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Common Hermitian solutions to some operator equations on Hilbert C^*-modules ☆

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We establish necessary and sufficient conditions for the existence of the general common Hermitian solution to the equations $A_1X = C_1,\ XB_1 = C_2,\ A_3XA_3^* = C_3,\ A_4XA_4^* = C_4$ for adjointable operators between Hilbert C^*-modules, and present an expression for the common Hermitian solution to the equations in terms of Moore–Penrose inverse of operators when the solvability conditions are satisfied. The findings of this paper extend some known results in the literature.

1. Introduction

Hermitian solutions to some matrix equations or some operator equations were investigated by many authors. For finite matrices, Khatri and Mitra [8] gave necessary and sufficient conditions for the existence of the common Hermitian solution to the equations

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$$A_1X = C_1, \quad XB_1 = C_2 \tag{1.1}$$

over the complex field $\mathbb{C}$, and presented an explicit expression for the general Hermitian solution to (1.1), by generalized inverses, when the solvability conditions were satisfied. Using the singular value decomposition (SVD), Yuan [16] investigated the general symmetric solution of (1.1) over the real number field $\mathbb{R}$. By the SVD, Dai and Lancaster in [2] considered the symmetric solution of equation

$$AXA^* = C \tag{1.2}$$

over $\mathbb{R}$, which was motivated and illustrated with an inverse problem of vibration theory. Größ in [6], Tian and Liu in [11] gave the solvability conditions for Hermitian solution and its expressions of (1.2) over $\mathbb{C}$ in terms of generalized inverses, respectively. By using the generalized SVD, Chang and Wang [1] examined the symmetric solution to the matrix equations

$$A_3XA_3^* = C_3, \quad A_4XA_4^* = C_4 \tag{1.3}$$

over $\mathbb{R}$. In [11], Tian and Liu established the solvability conditions for (1.3) to have a common Hermitian solution over $\mathbb{C}$ by the ranks of coefficient matrices. However, to our knowledge, the expression for the general Hermitian solution to (1.3) has not been available by generalized inverses so far. For operator equations, Phadke and Thakare [10] described the common Hermitian solution to Eq. (1.1) for Hilbert space operators. Dajić and Koliha revisited (1.1) and obtained some new results in [3]. Dajić and Koliha in [4] investigated the common Hermitian solution to Eq. (1.1) in rings with involution with applications to Hilbert space operators. Xu in [13] considered the solvability conditions for (1.1) to have a common Hermitian solution in the framework of Hilbert C^*-modules, gave an expression for the Hermitian solution to (1.1) when the solvability conditions were satisfied. To our knowledge, so far there has been little information on the common Hermitian solution to (1.3) for operators in the framework of Hilbert C^*-modules. Note that the Eqs. (1.1) and (1.3) for operators between Hilbert C^*-modules are special cases of the following equations

$$A_1X = C_1, \quad XB_1 = C_2, \quad A_3XA_3^* = C_3, \quad A_4XA_4^* = C_4 \tag{1.4}$$

for operators between Hilbert C^*-modules. Motivated by the work mentioned above, we in this paper aim to consider the common Hermitian solution to Eqs. (1.4) for operators between Hilbert C^*-modules.

The paper is organized as follows. We start with some basic concepts and results about the Hilbert C^*-modules in Section 2. We in Section 3 give some necessary and sufficient conditions for the existence of the common Hermitian solution to (1.4) for operators between Hilbert C^*-modules, and establish an expression for this solution when the solvability conditions are met. To conclude this paper, we in Section 4 propose some further research topics.

2. Preliminaries

Hilbert C^*-modules arose as generalizations of the notion Hilbert space. The basic idea was to consider modules over C^*-algebras instead of linear spaces and to allow the inner product to take values in a C^*-algebra. The structure was first used by Kaplansky [7] in 1952. For more details of C^*-algebra and Hilbert C^*-modules, we refer the readers to [9,12].

Let $\mathfrak{A}$ be a C^*-algebra. An inner-product $\mathfrak{A}$-module is a linear space E which is a right $\mathfrak{A}$-module (with a scalar multiplication satisfying $\lambda(xa) = x(\lambda a) = (\lambda x)a$ for $x \in E,\ a \in \mathfrak{A},\ \lambda \in \mathbb{C}$), together with a map $E \times E \to \mathfrak{A}$, $(x, y) \to \langle x, y\rangle$ such that

(1) $\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$;
(2) $\langle x, ya\rangle = \langle x, y\rangle a$;
(3) $\langle x, y\rangle = \langle y, x\rangle^*$;
(4) $\langle x, x\rangle \geqslant 0$, and $\langle x, x\rangle = 0 \Leftrightarrow x = 0$

for any $x, y, z \in E$, $\alpha, \beta \in \mathbb{C}$ and $a \in \mathfrak{A}$. An inner-product $\mathfrak{A}$-module E is called a (right) Hilbert $\mathfrak{A}$-module if it is complete with respect to the induced norm $\|x\| = |\langle x, x\rangle|^{1/2}$.

Assume that $\mathcal{H}$ and $\mathcal{K}$ are two Hilbert $\mathfrak{A}$-modules, and $\mathfrak{B}(\mathcal{H}, \mathcal{K})$ is the set of all maps $T : \mathcal{H} \to \mathcal{K}$ for which there is a map $T^* : \mathcal{K} \to \mathcal{H}$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$, for any $x \in \mathcal{H}$ and $y \in \mathcal{K}$. We know that any element T of $\mathfrak{B}(\mathcal{H},\mathcal{K})$ is a bounded linear operator. We call $\mathfrak{B}(\mathcal{H},\mathcal{K})$ the set of adjointable operators from $\mathcal{H}$ into $\mathcal{K}$. In case $\mathcal{H}=\mathcal{K}$, $\mathfrak{B}(\mathcal{H},\mathcal{H})$ which we abbreviate to $\mathfrak{B}(\mathcal{H})$, is a C^*-algebra and we use the notation $I_{\mathcal{H}}$ to denote the identity operator on $\mathcal{H}$. For any $A\in\mathfrak{B}(\mathcal{H},\mathcal{K})$, the notation $\mathcal{R}(A)$ and $\mathcal{N}(A)$ stand for the range of A and the null space of A, respectively. An operator $A\in\mathfrak{B}(\mathcal{H})$ is Hermitian (or self-adjoint) if $A^*=A$. Let $\mathfrak{B}(\mathcal{H})sa$ denote the set of all Hermitian elements of $\mathfrak{B}(\mathcal{H})$.

