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Strong differential subordination and superordination of analytic functions

M.P. Jeyaraman [a,*], T.K. Suresh [b]

[a] Department of Mathematics, L. N. Government College, Ponneri, Chennai 601 204, Tamilnadu, India
[b] Department of Mathematics, Easwari Engineering College, Chennai 600 089, Tamilnadu, India

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ABSTRACT

Strong differential subordination and superordination results are obtained for analytic functions in the open unit disk by investigating appropriate classes of admissible functions. New strong differential sandwich-type results are also obtained.

1. Introduction, definitions and preliminaries

Let $\mathcal{H}(\mathbb{U})$ denote the class of analytic functions in the open unit disk

$$\mathbb{U} := \{z \in \mathbb{C}\colon |z| < 1\}.$$

For $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$ and $a \in \mathbb{C}$, let

$$\mathcal{H}[a, n] = \{f\colon\ f \in \mathcal{H} \text{ and } f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\},$$

with $\mathcal{H}_0 \equiv \mathcal{H}[0, 1]$ and $\mathcal{H} \equiv \mathcal{H}[1, 1]$. Let $\mathcal{A}$ denote the class of all normalized analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \mathbb{U}).$$

Let f and F be members of $\mathcal{H}(\mathbb{U})$. The function f is said to be *subordinate* to F, or (equivalently) F is said to be *superordinate* to f, if there exists a Schwarz function w analytic in $\mathbb{U}$, with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1,$$

such that

$$f(z) = F(w(z)).$$

In such a case, we write

$$f \prec F \quad \text{or} \quad f(z) \prec F(z).$$

* Corresponding author.
E-mail addresses: jeyaraman_mp@yahoo.co.in (M.P. Jeyaraman), tksuresh73@yahoo.com (T.K. Suresh).

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If the function F is univalent in $\mathbb{U}$, then we have

$$f \prec F \iff f(0) = F(0) \quad \text{and} \quad f(\mathbb{U}) \subset F(\mathbb{U}).$$

Let $H(z,\zeta)$ be analytic in $\mathbb{U} \times \overline{\mathbb{U}}$ and let $f(z)$ be analytic and univalent in $\mathbb{U}$. Then the function $H(z,\zeta)$ is said to be *strongly subordinate* to $f(z)$, or $f(z)$ is said to be *strongly superordinate* to $H(z,\zeta)$, written as

$$H(z,\zeta) \prec\prec f(z),$$

if, for $\zeta \in \overline{\mathbb{U}}$, $H(z,\zeta)$ as a function of z is subordinate to $f(z)$. We note that

$$H(z,\zeta) \prec\prec f(z) \iff H(0,\zeta) = f(0) \quad \text{and} \quad H(\mathbb{U} \times \overline{\mathbb{U}}) \subset f(\mathbb{U}).$$

Definition 1.1. (See [8].) Let

$$\phi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$$

and let $h(z)$ be univalent in $\mathbb{U}$. If $p(z)$ is analytic in $\mathbb{U}$ and satisfies the following (second-order) differential subordination:

$$\phi\big(p(z), zp'(z), z^2p''(z); z, \zeta\big) \prec\prec h(z), \tag{1.1}$$

then $p(z)$ is called a solution of the strong differential subordination. The univalent function $q(z)$ is called a dominant of the solutions of the strong differential subordination or more simply a dominant if

$$p(z) \prec q(z)$$

for all $p(z)$ satisfying (1.1). A dominant $\tilde{q}(z)$ that satisfies

$$\tilde{q}(z) \prec q(z)$$

for all dominants $q(z)$ of (1.1) is said to be the best dominant.

Recently, Oros [6] introduced the following notion of strong differential superordination as the dual concept of strong differential subordination.

Definition 1.2. (See [5,6].) Let

$$\varphi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$$

and let $h(z)$ be analytic in $\mathbb{U}$. If

$$p(z) \quad \text{and} \quad \varphi\big(p(z), zp'(z), z^2p''(z); z, \zeta\big)$$

are univalent in $\mathbb{U}$ for $\zeta \in \overline{\mathbb{U}}$ and satisfy the following (second-order) strong differential superordination:

$$h(z) \prec\prec \varphi\big(p(z), zp'(z), z^2p''(z); z, \zeta\big), \tag{1.2}$$

then $p(z)$ is called a solution of the strong differential superordination. An analytic function $q(z)$ is called a subordinant of the solution of the strong differential superordination or more simply a subordinant if $q(z) \prec p(z)$ for all $p(z)$ satisfying (1.2). A univalent subordinant $\tilde{q}(z)$ that satisfies

$$q(z) \prec \tilde{q}(z)$$

for all subordinants $q(z)$ of (1.2) is said to be the best subordinant.

