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Maximal Subgroups of Direct Products

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We determine all maximal subgroups of the direct product G^n of n copies of a group G. If G is finite, we show that the number of maximal subgroups of G^n is a quadratic function of n if G is perfect, but grows exponentially otherwise. We deduce a theorem of Wiegold about the growth behaviour of the number of generators of G^n. © 1997 Academic Press

A group G is simple if and only if the diagonal subgroup of $G \times G$ is a maximal subgroup. This striking property is very easy to prove and raises the question of determining all the maximal subgroups of G^n, where G^n denotes the direct product of n copies of G. The first purpose of this article is to answer completely this question. We show in particular that if G is perfect, then any maximal subgroup of G^n is the inverse image of a maximal subgroup of G^2 for some projection $G^n \to G^2$ onto two factors.

If G is finite, we let $m(G^n)$ be the number of maximal subgroups of G^n. If $G = C_p$ is cyclic of prime order p, then $m(C_p^n) = (p^n - 1)/(p - 1)$, so that $m(C_p^n)$ is an exponential function of n. It follows easily that $m(G^n)$ grows exponentially if G is not perfect. In contrast, when G is perfect, the fact that any maximal subgroup of G^n comes from G^2 implies that $m(G^n)$ is a quadratic polynomial in n. We give in fact an explicit formula for $m(G^n)$ (in terms of numbers depending only on G).

The minimal number $d(H)$ of generators of a finite group H strongly depends on the number of maximal subgroups of H. For instance, if there is only one maximal subgroup, then H is cyclic (of prime power order) and $d(H) = 1$. So it is not surprising that the above results indicate that $d(G^n)$ behaves differently depending on whether G is perfect. It turns out that

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$d(G^n)$ grows logarithmically if G is perfect and linearly otherwise. This result is due to Wiegold [W1, W2] and we give here a new proof based on our study of maximal subgroups.

There is a general procedure for finding the maximal subgroups M of a finite group, due to Aschbacher and Scott [A–S] (see also the work of Gross and Kovacs). Although their assumption that M should be core-free could be realized in our case, our elementary methods do not make it necessary and this little note does not depend on their important work. We should perhaps apologize for the fact that this article is so elementary that it could be taught to undergraduates, but, as Schönberg said, there are still many musics to be written in C major.

1. SUBGROUPS OF DIRECT PRODUCTS

Let G and H be two groups. We first describe all subgroups of $G \times H$. The result is well known (and ought to appear in textbooks!), but we only found it in [Bo2, 3.1]. If $G \cong H$ and $\phi: G \to H$ is an isomorphism, then the graph $\Delta_\phi = \{(g, \phi(g)) \mid g \in G\}$ of ϕ is a subgroup of $G \times H$ embedded diagonally. If for instance $G = H$ and $\phi = \mathrm{id}$, then Δ_{id} is the diagonal subgroup of $G \times G$. It turns out that any subgroup of a direct product $G \times H$ is essentially obtained by such a procedure.

If S is a subgroup of $G \times H$, we define $S_1 = S \cap (G \times 1)$, $S_2 = S \cap (1 \times H)$, $\tilde{S}_1 = \mathrm{pr}_1(S)$, and $\tilde{S}_2 = \mathrm{pr}_2(S)$, where pr_1 and pr_2 denote the two projection maps. We identify S_1 with a subgroup of G and so $S_1 \le \tilde{S}_1$. In fact $S_1 \trianglelefteq \tilde{S}_1$ because $S_1 = S \cap (G \times 1) \trianglelefteq S$. Similarly we identify S_2 with a subgroup of H and we have $S_2 \trianglelefteq \tilde{S}_2$. Note that $S_1 \times S_2 \le S \le \tilde{S}_1 \times \tilde{S}_2$.

Now for any $g \in \tilde{S}_1$, there exists $h \in \tilde{S}_2$ such that $(g, h) \in S$ and the class $\bar{h} \in \tilde{S}_2/S_2$ is uniquely determined by g, because if $(g, h), (g, h') \in S$, then $(g, h)^{-1}(g, h') = (1, h^{-1}h') \in S_2$ so that $\bar{h} = \overline{h'}$. Moreover if $g \in S_1$, then $(g, 1) \in S$ and so $\bar{h} = 1$. So the class $\bar{h}$ only depends on the class $\bar{g} \in \tilde{S}_1/S_1$. This defines a group homomorphism $\phi: \tilde{S}_1/S_1 \to \tilde{S}_2/S_2$ mapping $\bar{g}$ to $\bar{h}$. Exchanging the role of the two factors of the product, we obtain similarly a group homomorphism ψ in the other direction and it follows easily that ϕ is an isomorphism and $\psi = \phi^{-1}$. Thus we have proved the following.