Let $\mathcal{H},\ \mathcal{K}$ be two Hilbert $\mathfrak{A}$-modules, $A\in\mathfrak{B}(\mathcal{H},\mathcal{K})$. The Moore–Penrose inverse $A^\dagger$ of A (if it exists) is defined as the unique element of $\mathfrak{B}(\mathcal{K},\mathcal{H})$ which satisfies the following four Penrose equations

$$AXA=A,\quad XAX=X,\quad (AX)^*=AX,\quad (XA)^*=XA.$$

For any $A\in\mathfrak{B}(\mathcal{H},\mathcal{K})$, the Moore–Penrose inverse $A^\dagger$ of A exists if and only if A has closed range [15]. In this case, $A^\dagger$ exists uniquely and $(A^*)^\dagger=(A^\dagger)^*$. Moreover, both $A^\dagger A$ and $AA^\dagger$ are idempotent and self-adjoint. For convenience, we use notations L_A and R_A to stand for $I_{\mathcal{H}}-A^\dagger A$ and $I_{\mathcal{K}}-AA^\dagger$ induced by A, respectively. Obviously, L_A and R_A are also idempotent and self-adjoint and $L_A=R_{A^*}$.

For any Hilbert C^*-modules $\mathcal{H}$ and $\mathcal{K}$, put

$$\mathcal{H}\oplus\mathcal{K}=\left\{\begin{pmatrix}h\\k\end{pmatrix}\middle|,\ h\in\mathcal{H},k\in\mathcal{K}\right\},$$

which is also a Hilbert C^*-module whose inner product is given by

$$\left\langle\begin{pmatrix}h_1\\k_1\end{pmatrix},\begin{pmatrix}h_2\\k_2\end{pmatrix}\right\rangle=\langle h_1,h_2\rangle+\langle k_1,k_2\rangle$$

for $h_i\in\mathcal{H}$ and $k_i\in\mathcal{K},\ i=1,2$.

Let $\mathcal{H}_1,\mathcal{H}_2$, and $\mathcal{H}_3$ be Hilbert $\mathfrak{A}$-modules, and $A_1\in\mathfrak{B}(\mathcal{H}_1,\mathcal{H}_3)$, $A_2\in\mathfrak{B}(\mathcal{H}_2,\mathcal{H}_3)$. Then $A=(A_1,A_2)\in\mathfrak{B}(\mathcal{H}_1\oplus\mathcal{H}_2,\mathcal{H}_3)$ is the partitioned operator defined as

$$A\begin{pmatrix}h_1\\h_2\end{pmatrix}=A_1h_1+A_2h_2\quad\text{for } h_i\in\mathcal{H}_i,\ i=1,2.$$

Similarly, one can define partitioned operators with more blocks, such as $A=(A_1,A_2,A_3,A_4)\in\mathfrak{B}(\mathcal{H}_1\oplus\mathcal{H}_2\oplus\mathcal{H}_3\oplus\mathcal{H}_4,\mathcal{H}_5)$ and a 4×4 partitioned operator $A_{4\times4}\in\mathfrak{B}(\mathcal{H}_1\oplus\mathcal{H}_2\oplus\mathcal{H}_3\oplus\mathcal{H}_4)$.

Lemma 2.1 [14]. *Let $A=(A_1,A_2)$ be a partitioned operator with $A_1\in\mathfrak{B}(\mathcal{H}_1,\mathcal{H}_3)$, $A_2\in\mathfrak{B}(\mathcal{H}_2,\mathcal{H}_3)$. Suppose A_1 has closed range, then $A^\dagger$ exists if and only if $(R_{A_1}A_2)^\dagger$ exists.*

In addition, since $\mathcal{R}(A_1A_1^\dagger)\subseteq\mathcal{R}(R_{A_1}A_2)^\perp$, where $\mathcal{R}(R_{A_1}A_2)^\perp$ is the orthogonal complement of $\mathcal{R}(R_{A_1}A_2)$, we have $(R_{A_1}A_2)^\dagger(A_1A_1^\dagger)=0$, i.e.,

$$(R_{A_1}A_2)^\dagger R_{A_1}=(R_{A_1}A_2)^\dagger. \tag{2.1}$$

3. Common Hermitian solution to (1.4)

In this section, we give some solvability conditions for (1.4) to possess common Hermitian solution and present an expression for this common Hermitian solution when the solvability conditions are met. Throughout this section, $\mathcal{H}$ and $\mathcal{K}_i$ $(i=1,2,3,4)$ are Hilbert $\mathfrak{A}$-modules. We have the following main result of this paper.

