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H-Accretive Operators and Resolvent Operator Technique for Solving Variational Inclusions in Banach Spaces

YA-PING FANG AND NAN-JING HUANG*
Department of Mathematics, Sichuan University
Chengdu, Sichuan 610064, P.R. China
nanjinghuang@hotmail.com

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Abstract—In this paper, we first introduce a new class of generalized accretive operators named H-accretive operators in Banach spaces. By studying the properties of H-accretive operators, we extend the concept of resolvent operators associated with the classical m-accretive operators to the new H-accretive operators. In terms of the new resolvent operator technique, we give the approximate solution for a class of variational inclusions involving H-accretive operators in Banach spaces. © 2004 Elsevier Ltd. All rights reserved.

Keywords—H-accretive operator, Resolvent operator technique, Variational inclusion, Iterative algorithm.

1. INTRODUCTION AND PRELIMINARIES

Variational inequalities and variational inclusions are among the most interesting and important mathematical problems and have been studied intensively in the past years since they have wide applications in mechanics, physics, optimization and control, nonlinear programming, economics and transportation equilibrium, and engineering sciences, etc. (see, for example, [1–20]). In the theory of variational inequalities and variational inclusions, the development of an efficient and implementable iterative algorithm is interesting and important. Various kinds of iterative algorithms to solve the variational inequalities and inclusions have been developed by many authors. For details, we can refer to [1–6,8–13,15–20] and the references therein. Among these methods, the resolvent operator techniques for solving variational inequalities and variational inclusions are interesting and important.

Recently, Huang and Fang [12] introduced a new class of maximal η-monotone mapping in Hilbert spaces, which is a generalization of the classical maximal monotone mapping, and studied

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*Author to whom all correspondence should be addressed.

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the properties of the resolvent operator associated with the maximal η-monotone mapping. They also introduced and studied a new class of nonlinear variational inclusions involving maximal η-monotone mapping in Hilbert spaces. For some related works, we refer to [8] and the references therein.

In this paper, we further generalize the resolvent operator technique by introducing a new class of H-accretive operators in Banach spaces. We extend the concept of resolvent operators associated with the classical m-accretive operators to the new H-accretive operators. By using the new resolvent operator technique, we study the approximate solution of a class of variational inclusions with H-accretive operators in Banach spaces.

In what follows, we always let X be a real Banach space with dual space X^*, $\langle \cdot, \cdot \rangle$ be the dual pair between X and X^*, and 2^X denote the family of all the nonempty subsets of X. The generalized duality mapping $J_q : X \to 2^{X^*}$ is defined by

$$J_q(x) = \left\{ f^* \in X^* : \langle x, f^* \rangle = \|x\|^q \text{ and } \|f^*\| = \|x\|^{q-1} \right\}, \qquad \forall\, x \in X,$$

where $q > 1$ is a constant. In particular, J_2 is the usual normalized duality mapping. It is known that, in general, $J_q(x) = \|x\|^{q-2} J_2(x)$, for all $x \neq 0$, and J_q is single-valued if X^* is strictly convex. In the sequel, unless otherwise specified, we always suppose that X is a real Banach space such that J_q is single-valued and $\mathcal{H}$ is a Hilbert space. If $X = \mathcal{H}$, then J_2 becomes the identity mapping of $\mathcal{H}$.

The modulus of smoothness of X is the function $\rho_X : [0, \infty) \to [0, \infty)$ defined by

$$\rho_X(t) = \sup \left\{ \frac{1}{2} \left(\|x + y\| + \|x - y\| \right) - 1 : \|x\| \leq 1,\ \|y\| \leq t \right\}.$$

A Banach space X is called uniformly smooth if

$$\lim_{t \to 0} \frac{\rho_X(t)}{t} = 0.$$

X is called q-uniformly smooth if there exists a constant $c > 0$, such that

$$\rho_X(t) \leq c t^q, \qquad q > 1.$$

Note that J_q is single-valued if X is uniformly smooth. In the study of characteristic inequalities in q-uniformly smooth Banach spaces, Xu [21] proved the following theorem.

