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A Christoffel–Darboux formula for multiple orthogonal polynomials

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Abstract

Bleher and Kuijlaars recently showed that the eigenvalue correlations from matrix ensembles with external source can be expressed by means of a kernel built out of special multiple orthogonal polynomials. We derive a Christoffel–Darboux formula for this kernel for general multiple orthogonal polynomials. In addition, we show that the formula can be written in terms of the solution of the Riemann–Hilbert problem for multiple orthogonal polynomials, which will be useful for asymptotic analysis.

Keywords: Christoffel–Darboux formula; Multiple orthogonal polynomials; Riemann–Hilbert problem

1. Introduction

Multiple orthogonal polynomials are polynomials that satisfy orthogonal conditions with respect to a number of weights, or more general with respect to a number of measures. Such polynomials were first introduced by Hermite in his proof of the transcendence of e, and were subsequently used in number theory and approximation theory, see e.g. [1,2,9,10,12], and the references cited therein. The motivation for the present work comes from a connection with random matrix theory. In the random matrix model considered in [3] the eigenvalue

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correlations are expressed in terms of a kernel built out of multiple orthogonal polynomials with respect to two weights

$$w_j(x) = e^{-V(x)+a_j x}, \qquad j = 1, 2, \qquad a_1 \neq a_2. \tag{1.1}$$

A Christoffel–Darboux formula was given in [3] which leads to a description of the kernel in terms of the Riemann–Hilbert problem for multiple orthogonal polynomials [14]. It is the aim of this paper to extend the Christoffel–Darboux formula to multiple orthogonal polynomials with respect to an arbitrary number of weights. We also allow more general weights than those in (1.1).

Let $m \geqslant 2$ be an integer, and let $w_1, w_2, \ldots, w_m$ be non-negative functions on $\mathbb{R}$ such that all moments $\int_{-\infty}^{\infty} x^k w_j(x)\, dx$ exist. Let $\vec{n} = (n_1, n_2, \ldots, n_m)$ be a vector of non-negative integers. The (monic) multiple orthogonal polynomial $P_{\vec{n}}$ of type II is a monic polynomial of degree $|\vec{n}|$ satisfying

$$\int P_{\vec{n}}(x) x^k w_j(x)\, dx = 0 \qquad \text{for } k = 0, \ldots, n_j - 1, \quad j = 1, \ldots, m. \tag{1.2}$$

Here we define, as usual, $|\vec{n}| = n_1 + n_2 + \cdots + n_m$.

We assume that the system is perfect, i.e., that for every $\vec{n} \in (\mathbb{N} \cup \{0\})^m$, the polynomial $P_{\vec{n}}$ exists and is unique, see [9]. This is for example the case when the weights form an Angelesco system or an AT system, see e.g. [13]. However, see Remark 1.3 for a relaxation of the perfectness assumption.

The multiple orthogonal polynomials of type I are polynomials $A_{\vec{n}}^{(k)}$ for $k = 1, \ldots, m$, where $A_{\vec{n}}^{(k)}$ has degree $\leqslant n_k - 1$, such that the function

$$Q_{\vec{n}}(x) = \sum_{k=1}^{m} A_{\vec{n}}^{(k)}(x) w_k(x) \tag{1.3}$$

satisfies

$$\int x^j Q_{\vec{n}}(x)\, dx = \begin{cases} 0 & \text{for } j = 0, \ldots, |\vec{n}| - 2, \\ 1 & \text{for } j = |\vec{n}| - 1. \end{cases} \tag{1.4}$$

The polynomials $A_{\vec{n}}^{(k)}$ exist, are unique, and they have full degree

$$\deg A_{\vec{n}}^{(k)} = n_k - 1,$$

since the system is perfect.

The usual monic orthogonal polynomials P_n on the real line with weight function $w(x)$ satisfy a three term recurrence relation and this gives rise to the basic Christoffel–Darboux formula

$$\sum_{j=0}^{n-1} \frac{1}{h_j} P_j(x) P_j(y) = \frac{1}{h_{n-1}} \frac{P_n(x) P_{n-1}(y) - P_{n-1}(x) P_n(y)}{x - y}, \tag{1.5}$$

where

$$h_j = \int P_j(x)x^j w(x)\,dx.$$

In order to generalize the formula (1.5) to multiple orthogonal polynomials, we consider a sequence of multi-indices $\vec{n}_0, \vec{n}_1, \ldots, \vec{n}_n$ such that for each $j = 0, 1, \ldots, n$,