Theorem 3.1. *Let $A_1,C_1\in\mathfrak{B}(\mathcal{H},\mathcal{K}_1)$, $B_1,C_2\in\mathfrak{B}(\mathcal{K}_2,\mathcal{H})$, $A_3\in\mathfrak{B}(\mathcal{H},\mathcal{K}_3)$, $A_4\in\mathfrak{B}(\mathcal{H},\mathcal{K}_4)$, $C_3\in\mathfrak{B}(\mathcal{K}_3)$, $C_4\in\mathfrak{B}(\mathcal{K}_4)$. Suppose that $A_1,\ B_1,\ A_3,\ A_4$ and $F,\ M,\ P,\ N$ have closed ranges, where $F=B_1^*L_{A_1}$, $M=SL_F$, $P=(TL_F)^*$, $N=P^*L_M$ and $S=A_3L_{A_1}$, $T=A_4L_{A_1}$. Let*

$$D=C_2^*-B_1^*A_1^\dagger C_1,\quad J=A_1^\dagger C_1+F^\dagger D, \tag{3.1}$$

$$G=C_3-A_3(J+L_{A_1}L_FJ^*)A_3^*, \tag{3.2}$$

$$Q=C_4-A_4[J+L_{A_1}L_FJ^*+M^\dagger G(M^\dagger)^*]A_4^*. \tag{3.3}$$

Then the following conditions are equivalent:

(1) *Eqs.* (1.4) *have a solution* $X \in \mathfrak{B}(\mathcal{H})sa$.
(2) $C_3 = C_3^*,\ C_4 = C_4^*$ *and*

$$A_1C_2 = C_1B_1, \quad A_1C_1^* = C_1A_1^*, \quad B_1^*C_2 = C_2^*B_1, \tag{3.4}$$
$$R_{A_1}C_1 = 0, \quad R_FD = 0, \tag{3.5}$$
$$R_MG = 0, \quad QL_P = 0, \quad R_NQR_N = 0. \tag{3.6}$$

(3) $C_3 = C_3^*,\ C_4 = C_4^*$, *the equalities in* (3.4) *hold and*

$$\mathcal{R}(C_1) \subseteq \mathcal{R}(A_1), \quad \mathcal{R}\begin{pmatrix} C_1 \\ C_2^* \end{pmatrix} \subseteq \mathcal{R}\begin{pmatrix} A_1 \\ B_1^* \end{pmatrix}, \tag{3.7}$$
$$\mathcal{R}\begin{pmatrix} C_1A_3^* \\ C_2^*A_3^* \\ C_3 \end{pmatrix} \subseteq \mathcal{R}\begin{pmatrix} A_1 \\ B_1^* \\ A_3 \end{pmatrix}, \tag{3.8}$$
$$\mathcal{R}\begin{pmatrix} C_1A_4^* \\ C_2^*A_4^* \\ C_4 \end{pmatrix} \subseteq \mathcal{R}\begin{pmatrix} A_1 \\ B_1^* \\ A_4 \end{pmatrix}, \tag{3.9}$$
$$\mathcal{R}(\phi R_\psi) \subseteq \mathcal{R}(\psi), \tag{3.10}$$

where

$$\psi = \begin{pmatrix} A_1 \\ B_1^* \\ A_3 \\ A_4 \end{pmatrix}, \quad \phi = \begin{pmatrix} 0 & 0 & 0 & C_1A_4^* \\ 0 & 0 & 0 & C_2^*A_4^* \\ -A_3C_1^* & -A_3C_2 & -C_3 & 0 \\ 0 & 0 & 0 & C_4 \end{pmatrix}. \tag{3.11}$$

In this case, the general Hermitian solution to (1.4) *can be expressed as*

$$X = J + L_{A_1}L_FJ^* + M^\dagger G(M^\dagger)^* + \frac{1}{2}[N^\dagger Q(R_N + I_{\mathcal{K}_4})P^\dagger + (P^\dagger)^*(R_N + I_{\mathcal{K}_4})Q(N^\dagger)^*] + U + U^*, \tag{3.12}$$

where

$$U = L_{A_1}L_FL_MVL_FL_{A_1} - N^\dagger NVPP^\dagger + \frac{1}{2}N^\dagger NVPNN^\dagger P^\dagger - \frac{1}{2}N^\dagger P^*V^*N^*P^\dagger \tag{3.13}$$

and $V \in \mathfrak{B}(\mathcal{H})$ *is arbitrary*.

Proof. (2) $\Rightarrow$ (1): Note that $A_1L_{A_1} = 0$. Then $A_1F^* = 0$, $FM^* = 0$ and

$$L_{A_1}L_F = (I_{\mathcal{H}} - A_1^\dagger A_1 - F^\dagger F) = L_FL_{A_1},$$
$$L_{A_1}L_M = (I_{\mathcal{H}} - A_1^\dagger A_1 - M^\dagger M) = L_ML_{A_1},$$
$$L_FL_M = (I_{\mathcal{H}} - F^\dagger F - M^\dagger M) = L_ML_F.$$

Hence it follows from (2.1) that

$$(N^*)^\dagger = (L_ML_FL_{A_1}A_4^*)^\dagger = (L_ML_FL_{A_1}A_4^*)^\dagger L_M = (L_FL_ML_{A_1}A_4^*)^\dagger L_M = (L_FL_ML_{A_1}A_4^*)^\dagger L_FL_M$$
$$= (L_{A_1}L_FL_MA_4^*)^\dagger L_FL_M = (L_{A_1}L_FL_MA_4^*)^\dagger L_{A_1}L_FL_M.$$

Therefore

$$(N)^{\dagger} = L_M L_F L_{A_1} N^{\dagger} = L_{A_1} L_F L_M N^{\dagger}.$$

Similarly,

$$F^{\dagger} = L_{A_1} F^{\dagger}, \quad M^{\dagger} = L_F L_{A_1} M^{\dagger} = L_{A_1} L_F M^{\dagger}, \quad (P^{\dagger})^* = L_F L_{A_1} (P^{\dagger})^* = L_{A_1} L_F (P^{\dagger})^*.$$

Accordingly

$$A_1 M^{\dagger} = A_1 (P^{\dagger})^* = A_1 N^{\dagger} = B_1^* M^{\dagger} = B_1^* (P^{\dagger})^* = B_1^* N^{\dagger} = 0, \quad B_1^* F^{\dagger} = FF^{\dagger}, \tag{3.14}$$

$$A_3 M^{\dagger} = MM^{\dagger}, \quad A_3 N^{\dagger} = 0, \quad A_4 M^{\dagger} = P^* M^{\dagger}, \quad A_4 (P^{\dagger})^* = P^{\dagger} P, \quad A_4 N^{\dagger} = NN^{\dagger}. \tag{3.15}$$

