Further properties of generalized and hypergeneralized projectors

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Received 10 October 2003; accepted 23 March 2004
Submitted by L. Elsner

Dedicated to Professor Götz Trenkler on the occasion of his 60th birthday

Abstract

Generalized and hypergeneralized projectors, introduced by Groß and Trenkler [Linear Algebra Appl. 264 (1997) 463], are revisited. Several properties of such matrices are established. Some of the results obtained extend and/or generalize those given in the reference above, and some others are solutions to new problems.

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AMS classification: 15A57; 15A09

Keywords: Generalized projector; Hypergeneralized projector; Idempotent matrix; Quadrupotent matrix; Partial isometry; EP matrix; Normal matrix

1. Introduction and preliminaries

Let Cm,n be the set of m × n complex matrices. The symbols K∗, ℜ(K), and r(K) will denote the conjugate transpose, range (column space), and rank, respectively, of

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0024-3795/$ - see front matter © 2004 Elsevier Inc. All rights reserved.
doi:10.1016/j.laa.2004.03.013
K ∈ C_{m,n}. Further, K^† will stand for the Moore–Penrose inverse of K, i.e., for the unique matrix satisfying the equations

\[ KK^\dagger K = K, \quad K^\dagger KK^\dagger = (KK^\dagger)^*, \quad K^\dagger K = (K^\dagger K)^*, \]

and I_n will be the identity matrix of order n. Moreover, C_{m,n}^{PI} will denote the subset of C_{m,n} comprising partial isometries, i.e.,

\[ C_{m,n}^{PI} = \{ K \in C_{m,n} : KK^*K = K \} = \{ K \in C_{m,n} : K^\dagger = K^* \}. \]

and C_{m,n}^U, C_{m,n}^N, and C_{m,n}^{EP} will stand for the subsets of C_{n,n} consisting of unitary, normal, and EP (range-Hermitian) matrices, respectively, i.e.,

\[ C_{n,n}^U = \{ K \in C_{n,n} : K K^* = I_n = K^* K \}, \]
\[ C_{n,n}^N = \{ K \in C_{n,n} : K K^* = K^* K \}, \]
\[ C_{n,n}^{EP} = \{ K \in C_{n,n} : R(K) = R(K^*) \} = \{ K \in C_{n,n} : K K^\dagger = K^\dagger K \}. \]

From (1.3)–(1.5) it is seen that

\[ C_{n,n}^U \subseteq C_{n,n}^N \subseteq C_{n,n}^{EP}. \]

The purpose of the present paper is to revisit generalized and hypergeneralized projectors. Definitions of these notions, introduced by Groß and Trenkler [5, pp. 465, 466], are restated below along with definitions of the classical notions of a projector and an orthogonal projector.

**Definition 1.** A matrix K ∈ C_{n,n} is called:

(a) orthogonal projector whenever K^2 = K = K^*, or, equivalently, K^2 = K = K^†,
(b) projector whenever K^2 = K,
(c) generalized projector whenever K^2 = K^*.
(d) hypergeneralized projector whenever K^2 = K^†.

The corresponding sets of matrices specified in Definition 1 will henceforth be denoted by C_{n,n}^{OP}, C_{n,n}^P, C_{n,n}^{GP}, and C_{n,n}^{HGP}, respectively. It can easily be observed that first premultiplying and then postmultiplying K^2 = K^* by K yields KK^* = K^3 = K^*K.