We denote by $\mathcal{Q}$ the class of functions q that are analytic and injective on $\overline{\mathbb{U}} \setminus E(q)$, where

$$E(q) = \Big\{\xi \in \partial\mathbb{U}\colon \lim_{z\to\xi} q(z) = \infty\Big\},$$

and are such that $q'(\xi) \neq 0$ for $\xi \in \partial\mathbb{U} \setminus E(q)$. Further, let the subclass of $\mathcal{Q}$ for which $q(0) = a$ be denoted by $\mathcal{Q}(a)$, $\mathcal{Q}(0) \equiv \mathcal{Q}_0$ and $\mathcal{Q}(1) \equiv \mathcal{Q}_1$.

Definition 1.3. (See [8].) Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{Q}$ and $n \in \mathbb{N}$. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions

$$\psi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$$

that satisfy the following admissibility condition:

$$\psi(r, s, t; z, \zeta) \notin \Omega$$

whenever

$$r = q(\xi), \qquad s = k\xi q'(\xi) \quad \text{and} \quad \Re\left(\frac{t}{s}+1\right) \geqslant k\Re\left\{\frac{\xi q''(\xi)}{q'(\xi)}+1\right\} \quad \big(z \in \mathbb{U};\ \xi \in \partial\mathbb{U} \setminus E(q);\ \zeta \in \overline{\mathbb{U}};\ k \geqslant n\big).$$

We simply write $\Psi_1[\Omega, q]$ as $\Psi[\Omega, q]$.

If

$$\psi : \mathbb{C}^2 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C},$$

then the admissibility condition reduces to

$$\psi\big(q(\xi), k\xi q'(\xi); z, \zeta\big) \notin \Omega$$

when

$$\big(z \in \mathbb{U};\ \xi \in \partial\mathbb{U} \setminus E(q);\ \zeta \in \overline{\mathbb{U}};\ k \geqslant n\big).$$

Definition 1.4. (See [6].) Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{H}[a, n]$ with $q'(z) \neq 0$. The class of admissible functions $\Psi'_n[\Omega, q]$ consists of those functions

$$\psi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$$

that satisfy the following admissibility condition:

$$\psi(r, s, t; \xi, \zeta) \in \Omega$$

whenever

$$r = q(z), \qquad s = \frac{zq'(z)}{m}, \quad \text{and} \quad \Re\left(\frac{t}{s}+1\right) \leqslant \frac{1}{m}\Re\left\{\frac{zq''(z)}{q'(z)}+1\right\} \quad (z \in \mathbb{U};\ \xi \in \partial\mathbb{U};\ \zeta \in \overline{\mathbb{U}};\ m \geqslant n \geqslant 1).$$

In particular, we write $\Psi'_1[\Omega, q]$ as $\Psi'[\Omega, q]$.

If

$$\psi : \mathbb{C}^2 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C},$$

then the admissibility condition reduces to

$$\psi\left(q(z), \frac{zq'(z)}{m}; \xi, \zeta\right) \in \Omega$$

when

$$\big(z \in \mathbb{U};\ \xi \in \partial\mathbb{U} \setminus E(q);\ \zeta \in \overline{\mathbb{U}};\ m \geqslant n \geqslant 1\big).$$

For the above two classes of admissible functions, G.I. Oros and G. Oros [8] proved the following result.

Lemma 1.1. *(See [8].) Let $\psi \in \Psi_n[\Omega, q]$ with $q(0) = a$. If $p \in \mathcal{H}[a, n]$ satisfies*

$$\psi\big(p(z), zp'(z), z^2 p''(z); z, \zeta\big) \in \Omega,$$

then

$$p(z) \prec q(z).$$

G.I. Oros [6], on the other hand proved Lemma 1.2.

Lemma 1.2. *(See [6].) Let $\psi \in \Psi'_n[\Omega, q]$ with $q(0) = a$. If $p \in \mathcal{Q}(a)$ and*

$$\psi\big(p(z), zp'(z), z^2 p''(z); z, \zeta\big)$$

is univalent in $\mathbb{U}$ for $\zeta \in \overline{\mathbb{U}}$, then

$$\Omega \subset \big\{\psi\big(p(z), zp'(z), z^2 p''(z); z, \zeta\big):\ z \in \mathbb{U},\ \zeta \in \overline{\mathbb{U}}\big\}$$

implies the following subordination relationship:

$$q(z) \prec p(z).$$

In this present investigation, by making use of the strong differential subordination results and the strong superordination results of G.I. Oros and G. Oros [6,8], we consider certain suitable classes of admissible functions and investigate some strong differential subordination and strong differential superordination properties of analytic functions. New strong differential sandwich-type results are also obtained. In recent years, several authors obtained many interesting results in strong differential subordination and superordination [1–3,6–9].

2. Subordination results

We first define the following class of admissible functions that are required in our first result.