THEOREM X. *Let X be a real uniformly smooth Banach space. Then, X is q-uniformly smooth if and only if there exists a constant $c_q > 0$, such that for all $x, y \in X$,*

$$\|x + y\|^q \leq \|x\|^q + q \langle y, J_q(x) \rangle + c_q \|y\|^q.$$

DEFINITION 1.1. *Let $T, H : X \to X$ be two single-valued operators. The operator T is said to be*

(i) *accretive if*

$$\langle Tx - Ty, J_q(x - y) \rangle \geq 0, \qquad \forall\, x, y \in X;$$

(ii) *strictly accretive if*

$$\langle Tx - Ty, J_q(x - y) \rangle \geq 0, \qquad \forall\, x, y \in X,$$

and the equality holds if and only $x = y$;

(iii) *strongly accretive if*

$$\langle Tx - Ty, J_q(x - y) \rangle \geq r \|x - y\|^q, \qquad \forall\, x, y \in X;$$

(iv) *strongly accretive with respect to H if there exists some constant $\gamma > 0$, such that*

$$\langle Tx - Ty, J_q(H(x) - H(y))\rangle \geq \gamma\|x - y\|^q, \qquad \forall\, x, y \in X;$$

(v) *Lipschitz continuous if there exists some constant $s > 0$, such that*

$$\|Tx - Ty\| \leq s\|x - y\|, \qquad \forall\, x, y \in X.$$

REMARK 1.1. If T and H are Lipschitz continuous with constants τ and s, respectively, and T is strongly accretive with respect to H with constant γ, then $\gamma \leq \tau s^{q-1}$.

EXAMPLE 1.1. Let $X = (-\infty, +\infty)$, $Tx = -x$ and $Hx = -2x$, for all $x \in X$. Then, T is strongly accretive with respect to H, but T is not strongly accretive.

Example 1.1 shows that the strong accretivity of T with respect to H is a generalization of the strong accretivity of T.

DEFINITION 1.2. *A multivalued operator $M : X \to 2^X$ is said to be*

(i) *accretive if*

$$\langle u - v, J_q(x - y)\rangle \geq 0, \qquad \forall\, x, y \in X, \quad u \in M(x), \quad v \in M(y);$$

(ii) *m-accretive if M is accretive and $(I + \lambda M)(X) = X$, for all $\lambda > 0$, where I denotes the identity mapping on X.*

REMARK 1.2.

(1) In Definitions 1.1 and 1.2, the number q ($q > 1$) can be replaced by 2 since $J_q(x) = \|x\|^{q-2} J_2(x)$, for all $x \in X$.
(2) If $X = \mathcal{H}$, then we can obtain the corresponding definitions of monotonicity, strict monotonicity, strong monotonicity, strong monotonicity with respect to H, and maximal monotonicity from Definitions 1.1 and 1.2.

2. H-ACCRETIVE OPERATORS

In this section, we shall introduce a new class of generalized accretive operators—H-accretive operators, and discuss some properties of H-accretive operators.

DEFINITION 2.1. *Let $H : X \to X$ be a single-valued operator and $M : X \to 2^X$ be a multivalued operator. We say that M is H-accretive if M is accretive and $(H + \lambda M)(X) = X$ holds, for all $\lambda > 0$.*

REMARK 2.1. If $H = I$, then Definition 2.1 reduces to the definition of m-accretive operator, and if $X = \mathcal{H}$ and $H = I$, then Definition 2.1 reduces to the definition of maximal monotone operator.

In the sequel, we show the existence of an H-accretive operator. First we recall some concepts.