In view of (1.4) and part (c) of Definition 1, this shows that

\[ C_{n,n}^{GP} \subseteq C_{n,n}^N. \]

The equality K^2 = K^* also entails

\[ KK^*K = K^4 = (K^*)^2 = (K^2)^* = (K^*)^* = K, \]

thus showing that

\[ C_{n,n}^{GP} \subseteq C_{n,n}^{PI} \quad \text{and} \quad C_{n,n}^{GP} \subseteq C_{n,n}^{QP}, \]

where C_{n,n}^{QP} denotes the set of quardipotent matrices of order n, i.e.,
\[ C_{\text{QP}}^n = \{ K \in C_{n,n} : K^4 = K \}. \]

Actually, it is known that inclusions (1.7) and (1.8) can be strengthened to
\[ C_{\text{GP}}^n = C_{\text{QP}}^n \cap C_{n,n} = C_{n,n} \cap C_{\text{N}}^n = C_{\text{QP}}^n \cap C_{\text{PI}}^{n,n}. \] (1.9)

The first and second characterizations of \( C_{\text{GP}}^n \) in (1.9) are inherent in [5, Theorem 1] while the third in [4, Theorem].

Furthermore, the first condition in (1.1) ensures that if \( K^2 = K^\dagger \), then \( K^4 = K \), i.e.,
\[ C_{\text{HGP}}^n \subseteq C_{\text{nP}}^n. \] (1.10)

Since, moreover, \( KK^\dagger = K^3 = K^\dagger K \), it follows that
\[ C_{\text{HGP}}^n \subseteq C_{\text{EP}}^n. \] (1.11)

Actually, it is known that inclusions (1.10) and (1.11) can be strengthened to the equality
\[ C_{\text{HGP}}^n = C_{\text{QP}}^n \cap C_{\text{EP}}^n. \] (1.12)

cf. part (a) ⇔ (d) of Theorem 2 in [5]. In view of (1.10), the statement
\[ C_{\text{GP}}^n = C_{\text{HGP}}^n \cap C_{\text{PI}}^{n,n}, \] (1.13)
constituting Corollary in [5, p. 466], can be considered as a more restrictive version of the last characterization of \( C_{\text{GP}}^n \) in (1.9).

Several further properties of generalized and hypergeneralized projectors are established in this paper. Some of the results obtained extend and/or generalize those given in [5], and some others are solutions to new problems.

2. Results and comments

It should be pointed out that in general the set of projectors \( C_{\text{P}}^n \) is not contained in either the set \( C_{\text{GP}}^n \) of generalized projectors or the set \( C_{\text{HGP}}^n \) of hypergeneralized projectors, and vice versa. A deeper insight into relationships between these sets is available through the theorem below.

**Theorem 1.** For any \( K \in C_{n,n} \), the following statements are equivalent:

(a) \( K \) is simultaneously a generalized projector and a projector,
(b) \( K \) is simultaneously a hypergeneralized projector and a projector,
(c) \( K \) is an orthogonal projector.

**Proof.** From Definition 1 it is immediately seen that (c) ⇒ (a) and (c) ⇒ (b). Conversely, combining the conditions \( K^2 = K^* \) and \( K^2 = K \), which constitute (a), leads to \( K = K^* \), and hence to (c). Similarly, combining the conditions \( K^2 = K^\dagger \) and \( K^2 = K \), which constitute (b), leads to \( K = K^\dagger \), and thus again to (c). □
In a supplement to Theorem 1 it can be noted that
\[ K \in \mathbb{C}^{\text{OP}}_n \Leftrightarrow K = GG^* \text{ for some } G \in \mathbb{C}^{\text{GP}}_n. \]
This is an alternative version of the statement in Remark of Groß and Trenkler [5, p. 465], which refers to a representation of the form \( K = G^*G \). Another observation of such a type is that
\[ K \in \mathbb{C}^{\text{OP}}_n \Leftrightarrow K = HH^\dagger \text{ for some } H \in \mathbb{C}^{\text{HGP}}_n, \]
where, similarly as above, \( K = HH^\dagger \) may be replaced by \( K = H^\dagger H \).

The following two lemmas are useful in investigating properties of generalized and hypergeneralized projectors.