$$|\vec{n}_j| = j, \qquad \vec{n}_{j+1} \geqslant \vec{n}_j, \tag{1.6}$$

where the inequality is taken componentwise. This means that we can go from $\vec{n}_j$ to $\vec{n}_{j+1}$ by increasing one of the components of $\vec{n}_j$ by 1. We view $\vec{n}_0, \vec{n}_1, \ldots, \vec{n}_n$ as a path from $\vec{n}_0 = \vec{0}$ (the all-zero vector) to an arbitrary multi-index $\vec{n} = \vec{n}_n$. This path will be fixed and all notions are related to this fixed path. Given such a path, we define the polynomials P_j and functions Q_j (with single index) as

$$P_j = P_{\vec{n}_j}, \qquad Q_j = Q_{\vec{n}_{j+1}}. \tag{1.7}$$

Our aim is to find a simplified expression for the sum

$$K_n(x, y) = \sum_{j=0}^{n-1} P_j(x)Q_j(y). \tag{1.8}$$

To do this, we introduce the following notation. We define for every multi-index $\vec{n}$ and every $k = 1, \ldots, m$,

$$h_{\vec{n}}^{(k)} = \int P_{\vec{n}}(x)x^{n_k} w_k(x)\,dx. \tag{1.9}$$

The numbers $h_{\vec{n}}^{(k)}$ are non-zero, since the system is perfect. We also use the standard basis vectors

$$\vec{e}_k = (0, \ldots, 0, 1, 0, \ldots, 0), \quad \text{where 1 is in the } k\text{th position}. \tag{1.10}$$

Our result is the following.

Theorem 1.1. *Let $n \in \mathbb{N}$ and let $\vec{n}_0, \vec{n}_1, \ldots, \vec{n}_n$ be multi-indices such that* (1.6) *holds. Let P_j and Q_j be as in* (1.7). *Then we have if $\vec{n} = \vec{n}_n$,*

$$(x - y)\sum_{j=0}^{n-1} P_j(x)Q_j(y) = P_{\vec{n}}(x)Q_{\vec{n}}(y) - \sum_{k=1}^{m} \frac{h_{\vec{n}}^{(k)}}{h_{\vec{n}-\vec{e}_k}^{(k)}} P_{\vec{n}-\vec{e}_k}(x)Q_{\vec{n}+\vec{e}_k}(y). \tag{1.11}$$

It is easy to see that (1.11) reduces to the classical Christoffel–Darboux formula (1.5) in case $m = 1$. For $m = 2$ the formula was proven in [3].

Remark 1.2. It follows from (1.11) that the kernel (1.8) only depends on the endpoint $\vec{n}$ of the chosen path from $\vec{0}$ to $\vec{n}$ and not on the particular path itself, since clearly the right-hand side of (1.11) only depends on $\vec{n}$.

This fact can be deduced from the fact that for any multi-index $\vec{k}$ and for $i \neq j$, we have

$$P_{\vec{k}}(x)Q_{\vec{k}+\vec{e}_i}(y) + P_{\vec{k}+\vec{e}_i}(x)Q_{\vec{k}+\vec{e}_i+\vec{e}_j}(y) = P_{\vec{k}}(x)Q_{\vec{k}+\vec{e}_j}(y) + P_{\vec{k}+\vec{e}_j}(x)Q_{\vec{k}+\vec{e}_i+\vec{e}_j}(y). \tag{1.12}$$

The relation (1.12) follows easily from Lemma 3.6 below.

Remark 1.3. For convenience we have assumed that the system is perfect, so that all multi-indices $\vec{n}$ are normal. (A multi-index is normal if $P_{\vec{n}}$ exists and is unique.) This assumption is not really necessary. A closer inspection of the proof of Theorem 1.1 in Section 3 reveals that, besides the normality of the multi-indices $\vec{n}_j$, for $j = 0, \ldots, n$, and $\vec{n} \pm \vec{e}_k$, for $k = 1, \ldots, m$, which appear in the statement of the theorem, we only use the normality of the multi-indices

$$\vec{n} + \vec{e}_1 + \vec{e}_2 + \cdots + \vec{e}_k, \quad k = 2, \ldots, m$$

and

$$\vec{n} + \vec{e}_1 + \vec{e}_2 + \cdots + \vec{e}_k + \vec{e}_j, \qquad k = 1, \ldots, m-2, \quad j = k+2, \ldots, m.$$

It might be possible to weaken the normality assumption even further, but we have not tried to do so.

Remark 1.4. In [11], Sorokin and Van Iseghem proved a Christoffel–Darboux formula for vector polynomials that have matrix orthogonality properties. As a special case this includes the multiple orthogonal polynomials of type I and type II, when one of the vector polynomials has only one component. In this special case, their Christoffel–Darboux formula comes down to the formula

$$(x-y)\sum_{j=0}^{n-1} P_j(x)Q_j(y) = P_n(x)Q_{n-1}(y) - \sum_{k=n}^{n+m-1}\sum_{j=0}^{n-1} c_{j,k}P_j(x)Q_k(y) \tag{1.13}$$

where the constants $c_{j,k}$ are such that

$$xP_k(x) = \sum_{j=0}^{k+1} c_{j,k}P_j(x),$$

see also (3.5) below. In the setting of [11] it holds that $c_{j,k} = 0$ if $k \geqslant j + m + 1$, so that the right-hand side of (1.13) has $1 + \frac{1}{2}m(m+1)$ terms. Note that in our formula (1.11) the right-hand side has only $1 + m$ terms.