DEFINITION 2.2. *A single-valued operator $H : \mathcal{H} \to \mathcal{H}$ is said to be*

(i) *coercive if*

$$\lim_{\|x\|\to\infty} \frac{\langle H(x), x\rangle}{\|x\|} = \infty;$$

(ii) *hemi-continuous if for any fixed $x, y, z \in \mathcal{H}$, the function $t \mapsto \langle H(x + ty), z\rangle$ is continuous at 0^+.*

DEFINITION 2.3. *A set-valued operator $A : X \to 2^X$ is said to be bounded if $A(B)$ is bounded for every bounded subset B of X.*

PROPOSITION 2.1. *Let $M : \mathcal{H} \to 2^{\mathcal{H}}$ be a maximal monotone operator and $H : \mathcal{H} \to \mathcal{H}$ be a bounded, coercive, hemi-continuous, and monotone operator. Then, M is H-accretive.*

PROOF. For every $\lambda > 0$, λM is maximal monotone since M is maximal monotone. Since H is bounded, coercive, hemi-continuous, and monotone, it follows from Corollary 32.26 of [22] that $H + \lambda M$ is surjective, i.e., $(H + \lambda M)(\mathcal{H}) = \mathcal{H}$ holds, for every $\lambda > 0$. Thus, M is an H-accretive operator. The proof is complete.

The following example shows that an m-accretive operator need not be H-accretive for some H.

EXAMPLE 2.1. Let $X = (-\infty, +\infty)$, $Mx = x$ and $H(x) = x^2$, for all $x \in X$. Then, it is easy to see that M is an m-accretive operator and the range of $H + M$ is $[-1/4, +\infty)$. Therefore, M is not H-accretive.

THEOREM 2.1. *Let $H : X \to X$ be a strictly accretive single-valued operator, $M : X \to 2^X$ be an H-accretive operator, $x, u \in X$ given points. If $\langle u - v, J_q(x - y)\rangle \geq 0$ holds, for all $(y, v) \in \operatorname{Graph} M$, then $u \in M(x)$, where $\operatorname{Graph} M = \{(x, u) \in X \times X : u \in M(x)\}$.*

PROOF. Let $x, u \in X$, such that

$$\langle u - v, J_q(x - y)\rangle \geq 0, \qquad \forall\, (v, y) \in \operatorname{Graph} M. \tag{2.1}$$

Since M is H-accretive, $(H+\lambda M)(X) = X$, for all $\lambda > 0$. Hence, there exists $(x_0, u_0) \in \operatorname{Graph} M$, such that

$$H(x_0) + \lambda u_0 = H(x) + \lambda u. \tag{2.2}$$

Equations (2.1) and (2.2) imply that

$$0 \leq \lambda \langle u - u_0, J_q(x - x_0)\rangle = \langle H(x_0) - H(x), J_q(x - x_0)\rangle .$$

Since H is strictly accretive, the inequalities above show that $x = x_0$. Again from (2.2), we have $u = u_0$. Hence, $(u, x) \in \operatorname{Graph} M$. The proof is complete.

THEOREM 2.2. *Let $H : X \to X$ be a strictly accretive operator and $M : X \to 2^X$ be an H-accretive operator. Then, the operator $(H + \lambda M)^{-1}$ is single-valued, where $\lambda > 0$ is a constant.*

PROOF. Let $u \in X$, $x, y \in (H+\lambda M)^{-1}(u)$. It follows that $-H(x)+u \in \lambda M(x)$ and $-H(y)+u \in \lambda M(y)$. Since M is accretive,

$$\langle (-H(x) + u) - (-H(y) + u), J_q(x - y)\rangle = \langle H(y) - H(x), J_q(x - y)\rangle \geq 0.$$

The strict accretiveness of H implies that $x = y$. Thus, $(H + \lambda M)^{-1}$ is single-valued. The proof is complete.

Based on Theorem 2.2, we can define the resolvent operator $R^H_{M,\lambda}$ associated with H and M as follows.