**Lemma 1.** Let \( K \in \mathbb{C}_{n,n} \) be of rank \( r(K) = k \). Then \( K \in \mathbb{C}^{\text{GP}}_n \) if and only if there exists \( U \in \mathbb{C}^U_n \) such that
\[
K = U \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} U^*, \tag{2.1}
\]
where \( E \) is a diagonal matrix with the diagonal elements \( e_{jj} \in \left\{ 1, -\frac{1}{2} - \frac{\sqrt{3}}{2}i, -\frac{1}{2} + \frac{\sqrt{3}}{2}i \right\}, j = 1, \ldots, k \) (and where the null matrices bordering \( E \) in (2.1) are absent when \( k = n \)).

**Lemma 2.** Let \( K \in \mathbb{C}_{n,n} \) be of rank \( r(K) = k \). Then \( K \in \mathbb{C}^{\text{HGP}}_n \) if and only if there exists \( U \in \mathbb{C}^U_n \) such that
\[
K = U \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix} U^*, \tag{2.2}
\]
where \( T \) is an upper triangular matrix with the diagonal elements \( t_{jj} \in \left\{ 1, -\frac{1}{2} - \frac{\sqrt{3}}{2}i, -\frac{1}{2} + \frac{\sqrt{3}}{2}i \right\}, j = 1, \ldots, k, \) satisfying \( T^3 = I_k \) (and where the null matrices bordering \( T \) in (2.2) are absent when \( k = n \)).

Lemma 1 is a quotation of part (c) ⇔ (d) of Theorem 1 in [5], while Lemma 2 is a corrected version of part (b) ⇔ (d) of Theorem 2 in the same paper. The correction consists in adding the condition \( T^3 = I_k \). Without it the result is invalid, as can be seen considering \( U = I_2 \) and
\[
T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},
\]
in which case \( UTU^* (= T) \) is a representation of \( K \) in the form (2.2), but \( K^\dagger (= K^{-1}) \neq K^2 \).

Alternative useful characterizations of generalized and hypergeneralized projectors can be obtained by referring to their singular value decompositions. They are given in the theorem below, accompanied (in view of (1.7), the first part of (1.8), and
(1.11)) by the same type characterizations of partial isometries, EP matrices, and normal matrices.

**Theorem 2.** Let $K \in \mathbb{C}_{n,n}$ of rank $r(K) = k$ have a representation of the form

$$K = UDV^*,$$  

(2.3)

where $U, V \in \mathbb{C}_{n,k}$ are such that $U^*U = I_k = V^*V$, and $D$ is a positive definite diagonal matrix. Moreover, let

$$W = V^*U.$$  

(2.4)

Then

(a) $K \in \mathbb{C}^{PI}_{n,n} \iff D = I_k$,

(b) $K \in \mathbb{C}^{EP}_{n} \iff W \in \mathbb{C}^{U}_{k}$,

(c) $K \in \mathbb{C}^{GP}_{n} \iff D = I_k, W \in \mathbb{C}^{U}_{k}, W^3 = I_k$,

(d) $K \in \mathbb{C}^{HGP}_{n} \iff W \in \mathbb{C}^{U}_{k}, (WD)^3 = I_k$.

**Proof.** According to (1.1), the Moore–Penrose inverse of $K$ having the singular value decomposition (2.3) is

$$K^* = VD^{-1}U^*.$$  

(2.5)

Hence it follows that $K^* = K^*$ if and only if $D^{-1} = D$, which is further equivalent to $D = I_k$. In view of (1.2), this establishes part (a).

On account of (2.3) and (2.5), the orthogonal projectors $P_K = KK^*$ and $P_{K^*} = K^*K$ onto $\mathcal{R}(K)$ and $\mathcal{H}(K^*)$, respectively, can be expressed as

$$P_K = UU^*$$ and $$P_{K^*} = VV^*.$$  

(2.6)

Consequently, from (1.5) it is seen that

$$K \in \mathbb{C}^{EP}_{n} \iff UU^* = VV^*.$$  

(2.7)