Suppose (2) holds and X has the form of (3.12), where U can be expressed as (3.13). It follows from (3.4) that

$$C_1 F^* = A_1 D^*, \quad C_2^* F^* = B_1^* D^*, \quad DF^* = FD^*. \tag{3.16}$$

By (3.16),

$$\begin{aligned} JL_F L_{A_1} &= (A_1^{\dagger} C_1 + F^{\dagger} D)(I_{\mathcal{H}} - A_1^{\dagger} A_1 - F^{\dagger} F) \\ &= A_1^{\dagger} C_1 - A_1^{\dagger} C_1 A_1^* (A_1^{\dagger})^* - A_1^{\dagger} C_1 F^* (F^{\dagger})^* + F^{\dagger} D - F^{\dagger} D A_1^* (A_1^{\dagger})^* - F^{\dagger} D F^* (F^{\dagger})^* \\ &= J - A_1^{\dagger} A_1 (A_1^{\dagger} C_1)^* - A_1^{\dagger} A_1 (F^{\dagger} D)^* - F^{\dagger} F (A_1^{\dagger} C_1)^* - F^{\dagger} F (F^{\dagger} D)^* \\ &= J - (A_1^{\dagger} A_1 + F^{\dagger} F) J^* \\ &= J - J^* + L_{A_1} L_F J^*, \end{aligned} \tag{3.17}$$

implying that $J + L_{A_1} L_F J^*$ is Hermitian. In view of $C_3 = C_3^*$, $C_4 = C_4^*$ and the definition of G, Q in (3.2) and (3.3), then $G = G^*$ and $Q = Q^*$. Note the expression of X in (3.12), then $X = X^*$, i.e., X is Hermitian. By (3.1), (3.4), (3.5) and (3.14),

$$A_1 X = A_1 A_1^{\dagger} C_1 = C_1, \quad XB_1 = (B_1^* X)^* = \left(D + B_1^* A_1^{\dagger} C_1 - R_F D\right)^* = C_2.$$

Noting $R_M G = 0$ yields

$$A_3 X A_3^* = A_3 [J + L_{A_1} L_F J^* + M^{\dagger} G (M^{\dagger})^*] A_3^* = C_3 - G + MM^{\dagger} G (M^{\dagger})^* M^* = C_3$$

by $ML_M = 0$, (3.2) and (3.15). It follows from $\mathcal{R}(N) \subset \mathcal{R}(P^*)$, i.e., $P^{\dagger} PN = N$ and $QL_P = 0$, $R_N Q R_N = 0$ that

$$\begin{aligned} &\frac{1}{2}[NN^{\dagger} Q (R_N + I_{\mathcal{K}_4}) P^{\dagger} P + P^{\dagger} P (R_N + I_{\mathcal{K}_4}) Q NN^{\dagger}] \\ &\quad = \frac{1}{2}[(I_{\mathcal{K}_4} - R_N) Q (R_N + I_{\mathcal{K}_4}) P^{\dagger} P + P^{\dagger} P (R_N + I_{\mathcal{K}_4}) Q (I_{\mathcal{K}_4} - R_N)] \\ &\quad = \frac{1}{2}(QR_N P^{\dagger} P + Q - R_N Q + P^{\dagger} P R_N Q + Q - QR_N) = Q. \end{aligned}$$

In view of (3.3) and (3.15),

$$\begin{aligned} A_4 X A_4^* &= A_4 [J + L_{A_1} L_F J^* + M^{\dagger} G (M^{\dagger})^*] A_4^* + \frac{1}{2}[NN^{\dagger} Q (R_N + I_{\mathcal{K}_4}) P^{\dagger} P + P^{\dagger} P (R_N + I_{\mathcal{K}_4}) Q NN^{\dagger}] \\ &= C_4 - Q + Q = C_4. \end{aligned}$$

(1) $\Rightarrow$ (3): Let $X_0 \in \mathfrak{B}(\mathcal{H})sa$ be a Hermitian solution to (1.4). Then

$$A_1 C_2 = A_1 X_0 B_1 = C_1 B_1, \quad A_1 C_1^* = A_1 X_0 A_1^* = C_1 A_1^*, \quad B_1^* C_2 = B_1^* X_0 B_1 = C_2^* B_1,$$

and

$$A_1 X_0 = C_1, \quad \begin{pmatrix} A_1 \\ B_1^* \end{pmatrix} X_0 = \begin{pmatrix} C_1 \\ C_2^* \end{pmatrix}$$

yielding (3.7),

$$\begin{pmatrix} A_1 \\ B_1^* \\ A_3 \end{pmatrix} X_0 A_3^* = \begin{pmatrix} C_1 A_3^* \\ C_2^* A_3^* \\ C_3 \end{pmatrix}, \quad \begin{pmatrix} A_1 \\ B_1^* \\ A_4 \end{pmatrix} X_0 A_4^* = \begin{pmatrix} C_1 A_4^* \\ C_2^* A_4^* \\ C_4 \end{pmatrix}$$

giving (3.8) and (3.9). It follows from $\psi Q + P\psi^* = \phi$, where

$$P = \begin{pmatrix} 0 & 0 & -X_0^* A_3^* & 0 \end{pmatrix}^*, \quad Q = \begin{pmatrix} 0 & 0 & 0 & X_0 A_4^* \end{pmatrix}$$

that $\psi Q R_\psi = \phi R_\psi$. Therefore (3.10) holds.