(1.1). LEMMA. *Any subgroup S of $G \times H$ is determined by a section $\tilde{S}_1/S_1$ of G, a section $\tilde{S}_2/S_2$ of H, and an isomorphism $\phi: \tilde{S}_1/S_1 \to \tilde{S}_2/S_2$. Specifically, S is the inverse image $\pi^{-1}(\Delta_\phi)$, where Δ_ϕ is the graph of ϕ and $\pi: \tilde{S}_1 \times \tilde{S}_2 \to \tilde{S}_1/S_1 \times \tilde{S}_2/S_2$ is the quotient map. Moreover, $\tilde{S}_1$ and $\tilde{S}_2$ are the projections of S on the two factors, while S_1 and S_2 are the intersections of S with the two factors.*

Conversely, it is obvious that any isomorphism of sections $\phi: \tilde{S}_1/S_1 \to \tilde{S}_2/S_2$ determines uniquely a subgroup S of $G \times H$ by the above procedure. In the special case where the two sections are trivial, then S has the form $S = S_1 \times S_2$ and such a subgroup will be called a *standard* subgroup of $G \times H$. More generally, a *standard* subgroup of $G_1 \times G_2 \times \cdots \times G_n$ is a subgroup of the form $S_1 \times S_2 \times \cdots \times S_n$, where $S_i \leq G_i$ for each i. For instance, it is easy to see that every centralizer is standard.

We shall need the following fact, which is slightly more general than the first sentence of the introduction, and which appears in [Hu, Satz 9.14].

(1.2). LEMMA. *Let $\phi: G \to H$ be an isomorphism and let Δ_ϕ be the graph of ϕ. Then the lattice of subgroups of $G \times H$ containing Δ_ϕ is isomorphic to the lattice of normal subgroups of G. In particular, Δ_ϕ is maximal if and only if G is simple* (*and hence H too*).

Proof. If $\Delta_\phi \leq S \leq G \times H$, we define $T = S \cap (G \times 1)$, identified with a subgroup of G. We have $T \trianglelefteq G$ because if $t \in T$ and $g \in G$, then $(gtg^{-1}, 1) = (g, \phi(g))(t, 1)(g, \phi(g))^{-1} \in S \cap (G \times 1) = T$. This defines the required map $S \mapsto T$. If conversely $T \trianglelefteq G$, we set $S = T\Delta_\phi$ and it is easy to check that this defines the inverse map. ∎

Now we turn to the description of the maximal subgroups of $G \times H$. The notation $M \underset{\max}{<} K$ will mean that M is a maximal subgroup of K. Let $S \underset{\max}{<} G \times H$, corresponding to $\phi: \tilde{S}_1/S_1 \to \tilde{S}_2/S_2$ as in (1.1). If $\tilde{S}_1 \neq G$, then $S \leq \tilde{S}_1 \times H < G \times H$, so $S = \tilde{S}_1 \times H$, and consequently S is standard, $S_1 = \tilde{S}_1$, and $S_1 \underset{\max}{<} G$. Similarly S is standard if $\tilde{S}_2 \neq H$. Now suppose that $\tilde{S}_1 = G$ and $\tilde{S}_2 = H$. We claim that G/S_1 $(\cong H/S_2)$ is a simple group. Indeed $S/(S_1 \times S_2)$ is equal to the graph of the isomorphism $\phi: G/S_1 \to H/S_2$. By the previous lemma, the maximality of S implies the simplicity of G/S_1. Thus we have proved the following.

(1.3). LEMMA. *Let S be a maximal subgroup of $G \times H$. Then*:

(a) *either S is standard* (*and so $S = S_1 \times H$ with $S_1 \underset{\max}{<} G$ or $S = G \times S_2$ with $S_2 \underset{\max}{<} H$*),

(b) *or S corresponds, by the construction in* (1.1), *to an isomorphism $\phi: G/S_1 \to H/S_2$ of simple groups*.

