [3] for details.

For $p = 2$, we have

$$\theta_{\Lambda_\ell(C)}(q) = cwe_C\big(\theta_{\Lambda_{0,0}}(q), \theta_{\Lambda_{1,0}}(q), \theta_{\Lambda_{0,1}}(q), \theta_{\Lambda_{1,1}}(q)\big).$$

Since $\theta_{\Lambda_{0,1}}(q) = \theta_{\Lambda_{1,1}}(q)$ (by Lemma 2), we can define the symmetric weight enumerator $swe_C$ by

$$swe_C(X, Y, Z) = cwe_C(X, Y, Z, Z)$$

to get

$$\theta_{\Lambda_\ell(C)}(q) = swe_C\big(\theta_{\Lambda_{0,0}}(q), \theta_{\Lambda_{1,0}}(q), \theta_{\Lambda_{0,1}}(q)\big).$$

These three theta functions are referred to as $A_d(q), C_d(q),$ and $G_d(q)$ in [2] and [7].

More generally, for odd $p$, the complete weight enumerator takes $p^2$ arguments corresponding to the $p^2$ lattices $\Lambda_{a,b}(q)$ and their theta functions. By Theorem 4, for $\ell$ large enough, there are only $\frac{(p+1)^2}{4}$ different theta functions among these $p^2$ lattices. As above with $p=2$, we define the symmetric weight enumerator of a code in terms of the complete weight enumerator, using the same variable for lattices that have the same theta series.

For the case where $p=3$, from Remark 2.2 in [6], we have four theta functions corresponding to the lattices $\Lambda_{a,b}$, namely $\theta_{\Lambda_{0,0}}(q)$, $\theta_{\Lambda_{1,0}}(q)$, $\theta_{\Lambda_{1,1}}(q)$, $\theta_{\Lambda_{0,1}}(q)$. We define the symmetric weight enumerator to be

$$swe_{\mathcal{C}}(X,Y,Z,W) = cwe_{\mathcal{C}}(X,Y,Y,Z,W,Z,Z,Z,W).$$

One then has

$$\begin{aligned}\theta_{\Lambda_\ell(C)}(q) &= cwe_{\mathcal{C}}\big(\theta_{\Lambda_{0,0}}(q), \theta_{\Lambda_{1,0}}(q), \dots, \theta_{\Lambda_{2,2}}(q)\big)\\ &= swe_{\mathcal{C}}\big(\theta_{\Lambda_{0,0}}(q), \theta_{\Lambda_{1,0}}(q), \theta_{\Lambda_{1,1}}(q), \theta_{\Lambda_{0,1}}(q)\big).\end{aligned}$$

**Example 8.** Let $\mathcal{C}_1$ be the length-2 repetition code $\{(x,x): x \in \mathcal{R}\}$ for $p=3$. The complete weight enumerator of this code is

$$cwe_{\mathcal{C}_1}(z_0,\dots,z_8) = z_0^2 + z_1^2 + \dots + z_8^2,$$

and the symmetric weight enumerator is

$$swe_{\mathcal{C}_1}(X,Y,Z,W) = X^2 + 2Y^2 + 4Z^2 + 2W^2.$$

Using a computational algebra package, one can calculate $\theta_{\Lambda_\ell(C)}(q)$ for each $\ell$. We display the cases when $\ell = 7, 11$.