Premultiplying the equality in (2.7) by $V^*$ and postmultiplying it by $V$, and then premultiplying by $U^*$ and postmultiplying by $U$ shows that $W$ satisfies $WW^* = I_k = W^*W$, i.e., $W \in \mathbb{C}^{U}_{k}$. Conversely, in view of (2.6), premultiplying and postmultiplying $WW^* = I_k$ by $V$ and $V^*$, respectively, yields

$$P_K P_K P_{K^*} = P_{K^*}.$$  

(2.8)

Hence it is clear that the product $P_K P_K$ of two orthogonal projectors is idempotent, for which it is necessary and sufficient that

$$P_K P_K = P_K P_{K^*};$$  

(2.9)

cf., e.g., Theorem 1 in [1] and a more general result in [3]. Consequently, from (2.8) and (2.9) it follows that

$$\mathcal{H}(K^*) = \mathcal{H}(P_{K^*}) = \mathcal{H}(P_K P_{K^*}) = \mathcal{H}(P_K P_{K^*}) = \mathcal{H}(P_K P_{K^*}) \subseteq \mathcal{H}(K).$$
Combining the inclusion \( \mathcal{R}(K^*) \subseteq \mathcal{R}(K) \) with \( r(K^*) = r(K) \) leads to the equality 
\[ P_K = P_{K^*}, \]
which in view of (2.6) is an alternative form of the right-hand side of (2.7). This concludes the proof of (b).

In view of (1.6), establishing part (c) reduces to answering the question of what should be added to the characterization of \( W \) given in (b) to obtain a set of necessary and sufficient conditions for the normality of \( K \), which in the case of \( K \) represented as in (2.3) means that 
\[ \text{UD} \text{U}^* = \text{VD} \text{V}^*. \] (2.10)

On account of (2.7), an equivalent version of (2.10) is obtained by premultiplying and postmultiplying it by \( V^* \) and \( U \), respectively. This leads to \( V^* \text{UD}^2 = \text{D}^2 \text{V}^* \text{U} \), i.e., to the second condition in the middle part of (c). The possibility of replacing \( \text{WD}^2 = \text{D}^2 \text{W} \) by \( \text{WD} = \text{DW} \) follows by analogous arguments as those used in the last section of the proof of Theorem 2.2 in [2].

Similarly, in view of (1.7) and the first part of (1.8), establishing part (d) consists in answering the question of what should be added to the conditions in (a) and (c), i.e., to \( D = I_k \) and \( W \in C_k \), to obtain a set of necessary and sufficient conditions for 
\[ \text{UDV}^* \text{UDV}^* = \text{VDU}^*, \]
which after substituting \( D = I_k \) simplifies to 
\[ \text{UWV}^* = \text{VU}^*. \] (2.11)

Premultiplying and postmultiplying (2.11) by \( V^* \) and \( U \), respectively, leads immediately to \( W^3 = I_k \), while the converse implication follows by premultiplying \( W^3 = I_k \) by \( V \), postmultiplying it by \( U^* \), and utilizing the equality \( \text{UU}^* = \text{VV}^* \) from (2.7), which is applicable on account of part (b) of this theorem ascertaining that if \( W \in C_k \), then \( K \in C^K \).

Finally, in view of (1.11), the proof of (e) reduces to deriving a condition which should supplement that in (b), i.e., \( W \in C_k \), to form a set of necessary and sufficient conditions for 
\[ \text{UDWDV}^* = \text{VD}^{-1} \text{U}^*. \] (2.12)

Premultiplying and postmultiplying (2.12) by \( V^* \) and \( \text{UD} \), respectively, yields \( (\text{WD})^3 = I_k \), while the converse implication follows by applying again \( \text{UU}^* = \text{VV}^* \) to \( (\text{WD})^3 = I_k \) premultiplied by \( V \) and postmultiplied by \( \text{D}^{-1} \text{U}^* \).