(3) $\Rightarrow$ (2): It is well known that if $\mathcal{R}(A)$ is closed, then $\mathcal{R}(A^*)$ is also closed. It follows from Lemma 2.1 and the fact $L_A = R_{A^*}$ that

$$\mathcal{R}(N) \text{ is closed} \Leftrightarrow \mathcal{R}(N^*) \text{ is closed} \Leftrightarrow (L_M P)^\dagger \text{ exists} \Leftrightarrow \left[L_F \begin{pmatrix} S^* & T^* \end{pmatrix}\right]^\dagger \text{ exists}$$
$$\Leftrightarrow \left[L_{A_1} \begin{pmatrix} B_1 & A_3^* & A_4^* \end{pmatrix}\right]^\dagger \text{ exists} \Leftrightarrow \begin{pmatrix} A_1^* & B_1 & A_3^* & A_4^* \end{pmatrix}^\dagger \text{ exists} \Leftrightarrow \psi^\dagger \text{ exists}.$$

Similarly, let

$$\varphi = \begin{pmatrix} A_1 \\ B_1^* \\ A_3 \end{pmatrix} \in \mathfrak{B}(\mathcal{H}, \mathcal{K}_1 \oplus \mathcal{K}_2 \oplus \mathcal{K}_3), \quad \eta = \begin{pmatrix} A_1 \\ B_1^* \\ A_4 \end{pmatrix} \in \mathfrak{B}(\mathcal{H}, \mathcal{K}_1 \oplus \mathcal{K}_2 \oplus \mathcal{K}_4),$$
$$\xi = \begin{pmatrix} A_1 \\ B_1^* \end{pmatrix} \in \mathfrak{B}(\mathcal{H}, \mathcal{K}_1 \oplus \mathcal{K}_2).$$

Then $\varphi^\dagger \in \mathfrak{B}(\mathcal{K}_1 \oplus \mathcal{K}_2 \oplus \mathcal{K}_3, \mathcal{H})$, $\eta^\dagger \in \mathfrak{B}(\mathcal{K}_1 \oplus \mathcal{K}_2 \oplus \mathcal{K}_4, \mathcal{H})$ and $\xi^\dagger \in \mathfrak{B}(\mathcal{K}_1 \oplus \mathcal{K}_2, \mathcal{H})$ exist.

Suppose (3.7)–(3.10) hold. It follows from the well known fact

$$\mathcal{R}(B) \subseteq \mathcal{R}(A) \Longleftrightarrow AA^\dagger B = B \tag{3.18}$$

that $R_{A_1} C_1 = 0$,

$$R_\varphi \begin{pmatrix} C_1 A_3^* \\ C_2^* A_3^* \\ C_3 \end{pmatrix} = 0, \quad R_\eta \begin{pmatrix} C_1 A_4^* \\ C_2^* A_4^* \\ C_4 \end{pmatrix} = 0, \quad R_\xi \begin{pmatrix} C_1 \\ C_2^* \end{pmatrix} = 0 \tag{3.19}$$

and

$$R_\psi \phi L_{\psi^*} = R_\psi \phi R_\psi = 0. \tag{3.20}$$

Suppose $\varphi^\dagger = \begin{pmatrix} K_1 & K_2 & K_3 \end{pmatrix}$. Then

$$R_\varphi = I_{\mathcal{K}_1 \oplus \mathcal{K}_2 \oplus \mathcal{K}_3} - \varphi\varphi^\dagger = \begin{pmatrix} I_{\mathcal{K}_1} - A_1 K_1 & -A_1 K_2 & -A_1 K_3 \\ -B_1^* K_1 & I_{\mathcal{K}_2} - B_1^* K_2 & -B_1^* K_3 \\ -A_3 K_1 & -A_3 K_2 & I_{\mathcal{K}_3} - A_3 K_3 \end{pmatrix}.$$

Substituting R_φ above into the first equality in (3.19) gives

$$A_1 K_1 C_1 A_3^* + A_1 K_2 C_2^* A_3^* + A_1 K_3 C_3 = C_1 A_3^*, \tag{3.21}$$
$$B_1^* K_1 C_1 A_3^* + B_1^* K_2 C_2^* A_3^* + B_1^* K_3 C_3 = C_2^* A_3^*, \tag{3.22}$$
$$A_3 K_1 C_1 A_3^* + A_3 K_2 C_2^* A_3^* + A_3 K_3 C_3 = C_3. \tag{3.23}$$

Multiplying (3.21) by $(-B_1^* A_1^\dagger)$, $(-A_3 A_1^\dagger)$ from left side and adding them to (3.22), (3.23), respectively, we have

$$F K_1 C_1 A_3^* + F K_2 C_2^* A_3^* + F K_3 C_3 = D A_3^*, \tag{3.24}$$
$$S K_1 C_1 A_3^* + S K_2 C_2^* A_3^* + S K_3 C_3 = C_3 - A_3 A_1^\dagger C_1 A_3^*. \tag{3.25}$$

Multiplying (3.24) by $(-SF^{\dagger})$ from left side and adding it to (3.25) yields

$$MK_1C_1A_3^* + MK_2C_2^*A_3^* + MK_3C_3 = C_3 - A_3A_1^{\dagger}C_1A_3^* - SF^{\dagger}DA_3^*. \tag{3.26}$$

Note $R_MM = 0$ and (3.2). Multiplying (3.26) by R_M from left side gives $R_MG = 0$.

Similarly, one can easily show that $QL_P = 0$ and $R_FD = 0$ by the second equality and the third equality in (3.19), respectively.