Another Christoffel–Darboux formula for multiple orthogonal polynomials similar to the one in [11] has been given recently in [6].

Remark 1.5. As mentioned before, the formula (1.11) is useful in the theory of random matrices. Brézin and Hikami [5] studied a random matrix model with external source given by the probability measure

$$\frac{1}{Z_n} e^{-Tr(V(M)-AM)}\, dM \tag{1.14}$$

defined on the space of $n \times n$ Hermitian matrices M. Here we have that $V(x) = \frac{1}{2}x^2$, A is a fixed Hermitian matrix (the external source), and Z_n is a normalizing constant. For this case, we can write $M = H + A$, where H is a random matrix from the Gaussian unitary ensemble and A is deterministic. Zinn–Justin [15] considered the case of an arbitrary polynomial V.

The k-point correlation function $R_k(\lambda_1, \dots, \lambda_k)$ of the (random) eigenvalues of a matrix from the ensemble (1.14) can be expressed as a $k \times k$ determinant involving a kernel $K_n(x, y)$

$$R_k(\lambda_1, \dots, \lambda_k) = \det(K_n(\lambda_i, \lambda_j))_{1 \leqslant i,j \leqslant k}, \tag{1.15}$$

see [15]. Suppose that the external source A has m distinct eigenvalues $\alpha_1, \dots, \alpha_m$ with respective multiplicities $n_1, \dots, n_m$. Let $\vec{n} = (n_1, \dots, n_m)$. Then it was shown in [3] that the kernel K_n has the form (1.8) built out of the multiple orthogonal polynomials associated with the weights

$$w_j(x) = e^{-(V(x) - \alpha_j x)}, \qquad j = 1, \dots, m.$$

The Christoffel–Darboux formula (1.11) gives a compact expression for the kernel.

There is another expression for the kernel (1.8) in terms of the solution of a Riemann–Hilbert problem. This will be especially useful for the asymptotic analysis of the matrix model (1.14). We will discuss this in the next section. The proof of Theorem 1.1 is presented in Section 3.

2. Link with the Riemann–Hilbert problem

Van Assche et al. [14] found a Riemann–Hilbert problem that characterizes the multiple orthogonal polynomials. This is an extension of the Riemann–Hilbert problem for orthogonal polynomials due to Fokas et al. [8]. We seek $Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{(m+1)\times(m+1)}$ such that

1. Y is analytic on $\mathbb{C} \setminus \mathbb{R}$,
2. for $x \in \mathbb{R}$, we have $Y_+(x) = Y_-(x)S(x)$, where

$$S(x) = \begin{bmatrix} 1 & w_1(x) & w_2(x) & \cdots & w_m(x) \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \tag{2.1}$$

3. as $z \to \infty$, we have that

$$Y(z) = \left(I + O\left(\frac{1}{z}\right)\right) \begin{bmatrix} z^n & 0 & 0 & \cdots & 0 \\ 0 & z^{-n_1} & 0 & \cdots & 0 \\ 0 & 0 & z^{-n_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & z^{-n_m} \end{bmatrix}, \tag{2.2}$$

where I denotes the $(m+1) \times (m+1)$ identity matrix.

This Riemann–Hilbert problem has a unique solution given by:

$$Y(z) = \begin{bmatrix} P_{\vec{n}}(z) & \vec{R}_{\vec{n}}(z) \\ c_1 P_{\vec{n}-\vec{e}_1}(z) & c_1 \vec{R}_{\vec{n}-\vec{e}_1}(z) \\ c_2 P_{\vec{n}-\vec{e}_2}(z) & c_2 \vec{R}_{\vec{n}-\vec{e}_2}(z) \\ \vdots & \vdots \\ c_m P_{\vec{n}-\vec{e}_m}(z) & c_m \vec{R}_{\vec{n}-\vec{e}_m}(z) \end{bmatrix}, \tag{2.3}$$

where $P_{\vec{n}}(z)$ is the multiple orthogonal polynomial of type II with respect to the weights $w_1, \dots, w_m$ and $\vec{R}_{\vec{n}} = (R_{\vec{n},1}, R_{\vec{n},2}, \dots, R_{\vec{n},m})$ is the vector containing the Cauchy transforms

$$R_{\vec{n},j}(z) = \frac{1}{2\pi i} \int \frac{P_{\vec{n}}(x) w_j(x)}{x - z}\, dx$$

and

$$c_j = -\frac{2\pi i}{h^{(j)}_{\vec{n}-\vec{e}_j}}, \qquad j = 1, \dots, m. \tag{2.4}$$