Definition 2.1. Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{Q}_1 \cap \mathcal{H}$. The class of admissible function $\Phi_s[\Omega, q]$ consists of those functions

$$\phi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$$

that satisfy the admissibility condition:

$$\phi(u, v, w; z, \zeta) \notin \Omega$$

whenever

$$u = q(\xi), \qquad v = k\frac{\xi q'(\xi)}{q(\xi)} \quad \bigl(q(\xi) \neq 0\bigr),$$

and

$$\Re\left\{\frac{w - 2v + uv(1+2v)}{uv}\right\} \geqslant k\Re\left\{\frac{\xi q''(\xi)}{q'(\xi)} + 1\right\} \quad \bigl(z \in \mathbb{U};\ \xi \in \partial\mathbb{U} \setminus E(q);\ \zeta \in \overline{\mathbb{U}};\ k \geqslant 1\bigr).$$

Theorem 2.1. *Let $\phi \in \Phi_s[\Omega, q]$. If $f \in \mathcal{A}$ satisfies*

$$\left\{\phi\left(\frac{f(z)}{zf'(z)}, \frac{zf'(z)}{f(z)} - \left(1 + \frac{zf''(z)}{f'(z)}\right), z^3\left(\frac{f''(z)}{f'(z)}\right)^2\left(\frac{f(z)}{z^2 f''(z)}\right)'; z, \zeta\right) : z \in \mathbb{U},\ \zeta \in \overline{\mathbb{U}}\right\} \subset \Omega, \tag{2.1}$$

then

$$\frac{f(z)}{zf'(z)} \prec q(z).$$

Proof. Define the function p in $\mathbb{U}$ by

$$p(z) := \frac{f(z)}{zf'(z)}. \tag{2.2}$$

A simple calculation yields

$$\frac{zf'(z)}{f(z)} - \left(1 + \frac{zf''(z)}{f'(z)}\right) = \frac{zp'(z)}{p(z)}. \tag{2.3}$$

Further computations show that

$$z^3\left(\frac{f''(z)}{f'(z)}\right)^2\left(\frac{f(z)}{z^2 f''(z)}\right)' = z^2 p''(z) + \frac{2zp'(z)}{p(z)}\bigl(1 - zp'(z)\bigr). \tag{2.4}$$

We now define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u = r, \qquad v = \frac{s}{r}, \qquad w = t + \frac{2s}{r}(1 - s). \tag{2.5}$$

Let

$$\psi(r, s, t; z, \zeta) = \phi(u, v, w; z, \zeta) = \phi\left(r, \frac{s}{r}, t + \frac{2s}{r}(1 - s); z, \zeta\right). \tag{2.6}$$

The proof will make use of Lemma 1.1. Using (2.2), (2.3), and (2.4), from (2.6) we obtain

$$\psi\bigl(p(z), zp'(z), z^2 p''(z); z, \zeta\bigr) = \phi\left(\frac{f(z)}{zf'(z)}, \frac{zf'(z)}{f(z)} - \left(1 + \frac{zf''(z)}{f'(z)}\right), z^3\left(\frac{f''(z)}{f'(z)}\right)^2\left(\frac{f(z)}{z^2 f''(z)}\right)'; z, \zeta\right). \tag{2.7}$$

Hence (2.1) becomes

$$\psi\bigl(p(z), zp'(z), z^2 p''(z); z, \zeta\bigr) \in \Omega.$$

A computation using (2.5) yields

$$\frac{t}{s} + 1 = \frac{w - 2v + uv(1+2v)}{uv}.$$

Thus the admissibility condition for $\phi \in \Phi_s[\Omega, q]$ in Definition 2.1 is equivalent to the admissibility condition for ψ as given in Definition 1.3. Hence $\psi \in \Psi[\Omega, q]$ and by Lemma 1.1

$$p(z) \prec q(z)$$

or, equivalently,

$$\frac{f(z)}{zf'(z)} \prec q(z),$$

which evidently completes the proof of Theorem 2.1. □

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping h of $\mathbb{U}$ onto Ω. In this case, the class $\Phi_s[h(\mathbb{U}), q]$ is written as $\Phi_s[h, q]$. The following result is an immediate consequence of Theorem 2.1.

Theorem 2.2. *Let $\phi \in \Phi_s[h, q]$. If $f \in \mathcal{A}$ satisfies*

$$\phi\left(\frac{f(z)}{zf'(z)}, \frac{zf'(z)}{f(z)} - \left(1 + \frac{zf''(z)}{f'(z)}\right), z^3\left(\frac{f''(z)}{f'(z)}\right)^2\left(\frac{f(z)}{z^2 f''(z)}\right)'; z, \zeta\right) \prec\prec h(z), \tag{2.8}$$

then

$$\frac{f(z)}{zf'(z)} \prec q(z).$$

Our next result in an extension of Theorem 2.1 to the case in which the behavior of q on $\partial\mathbb{U}$ is not known.