Now we want to show that $R_NQR_N = 0$. Assume that

$$\psi^{\dagger} = \begin{pmatrix} K_4 & K_5 & K_6 & K_7 \end{pmatrix} \in \mathfrak{B}(\mathcal{K}_1 \oplus \mathcal{K}_2 \oplus \mathcal{K}_3 \oplus \mathcal{K}_4, \mathcal{H}).$$

Then

$$\begin{aligned} R_{\psi} &= I_{\mathcal{K}_1\oplus\mathcal{K}_2\oplus\mathcal{K}_3\oplus\mathcal{K}_4} - \psi\psi^{\dagger} \\ &= \begin{pmatrix} I_{\mathcal{K}_1} - A_1K_4 & -A_1K_5 & -A_1K_6 & -A_1K_7 \\ -B_1^*K_4 & I_{\mathcal{K}_2} - B_1^*K_5 & -B_1^*K_6 & -B_1^*K_7 \\ -A_3K_4 & -A_3K_5 & I_{\mathcal{K}_3} - A_3K_6 & -A_3K_7 \\ -A_4K_4 & -A_4K_5 & -A_4K_6 & I_{\mathcal{K}_4} - A_4K_7 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} L_{\psi^*} &= I_{\mathcal{K}_1\oplus\mathcal{K}_2\oplus\mathcal{K}_3\oplus\mathcal{K}_4} - (\psi^*)^{\dagger}\psi^* \\ &= \begin{pmatrix} I_{\mathcal{K}_1} - K_4^*A_1^* & -K_4^*B_1 & -K_4^*A_3^* & -K_4^*A_4^* \\ -K_5^*A_1^* & I_{\mathcal{K}_2} - K_5^*B_1 & -K_5^*A_3^* & -K_5^*A_4^* \\ -K_6^*A_1^* & -K_6^*B_1 & I_{\mathcal{K}_3} - K_6^*A_3^* & -K_6^*A_4^* \\ -K_7^*A_1^* & -K_7^*B_1 & -K_7^*A_3^* & I_{\mathcal{K}_4} - K_7^*A_4^* \end{pmatrix}. \end{aligned}$$

Substituting R_{ψ} and L_{ψ^*} above into (3.20) yields

$$A_1K_6H_1 - [(I_{\mathcal{K}_1} - A_1K_4)C_1A_4^* - A_1K_5C_2^*A_4^* - A_1K_7C_4]K_7^*A_1^* = 0, \tag{3.27}$$
$$B_1^*K_6H_1 - \left[(I_{\mathcal{K}_2} - B_1^*K_5)C_2^*A_4^* - B_1^*K_4C_1A_4^* - B_1^*K_7C_4\right]K_7^*A_1^* = 0, \tag{3.28}$$
$$(A_3K_6 - I_{\mathcal{K}_3})H_1 + (A_3K_4C_1A_4^* + A_3K_5C_2^*A_4^* + A_3K_7C_4)K_7^*A_1^* = 0, \tag{3.29}$$
$$A_4K_6H_1 - \left[(I_{\mathcal{K}_4} - A_4K_7)C_4 - A_4K_4C_1A_4^* - A_4K_5C_2^*A_4^*\right]K_7^*A_1^* = 0, \tag{3.30}$$

and

$$\begin{aligned} &A_1K_6H_2 - \left[(I_{\mathcal{K}_1} - A_1K_4)C_1A_4^* - A_1K_5C_2^*A_4^* - A_1K_7C_4\right]K_7^*B_1 = 0, \\ &B_1^*K_6H_2 - \left[(I_{\mathcal{K}_2} - B_1^*K_5)C_2^*A_4^* - B_1^*K_4C_1A_4^* - B_1^*K_7C_4\right]K_7^*B_1 = 0, \\ &(A_3K_6 - I_{\mathcal{K}_3})H_2 + \left(A_3K_4C_1A_4^* + A_3K_5C_2^*A_4^* + A_3K_7C_4\right)K_7^*B_1 = 0, \\ &A_4K_6H_2 - \left[(I_{\mathcal{K}_4} - A_4K_7)C_4 - A_4K_4C_1A_4^* - A_4K_5C_2^*A_4^*\right]K_7^*B_1 = 0, \end{aligned} \tag{3.31}$$

$$\begin{aligned} &A_1K_6H_3 - \left[(I_{\mathcal{K}_1} - A_1K_4)C_1A_4^* - A_1K_5C_2^*A_4^* - A_1K_7C_4\right]K_7^*A_3^* = 0, \\ &B_1^*K_6H_3 - \left[(I_{\mathcal{K}_2} - B_1^*K_5)C_2^*A_4^* - B_1^*K_4C_1A_4^* - B_1^*K_7C_4\right]K_7^*A_3^* = 0, \\ &(A_3K_6 - I_{\mathcal{K}_3})H_3 + \left(A_3K_4C_1A_4^* + A_3K_5C_2^*A_4^* + A_3K_7C_4\right)K_7^*A_3^* = 0, \\ &A_4K_6H_3 - \left[(I_{\mathcal{K}_4} - A_4K_7)C_4 - A_4K_4C_1A_4^* - A_4K_5C_2^*A_4^*\right]K_7^*A_3^* = 0, \end{aligned} \tag{3.32}$$

$$\begin{aligned} &A_1K_6H_4 - \left[(I_{\mathcal{K}_1} - A_1K_4)C_1A_4^* - A_1K_5C_2^*A_4^* - A_1K_7C_4\right]\left(K_7^*A_4^* - I_{\mathcal{K}_4}\right) = 0, \\ &B_1^*K_6H_4 - \left[(I_{\mathcal{K}_2} - B_1^*K_5)C_2^*A_4^* - B_1^*K_4C_1A_4^* - B_1^*K_7C_4\right]\left(K_7^*A_4^* - I_{\mathcal{K}_4}\right) = 0, \\ &(A_3K_6 - I_{\mathcal{K}_3})H_4 + \left(A_3K_4C_1A_4^* + A_3K_5C_2^*A_4^* + A_3K_7C_4\right)\left(K_7^*A_4^* - I_{\mathcal{K}_4}\right) = 0, \\ &A_4K_6H_4 - \left[(I_{\mathcal{K}_4} - A_4K_7)C_4 - A_4K_4C_1A_4^* - A_4K_5C_2^*A_4^*\right]\left(K_7^*A_4^* - I_{\mathcal{K}_4}\right) = 0, \end{aligned} \tag{3.33}$$