$$\begin{aligned}\theta_{\Lambda_7(C)}(q) &= 1 + 2q^2 + 4q^4 + 4q^5 + 10q^8 + 4q^9 + 16q^{10} + 8q^{11} + 8q^{13} + 2q^{14}\\ &\quad + 24q^{16} + 12q^{17} + 12q^{18} + 16q^{19} + 28q^{20} + 20q^{22} + 16q^{23} + 16q^{25}\\ &\quad + 28q^{26} + 16q^{27} + 4q^{28} + 20q^{29} + 24q^{31} + 42q^{32} + 32q^{34} + 4q^{35}\\ &\quad + 28q^{36} + 24q^{37} + 40q^{38} + 56q^{40} + 28q^{41} + 32q^{43} + 56q^{44} + 24q^{45}\\ &\quad + 52q^{46} + 32q^{47} + 62q^{50} + \cdots,\\ \theta_{\Lambda_{11}(C)}(q) &= 1 + 2q^2 + 4q^5 + 4q^6 + 2q^8 + 4q^9 + 8q^{10} + 8q^{12} + 8q^{15} + 8q^{16}\\ &\quad + 4q^{17} + 24q^{18} + 4q^{20} + 8q^{21} + 2q^{22} + 20q^{24} + 16q^{25} + 12q^{26}\\ &\quad + 24q^{27} + 16q^{28} + 12q^{29} + 24q^{30} + 8q^{31} + 10q^{32} + 8q^{34} + 8q^{35}\\ &\quad + 36q^{36} + 8q^{38} + 16q^{39} + 8q^{40} + 20q^{41} + 24q^{42} + 16q^{43} + 32q^{45}\\ &\quad + 8q^{46} + 8q^{47} + 40q^{48} + 8q^{49} + 22q^{50} + \cdots.\end{aligned}$$

It will be the goal of our next section to study how the corresponding theta function of a given code differs for different levels $\ell$.

## 4. Theta functions and the corresponding complete weight enumerator polynomials

For a fixed prime $p$, let $C$ be a linear code over $\mathcal{R} = \mathbb{F}_{p^2}$ or $\mathbb{F}_p \times \mathbb{F}_p$ of length $n$ and dimension $k$. An admissible level $\ell$ is an integer $\ell$ such that $\mathcal{O}_K/p\mathcal{O}_K$ is isomorphic to $\mathcal{R}$. For an admissible $\ell$, let $\Lambda_\ell(C)$ be the corresponding lattice as in the previous section. Then, the **level $\ell$ theta function** $\theta_{\Lambda_\ell(C)}(q)$ of the lattice $\Lambda_\ell(C)$ is determined by the symmetric weight enumerator $cwe_C$ of $C$ evaluated on the theta functions defined on certain cosets of $\mathcal{O}_K/p\mathcal{O}_K$. We consider the following questions. How do the theta functions $\theta_{\Lambda_\ell(C)}(q)$ of the same code $C$ differ for different levels $\ell$? Can non-equivalent codes give the same theta functions for all levels $\ell$?

We give a satisfactory answer to the first question (cf. Theorem 11, Lemma 12) and for the second question we conjecture that:

**Conjecture 9.** *Let $C$ be a code of size $n$ defined over $\mathcal{R}$ and $\theta_{\Lambda_\ell(C)}(q)$ be its corresponding theta function for level $\ell$. Then, for large enough $\ell$, there is a unique symmetric weight enumerator polynomial which corresponds to $\theta_{\Lambda_\ell(C)}(q)$.*

Let $C$ be a code defined over $\mathcal{R}$ for a fixed $p > 2$. Let the complete weight enumerator of $C$ be the degree $n$ polynomial $cwe_C = f(x_1, \dots, x_{p^2})$. Then from Lemma 6 we have that

$$\theta_{\Lambda_\ell(\mathcal{C})}(q) = f\big(\theta_{\Lambda_{0,0}}(q), \dots, \theta_{\Lambda_{p-1,p-1}}(q)\big)$$

for a given $\ell$. First we want to address how $\theta_{\Lambda_\ell(\mathcal{C})}(q)$ and $\theta_{\Lambda_{\ell'}(\mathcal{C})}(q)$ differ for different $\ell$ and $\ell'$. The proof of the following remark is elementary.

**Remark 10.** For $n \neq 0$, $Q_d(m,n) \geqslant d$.

Then we have the following theorem.

**Theorem 11.** *Let $C$ be a code defined over $\mathcal{R}$. For all admissible $\ell, \ell'$ with $\ell < \ell'$ the following holds*

$$\theta_{\Lambda_\ell(C)}(q) = \theta_{\Lambda_{\ell'}(C)}(q) + \mathcal{O}\big(q^{\frac{\ell+1}{4}}\big).$$

**Proof.** From Section 3, we have the map $\rho_{\ell,p} : \mathcal{O}_K \to \mathcal{O}_K/p\mathcal{O}_K \to \mathcal{R}$ and

$$\Lambda_\ell(C) = \big\{u = (u_1, \dots, u_n) \in \mathcal{O}_K^n\colon\ \big(\rho_{\ell,p}(u_1), \dots, \rho_{\ell,p}(u_n)\big) \in C\big\}.$$