This lemma shows that we need to know the maximal normal subgroups of each factor for the determination of maximal subgroups. In order to apply this to $G \times G^{n-1}$, we need to know the maximal normal subgroups of G^{n-1}. So we first have to understand the normal subgroups of a direct product. See [Mi] for related results.

(1.4). LEMMA. *Let S be a subgroup of $G \times H$, corresponding to an isomorphism $\phi: \tilde{S}_1/S_1 \to \tilde{S}_2/S_2$ as in* (1.1).

(a) *$S \trianglelefteq G \times H$ if and only if $\tilde{S}_1/S_1$ is centralized by G and $\tilde{S}_2/S_2$ is centralized by H (so in particular all those subgroups are normal).*

(b) *If S is a maximal normal subgroup of $G \times H$, then either S is standard or $(G \times H)/S$ has prime order.*

Proof. (a) If $S \trianglelefteq G \times H$, then $\tilde{S}_1$ and S_1 are normal in G. If $(u, v) \in S$, then $\phi(\bar{u}) = \bar{v}$. For any $g \in G$, we obtain $(gug^{-1}, v) = (g, 1)(u, v)(g, 1)^{-1} \in S$, so that $\phi(\overline{gug^{-1}}) = \bar{v}$. Since ϕ is an isomorphism, it follows that $\overline{gug^{-1}} = \bar{u}$, showing that $\tilde{S}_1/S_1$ is centralized by G. The proof for the other factor is the same.

(b) We have $S \leq \tilde{S}_1 \times \tilde{S}_2 \trianglelefteq G \times H$, so either $S = \tilde{S}_1 \times \tilde{S}_2$ and S is standard, or $\tilde{S}_1 \times \tilde{S}_2 = G \times H$. In this latter case, then by (a) G/S_1 is centralized by G and is therefore abelian. It follows that $(G \times H)/S$ is abelian, hence of prime order by maximality of S. ∎

The next result immediately follows by induction.

(1.5). Corollary. *Any maximal normal subgroup of G^n is either standard or of prime index. In particular, if G is a finite group, then G is perfect if and only if any maximal normal subgroup of G^n is standard.*

Finally we obtain the description of the maximal subgroups of G^n.

(1.6). Proposition. *Let M be a maximal subgroup of G^n with $n \geq 2$. Then one of the following cases holds (or both):*

(a) *either M is a normal subgroup of prime index,*

(b) *or $M = \pi^{-1}(S)$, where $S \underset{\max}{<} G^2$ and $\pi \colon G^n \to G^2$ is one of the projections on two factors.*

Moreover, in the second case, S is either standard (so that M is standard too) or S corresponds, by the construction in (1.1), to an isomorphism $\phi \colon G/S_1 \to G/S_2$ of simple groups.

Proof. We proceed by induction. If $n = 2$, then (b) holds trivially. Assume that $n \geq 3$ and apply Lemma 1.3 to the direct product $G^n = G \times G^{n-1}$. If the first case of Lemma 1.3 occurs, M is standard in this decomposition. Then either $M = S \times G^{n-1}$, so M is standard in G^n and we are in case (b), or $M = G \times T$ with $T \underset{\max}{<} G^{n-1}$ and the result follows by induction. If the second case of Lemma 1.3 occurs, then M corresponds to an isomorphism of simple groups $\phi \colon G/S_1 \to G^{n-1}/S_2$. If this simple group has order p, then we are in case (a). If G^{n-1}/S_2 is a non-abelian simple group, then S_2 is standard in G^{n-1} by Corollary 1.5. Thus $S_2 = U \times G^{n-2}$ and $M \geq S_1 \times S_2 \geq 1 \times 1 \times G^{n-2}$, so we are in case (b).