DEFINITION 2.4. *Let $H : X \to X$ be a strictly accretive operator and $M : X \to 2^X$ be an H-accretive operator. The resolvent operator $R^H_{M,\lambda} : X \to X$ associated with H and M is defined by*

$$R^H_{M,\lambda}(u) = (H + \lambda M)^{-1}(u), \qquad \forall\, u \in H.$$

THEOREM 2.3. *Let $H : X \to X$ be a strongly accretive operator with constant r and $M : X \to 2^X$ be an H-accretive operator. Then, the resolvent operator $R^H_{M,\lambda} : X \to X$ is Lipschitz continuous with constant $1/r$, i.e.,*

$$\left\| R^H_{M,\lambda}(u) - R^H_{M,\lambda}(v) \right\| \leq \frac{1}{r\|u - v\|}, \qquad \forall\, u, v \in X.$$

PROOF. Let u, v be any given points in X. It follows that

$$R^H_{M,\lambda}(u) = (H + \lambda M)^{-1}(u)$$

and

$$R^H_{M,\lambda}(v) = (H + \lambda M)^{-1}(v).$$

This implies that

$$\frac{1}{\lambda}\left(u - H\left(R^H_{M,\lambda}(u)\right)\right) \in M\left(R^H_{M,\lambda}(u)\right)$$

and

$$\frac{1}{\lambda}\left(v - H\left(R^H_{M,\lambda}(v)\right)\right) \in M\left(R^H_{M,\lambda}(v)\right).$$

Since M is accretive,

$$\begin{aligned}
&\frac{1}{\lambda}\left\langle u - H\left(R^H_{M,\lambda}(u)\right) - \left(v - H\left(R^H_{M,\lambda}(v)\right)\right), J_q\left(R^H_{M,\lambda}(u) - R^H_{M,\lambda}(v)\right)\right\rangle \\
&\qquad = \frac{1}{\lambda}\left\langle u - v - \left(H\left(R^H_{M,\lambda}(u)\right) - \left(R^H_{M,\lambda}(v)\right)\right), J_q\left(R^H_{M,\lambda}(u) - R^H_{M,\lambda}(v)\right)\right\rangle \geq 0.
\end{aligned}$$

The inequality above implies that

$$\begin{aligned}
\|u - v\| \cdot \left\|R^H_{M,\lambda}(u) - R^H_{M,\lambda}(v)\right\|^{q-1} &= \|u - v\| \cdot \left\|J_q\left(R^H_{M,\lambda}(u) - R^H_{M,\lambda}(v)\right)\right\| \\
&\geq \left\langle u - v, J_q\left(R^H_{M,\lambda}(u) - R^H_{M,\lambda}(v)\right)\right\rangle \\
&\geq \left\langle H\left(R^H_{M,\lambda}(u)\right) - H\left(R^H_{M,\lambda}(v)\right), J_q\left(R^H_{M,\lambda}(u) - R^H_{M,\lambda}(v)\right)\right\rangle \\
&\geq r\left\|R^H_{M,\lambda}(u) - R^H_{M,\lambda}(v)\right\|^q.
\end{aligned}$$

Hence,

$$\left\|R^H_{M,\lambda}(u) - R^H_{M,\lambda}(v)\right\| \leq \frac{1}{r\|u - v\|}, \qquad \forall\, u, v \in H.$$

This completes the proof.

3. VARIATIONAL INCLUSIONS

Let $A, H : X \to X$ be two single-valued operators and $M : X \to 2^X$ be a multivalued operator. We consider the following problem of finding $u \in X$, such that

$$0 \in A(u) + M(u), \tag{3.1}$$

which is called the generalized variational inclusion and has been studied by many authors in the setting of Hilbert spaces when M is maximal monotone and A is strongly monotone. It is easy to see that problem (3.1) includes many variational inequality (inclusion) and complementarity problems as special cases.

In this section, we shall study problem (3.1) when M is H-accretive and A is strongly accretive with respect to H. To obtain the approximate solution of problem (3.1), we first give a characterization the solution of problem (3.1) by using the resolvent operator $R^H_{M,\lambda}$.

LEMMA 3.1. *Let $H : X \to X$ be a strictly accretive operator and $M : X \to 2^X$ be H-accretive. Then, $u \in X$ is a solution of problem (3.1) if and only if*

$$u = R^H_{M,\lambda}[H(u) - \lambda A(u)].$$

PROOF. The conclusion follows directly from Definition 2.4 and some simple arguments.

Based on Lemma 3.1, we construct the following iterative algorithm for solving problem (3.1).