Theorem 2.3. *Let h and q be univalent in $\mathbb{U}$ with $q(0) = 0$, and set $q_\rho(z) = q(\rho z)$ and $h_\rho(z) = h(\rho z)$. Let $\phi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$ satisfies one of the following conditions*:

(i) *$\phi \in \Phi_s[h, q_\rho]$ for some $\rho \in (0, 1)$, or*
(ii) *there exists $\rho_0 \in (0, 1)$ such that $\phi \in \Phi_s[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.*

If $f \in \mathcal{A}$ satisfies (2.8), *then*

$$\frac{f(z)}{zf'(z)} \prec q(z).$$

Proof. The proof of Theorem 2.3 is similar to that of a known result [4, Theorem 2.3d, p. 30] and so it is omitted here. □

Our next theorem yields the best dominant of the strong differential subordination (2.8).

Theorem 2.4. *Let h be univalent in $\mathbb{U}$, and $\phi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$. Suppose that the following differential equation*

$$\phi\left(q(z), \frac{zq'(z)}{q(z)}, z^2 q''(z) + \frac{2zq'(z)}{q(z)}\bigl(1 - zq'(z)\bigr); z, \zeta\right) = h(z) \tag{2.9}$$

has a solution q with $q(0) = 1$ and satisfies one of the following conditions:

(i) *$q \in \mathcal{Q}_1$ and $\phi \in \Phi_s[h, q]$,*
(ii) *q is univalent in $\mathbb{U}$ and $\phi \in \Phi_s[h, q_\rho]$ for some $\rho \in (0, 1)$, or*
(iii) *q is univalent in $\mathbb{U}$ and there exists $\rho_0 \in (0, 1)$ such that $\phi \in \Phi_s[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.*

If $f \in \mathcal{A}$ satisfies (2.8), *then*

$$\frac{f(z)}{zf'(z)} \prec q(z),$$

and q is the best dominant.

Proof. Following the same arguments as in [4, Theorem 2.3e, p. 31], we deduce that q is a dominant from Theorems 2.2 and 2.3. Since q satisfies (2.9), it is also a solution of (2.8) and therefore q will be dominated by all dominants. Hence q is the best dominant. □

We will apply Theorem 2.1 to a specific case for $q(z) = 1 + Mz$, $M > 0$.

In the particular case $q(z) = 1 + Mz$, $M > 0$, and in view of Definition 2.1, the class of admissible functions $\Phi_1[\Omega, q]$, denoted by $\Phi_1[\Omega, M]$, is described below.

Definition 2.2. Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\Phi_1[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$ such that

$$\phi\left(1 + Me^{i\theta}, \frac{kMe^{i\theta}}{1 + Me^{i\theta}}, L + \frac{2kMe^{i\theta}}{1 + Me^{i\theta}}\left(1 - kMe^{i\theta}\right) : z, \zeta\right) \notin \Omega, \tag{2.10}$$

whenever $z \in \mathbb{U}$, $\theta \in \mathbb{R}$ and $\Re\{Le^{-i\theta}\} \geqslant (k-1)kM$ for all θ, $\zeta \in \overline{\mathbb{U}}$ and $k \geqslant 1$.

Corollary 2.5. *Let $\phi \in \Phi_1[\Omega, M]$. If $f \in \mathcal{A}$ satisfies*

$$\phi\left(\frac{f(z)}{zf'(z)}, \frac{zf'(z)}{f(z)} - \left(1 + \frac{zf''(z)}{f'(z)}\right), z^3\left(\frac{f''(z)}{f'(z)}\right)^2\left(\frac{f(z)}{z^2 f''(z)}\right)'; z, \zeta\right) \in \Omega$$

then

$$\left|\frac{f(z)}{zf'(z)} - 1\right| < M.$$

For the special case $\Omega = q(\mathbb{U}) = \{w\colon |w - 1| < M\}$, the class $\Phi_1[\Omega, M]$ is simply denoted by $\Phi_1[M]$.

Corollary 2.6. *Let $\phi \in \Phi_1[M]$. If $f \in \mathcal{A}$ satisfies*

$$\left|\phi\left(\frac{f(z)}{zf'(z)}, \frac{zf'(z)}{f(z)} - \left(1 + \frac{zf''(z)}{f'(z)}\right), z^3\left(\frac{f''(z)}{f'(z)}\right)^2\left(\frac{f(z)}{z^2 f''(z)}\right)'; z, \zeta\right) - 1\right| < M,$$

then

$$\left|\frac{f(z)}{zf'(z)} - 1\right| < M.$$

Example 1. The functions

$$\phi_1(u, v, w; z, \zeta) := u(v + 1) \quad \text{and} \quad \phi_2(u, v, w; z, \zeta) := \alpha u(v + 1) + (1 - \alpha)u$$

satisfy the admissibility condition (2.10) and hence Corollary 2.5, yields

$$\left|\left(1 - \frac{f(z)f''(z)}{(f'(z))^2}\right) - 1\right| < M \quad \Rightarrow \quad \left|\frac{f(z)}{zf'(z)} - 1\right| < M.$$

$$\left|\left\{\alpha\left(1 - \frac{f(z)f''(z)}{(f'(z))^2}\right) + (1 - \alpha)\frac{f(z)}{zf'(z)}\right\} - 1\right| < M \quad \Rightarrow \quad \left|\frac{f(z)}{zf'(z)} - 1\right| < M.$$

Now, we introduce the following class of admissible function.