The additional assertion follows either from the proof or directly from Lemma 1.3. ∎

(1.7). COROLLARY. *If G is perfect, every maximal subgroup of G^n is the inverse image of a maximal subgroup of G^2 for some projection $G^n \to G^2$ on two factors.*

2. THE NUMBER OF MAXIMAL SUBGROUPS

From now on G will be a finite group. Let G_s be the largest semi-simple abelian quotient of G, so $G_s \cong \prod_{i=1}^{r} (C_{p_i})^{m_i}$, a direct product of groups of prime order, where the p_i's are distinct primes. Since the number of hyperplanes in the $\mathbb{F}_p$-vector space $\mathbb{F}_p^k$ is equal to $(p^k - 1)/(p - 1)$, the number of subgroups of index p_i in $G_s^n \cong \prod_{i=1}^{r} (C_{p_i})^{nm_i}$ is equal to $(p_i^{nm_i} - 1)/(p_i - 1)$, and this is also the number of normal subgroups of index p_i in G^n. So the total number of subgroups of G^n which are maximal and normal (hence of prime index) is equal to

$$\sum_{i=1}^{r} \frac{p_i^{nm_i} - 1}{p_i - 1}.$$

Let a be the number of non-normal maximal subgroups of G. Then the number of non-normal maximal subgroups of G^n which are standard is equal to an. Finally let b be the number of triples (S_1, S_2, ϕ) such that $S_i \trianglelefteq G$, G/S_i is non-abelian simple, and $\phi\colon G/S_1 \xrightarrow{\sim} G/S_2$ is an isomorphism. For instance $b = |\mathrm{Aut}(G)|$ if G is non-abelian simple. By Lemma 1.3, b is the number of non-normal maximal subgroups of G^2 which are not standard. Therefore the total number of non-normal maximal subgroups of G^n which come from some quotient G^2 but are not standard is equal to $b\binom{n}{2}$.

By Proposition 1.6, we have counted above all the maximal subgroups of G^n and therefore we have proved the following.

(2.1). PROPOSITION. *Let G be a finite group. With the notation above, the number $m(G^n)$ of maximal subgroups of G^n is equal to*

$$m(G^n) = an + b\binom{n}{2} + \sum_{i=1}^{r} \frac{p_i^{nm_i} - 1}{p_i - 1}.$$

(2.2). COROLLARY. *If G is perfect, then $m(G^n)$ is a quadratic polynomial in n. If G is not perfect, then $m(G^n)$ grows exponentially.*

It is tempting to introduce the generating function $\sum_{n \geq 0} m(G^n) X^n$. By standard results on generating functions (see [St, 4.1, 4.3]), one easily obtains

$$\sum_{n\geq 0} m(G^n)X^n = \frac{aX + (b-a)X^2}{(1-X)^3} + \sum_{i=1}^{r} \frac{p_i^{m_i}-1}{p_i-1} \frac{X}{(1-p_i^{m_i}X)(1-X)}.$$

3. THE NUMBER OF GENERATORS

Our analysis of maximal subgroups of G^n can be used to prove a theorem of Wiegold [W1–W4] on the growth behaviour of the minimal number $d(G^n)$ of generators of G^n, where G is finite. First we recall the following easy fact.

(3.1). LEMMA. *For any finite group G, we have* $\log_{|G|} n \leq d(G^n) \leq dn$, *where* $d = d(G)$.

Proof. If one considers the elements of G^n having a generator of G in some component and 1 elsewhere, one obtains dn elements which clearly generate G^n. Therefore $d(G^n) \leq dn$. Now consider a k-tuple of elements of G^n and assume that $k < \log_{|G|} n$. This k-tuple can be viewed as a $(k \times n)$-matrix with entries in G and since $|G|^k < n$ by the assumption, we necessarily have two columns equal, say the ith and the jth columns. Therefore, if $\pi\colon G^n \to G^2$ denotes the projection onto the ith and jth components, the image of the k-tuple under π lies in the diagonal subgroup of G^2 and hence does not generate G^2. It follows that the k-tuple does not generate G^n and so $k < d(G^n)$. Therefore $\log_{|G|} n \leq d(G^n)$ as required. ∎

Using a refinement of this proof, Meier and Wiegold [M–W] showed that the lower bound can be slightly improved: $\log_{|G|} n + \log_{|G|} |\mathrm{Aut}(G)| \leq d(G^n)$. Now we recall that the growth behaviour is linear in the non-perfect case.

(3.2). LEMMA. *If G is not perfect, then* $d_{ab} n \leq d(G^n) \leq dn$, *where* $d = d(G)$ *and* $d_{ab} = d(G_{ab})$.