Van Assche et al. [14] also gave a Riemann–Hilbert problem that characterizes the multiple orthogonal polynomials of type I. Here we seek $X : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{(m+1)\times(m+1)}$ such that

1. X is analytic on $\mathbb{C} \setminus \mathbb{R}$,
2. for $x \in \mathbb{R}$, we have $X_+(x) = X_-(x) U(x)$, where

$$U(x) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -w_1(x) & 1 & 0 & \cdots & 0 \\ -w_2(x) & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -w_m(x) & 0 & 0 & \cdots & 1 \end{bmatrix}, \tag{2.5}$$

3. as $z \to \infty$, we have

$$X(z) = \left(I + O\left(\frac{1}{z}\right)\right) \begin{bmatrix} z^{-n} & 0 & 0 & \cdots & 0 \\ 0 & z^{n_1} & 0 & \cdots & 0 \\ 0 & 0 & z^{n_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & z^{n_m} \end{bmatrix}. \tag{2.6}$$

This Riemann–Hilbert problem also has a unique solution and it is given by

$$X(z) = \begin{bmatrix} \int Q_{\vec{n}}(x) \frac{dx}{z-x} & 2\pi i \vec{A}_{\vec{n}}(z) \\ k_1 \frac{1}{2\pi i} \int Q_{\vec{n}+\vec{e}_1}(x) \frac{dx}{z-x} & k_1 \vec{A}_{\vec{n}+\vec{e}_1}(z) \\ k_2 \frac{1}{2\pi i} \int Q_{\vec{n}+\vec{e}_2}(x) \frac{dx}{z-x} & k_2 \vec{A}_{\vec{n}+\vec{e}_2}(z) \\ \vdots & \vdots \\ k_m \frac{1}{2\pi i} \int Q_{\vec{n}+\vec{e}_m}(x) \frac{dx}{z-x} & k_m \vec{A}_{\vec{n}+\vec{e}_m}(z) \end{bmatrix}, \tag{2.7}$$

where $\vec{A}_{\vec{n}} = (A_{\vec{n}}^{(1)}, A_{\vec{n}}^{(2)}, \ldots, A_{\vec{n}}^{(m)})$ is the vector of multiple orthogonal polynomials of type I with respect to $w_1, \ldots, w_m$, $Q_{\vec{n}}(z) = \sum_{k=1}^{m} A_{\vec{n}}^{(k)}(x) w_k(x)$ and

$$k_j = h_{\vec{n}}^{(j)}, \quad j = 1, \ldots, m. \tag{2.8}$$

It is now possible to write the kernel $K_n(x, y)$ in terms of the solutions of the two Riemann–Hilbert problems, see also [3]. First, we observe that $X = Y^{-t}$. If we look at the $j+1, 1$-entry of the product $Y^{-1}(y)Y(x) = X^t(y)Y(x)$, where $j = 1, \ldots, m$, then we find by (2.3) and (2.7)

$$\begin{aligned} [Y^{-1}(y)Y(x)]_{j+1,1} &= \begin{bmatrix} 2\pi i A_{\vec{n}}^{(j)}(y) & k_1 A_{\vec{n}+\vec{e}_1}^{(j)}(y) & \cdots & k_m A_{\vec{n}+\vec{e}_m}^{(j)}(y) \end{bmatrix} \begin{bmatrix} P_{\vec{n}}(x) \\ c_1 P_{\vec{n}-\vec{e}_1}(x) \\ c_2 P_{\vec{n}-\vec{e}_2}(x) \\ \vdots \\ c_m P_{\vec{n}-\vec{e}_m}(x) \end{bmatrix} \\ &= 2\pi i \left(P_{\vec{n}}(x) A_{\vec{n}}^{(j)}(y) - \sum_{k=1}^{m} \frac{h_{\vec{n}}^{(k)}}{h_{\vec{n}-\vec{e}_k}^{(k)}} P_{\vec{n}-\vec{e}_k}(x) A_{\vec{n}+\vec{e}_k}^{(j)}(y) \right). \end{aligned} \tag{2.9}$$

where in the last step we used the expressions (2.4) and (2.8) for the constants c_j and k_j. Multiplying (2.9) by $w_j(y)$, dividing by $2\pi i$, and summing over $j = 1, \ldots, m$, we obtain the right-hand side of (1.11). Therefore we see that

$$\begin{aligned} (x-y)K_n(x,y) &= \frac{1}{2\pi i} \sum_{j=1}^{m} w_j(y) [Y^{-1}(y)Y(x)]_{j+1,1} \\ &= \frac{1}{2\pi i} \begin{bmatrix} 0 & w_1(y) & \cdots & w_m(y) \end{bmatrix} Y^{-1}(y)Y(x) \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \end{aligned} \tag{2.10}$$

It is clear that the right-hand side of (2.10) is 0 for $x = y$, which is not obvious at all for the right-hand side of (1.11).