ALGORITHM 3.1. For any $u_0 \in X$, the iterative $\{u_n\} \subset X$ is defined by

$$u_{n+1} = R^H_{M,\lambda}[H(u_n) - \lambda A(u_n)], \qquad n = 0, 1, 2, \ldots. \tag{3.2}$$

THEOREM 3.1. *Let X be a q-uniformly smooth Banach space and $H : X \to X$ be a strongly accretive and Lipschitz continuous operator with constants γ and τ, respectively. Let $A : X \to X$ be Lipschitz continuous and strongly accretive with respect to H with constants s and r, respectively. Assume that $M : X \to 2^X$ is an H-accretive operator and there exist $\lambda > 0$, such that*

$$\tau^q - q\lambda r + c_q \lambda^q s^q \leq \gamma^q, \tag{3.3}$$

where $c_q > 0$ is the same as in Theorem X. Then, the iterative sequence $\{u_n\}$ generated by Algorithm 3.1 converges strongly to the unique solution of problem (3.1).

PROOF. It follows from (3.2) and Theorem 2.3 that

$$\begin{aligned} \|u_{n+1} - u_n\| &= \left\|R^H_{M,\lambda}[H(u_n) - \lambda A(u_n)] - R^H_{M,\lambda}[H(u_{n-1}) - \lambda A(u_{n-1})]\right\| \\ &\leq \frac{1}{\gamma \|H(u_n) - H(u_{n-1}) - \lambda(A(u_n) - A(u_{n-1}))\|}. \end{aligned} \tag{3.4}$$

By assumptions and Theorem X, one has

$$\begin{aligned} &\|H(u_n) - H(u_{n-1}) - \lambda(A(u_n) - A(u_{n-1}))\|^q \\ &\quad \leq \|H(u_n) - H(u_{n-1})\|^q - q\lambda \langle A(u_n) - A(u_{n-1}), J_q(H(u_n) - H(u_{n-1}))\rangle \\ &\qquad + \lambda^q c_q \|A(u_n) - A(u_{n-1})\|^q \\ &\quad \leq (\tau^q - q\lambda r + c_q \lambda^q s^q)\|u_n - u_{n-1}\|^q. \end{aligned} \tag{3.5}$$

Equations (3.4) and (3.5) imply that

$$\|u_{n+1} - u_n\| \leq k\|u_n - u_{n-1}\|,$$

where

$$k = \frac{1}{\gamma \sqrt[q]{\tau^q - q\lambda r + c_q \lambda^q s^q}}.$$

By (3.3), we know that $0 \leq k < 1$. Hence, $\{u_n\}$ is a Cauchy sequence. Let $u_n \to u$ as $n \to \infty$. It follows from (3.2) that

$$u = R^H_{M,\lambda}\left[H(u) - \lambda A(u)\right]. \tag{3.6}$$

By Lemma 3.1, u is a solution of problem (3.1).

Let u^* be another solution of problem (3.1). Then, Lemma 3.1 implies that

$$u^* = R^H_{M,\lambda}\left[H(u^*) - \lambda A(u^*)\right]. \tag{3.7}$$

By (3.6), (3.7), and the similar arguments above, we have

$$\|u - u^*\| \leq k\|u - u^*\|,$$

where

$$k = \frac{1}{\gamma \sqrt[q]{\tau^q - q\lambda r + c_q \lambda^q s^q}}.$$

Since $0 \leq k < 1$, we know $u^* = u$, and so u is the unique solution of problem (3.1). This completes the proof.

REMARK 3.1. If X is 2-uniformly smooth and there exists $\lambda > 0$, such that

$$\left|\lambda - \frac{r}{c_2 s^2}\right| < \frac{\sqrt{r^2 - c_2 s^2(\tau^2 - \gamma^2)}}{c_2 s^2}, \qquad r^2 > c_2 s^2\left(\tau^2 - \gamma^2\right),$$

then (3.3) holds. We note that Hilbert spaces and L_p (or l_q) spaces $(2 \leq q < \infty)$ are 2-uniformly smooth.