Definition 2.3. Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{Q}_0 \cap \mathcal{H}_0$. The class of admissible functions $\Phi_H[\Omega, q]$ consists of those functions

$$\phi : \mathbb{C}^2 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$$

that satisfy the admissibility condition:

$$\phi\left(q(\xi), k\xi q'(\xi) + q(\xi); z, \zeta\right) \notin \Omega \quad \left(z \in \mathbb{U};\ \xi \in \partial\mathbb{U} \setminus E(q);\ \zeta \in \overline{\mathbb{U}};\ k \geqslant 1\right). \tag{2.11}$$

Theorem 2.7. *Let $\phi \in \Phi_H[\Omega, q]$ and $f \in \mathcal{A}$. If*

$$\left\{\phi\left(\frac{f(z)}{zf'(z)}, 1 - \frac{f(z)f''(z)}{(f'(z))^2}\right);\ z \in \mathbb{U},\ \zeta \in \overline{\mathbb{U}}\right\} \subset \Omega, \tag{2.12}$$

then

$$\frac{f(z)}{zf'(z)} \prec q(z).$$

Proof. Define the function p by

$$p(z) := \frac{f(z)}{zf'(z)}. \tag{2.13}$$

Clearly $p \in \mathcal{A}$, and a simple computation yields

$$1 - \frac{f(z)f''(z)}{(f'(z))^2} = zp'(z) + p(z). \tag{2.14}$$

Define the transformation from $\mathbb{C}^2 \times \mathbb{U} \times \overline{\mathbb{U}}$ to $\mathbb{C}$ by

$$u = r, \qquad v = s + r.$$

Let

$$\psi(r, s; z, \zeta) = \phi(u, v; z, \zeta) = \phi(r, s + r; z, \zeta). \tag{2.15}$$

The proof will make use of Lemma 1.1. Using (2.13) and (2.14), from (2.15), we obtain

$$\psi\big(p(z), zp'(z); z, \zeta\big) = \phi\left(\frac{f(z)}{zf'(z)}, 1 - \frac{f(z)f''(z)}{(f'(z))^2}; z, \zeta\right). \tag{2.16}$$

Hence (2.12) becomes

$$\psi\big(p(z), zp'(z); z, \zeta\big) \in \Omega.$$

From (2.15), we see that the admissibility condition for $\phi \in \Phi_H[\Omega, q]$ in Definition 2.3 is equivalent to the admissibility condition for ψ as given in Definition 1.3. Hence $\psi \in \Psi[\Omega, q]$ and by Lemma 1.1,

$$p(z) \prec q(z)$$

or, equivalently,

$$\frac{f(z)}{zf'(z)} \prec q(z). \qquad \square$$

We will denote by $\Phi_H[h, q]$ the class $\Phi_H[h(\mathbb{U}), q]$, where h is the conformal mapping of $\mathbb{U}$ onto $\Omega \neq \mathbb{C}$. The following result is an immediate consequence of Theorem 2.7, which we state without proof.

Theorem 2.8. *Let $\phi \in \Phi_H[h, q]$. If $f \in \mathcal{A}$ satisfies*

$$\phi\left(\frac{f(z)}{zf'(z)}, 1 - \frac{f(z)f''(z)}{(f'(z))^2}; z, \zeta\right) \prec\prec h(z), \tag{2.17}$$

then

$$\frac{f(z)}{zf'(z)} \prec q(z). \tag{2.18}$$

We extend Theorem 2.8 to the case where the behavior of q on $\partial\mathbb{U}$ is not known.

Theorem 2.9. *Let $\Omega \subset \mathbb{C}$ and let q be univalent in $\mathbb{U}$ with $q(0) = 0$. Let $\phi \in \Phi_H[h, q_\rho]$ for some $\rho \in (0, 1)$ where $q_\rho(z) = q(\rho z)$. If $f \in \mathcal{A}$ satisfies* (2.12)*, then* (2.18) *holds.*

With $q(z) = 1 + Mz$, we get the following:

Corollary 2.10. *Let Ω be a set in $\mathbb{C}$, $q(z) = 1 + Mz$, $M > 0$, and $\phi : \mathbb{C}^2 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$ satisfy*

$$\phi\big(1 + Me^{i\theta}, 1 + (k + 1)Me^{i\theta}; z, \zeta\big) \notin \Omega$$

whenever $z \in \mathbb{U}$, $\theta \in \mathbb{R}$, $\zeta \in \overline{\mathbb{U}}$ and $k \geqslant 1$. Let $f \in \mathcal{A}$. If

$$\phi\left(\frac{f(z)}{zf'(z)}, 1 - \frac{f(z)f''(z)}{(f'(z))^2}; z, \zeta\right) \in \Omega,$$