We denote $u_i = a_i - b_i\omega_\ell$ for $a_i, b_i \in \mathbb{Z}$ with $i = 1, \dots, n$ and $d = \frac{\ell+1}{4}$. Then

$$\begin{aligned}
\theta_{\Lambda_\ell(C)}(q) &= \sum_{u\in\Lambda_\ell(C)} q^{u\cdot\bar{u}} = \sum_{u\in\Lambda_\ell(C)} q^{u_1\bar{u_1}+\cdots+u_n\bar{u_n}} \\
&= \sum_{u\in\Lambda_\ell(C)} q^{Q_d(a_1,b_1)+\cdots+Q_d(a_n,b_n)} \\
&= \sum_{\substack{u\in\Lambda_\ell(C),\\ b_i=0 \text{ for all } i}} q^{Q_d(a_1,b_1)+\cdots+Q_d(a_n,b_n)} + \sum_{\substack{u\in\Lambda_\ell(C),\\ b_i\neq 0 \text{ for some } i}} q^{Q_d(a_1,b_1)+\cdots+Q_d(a_n,b_n)} \\
&= \sum_{\substack{u\in\Lambda_\ell(C),\\ b_i=0 \text{ for all } i}} q^{a_1^2+\cdots+a_n^2} + \sum_{\substack{u\in\Lambda_\ell(C),\\ b_i\neq 0 \text{ for some } i}} q^{Q_d(a_1,b_1)+\cdots+Q_d(a_n,b_n)}.
\end{aligned}$$

Note that this first summation does not depend on $d$ (or $\ell$). In the second summation, each term's exponent contains a term of the form $Q_d(a_i, b_i)$ where $b_i \neq 0$. By the lemma above, we have $Q_d(a_i, b_i) \geqslant d$. Since all of the terms in the exponent are added, each term in the second summation has exponent at least $d$. Hence, the second summation is $\mathcal{O}(q^d)$. Thus, we have

$$\theta_{\Lambda_\ell(C)}(q) = \sum_{\substack{u \in \Lambda_\ell(C),\\ b_i=0 \text{ for all } i}} q^{a_1^2+\cdots+a_n^2} + \mathcal{O}\big(q^d\big).$$

Similarly,

$$\theta_{\Lambda_{\ell'}(C)}(q) = \sum_{\substack{u \in \Lambda_{\ell'}(C),\\ b_i=0 \text{ for all } i}} q^{a_1^2+\cdots+a_n^2} + \mathcal{O}\big(q^{d'}\big).$$

For admissible $\ell$, $\ell'$ with $\ell < \ell'$, we conclude that

$$\theta_{\Lambda_\ell(C)}(q) - \theta_{\Lambda_{\ell'}(C)}(q) = \mathcal{O}\big(q^d\big).$$

This completes the proof. □

We have the following lemma.

**Lemma 12.** *Let $C$ be a code of size $n$ defined over $\mathcal{R}$ and $\theta(q) = \sum \lambda_i q^i$ be its theta function for level $\ell$. Then, there exists a bound $B_{\ell,p,n}$ such that $\theta(q)$ is uniquely determined by its first $B_{\ell,p,n}$ coefficients.*

**Proof.** We want to show that if $\theta(q) - \theta'(q) = \mathcal{O}(q^{B_{\ell,p,n}})$, then $\theta(q) = \theta'(q)$. Fix $p, n, \ell$. There are finitely many codes $C$ over $\mathcal{R}$ of length $n$. Denote them by $C_1, \ldots, C_m$, for some integer $m$. To each code $C_i$, there is a corresponding theta function $\theta_{C_i}(q)$.