The characterizations comprised in Theorem 2 enable establishing certain properties of generalized and hypergeneralized projectors in a very simple way. For instance, it is seen that parts (a), (d), and (e) of this theorem lead straightforwardly to
\begin{align*}
K \in C^H \cap C^{PI} & \iff W \in C^U_k, \quad (\text{WD})^3 = I_k, \quad D = I_k \\
\iff D = I_k, \quad W \in C_k, \quad W^3 = I_k & \iff K \in C^P \cap C_n
\end{align*}
(2.13)
which is a confirmation of Corollary in [5, p. 466], restated in this paper as (1.13). Moreover, since \( K \in C_{n,n} \) of the form (2.3) satisfies
\[
\mathbf{K}^4 = \mathbf{K} \Leftrightarrow \mathbf{U} \mathbf{D} \mathbf{W} \mathbf{D} \mathbf{W} \mathbf{V}^* = \mathbf{U} \mathbf{D} \mathbf{V}^* \Leftrightarrow (\mathbf{W} \mathbf{D})^3 = \mathbf{I}_k,
\]
(2.14)

it follows from parts (b) and (e) of Theorem 2 that
\[
\mathbf{K} \in \mathbb{C}^\text{QP}_n \cap \mathbb{C}^\text{EP}_n \Leftrightarrow (\mathbf{W} \mathbf{D})^3 = \mathbf{I}_k, \quad \mathbf{W} \in \mathbb{C}_k \Leftrightarrow \mathbf{K} \in \mathbb{C}^\text{HGP}_n,
\]
(2.15)

which is actually the part (a) \(\Leftrightarrow\) (e) of Theorem 2 in [5], restated in this paper as (1.12). The result (2.15) can be combined with (2.13), leading to
\[
\mathbf{K} \in \mathbb{C}^\text{HP}_n \Leftrightarrow \mathbf{K} \in \mathbb{C}^\text{QP}_n \cap \mathbb{C}^\text{PI}_n \cap \mathbb{C}^\text{EP}_n,
\]
which is another characterization of \(\mathbb{C}^\text{GP}_n\), supplementing those given in (1.9). On the other hand, it can be generalized by referring to the concept of a weak-EP matrix specified in the following.

**Definition 2.** A matrix \(\mathbf{K} \in \mathbb{C}_{n,n}\) is said to be weak-EP matrix whenever
\[
\mathbf{P}_\mathbf{K} \mathbf{P}_\mathbf{K}^* = \mathbf{P}_\mathbf{K}^* \mathbf{P}_\mathbf{K}.
\]
The set of all matrices satisfying Definition 2 will henceforth be denoted by \(\mathbb{C}_n^\text{WEP}\).

According to (1.5), \(\mathbf{K} \in \mathbb{C}^\text{EP}_n \Leftrightarrow \mathbf{P}_\mathbf{K} = \mathbf{P}_\mathbf{K}^*\), and thus it is obvious that
\[
\mathbb{C}_n^\text{EP} \subseteq \mathbb{C}_n^\text{WEP}.
\]
A deeper insight into this relationship is provided by the following.

**Lemma 3.** A matrix \(\mathbf{K} \in \mathbb{C}_{n,n}\) is EP if and only if it is weak-EP and has the index not greater than one, i.e.,
\[
r(\mathbf{K}^2) = r(\mathbf{K}).
\]
(2.17)