then

$$\left|\frac{f(z)}{zf'(z)} - 1\right| < M.$$

In the special case $\Omega = q(\mathbb{U}) = \{w\colon\ |w-1| < M\}$, Corollary 2.10 gives the following: Let $\phi : \mathbb{C}^2 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$ satisfy

$$\left|\phi\big(1+Me^{i\theta}, 1+(k+1)Me^{i\theta}; z, \zeta\big) - 1\right| \geqslant M,$$

whenever $z \in \mathbb{U}$, $\theta \in \mathbb{R}$, $\zeta \in \overline{\mathbb{U}}$ and $k \geqslant 1$; if $f \in \mathcal{A}$ satisfies

$$\left|\phi\left(\frac{f(z)}{zf'(z)}, 1 - \frac{f(z)f''(z)}{(f'(z))^2}\right) - 1\right| < M,$$

then

$$\left|\frac{f(z)}{zf'(z)} - 1\right| < M.$$

With $\phi(u, v; z) = \alpha v + (1-\alpha)u$, we get the following:

Example 2. If $f \in \mathcal{A}$ satisfies

$$\left|\left\{\alpha\left(1 - \frac{f(z)f''(z)}{(f'(z))^2}\right) + (1-\alpha)\frac{f(z)}{zf'(z)}\right\} - 1\right| < M,$$

then

$$\left|\frac{f(z)}{zf'(z)} - 1\right| < M.$$

3. Superordination and sandwich-type results

In this section, we investigate the dual problem of strong differential subordination (that is, strong differential superordination). For this purpose, the class of admissible functions is given in the following definition.

Definition 3.1. Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{H}$ with $zq'(z) \neq 0$. The class of admissible functions $\Phi_s'[\Omega, q]$ consists of those functions

$$\phi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$$

that satisfy the admissibility condition:

$$\phi(u, v, w; \xi, \zeta) \in \Omega$$

whenever

$$u = q(z), \qquad v = \frac{zq'(z)}{mq(z)} \quad \big(q(z) \neq 0,\ zq'(z) \neq 0\big),$$

and

$$\Re\left\{\frac{w - 2v + uv(1+2v)}{uv}\right\} \leqslant \frac{1}{m}\Re\left\{\frac{zq''(z)}{q'(z)} + 1\right\} \quad (z \in \mathbb{U};\ \xi \in \partial\mathbb{U};\ \zeta \in \overline{\mathbb{U}};\ m \geqslant 1).$$

Theorem 3.1. *Let $\phi \in \Phi_s'[\Omega, q]$. If $f \in \mathcal{A}$, $\frac{f(z)}{zf'(z)} \in \mathcal{Q}_1$ and*

$$\phi\left(\frac{f(z)}{zf'(z)}, \frac{zf'(z)}{f(z)} - \left(1 + \frac{zf''(z)}{f'(z)}\right), z^3\left(\frac{f''(z)}{f'(z)}\right)^2\left(\frac{f(z)}{z^2 f''(z)}\right)'; z, \zeta\right)$$

is univalent in $\mathbb{U}$, then

$$\Omega \subset \left\{\phi\left(\frac{f(z)}{zf'(z)}, \frac{zf'(z)}{f(z)} - \left(1 + \frac{zf''(z)}{f'(z)}\right), z^3\left(\frac{f''(z)}{f'(z)}\right)^2\left(\frac{f(z)}{z^2 f''(z)}\right)'; z\right)\colon\ z, \zeta\right\} \tag{3.1}$$

implies

$$q(z) \prec \frac{f(z)}{zf'(z)}. \tag{3.2}$$

Proof. With $p(z)=\frac{f(z)}{zf'(z)}$ and

$$\psi(r,s,t;z,\zeta)=\phi\left(r,\frac{s}{r},t+\frac{2s}{r}(1-s);\xi,\zeta\right)=\phi(u,v,w;\xi,\zeta),$$

Eqs. (2.7) and (3.1) yield

$$\Omega\subset\left\{\psi\left(p(z),zp'(z),z^2p''(z);z,\zeta\right):\ z\in\mathbb{U},\ \zeta\in\overline{\mathbb{U}}\right\}.$$

Since

$$\frac{t}{s}+1=\frac{w-2v+uv(1+2v)}{uv},$$

the admissibility condition for $\phi\in\Phi_s'[\Omega,q]$ in Definition 3.1 is equivalent to the admissibility condition for ψ as given in Definition 1.4. Hence $\psi\in\Psi'[\Omega,q]$, and by Lemma 1.2

$$q(z)\prec p(z)$$

or

$$q(z)\prec\frac{f(z)}{zf'(z)}.\qquad\Box$$

If $\Omega\neq\mathbb{C}$ is a simply connected domain, then $\Omega=h(\mathbb{U})$ for some conformal mapping h of $\mathbb{U}$ onto Ω with $\Phi_s'[h(\mathbb{U}),q]$ as $\Phi_s'[h,q]$, Theorem 3.1 can be written in the following form.