Let

$$S = \big\{ r \in \mathbb{Z}_{\geqslant 0}\colon\ \theta_{C_i}(q) - \theta_{C_j}(q) = \mathcal{O}\big(q^r\big) \text{ and } \theta_{C_i}(q) \neq \theta_{C_j}(q) \big\}$$

and let $B_{\ell,p,n} = 1 + \max S$. Since $S$ is finite, $B_{\ell,p,n}$ is well defined. Furthermore, if $\theta_{C_i}(q) - \theta_{C_j}(q) = \mathcal{O}(q^m)$ for some $m \geqslant B_{\ell,p,n}$, this implies that $m \notin S$, so we must have $\theta_{C_i}(q) = \theta_{C_j}(q)$. □

For notation, when $p$ and $n$ are fixed, we will let $B_\ell = B_{\ell,p,n}$.

To extend the theory for $p = 2$ to $p > 2$ we have to find a relation between the theta function $\theta_{\Lambda_\ell(C)}(q)$ and the number of symmetric weight enumerator polynomials corresponding to it.

Fix an odd prime $p$ and let $C$ be a given code of length $n$ over $\mathcal{R}$. Choose an admissable value of $\ell$ such that there are $\frac{(p+1)^2}{4}$ independent theta functions (as in Theorem 4). Then, the symmetric weight enumerator of $C$ has degree $n$ and $r = \frac{(p+1)^2}{4}$ variables $x_1, \ldots, x_r$.

**Lemma 13.** *A homogeneous polynomial of degree $n$ in $r = \frac{(p+1)^2}{4}$ variables has $s := \frac{(n-1+\frac{(p+1)^2}{4})!}{n!\cdot(\frac{(p+1)^2}{4}-1)!}$ monomials.*

**Proof.** We need to count the number of monomials of a homogeneous degree $n$ polynomial in $r = \frac{(p+1)^2}{4}$ variables which is

$$s = \frac{(n+r-1)!}{n!(r-1)!} = \frac{(n-1+\frac{(p+1)^2}{4})!}{n!\cdot(\frac{(p+1)^2}{4}-1)!}$$

This completes the proof. □

Denote by $P(x_1,\dots,x_r)$ a generic $r$-nary degree $n$ homogeneous polynomial. Assume that there is a length $n$ code $C$ defined over $\mathcal{R}$ such that $P(x_1,\dots,x_r)$ is the symmetric weight enumerator polynomial. In other words,

$$swe_C(x_1,\dots,x_r) = P(x_1,\dots,x_r).$$

Fix the level $\ell$. Then, by replacing $x_1, x_2, \dots, x_r$ with the $r$ different theta series corresponding to the lattices $\Lambda_{a,b}(q)$, we compute the left side of the above equation as a series $\sum_{i=0}^{\infty} \lambda_i q^i$. By equating both sides of $\sum_{i=0}^{\infty} \lambda_i q^i = P(x_1,\dots,x_r)$, we get a linear system of equations. Since the first $\lambda_0,\dots,\lambda_{B_{\ell-1}}$ determine all the coefficients of the theta series then we have to pick $B_\ell$ equations (these equations are not necessarily independent).

Consider the coefficients of the polynomial $P(x_1,\dots,x_r)$ as parameters $c_1,\dots,c_s$. Then, the linear map

$$L_\ell : \mathbb{C}^s \to \mathbb{C}^{B_{\ell-1}}$$

$$(c_1,\dots,c_s) \mapsto (\lambda_0,\dots,\lambda_{B_{\ell-1}})$$

has an associated matrix $M_\ell$. For a fixed value of $(\lambda_0,\dots,\lambda_{B_{\ell-1}})$, determining th rank of the matrix $M_\ell$ would determine the number of polynomials giving the same theta series. There is a unique symmetric weight enumerator corresponding to a given theta function when

$$\operatorname{null}(M_\ell) = s - \operatorname{rank}(M_\ell) = 0.$$

**Conjecture 14.** *For* $\ell \geqslant \frac{p(n+1)(n+2)}{n} - 1$ *we have* $\operatorname{null} M_\ell = 0$, *or in other words*

$$\operatorname{rank}(M_\ell) = \frac{(n-1+\frac{(p+1)^2}{4})!}{n!\cdot(\frac{(p+1)^2}{4}-1)!}.$$

The choice of $\ell$ is taken from experimental results for primes $p = 2$ and 3. More details are given in the next section.