**Proof.** It is clear that if \(\mathbf{P}_\mathbf{K} = \mathbf{P}_\mathbf{K}^*\), i.e., \(\mathbf{K} \mathbf{K}^\dagger = \mathbf{K}^\dagger \mathbf{K}\), then
\[
r(\mathbf{K}^2) = r(\mathbf{K}^2 \mathbf{K}^\dagger) = r(\mathbf{K} \mathbf{K}^\dagger \mathbf{K}) = r(\mathbf{K}),
\]
which shows that every \(\mathbf{K} \in \mathbb{C}^\text{EP}_n\) satisfies (2.17). In view of (2.16), this completes the proof of the “only if” part. Conversely, from equality (2.9) defining \(\mathbf{K} \in \mathbb{C}_n^\text{WEP}\) it is seen that \(\mathcal{R}(\mathbf{P}_\mathbf{K} \mathbf{P}_\mathbf{K}^*) \subseteq \mathcal{R}(\mathbf{P}_\mathbf{K}^*)\). Moreover, (2.17) leads to
\[
r(\mathbf{P}_\mathbf{K}^*) = r(\mathbf{K}) = r(\mathbf{K}^2) = r(\mathbf{K}^\dagger \mathbf{K} \mathbf{K}^\dagger) = r(\mathbf{P}_\mathbf{K} \mathbf{P}_\mathbf{K}^*) = r(\mathbf{P}_\mathbf{K}^* \mathbf{P}_\mathbf{K}^*),
\]
thus implying \(\mathcal{R}(\mathbf{P}_\mathbf{K} \mathbf{P}_\mathbf{K}^*) = \mathcal{R}(\mathbf{P}_\mathbf{K}^*)\). Analogous arguments show that \(\mathcal{R}(\mathbf{P}_\mathbf{K} \mathbf{P}_\mathbf{K}) = \mathcal{R}(\mathbf{P}_\mathbf{K}^*)\), and since there is one-to-one correspondence between orthogonal projectors and subspaces onto which they project, it follows that \(\mathcal{R}(\mathbf{K}) = \mathcal{R}(\mathbf{K}^*), i.e., \mathbf{K} \in \mathbb{C}_{n}^\text{EP}\), as desired. \(\square\)

**Theorem 3.** For any \(\mathbf{K} \in \mathbb{C}_{n,n}\),
\[
\mathbf{K} \in \mathbb{C}^\text{HGP}_n \Leftrightarrow \mathbf{K} \in \mathbb{C}^\text{QP}_n \cap \mathbb{C}^\text{WEP}_n.
\]

**Proof.** In view of (2.16), the necessity is an immediate consequence of (2.15). Conversely, from (2.6) it follows that if \(\mathbf{K}\) is decomposed as in (2.3), then it is a weak-EP matrix if and only if
\[ U^*V^* = VV^*U^*, \]

which on account of (2.4) leads to \( WW^* = W \). This means that \( W \in \mathbb{C}^{P^I}_{k,k} \) or, alternatively, \( WW^* \in \mathbb{C}^{QP}_{k} \). On the other hand, from (2.14) it is seen that the condition \( K \in \mathbb{C}^{QP}_{n} \) entails \( r(W) = k \). Consequently, \( WW^* \) is nonsingular, and since the only nonsingular projector is the identity matrix, it follows that \( WW^* = I_k \), i.e., \( W \in \mathbb{C}^U_{k} \). In view of part (e) of Theorem 2, combining this condition with \((WD)^3 = I_k \) shows that \( K \) is a hypergeneralized projector, thus completing the proof. \( \square \)

By analogy to Definition 2 we introduce the concept of weak-normality.

**Definition 3.** A matrix \( K \in \mathbb{C}_{n,n} \) is said to be weak-normal whenever

\[ KK^*K = K^*KKK^*. \]

The set of all matrices satisfying Definition 3 will henceforth be denoted by \( \mathbb{C}^{WN}_{n} \). In view of (1.4), it is obvious that \( \mathbb{C}^{N}_{n} \subseteq \mathbb{C}^{WN}_{n} \). The result (1.9) asserts, in particular, that if \( K \in \mathbb{C}_{n,n} \) is quadripotent and normal, then it is a generalized projector. The theorem below shows that replacing the normality requirement in this statement by the weak-normality leads to a new property of hypergeneralized projectors.