In [4] the Riemann–Hilbert problem (2.1) and (2.2) is analyzed in the limit $n \to \infty$ for the special case of $m = 2$, $n_1 = n_2$, and weights

$$w_1(x) = e^{-n(\frac{1}{2}x^2 - ax)}, \qquad w_2(x) = e^{-n(\frac{1}{2}x^2 + ax)}.$$

The corresponding multiple orthogonal polynomials are known as multiple Hermite polynomials [2,13]. The Deift/Zhou steepest descent method for Riemann–Hilbert problems can be applied to the asymptotic analysis of (2.1) and (2.2), see [7] and references cited therein.

3. Proof of Theorem 1.1

For the proof we are going to extend the path $\vec{n}_0, \vec{n}_1, \ldots, \vec{n}_n$ by defining

$$\vec{n}_{n+k} - \vec{n}_{n+k-1} = \vec{e}_k, \qquad k = 1, 2, \ldots, m. \tag{3.1}$$

We will also extend the definition (1.7) by putting $P_j = P_{\vec{n}_j}$ and $Q_{j-1} = Q_{\vec{n}_j}$ for $j = n+1, \ldots, n+m$.

3.1. Biorthogonality and recurrence relations

The multiple orthogonal polynomials satisfy a biorthogonality relation.

Lemma 3.1. *We have*

$$\int P_k(x) Q_j(x)\, dx = \delta_{j,k},$$

where $\delta_{j,k}$ is the Kronecker delta.

Proof. This is immediate from the definitions (1.7), the orthogonality conditions (1.4) of the function Q_j and (1.2) of the polynomial P_k and the fact that P_k is a monic polynomial. □

Because $xP_k(x)$ is a polynomial of degree $k+1$, we can expand $xP_k(x)$ as

$$xP_k(x) = \sum_{j=0}^{k+1} c_{j,k} P_j(x). \tag{3.2}$$

The coefficients can be calculated by Lemma 3.1 by multiplying both sides of (3.2) with $Q_j(x)$ and integrating over the real line. That gives us

$$c_{j,k} = \int x P_k(x) Q_j(x)\, dx. \tag{3.3}$$

The coefficients $c_{j,k}$ are 0 if $j \geqslant k+2$.

Because of (3.1) we can write $yQ_j(y)$ with $j \leqslant n-1$ as a linear combination of $Q_0, \ldots, Q_{n+m-1}$ and we have by Lemma 3.1

$$yQ_j(y) = \sum_{k=0}^{n+m-1} c_{j,k} Q_k(y) \qquad \text{for } j = 0, \ldots, n-1. \tag{3.4}$$

Using the expansions (3.2) and (3.4) for $xP_k(x)$ and $yQ_j(y)$ we can write

$$\begin{aligned}(x-y)\sum_{k=0}^{n-1} P_k(x)Q_k(y) &= \sum_{k=0}^{n-1} xP_k(x)Q_k(y) - \sum_{k=0}^{n-1} P_k(x)yQ_k(y) \\ &= \sum_{k=0}^{n-1}\sum_{j=0}^{k+1} c_{j,k}P_j(x)Q_k(y) - \sum_{k=0}^{n-1}\sum_{j=0}^{n+m-1} c_{k,j}P_k(x)Q_j(y).\end{aligned}$$

A lot of terms cancel. Since $c_{j,k} = 0$ for $j \geqslant k+2$, and $c_{n,n-1} = 1$, what remains is

$$(x-y)\sum_{k=0}^{n-1} P_k(x)Q_k(y) = P_{\vec{n}}(x)Q_{\vec{n}}(y) - \sum_{k=n}^{n+m-1}\sum_{j=0}^{n-1} c_{j,k}P_j(x)Q_k(y). \tag{3.5}$$

We also used the fact that $P_n = P_{\vec{n}}$ and $Q_{n-1} = Q_{\vec{n}}$. Note that (3.5) corresponds to the Christoffel–Darboux formula of [11], as mentioned in the introduction.

In the rest of the proof we are going to show that

$$\sum_{k=n}^{n+m-1} \sum_{j=0}^{n-1} c_{j,k} P_j(x) Q_k(y) = \sum_{k=1}^{m} \frac{h_{\vec{n}}^{(k)}}{h_{\vec{n}-\vec{e}_k}^{(k)}} P_{\vec{n}-\vec{e}_k}(x) Q_{\vec{n}+\vec{e}_k}(y) \tag{3.6}$$

so that (3.5) then leads to our desired formula (1.11).