where

$$\begin{aligned} H_1 &= A_3C_1^*(I_{\mathcal{K}_1} - K_4^*A_1^*) - A_3C_2K_5^*A_1^* - C_3K_6^*A_1^*, \\ H_2 &= A_3C_1^*(-K_4^*B_1) + A_3C_2(I_{\mathcal{K}_2} - K_5^*B_1) - C_3K_6^*B_1, \\ H_3 &= A_3C_1^*(-K_4^*A_3^*) - A_3C_2K_5^*A_3^* + C_3(I_{\mathcal{K}_3} - K_6^*A_3^*), \\ H_4 &= -A_3C_1^*K_4^*A_4^* - A_3C_2K_5^*A_4^* - C_3K_6^*A_4^*. \end{aligned}$$

Multiplying (3.27) by $(-B_1^*A_1^\dagger)$, $(-A_3A_1^\dagger)$, $(-A_4A_1^\dagger)$ from left side and adding them to (3.28)–(3.30), respectively, we can get

$$\begin{aligned}
&FK_6H_1 + [-DA_4^* + FK_4C_1A_4^* + FK_5C_2^*A_4^* + FK_7C_4]K_7^*A_1^* = 0,\\
&(SK_6 - I_{\mathcal{K}_3})H_1 + [A_3A_1^\dagger C_1A_4^* + SK_4C_1A_4^* + SK_5C_2^*A_4^* + SK_7C_4]K_7^*A_1^* = 0,\\
&TK_6H_1 + [A_4A_1^\dagger C_1A_4^* - C_4 + TK_4C_1A_4^* + TK_5C_2^*A_4^* + TK_7C_4]K_7^*A_1^* = 0.
\end{aligned} \tag{3.34}$$

Similarly, the equalities in (3.31)–(3.33) gives

$$\begin{aligned}
&FK_6H_2 + (-DA_4^* + FK_4C_1A_4^* + FK_5C_2^*A_4^* + FK_7C_4)K_7^*B_1 = 0,\\
&(SK_6 - I_{\mathcal{K}_3})H_2 + (A_3A_1^\dagger C_1A_4^* + SK_4C_1A_4^* + SK_5C_2^*A_4^* + SK_7C_4)K_7^*B_1 = 0,\\
&TK_6H_2 + (A_4A_1^\dagger C_1A_4^* - C_4 + TK_4C_1A_4^* + TK_5C_2^*A_4^* + TK_7C_4)K_7^*B_1 = 0,
\end{aligned} \tag{3.35}$$

$$\begin{aligned}
&FK_6H_3 + (-DA_4^* + FK_4C_1A_4^* + FK_5C_2^*A_4^* + FK_7C_4)K_7^*A_3^* = 0,\\
&(SK_6 - I_{\mathcal{K}_3})H_3 + (A_3A_1^\dagger C_1A_4^* + SK_4C_1A_4^* + SK_5C_2^*A_4^* + SK_7C_4)K_7^*A_3^* = 0,\\
&TK_6H_3 + (A_4A_1^\dagger C_1A_4^* - C_4 + TK_4C_1A_4^* + TK_5C_2^*A_4^* + TK_7C_4)K_7^*A_3^* = 0,
\end{aligned} \tag{3.36}$$

and

$$\begin{aligned}
&FK_6H_4 + (-DA_4^* + FK_4C_1A_4^* + FK_5C_2^*A_4^* + FK_7C_4)(K_7^*A_4^* - I_{\mathcal{K}_4}) = 0,\\
&(SK_6 - I_{\mathcal{K}_3})H_4 + (A_3A_1^\dagger C_1A_4^* + SK_4C_1A_4^* + SK_5C_2^*A_4^* + SK_7C_4)(K_7^*A_4^* - I_{\mathcal{K}_4}) = 0,\\
&TK_6H_4 + (A_4A_1^\dagger C_1A_4^* - C_4 + TK_4C_1A_4^* + TK_5C_2^*A_4^* + TK_7C_4)(K_7^*A_4^* - I_{\mathcal{K}_4}) = 0.
\end{aligned} \tag{3.37}$$

Multiplying the first equality in (3.34) by $-(A_1^\dagger)^*B_1$, $-(A_1^\dagger)^*A_3^*$, $-(A_1^\dagger)^*A_4^*$ from right side and adding them to the first equality in (3.35)–(3.37), respectively, we have

$$\begin{aligned}
&FK_6(A_3D^* - A_3C_1^*K_4^*F^* - A_3C_2K_5^*F^* - C_3K_6^*F^*) + H_5K_7^*F^* = 0,\\
&FK_6[C_3 - A_3(A_1^\dagger C_1)^*A_3^* - A_3C_1^*K_4^*S^* - A_3C_2K_5^*S^* - C_3K_6^*S^*] + H_5K_7^*S^* = 0,\\
&FK_6[-A_3(A_1^\dagger C_1)^*A_4^* - A_3C_1^*K_4^*T^* - A_3C_2K_5^*T^* - C_3K_6^*T^*] + H_5(K_7^*T^* - I_{\mathcal{K}_4}) = 0,
\end{aligned} \tag{3.38}$$

where

$$H_5 = -DA_4^* + FK_4C_1A_4^* + FK_5C_2^*A_4^* + FK_7C_4.$$

Likewise, it follows from the second equality in (3.34)–(3.37) and the third equality in (3.34)–(3.37) that