Proof. It is obvious that $d(G^n) \geq d(G_{ab}^n)$. Now $G_{ab} = C_{m_1} \times \cdots \times C_{m_k}$ where C_{m_i} is cyclic of order m_i and m_{i-1} divides m_i for each $i \geq 2$. We have $d(G_{ab}) = k$ and similarly $d(G_{ab}^n) = kn$. Thus $d(G^n) \geq kn = d_{ab} n$. ∎

If G is not perfect, it is not hard to prove that $\lim_{n\to\infty}(d(G^n))/n = d_{ab}$. But in fact, by [W2], the limit is already reached for n large enough: $d(G^n) = d_{ab} n$. As a result, the generating function $\sum_{n\geq 1} d(G^n)X^n$ is a rational function with denominator $(1-X)^2$ (see [St, 4.3]).

In contrast, if G is perfect, Wiegold [W1–W4] proved that the growth behaviour of $d(G^n)$ is logarithmic. His successive proofs proceed by reduction to the largest semi-simple quotient of G and used results of Hall [Ha]. We give here a direct proof based on our study of maximal subgroups. This proof immediately gives bounds (obtained in [W3]) which turn out to give the correct asymptotic behaviour. We first need a lemma.

(3.3). LEMMA. *Let X_k be the set of all k-tuples of elements of G which generate G. If k is large enough, then $|X_k| \geq \frac{1}{2}|G|^k$.*

Proof. Let $M_1, \ldots, M_r$ be the set of all maximal subgroups of G. If a k-tuple does not generate G, it is contained in some M_i. Therefore

$$|G^k - X_k| \leq \sum_{i=1}^{r} |M_i|^k = |G|^k \sum_{i=1}^{r} \frac{1}{|G:M_i|^k} \leq |G|^k \frac{r}{2^k},$$

and so

$$|X_k| \geq |G|^k \left(1 - \frac{r}{2^k}\right) \geq \frac{1}{2}|G|^k$$

if k is large enough. ∎

(3.4). PROPOSITION. *Let G be a perfect finite group and let s be the order of the smallest simple quotient of G. Then there exists a constant C such that*

$$\frac{\log n}{\log s} \leq d(G^n) \leq \frac{\log n}{\log s} + C \qquad \textit{for } n \textit{ large enough}.$$

An explicit value of C is given by $C = \log 2b/\log s + 1$, where b is the number of triples (S_1, S_2, ϕ) such that $S_i \trianglelefteq G$, G/S_i is simple (non-abelian), and $\phi\colon G/S_1 \xrightarrow{\sim} G/S_2$ is an isomorphism.

Proof. Let S be a simple quotient of G of order s. Clearly $d(G^n) \geq d(S^n)$. Applying Lemma 3.1 to S^n, we obtain the lower bound

$$d(G^n) \geq d(S^n) \geq \log_s n = \frac{\log n}{\log s}.$$

In order to find an upper bound for $d(G^n)$, we consider the set X_k of all k-tuples of elements of G which generate G. We shall say that $x, y \in X_k$ are *neighbours* if there exists a triple (S_1, S_2, ϕ) as in the statement such that $\phi(\bar{x}) = \bar{y}$ (where $\bar{x} \in (G/S_1)^k$ and $\bar{y} \in (G/S_2)^k$ denote the images of x and y). In that case x and yz are also neighbours if $z \in (S_2)^k$. Therefore the number of neighbours of $x \in X_k$ is at most bm^k, where b is the number of such triples and m is the maximal possible order of a maximal normal subgroup of G (in other words, $s = |G|/m$, where s is as in the statement).

Assume that $nbm^k \leq |X_k|$. Then we can find n elements of X_k such that any two of them are not neighbours. We view these n k-tuples as a $(k \times n)$-matrix with entries in G, hence also as a k-tuple of elements of G^n. We claim that this k-tuple in G^n is not contained in any maximal subgroup of G^n. By Proposition 1.6 and the fact that G is perfect, every maximal subgroup of G^n has the form $\pi^{-1}(S)$, where $\pi: G^n \to G^2$ is some projection on two factors and $S \underset{\max}{<} G^2$ either corresponds to an isomorphism of simple groups $\phi: G/S_1 \xrightarrow{\sim} G/S_2$, via the construction of (1.1), or is standard in G^2, so that $\pi^{-1}(S)$ is standard in G^n. Every column of our $(k \times n)$-matrix is a k-tuple in X_k and hence generates G. Thus our k-tuple in G^n is not contained in a standard proper subgroup of G^n. Any two distinct columns are not neighbours, so if $\pi: G^n \to G^2$ is any projection on two factors, the image of our k-tuple under π is not contained in any maximal subgroup S of G^2 corresponding to an isomorphism $\phi: G/S_1 \xrightarrow{\sim} G/S_2$ via the construction of (1.1). It follows that our k-tuple in G^n is not contained in any maximal subgroup of G^n, as required. This shows that our k-tuple generates G^n and so $d(G^n) \leq k$.