3.2. *The vector space generated by the polynomials $P_{\vec{n}-\vec{e}_1}, \ldots, P_{\vec{n}-\vec{e}_m}$*

For fixed y, the right-hand side of (3.6) belongs to the vector space spanned by the polynomials of $P_{\vec{n}-\vec{e}_1}, \ldots, P_{\vec{n}-\vec{e}_m}$. In this part of the proof, we characterize this vector space and show that the left-hand side of (3.6) also belongs to this vector space V.

Lemma 3.2. *The polynomials $P_{\vec{n}-\vec{e}_1}, \ldots, P_{\vec{n}-\vec{e}_m}$ are a basis of the vector space V of all polynomials π of degree $\leqslant n-1$ satisfying*

$$\int \pi(x) x^i w_j(x)\, dx = 0, \quad i = 0, \ldots, n_j - 2, \quad j = 1, \ldots, m. \tag{3.7}$$

Proof. By the orthogonality properties (1.2) of the polynomials $P_{\vec{n}-\vec{e}_i}$ for $i = 1, \ldots, m$, it is obvious that they belong to V. We are first going to show that the polynomials $P_{\vec{n}-\vec{e}_i}$ are linearly independent. Suppose that

$$a_1 P_{\vec{n}-\vec{e}_1} + a_2 P_{\vec{n}-\vec{e}_2} + \cdots + a_m P_{\vec{n}-\vec{e}_m} = 0 \tag{3.8}$$

for some coefficients a_j. Multiplying (3.8) with $w_j(x) x^{n_j - 1}$, and integrating over the real line, we obtain $a_j h_{\vec{n}-\vec{e}_j}^{(j)} = 0$. Since $h_{\vec{n}-\vec{e}_j}^{(j)} \neq 0$, we get $a_j = 0$ for $j = 1, \ldots, m$, which shows that the polynomials are linearly independent.

Suppose next that π belongs to V. Put

$$b_j = \frac{1}{h_{\vec{n}-\vec{e}_j}^{(j)}} \int \pi(x) x^{n_j - 1} w_j(x)\, dx$$

and define the polynomial π_1 by

$$\pi_1 = b_1 P_{\vec{n}-\vec{e}_1} + b_2 P_{\vec{n}-\vec{e}_2} + \cdots + b_m P_{\vec{n}-\vec{e}_m}. \tag{3.9}$$

Then $\pi_1 - \pi$ belongs to V and

$$\int (\pi_1(x) - \pi(x))\, x^{n_j - 1} w_j(x)\, dx = 0, \qquad j = 1, \ldots, m. \tag{3.10}$$

This means that $\pi_1 - \pi$ satisfies the conditions

$$\int (\pi_1(x) - \pi(x)) x^i w_j(x)\, dx = 0, \qquad i = 0, \ldots, n_j - 1, \quad j = 1, \ldots, m. \tag{3.11}$$

Because $\pi_1 - \pi$ is a polynomial of degree $\leqslant n-1$ and the system is perfect, it follows from (3.11) that $\pi_1 - \pi = 0$. Therefore $\pi = \pi_1$, and π can be written as a linear combination of the polynomials $P_{\vec{n}-\vec{e}_1}, \ldots, P_{\vec{n}-\vec{e}_m}$. □

The lemma follows.

Lemma 3.3. *For every* $k = n, \ldots, n+m-1$, *we have that the polynomial*

$$\pi_k(x) = \sum_{j=0}^{n-1} c_{j,k} P_j(x) \tag{3.12}$$

belongs to the vector space V.

Proof. Clearly π_k is a polynomial of degree $n-1$. Using (3.2) we see that

$$\pi_k(x) = x P_k(x) - \sum_{j=n}^{k+1} c_{j,k} P_j(x). \tag{3.13}$$

The representation (3.13) of π_k and the orthogonality conditions (1.2) show that π_k satisfies the relations (3.7), so that π_k belongs to V by Lemma 3.2. □

Because of Lemma 3.3, the left-hand side of (3.6) belongs to V for every y, and so by Lemma 3.2, we can write

$$\sum_{k=n}^{n+m-1} \sum_{j=0}^{n-1} c_{j,k} P_j(x) Q_k(y) = \sum_{j=1}^{m} \phi_j(y) P_{\vec{n}-\vec{e}_j}(x) \tag{3.14}$$

for certain functions $\phi_j(y)$. The next lemma gives an expression for ϕ_j. We use the notation $\vec{s}_0 = \vec{0}$ (all-zero vector) and

$$\vec{s}_j = \sum_{k=1}^{j} \vec{e}_k, \qquad j = 1, \ldots, m.$$

Lemma 3.4. *We have for* $j = 1, \ldots, m$,

$$h^{(j)}_{\vec{n}-\vec{e}_j} \phi_j(y) = \sum_{i=1}^{j} h^{(j)}_{\vec{n}+\vec{s}_{i-1}} Q_{\vec{n}+\vec{s}_i}(y). \tag{3.15}$$