Theorem 3.2. *Let $q\in\mathcal{H}$, h be analytic in $\mathbb{U}$ and $\phi\in\Phi_s'[h,q]$. If $f\in\mathcal{A}$, $\frac{f(z)}{zf'(z)}\in\mathcal{Q}_1$ and*

$$\phi\left(\frac{f(z)}{zf'(z)},\frac{zf'(z)}{f(z)}-\left(1+\frac{zf''(z)}{f'(z)}\right),z^3\left(\frac{f''(z)}{f'(z)}\right)^2\left(\frac{f(z)}{z^2f''(z)}\right)';z,\zeta\right)$$

is univalent in $\mathbb{U}$, then

$$h(z)\prec\prec\phi\left(\frac{f(z)}{zf'(z)},\frac{zf'(z)}{f(z)}-\left(1+\frac{zf''(z)}{f'(z)}\right),z^3\left(\frac{f''(z)}{f'(z)}\right)^2\left(\frac{f(z)}{z^2f''(z)}\right)';z,\zeta\right)\tag{3.3}$$

implies

$$q(z)\prec\frac{f(z)}{zf'(z)}.$$

Theorems 3.1 and 3.2 can only be used to obtain subordinants of differential superordination of the form (3.1) or (3.3). The following theorem proves the existence of the best subordinant of (3.3) for an appropriate ϕ.

Theorem 3.3. *Let h be analytic in $\mathbb{U}$ and $\phi:\mathbb{C}^3\times\mathbb{U}\times\overline{\mathbb{U}}\to\mathbb{C}$. Suppose that the differential equation*

$$\phi\left(q(z),\frac{zq'(z)}{q(z)},z^2q''(z)+\frac{2zq'(z)}{q(z)}\left(1-zq'(z)\right);z,\zeta\right)=h(z)\tag{3.4}$$

has a solution $q\in\mathcal{Q}_1$. If $\phi\in\Phi_s'[h,q]$, $f\in\mathcal{A}$, $\frac{f(z)}{zf'(z)}\in\mathcal{Q}_1$ and

$$\phi\left(\frac{f(z)}{zf'(z)},\frac{zf'(z)}{f(z)}-\left(1+\frac{zf''(z)}{f'(z)}\right),z^3\left(\frac{f''(z)}{f'(z)}\right)^2\left(\frac{f(z)}{z^2f''(z)}\right)';z,\zeta\right)$$

is univalent in $\mathbb{U}$, then

$$h(z)\prec\prec\phi\left(\frac{f(z)}{zf'(z)},\frac{zf'(z)}{f(z)}-\left(1+\frac{zf''(z)}{f'(z)}\right),z^3\left(\frac{f''(z)}{f'(z)}\right)^2\left(\frac{f(z)}{z^2f''(z)}\right)';z,\zeta\right)$$

implies

$$q(z)\prec\frac{f(z)}{zf'(z)},$$

and q is the best subordinant.

Proof. The proof is similar to that of Theorem 2.4, and so it is being omitted here. □

By combining Theorems 2.2 and 3.2, we obtain the following sandwich-type theorem.

Corollary 3.4. *Let h_1 and q_1 be analytic functions in $\mathbb{U}$, h_2 be univalent function in $\mathbb{U}$, $q_2 \in \mathcal{Q}_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \Phi_s[h_2, q_2] \cap \Phi'_s[h_1, q_1]$. If $f \in \mathcal{A}$, $\frac{f(z)}{zf'(z)} \in \mathcal{H} \cap \mathcal{Q}_1$ and*

$$\phi\left(\frac{f(z)}{zf'(z)}, \frac{zf'(z)}{f(z)} - \left(1 + \frac{zf''(z)}{f'(z)}\right), z^3\left(\frac{f''(z)}{f'(z)}\right)^2\left(\frac{f(z)}{z^2 f''(z)}\right)'; z, \zeta\right)$$

is univalent in $\mathbb{U}$, then

$$h_1(z) \prec\prec \phi\left(\frac{f(z)}{zf'(z)}, \frac{zf'(z)}{f(z)} - \left(1 + \frac{zf''(z)}{f'(z)}\right), z^3\left(\frac{f''(z)}{f'(z)}\right)^2\left(\frac{f(z)}{z^2 f''(z)}\right)'; z, \zeta\right) \prec\prec h_2(z)$$

implies

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z).$$

Definition 3.2. Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{H}$. The class of admissible function $\Phi'_H[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^2 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$ that satisfy the admissibility condition

$$\phi\left(q(z), \frac{zq'(z)}{m} + q(z); z, \zeta\right) \in \Omega \quad (z \in \mathbb{U};\ \xi \in \partial\mathbb{U};\ \zeta \in \overline{\mathbb{U}};\ m \geqslant 1).$$

Now, we will give the dual result of Theorem 2.7 for differential superordination.