Thus we have shown that $d(G^n) \leq k$ if $|X_k| \geq nbm^k = nb(|G|^k/s^k)$. By Lemma 3.3 this holds in particular if $\frac{1}{2}|G|^k \geq nb(|G|^k/s^k)$, that is, $2nb \leq s^k$, or $k \geq \log 2nb/\log s$, provided n is large enough (so that k is large enough too). Taking $k = [\log 2nb/\log s] + 1$ (the smallest integer $>$ $\log 2nb/\log s$), we obtain

$$d(G^n) \leq \left[\frac{\log 2nb}{\log s}\right] + 1 \leq \frac{\log 2nb}{\log s} + 1 = \frac{\log n}{\log s} + \frac{\log 2b}{\log s} + 1,$$

as was to be shown. ∎

Remarks. 1. It follows immediately from the proposition that $\lim_{n \to \infty} d(G^n)/\log n = 1/\log s$. The logarithmic growth of $d(G^n)$ also implies that the generating function $\sum_{n \geq 1} d(G^n)X^n$ is not a rational function (see [St, 4.1]), contrary to the non-perfect case. We deduce that G is perfect if and only if $\sum_{n \geq 1} d(G^n)X^n$ is not a rational function.

2. By a result of Gaschütz (see [Ga, Satz 3; W2, 1.1]), the value of $d(G^n)$ does not change by passing to the semi-simple quotient $\overline{G}$ of G, provided n is large enough to ensure that $d(\overline{G}^n) \geq d(G)$. This provides a reduction to direct products of simple groups. If G is perfect, the smallest of those simple groups has the quickest growth because of the denominator. This is the basis of Wiegold's approach.

3. For specific values of n, the upper bound can easily be improved if the estimate of $|X_k|$ in Lemma 3.3 is improved. For instance, we have $|X_k| \geq |G|^k(1 - e/r)$, where e is the number of conjugacy classes of maximal subgroups and r is the minimal possible index of such a subgroup (and this holds for every $k \geq 2$) and so, in Proposition 3.4, $2b$ can be replaced by $(1 - e/r)^{-1}b$. Applying this to the group $G = M_{24}$, we easily obtain $d(G^n) = 2$ if $n \leq 232{,}891{,}477$ (but the maximal value of n with $d(G^n) = 2$ is probably much higher).

Explicit values of $|X_k|$ of course give optimal results. By the work of Hall [Ha], the value of $|X_k|$ (which is the Eulerian function $\phi_k(G)$ in [Ha]) can be explicitly computed using Möbius inversion. This can be done particularly well for small (simple) groups. When G is simple, Hall observed a connection between $\phi_k(G)$ and $d(G^k)$ and this is the starting point of Wiegold's approach, as well as an excellent method for exact computations of $d(G^k)$, as in [Ha], [W4], and [E-W]. We note that Bouc has a formula for $\phi_k(G)$ when G is soluble [Bo1] and that he introduced a polynomial which generalizes the function $\phi_k(G)$ [Bo2, p. 709].

4. For the direct product of infinitely many *non-isomorphic* groups, it may happen that there is no growth at all. For instance, the direct product of n pairwise non-isomorphic simple groups can always be generated by two elements, by simply choosing two generators in each component. Indeed the subgroup generated by these two n-tuples surjects onto each component and is therefore the whole group, by repeated applications of Lemma 1.1 (using the fact that the factors have no isomorphic quotients).

5. One can also have a *bounded* number of generators if one considers iterated wreath products, as for instance in [Bh] for alternating groups. Recently, Burger [Bu] has obtained such a result for much more general iterated wreath products.

6. The method used in the proof of Proposition 3.4 was already used by Kantor and Lubotzky [K−W, Prop. 6] in the case of a non-abelian simple group.

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