REFERENCES

1. R.P. Agarwal, Y.J. Cho and N.J. Huang, Sensitivity analysis for strongly nonlinear quasi-variational inclusions, *Appl. Math. Lett.* **13** (6), 19–24, (2000).
2. R.P. Agarwal, N.J. Huang and Y.J. Cho, Generalized nonlinear mixed implicit quasi-variational inclusions with set-valued mappings, *J. Inequal. Appl.* **7** (6), 807–828, (2002).
3. R. Ahmad and Q.H. Ansari, An iterative algorithm for generalized nonlinear variational inclusions, *Appl. Math. Lett.* **13** (5), 23–26, (2000).
4. S.S. Chang, Y.J. Cho and H.Y. Zhou, *Iterative Methods for Nonlinear Operator Equations in Banach Spaces*, Nova Sci. Publ., New York, (2002).
5. J.Y. Chen, N.C. Wong and J.C. Yao, Algorithm for generalized co-complementarity problems in Banach spaces, *Computers Math. Applic.* **43** (1/2), 49–54, (2002).
6. X.P. Ding and C.L. Luo, Perturbed proximal point algorithms for generalized quasi-variational-like inclusions, *J. Comput. Appl. Math.* **210**, 153–165, (2000).
7. F. Giannessi and A. Maugeri, *Variational Inequalities and Network Equilibrium Problems*, Plenum, New York, (1995).
8. N.J. Huang, Nonlinear implicit quasi-variational inclusions involving generalized m-accretive mappings, *Arch. Inequal. Appl.* (to appear).
9. N.J. Huang, Generalized nonlinear variatonal inclusions with noncompact valued mappings, *Appl. Math. Lett.* **9** (3), 25–29, (1996).
10. N.J. Huang, Mann and Ishikawa type perturbed iterative algorithms for generalized nonlinear implicit quasi-variational inclusions, *Computers Math. Applic.* **35** (10), 1–7, (1998).
11. N.J. Huang, A new completely general class of variational inclusions with noncompact valued mappings, *Computers Math. Applic.* **35** (10), 9–14, (1998).
12. N.J. Huang and Y.P. Fang, A new class of general variational inclusions involving maximal η-monotone mappings, *Publ. Math. Debrecen* **62** (1-2), 83–98, (2003).
13. C.R. Jou and J.C. Yao, Algorithm for generalized multivalued variational inequalities in Hilbert spaces, *Computers Math. Applic.* **25** (9), 7–16, (1993).
14. J.S. Jung and C.H. Morales, The Mann process for perturbed m-accretive operators in Banach spaces, *Nonlinear Anal. T.M.A.* **46** (2), 231–243, (2001).
15. K.R. Kazmi, Mann and Ishikawa type perturbed iterative algorithms for generalized quasivariational inclusions, *J. Math. Anal. Appl.* **209**, 572–584, (1997).
16. C.H. Lee, Q.H. Ansari and J.C. Yao, A perturbed algorithm for strongly nonlinear variational-like inclusions, *Bull. Austral. Math. Soc.* **62**, 417–426, (2000).
17. L.W. Liu and Y.Q. Li, On generalized set-valued variational inclusions, *J. Math. Anal. Appl.* **261** (1), 231–240, (2001).
18. S.H. Shim, S.M. Kang, N.J. Huang and Y.J. Cho, Perturbed iterative algorithms with errors for completely generalized strongly nonlinear implicit quasivariational inclusions, *J. Inequal. Appl.* **5** (4), 381–395, (2000).
19. A.H. Siddiqi, R. Ahmad and S. Husain, A perturbed algorithm for generalized nonlinear quasi-variational inclusions, *Math. Comput. Appl.* **3** (3), 177–184, (1998).
20. G.X.Z. Yuan, *KKM Theory and Applications in Nonlinear Analysis*, Marcel Dekker, New York, (1999).
21. H.K. Xu, Inequalities in Banach spaces with applications, *Nonlinear Anal.* **16** (12), 1127–1138, (1991).
22. E. Zeidler, *Nonlinear Functional Analysis and its Applications II: Monotone Operators*, Springer-Verlag, Berlin, (1985).