Theorem 3.5. *Let $\phi \in \Phi'_H[\Omega, q]$. If $f \in \mathcal{A}$, $\frac{f(z)}{zf'(z)} \in \mathcal{Q}_0$ and $\phi(\frac{f(z)}{zf'(z)}, 1 - \frac{f(z)f''(z)}{(f'(z))^2}; z, \zeta)$ is univalent in $\mathbb{U}$, then*

$$\Omega \subset \left\{\phi\left(\frac{f(z)}{zf'(z)}, 1 - \frac{f(z)f''(z)}{(f'(z))^2}; z, \zeta\right) :\ z \in \mathbb{U},\ \zeta \in \overline{\mathbb{U}}\right\} \tag{3.5}$$

implies

$$q(z) \prec \frac{f(z)}{zf'(z)}.$$

Proof. With $p(z) = \frac{f(z)}{zf'(z)}$, and

$$\psi(r, s; z, \zeta) = \phi(r, s + r; \xi, \zeta) = \phi(u, v; \xi, \zeta), \tag{3.6}$$

from (2.16) and (3.6), we have

$$\Omega \subset \left\{\psi\left(p(z), zp'(z); z \in \mathbb{U}, \zeta \in \overline{\mathbb{U}}\right)\right\}.$$

From (3.6), we see that the admissibility condition for $\Phi'_H[\Omega, q]$ in Definition 3.2 is equivalent to the admissibility condition for ψ as given in Definition 1.4. Hence $\psi \in \Psi'[\Omega, q]$, and by Lemma 1.2,

$$q(z) \prec p(z)$$

or

$$q(z) \prec \frac{f(z)}{zf'(z)}. \qquad \square$$

Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 3.5.

Theorem 3.6. *Let $q \in \mathcal{H}_0$, let h be analytic in $\mathbb{U}$ and $\phi \in \Phi'_H[h, q]$. Let $f \in \mathcal{A}$. If $\frac{f(z)}{zf'(z)} \in \mathcal{Q}_0$ and $\phi(\frac{f(z)}{zf'(z)}, 1 - \frac{f(z)f''(z)}{(f'(z))^2}; z, \zeta)$ is univalent in $\mathbb{U}$, then*

$$h(z) \prec\prec \phi\left(\frac{f(z)}{zf'(z)}, 1 - \frac{f(z)f''(z)}{(f'(z))^2}; z, \zeta\right)$$

implies

$$q(z) \prec \frac{f(z)}{zf'(z)}.$$

Combining Theorems 2.8 and 3.6, we obtain the following sandwich-type theorem.

Corollary 3.7. *Let h_1 and q_1 be analytic functions in $\mathbb{U}$, h_2 be univalent function in $\mathbb{U}$, $q_2 \in \mathcal{Q}_0$ with $q_1(0) = q_2(0) = 0$ and $\phi \in \Phi_H[h_2, q_2] \cap \Phi'_H[h_1, q_1]$. If $f \in \mathcal{A}$, $\frac{f(z)}{zf'(z)} \in \mathcal{H}_0 \cap \mathcal{Q}_0$ and*

$$\phi\left(\frac{f(z)}{zf'(z)}, 1 - \frac{f(z)f''(z)}{(f'(z))^2}; z, \zeta\right)$$

is univalent in $\mathbb{U}$, then

$$h_1(z) \prec\prec \phi\left(\frac{f(z)}{zf'(z)}, 1 - \frac{f(z)f''(z)}{(f'(z))^2}; z\right) \prec\prec h_2(z) \tag{3.7}$$

implies

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z).$$

References

[1] J.A. Antonino, S. Romaguera, Strong differential subordination to Briot–Bouquet differential equations, J. Differential Equations 114 (1) (1994) 101–105.
[2] J.A. Antonino, Strong differential subordination and applications to univalency conditions, J. Korean Math. Soc. 43 (2) (2006) 311–322.
[3] N.E. Cho, Oh. Sang Kwon, H.M. Srivastava, Strong differential subordination and superordination for multivalently meromorphic functions involving the Liu–Srivastava operator, Integral Transforms Spec. Funct. 21 (8) (2010) 589–601.
[4] S.S. Miller, P.T. Mocanu, Differential Subordination: Theory and Applications, Ser. Monogr. Textbooks Pure Appl. Math., vol. 225, Marcel Dekker, New York, Basel, 2000.
[5] S.S. Miller, P.T. Mocanu, Subordinants of differential superordinations, Complex Var. Theory Appl. 48 (10) (2003) 815–826.
[6] G.I. Oros, Strong differential superordination, Acta Univ. Apulensis Math. Inform. 19 (2009) 101–106.
[7] G.I. Oros, First order strong differential superordination, Gen. Math. 15 (2–3) (2007) 77–87.
[8] G.I. Oros, G. Oros, Strong differential subordination, Turkish J. Math. 33 (2009) 249–257.
[9] G.I. Oros, G. Oros, First order linear strong differential subordination, Gen. Math. 15 (2–3) (2007) 98–107.