It is obvious that Conjecture 14 implies Conjecture 9. If Conjecture 9 is true then for large enough $\ell$ there would be a one to one correspondence between the symmetric weight enumerator polynomials and the corresponding theta functions. Perhaps, more interesting is to find $\ell$ and $p$ for which there is not a one to one such correspondence. Consider the map

$$\Phi(\ell, p) = \big(\lambda_0(\ell, p), \dots, \lambda_{B_\ell - 1}(\ell, p)\big),$$

where $\lambda_0,\dots,\lambda_{B_\ell-1}$ are now functions in $\ell$ and $p$. Let $V$ be the variety given by the Jacobian of the map $\Phi$. Finding integer points $\ell, p$ on this variety such that $\ell$ and $p$ satisfy our assumptions would give us values for $\ell, p$ when the above correspondence is not one to one. However, it seems quite hard to get explicit description of the map $\Phi$. Next, we will try to shed some light over the above conjectures for fixed small primes $p$.

## 5. Bounds for small primes

In [7] we determine explicit bounds for the above theorems for prime $p = 2$. In this section we give some computation evidence for the generalization of the result for $p = 3$. We recall the theorem for $p = 2$.

**Theorem 15.** *(See [7, Thm. 2].) Let $p = 2$ and $C$ be a code of size $n$ defined over $\mathcal{R}$ and $\theta_{\Lambda_\ell(C)}(q)$ be its corresponding theta function for level $\ell$. Then the following hold*:

(i) *For* $\ell < \frac{2(n+1)(n+2)}{n} - 1$ *there is a* $\delta$*-dimensional family of symmetric weight enumerator polynomials corresponding to* $\theta_{\Lambda_\ell(C)}(q)$*, where* $\delta \geqslant \frac{(n+1)(n+2)}{2} - \frac{n(\ell+1)}{4} - 1$.

(ii) *For* $\ell \geqslant \frac{2(n+1)(n+2)}{n} - 1$ *and* $n < \frac{\ell+1}{4}$ *there is a unique symmetric weight enumerator polynomial which corresponds to* $\theta_{\Lambda_\ell(C)}(q)$.

These results were obtained by using the explicit expression of theta in terms of the symmetric weight enumerator valuated on the theta functions of the cosets.

Next we want to find explicit bounds for $p = 3$ as in the case of $p = 2$. In the case of $p = 3$ it is enough to consider four theta functions, $\theta_{\Lambda_{0,0}}(q)$, $\theta_{\Lambda_{1,0}}(q)$, $\theta_{\Lambda_{0,1}}(q)$, and $\theta_{\Lambda_{1,1}}(q)$ since $\theta_{\Lambda_{2,0}}(q) = \theta_{\Lambda_{1,0}}(q)$, $\theta_{\Lambda_{2,2}}(q) = \theta_{\Lambda_{1,1}}(q)$ and $\theta_{\Lambda_{0,2}}(q) = \theta_{\Lambda_{1,2}}(q) = \theta_{\Lambda_{2,1}}(q) = \theta_{\Lambda_{0,1}}(q)$. If we are given a code $C$ and its symmetric weight enumerator polynomial, then we can find the theta function of the lattice constructed from $C$ using Construction A. Let $\theta(q) = \sum_{i=0}^{\infty} \lambda_i q^i$ be the theta series for level $\ell$ and

$$p(x, y, z, w) = \sum_{i+j+k+m=n} c_{i,j,k} x^i y^j z^k w^m$$

be a degree $n$ generic 4-nary homogeneous polynomial. We would like to find out how many polynomials $p(x, y, z, w)$ correspond to $\theta(q)$ for a fixed $\ell$. For a given $\ell$ find $\theta_{\Lambda_{0,0}}(q)$, $\theta_{\Lambda_{1,0}}(q)$, $\theta_{\Lambda_{0,1}}(q)$ and $\theta_{\Lambda_{1,1}}(q)$ and substitute them in the $p(x, y, z, w)$. Hence, $p(x, y, z, w)$ is now written as a series in $q$. We get infinitely many equations by equating the corresponding coefficients of the two sides of the equation

$$p\big(\theta_{\Lambda_{0,0}}(q), \theta_{\Lambda_{1,0}}(q), \theta_{\Lambda_{0,1}}(q), \theta_{\Lambda_{1,1}}(q)\big) = \sum_{i=0}^{\infty} \lambda_i q^i.$$