Proof. Rewriting the left-hand side of (3.14) using (3.12) and (3.13) we obtain

$$\sum_{j=1}^{m} \phi_j(y) P_{\vec{n}-\vec{e}_j}(x) = \sum_{k=n}^{n+m-1} x P_k(x) Q_k(y) - \sum_{k=n}^{n+m-1} \sum_{j=n}^{k+1} c_{j,k} P_j(x) Q_k(y). \tag{3.16}$$

Now multiply (3.16) with $x^{n_j-1} w_j(x)$ and integrate with respect to x. Then the left-hand side gives

$$h^{(j)}_{\vec{n}-\vec{e}_j} \phi_j(y). \tag{3.17}$$

The second sum in the right-hand side of (3.16) gives no contribution to the integral because of orthogonality, and the first sum gives

$$\sum_{k=n}^{n+m-1} \left(\int P_k(x) x^{n_j} w_j(x)\, dx \right) Q_k(y)$$
$$= \sum_{i=1}^{m} \left(\int P_{n+i-1}(x) x^{n_j} w_j(x)\, dx \right) Q_{n+i-1}(y). \tag{3.18}$$

Because of the choice (3.1) and the definition (1.7) we have

$$P_{n+i-1} = P_{\vec{n}+\vec{s}_{i-1}}, \quad Q_{n+i-1} = Q_{\vec{n}+\vec{s}_i} \qquad i = 1, \dots, m.$$

Then we see that the integral in the right-hand side of (3.18) is zero if $i \geqslant j+1$ and otherwise it is equal to $h^{(j)}_{\vec{n}+\vec{s}_{i-1}}$. Then (3.15) follows. □

3.3. *Completion of the proof of Theorem 1.1*

In view of (3.14) and (3.15) it remains to prove that

$$h^{(j)}_{\vec{n}} Q_{\vec{n}+\vec{e}_j}(y) = \sum_{i=1}^{j} h^{(j)}_{\vec{n}+\vec{s}_{i-1}} Q_{\vec{n}+\vec{s}_i}(y) \tag{3.19}$$

for $j = 1, \dots, m$, and then (3.6) follows.

To establish (3.19) we need some properties of the numbers $h^{(j)}_{\vec{n}}$ and relations between Q-functions with different multi-indices. We already noted that $h^{(j)}_{\vec{n}} \neq 0$. We express the leading coefficients of the polynomials $A^{(j)}_{\vec{n}}$ in terms of these numbers.

Lemma 3.5. *The leading coefficient of $A^{(j)}_{\vec{n}+\vec{e}_j}$ is equal to $\frac{1}{h^{(j)}_{\vec{n}}}$.*

Proof. Because of the orthogonality conditions (1.2) and (1.4) we have that

$$1 = \int P_{\vec{n}}(x) Q_{\vec{n}+\vec{e}_j}(x)\, dx$$
$$= \int P_{\vec{n}}(x) A^{(j)}_{\vec{n}+\vec{e}_j}(x) w_j(x)\, dx$$
$$= (\text{leading coefficient of } A^{(j)}_{\vec{n}+\vec{e}_j}) \int P_{\vec{n}}(x) x^{n_j} w_j(x)\, dx$$
$$= (\text{leading coefficient of } A^{(j)}_{\vec{n}+\vec{e}_j}) h^{(j)}_{\vec{n}}$$

and the lemma follows. □

Lemma 3.6. *Let $j \neq k$. Then we have for every multi-index $\vec{n}$ that*

$$P_{\vec{n}}(x) = \frac{h^{(k)}_{\vec{n}}}{h^{(k)}_{\vec{n}+\vec{e}_j}} (P_{\vec{n}+\vec{e}_j} - P_{\vec{n}+\vec{e}_k}) = -\frac{h^{(j)}_{\vec{n}}}{h^{(j)}_{\vec{n}+\vec{e}_k}} (P_{\vec{n}+\vec{e}_j} - P_{\vec{n}+\vec{e}_k}) \tag{3.20}$$

and

$$Q_{\vec{n}} = \frac{h^{(k)}_{\vec{n}-\vec{e}_j-\vec{e}_k}}{h^{(k)}_{\vec{n}-\vec{e}_k}}(Q_{\vec{n}-\vec{e}_j} - Q_{\vec{n}-\vec{e}_k}) = -\frac{h^{(j)}_{\vec{n}-\vec{e}_j-\vec{e}_k}}{h^{(j)}_{\vec{n}-\vec{e}_j}}(Q_{\vec{n}-\vec{e}_j} - Q_{\vec{n}-\vec{e}_k}). \tag{3.21}$$