**Theorem 4.** For any \( K \in \mathbb{C}_{n,n} \),

\[ K \in \mathbb{C}^{QP}_{n} \cap \mathbb{C}^{WN}_{n} \Rightarrow K \in \mathbb{C}^{HGP}_{n}. \]

**Proof.** If \( K \) is quadripotent and weak-normal, then

\[ \mathcal{R}(K) = \mathcal{R}(KK^*) = \mathcal{R}[K(K^*)^4] \subseteq \mathcal{R}(KK^*K^*K) = \mathcal{R}(KK^*K^*K) \]

\[ = \mathcal{R}(K^*KK^*) \subseteq \mathcal{R}(K^*), \]

and since \( r(K) = r(K^*) \), it is seen that \( \mathcal{R}(K) = \mathcal{R}(K^*) \), i.e., \( K \in \mathbb{C}^{EP}_{n} \). Consequently, on account of (1.12), it follows that \( K \in \mathbb{C}^{HGP}_{n} \). \( \square \)

In view of \((K^2)^* = (K^*)^2 \) and \((K^3)^* = (K^*)^3 \), it is clear from parts (c) and (d) of Definition 1 that

\[ K \in \mathbb{C}^{GP}_{n} \Leftrightarrow K^* \in \mathbb{C}^{GP}_{n} \quad \text{and} \quad K \in \mathbb{C}^{HGP}_{n} \Leftrightarrow K^* \in \mathbb{C}^{HGP}_{n}. \quad (2.18) \]

It appears that analogous equivalences are valid also when the conjugate transpose \( K^* \) is replaced by the Moore–Penrose inverse \( K^\dagger \).

**Theorem 5.** For any \( K \in \mathbb{C}_{n,n} \),

\[ K \in \mathbb{C}^{GP}_{n} \Leftrightarrow K^\dagger \in \mathbb{C}^{GP}_{n} \quad \text{and} \quad K \in \mathbb{C}^{HGP}_{n} \Leftrightarrow K^\dagger \in \mathbb{C}^{HGP}_{n}. \]
Proof. From (2.5) it is clear that if $K$ is represented as in (2.3), then $(K^\dagger)^* = UD^{-1}V^*$. Consequently, $(K^\dagger)^* \in C_n^{GP}$ and $(K^\dagger)^* \in C_n^{HGP}$ if and only if the conditions on the right-hand sides of parts (d) and (e) of Theorem 2 are satisfied with the replacement of $D = I_k$ by $D^{-1} = I_k$ in the former case and of $(WD)^3 = I_k$ by $(WD^{-1})^3 = I_k$ in the latter. But $D^{-1} = I_k \iff D = I_k$ and, for $W \in C_n^U$, $(WD^{-1})^3 = I_k \iff (WD)^3 = I_k$, thus showing that the sets of conditions characterizing $(K^\dagger)^* \in C_n^{GP}$ and $(K^\dagger)^* \in C_n^{HGP}$ are equivalent to those characterizing $K \in C_n^{GP}$ and $K \in C_n^{HGP}$. Invoking (2.18), this observation concludes the proof. □

Clearly, an alternative proof of Theorem 5 can be obtained utilizing characterizations of generalized and hypergeneralized projectors given in Lemmas 1 and 2. This is a consequence of the fact that the Moore–Penrose inverses of $K$ represented as in (2.1) and (2.2) are

$$K^\dagger = U \begin{pmatrix} E^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} U^*$$ and $$K^\dagger = U \begin{pmatrix} T^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} U^*,$$

respectively, where $E^{-1}$ is the diagonal matrix and $T^{-1}$ is an upper triangular matrix with the diagonal elements $e_{jj}^{-1}$ and $t_{jj}^{-1}$, $j = 1, \ldots, k$. Since the reciprocals of the numbers constituting the set $\left\{ 1, -\frac{1}{2} - \frac{\sqrt{3}}{2}i, -\frac{1}{2} + \frac{\sqrt{3}}{2}i \right\}$ form the same set, the result follows. Moreover, notice that the second part of Theorem 5 is inherent in Theorem 2 of Groß and Trenkler [5] as the part (c) $\iff$ (d).

Acknowledgement

The authors thank a referee for suggestions which resulted in improving some fragments of proofs of Theorems 2 and 3.

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