Since the first $\lambda_0, \ldots, \lambda_{B_\ell - 1}$ determine all the coefficients of the theta series then it is enough to pick the first $B_\ell$ equations. The linear map

$$L_\ell : (c_1, \ldots, c_{20}) \mapsto (\lambda_0, \ldots, \lambda_{B_\ell - 1})$$

has an associated matrix $M_\ell$. If the nullity of $M_\ell$ is zero then we have a unique polynomial that corresponds to the given theta series. We have calculated the nullity of the matrix and $B_\ell$ for small $n$ and $\ell$.

**Example 16** *(The case* $p = 3, n = 3$*).* The generic homogeneous polynomial is given by

$$\begin{aligned} P(x, y, z) = {} & c_1 x^3 + c_2 x^2 y + c_3 x^2 z + c_4 x^2 w + c_5 x y^2 + c_6 x z^2 + c_7 x w^2 + c_8 x y z \\ & + c_9 x y w + c_{10} x z w + c_{11} y^3 + c_{12} y^2 z + c_{13} y^2 w + c_{14} y z^2 + c_{15} y w^2 \\ & + c_{16} y z w + c_{17} z^3 + c_{18} z^2 w + c_{19} z w^2 + c_{20} w^3. \end{aligned} \tag{18}$$

The system of equations can be written by the form of

$$A\vec{c} = \vec{\lambda}$$

where $\vec{c} = (c_1 \quad c_2 \quad \cdots \quad c_{20})^t$, $\vec{\lambda} = (\lambda_0 \quad \lambda_1 \quad \cdots \quad \lambda_{15})^t$. In the case of $\ell = 7$ the matrix $M_7$ has $\mathrm{null}(M_7) = 4$. We have a positive dimension family of solution set. The case of $\ell = 11$ the matrix $M_{11}$ has $\mathrm{null}(M_{11}) = 1$. For any case where $\ell \geqslant 19$ the nullity of the matrix is 0. Hence, for every given theta series, there is a unique symmetric weight enumerator polynomial.

We summarize the results in the following table:

| $\ell$ | $n=3$ | | $n=4$ | | $n=5$ | |
|---|---|---|---|---|---|---|
| | $B_\ell$ | null $M_\ell$ | $B_\ell$ | null $M_\ell$ | $B_\ell$ | null $M_\ell$ |
| 7 | 16 | 4 | 26 | 9 | 33 | 24 |
| 11 | 19 | 1 | 30 | 5 | 42 | 14 |
| 19 | 22 | 0 | 38 | 0 | 60 | 0 |
| 23 | 25 | 0 | 37 | 0 | 58 | 0 |
| 31 | 31 | 0 | 41 | 0 | 60 | 0 |
| 35 | 34 | 0 | 48 | 0 | 61 | 0 |
| 43 | 40 | 0 | 55 | 0 | 69 | 0 |
| 47 | 43 | 0 | 60 | 0 | 74 | 0 |
| 55 | 49 | 0 | 70 | 0 | 86 | 0 |
| 59 | 52 | 0 | 75 | 0 | 92 | 0 |

It seems from the table that the same bound of $B_\ell = \frac{2(n+1)(n+2)}{n}$ as for $p=2$ holds also for $p=3$, $n=3$.

We have the following conjecture for general $p, n$ and $\ell$.

**Conjecture 17.** *For a given theta function $\theta_{\Lambda_\ell(C)}(q)$ of a code $C$ for level $\ell$ there is a unique symmetric weight enumerator polynomial corresponding to $\theta_{\Lambda_\ell(C)}(q)$ if $\ell \geqslant \frac{p(n+1)(n+2)}{n}$.*

It is interesting to consider such question for lattices $\mathcal{O}_K/p\mathcal{O}_K$ independently of the connection to coding theory. What is the meaning of the bound $B_\ell$ for the ring $\mathcal{O}_K/p\mathcal{O}_K$? Do the theta functions defined here correspond to any modular forms? Is there any difference between the cases when the ring is $\mathbb{F}_p \times \mathbb{F}_p$ or $\mathbb{F}_{p^2}$?

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