Proof. We know that $P_{\vec{n}}$ is a polynomial of degree $|\vec{n}|$ that satisfies the orthogonality conditions (1.2). It is easy to see that $P_{\vec{n}+\vec{e}_j} - P_{\vec{n}+\vec{e}_k}$ is a polynomial of degree $|\vec{n}|$ that satisfies these same conditions. Because the system is perfect, we then have that

$$\gamma P_{\vec{n}}(x) = P_{\vec{n}+\vec{e}_j}(x) - P_{\vec{n}+\vec{e}_k}(x), \tag{3.22}$$

for some $\gamma \in \mathbb{R}$. Multiplying (3.22) with $x^{n_k} w_k(x)$ and integrating over the real line, we find that

$$\gamma h^{(k)}_{\vec{n}} = h^{(k)}_{\vec{n}+\vec{e}_j} - 0 = h^{(k)}_{\vec{n}+\vec{e}_j}.$$

This proves the first equality of (3.20). The second equality follows by interchanging *j* and *k*.

Next we show (3.21). It is easy to see that $Q_{\vec{n}-\vec{e}_j} - Q_{\vec{n}-\vec{e}_k}$ satisfies the same orthogonality conditions (1.4) as $Q_{\vec{n}}$. Since the degrees of the polynomials $A^{(i)}_{\vec{n}-\vec{e}_j} - A^{(i)}_{\vec{n}-\vec{e}_k}$ do not exceed the degrees of $A^{(i)}_{\vec{n}}$ for $i = 1, \ldots, m$, it follows that

$$\gamma Q_{\vec{n}} = Q_{\vec{n}-\vec{e}_j} - Q_{\vec{n}-\vec{e}_k}, \tag{3.23}$$

for some $\gamma \in \mathbb{R}$. To compute γ, we are going to compare the leading coefficients of the polynomials that come with $w_k(x)$. Using Lemma 3.5, we find that

$$\gamma \frac{1}{h^{(k)}_{\vec{n}-\vec{e}_k}} = \frac{1}{h^{(k)}_{\vec{n}-\vec{e}_j-\vec{e}_k}} - 0 = \frac{1}{h^{(k)}_{\vec{n}-\vec{e}_j-\vec{e}_k}}.$$

This proves the first equality of (3.21). The second equality follows by interchanging *j* and *k*. □

Now we are ready to complete the proof of Theorem 1.1.

Proof of Theorem 1.1

In view of what was said before, it suffices to prove (3.19). Fix $j = 1, \ldots, m$. We are going to prove by induction that for $k = 0, \ldots, j-1$,

$$h^{(j)}_{\vec{n}} Q_{\vec{n}+\vec{e}_j} = \sum_{i=1}^{k} h^{(j)}_{\vec{n}+\vec{s}_{i-1}} Q_{\vec{n}+\vec{s}_i} + h^{(j)}_{\vec{n}+\vec{s}_k} Q_{\vec{n}+\vec{s}_k+\vec{e}_j}. \tag{3.24}$$

For $k = 0$, the sum in the right-hand side of (3.24) is an empty sum, and then the equality (3.24) is clear.

Suppose that (3.24) holds for some $k \leqslant j-2$. Taking (3.21) with $\vec{n}+\vec{s}_{k+1}+\vec{e}_j$ instead of $\vec{n}$ and $k+1$ instead of k, we get

$$Q_{\vec{n}+\vec{s}_{k+1}+\vec{e}_j} = -\frac{h^{(j)}_{\vec{n}+\vec{s}_k}}{h^{(j)}_{\vec{n}+\vec{s}_{k+1}}}\left(Q_{\vec{n}+\vec{s}_{k+1}} - Q_{\vec{n}+\vec{s}_k+\vec{e}_j}\right).$$

Thus

$$h^{(j)}_{\vec{n}+\vec{s}_k} Q_{\vec{n}+\vec{s}_k+\vec{e}_j} = h^{(j)}_{\vec{n}+\vec{s}_k} Q_{\vec{n}+\vec{s}_{k+1}} + h^{(j)}_{\vec{n}+\vec{s}_{k+1}} Q_{\vec{n}+\vec{s}_{k+1}+\vec{e}_j} \tag{3.25}$$

and using the induction hypothesis (3.24) we obtain (3.24) with k replaced by $k+1$.

So (3.24) holds for every $k = 0, 1, \ldots, j-1$. Taking $k = j-1$ in (3.24), we obtain (3.19) and this completes the proof of Theorem 1.1. □

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