$$\begin{aligned}
&(SK_6 - I_{\mathcal{K}_3})(A_3D^* - A_3C_1^*K_4^*F^* - A_3C_2K_5^*F^* - C_3K_6^*F^*) + H_6K_7^*F^* = 0,\\
&(SK_6 - I_{\mathcal{K}_3})[C_3 - A_3(A_1^\dagger C_1)^*A_3^* - A_3C_1^*K_4^*S^* - A_3C_2K_5^*S^* - C_3K_6^*S^*] + H_6K_7^*S^* = 0,\\
&(SK_6 - I_{\mathcal{K}_3})[-A_3(A_1^\dagger C_1)^*A_4^* - A_3C_1^*K_4^*T^* - A_3C_2K_5^*T^* - C_3K_6^*T^*] + H_6(K_7^*T^* - I_{\mathcal{K}_4}) = 0
\end{aligned} \tag{3.39}$$

and

$$\begin{aligned}
&TK_6(A_3D^* - A_3C_1^*K_4^*F^* - A_3C_2K_5^*F^* - C_3K_6^*F^*) + H_7K_7^*F^* = 0,\\
&TK_6[C_3 - A_3(A_1^\dagger C_1)^*A_3^* - A_3C_1^*K_4^*S^* - A_3C_2K_5^*S^* - C_3K_6^*S^*] + H_7K_7^*S^* = 0,\\
&TK_6[-A_3(A_1^\dagger C_1)^*A_4^* - A_3C_1^*K_4^*T^* - A_3C_2K_5^*T^* - C_3K_6^*T^*] + H_7(K_7^*T^* - I_{\mathcal{K}_4}) = 0,
\end{aligned} \tag{3.40}$$

where

$$\begin{aligned}
&H_6 = A_3A_1^\dagger C_1A_4^* + SK_4C_1A_4^* + SK_5C_2^*A_4^* + SK_7C_4,\\
&H_7 = A_4A_1^\dagger C_1A_4^* - C_4 + TK_4C_1A_4^* + TK_5C_2^*A_4^* + TK_5C_4.
\end{aligned}$$

Let

$$H_8 = A_3D^* - A_3C_1^*K_4^*F^* - A_3C_2K_5^*F^* - C_3K_6^*F^*.$$

Then multiplying the first equality in (3.38) by $-SF^{\dagger}$, $-TF^{\dagger}$ from left side and adding them to the first equality in (3.39) and (3.40), respectively, we obtain

$$\begin{aligned}&(MK_6 - I_{\mathcal{K}_3})H_8 + (A_3JA_4^* + MK_4C_1A_4^* + MK_5C_2^*A_4^* + MK_7C_4)K_7^*F^* = 0,\\ &P^*K_6H_8 + (A_4JA_4^* - C_4 + P^*K_4C_1A_4^* + P^*K_5C_2^*A_4^* + P^*K_7C_4)K_7^*F^* = 0.\end{aligned} \tag{3.41}$$

In the same way, from the second equality in (3.38)–(3.40) and the third equality in (3.38)–(3.40), we derive

$$\begin{aligned}&(MK_6 - I_{\mathcal{K}_3})H_9 + (A_3JA_4^* + MK_4C_1A_4^* + MK_5C_2^*A_4^* + MK_7C_4)K_7^*S^* = 0,\\ &P^*K_6H_9 + (A_4JA_4^* - C_4 + P^*K_4C_1A_4^* + P^*K_5C_2^*A_4^* + P^*K_7C_4)K_7^*S^* = 0\end{aligned} \tag{3.42}$$

and

$$\begin{aligned}&(MK_6 - I_{\mathcal{K}_3})H_{10} + (A_3JA_4^* + MK_4C_1A_4^* + MK_5C_2^*A_4^* + MK_7C_4)(K_7^*T^* - I_{\mathcal{K}_4}) = 0,\\ &P^*K_6H_{10} + (A_4JA_4^* - C_4 + P^*K_4C_1A_4^* + P^*K_5C_2^*A_4^* + P^*K_7C_4)(K_7^*T^* - I_{\mathcal{K}_4}) = 0,\end{aligned} \tag{3.43}$$

where

$$H_9 = C_3 - A_3(A_1^{\dagger}C_1)^*A_3^* - A_3C_1^*K_4^*S^* - A_3C_2K_5^*S^* - C_3K_6^*S^*,$$
$$H_{10} = -A_3(A_1^{\dagger}C_1)^*A_4^* - A_3C_1^*K_4^*T^* - A_3C_2K_5^*T^* - C_3K_6^*T^*.$$

Then multiplying the first equality in (3.41) by $-(F^{\dagger})^*S^*$, $-(F^{\dagger})^*T^*$ from right side and adding them to the first equality in (3.42) and (3.43), respectively, we have

$$(MK_6 - I_{\mathcal{K}_3})(C_3 - A_3J^*A_3^* - A_3C_1^*K_4^*M^* - A_3C_2K_5^*M^* - C_3K_6^*M^*) + H_{11}K_7^*M^* = 0, \tag{3.44}$$
$$(MK_6 - I_{\mathcal{K}_3})(-A_3J^*A_4^* - A_3C_1^*K_4^*P - A_3C_2K_5^*P - C_3K_6^*P) + H_{11}(K_7^*P - I_{\mathcal{K}_4}) = 0, \tag{3.45}$$

where

$$H_{11} = A_3JA_4^* + MK_4C_1A_4^* + MK_5C_2^*A_4^* + MK_7C_4.$$

Similarly, it follows from the second equality in (3.41)–(3.43) that

$$P^*K_6(C_3 - A_3J^*A_3^* - A_3C_1^*K_4^*M^* - A_3C_2K_5^*M^* - C_3K_6^*M^*) + H_{12}K_7^*M^* = 0, \tag{3.46}$$
$$P^*K_6(-A_3J^*A_4^* - A_3C_1^*K_4^*P - A_3C_2K_5^*P - C_3K_6^*P) + H_{12}(K_7^*P - I_{\mathcal{K}_4}) = 0, \tag{3.47}$$

where

$$H_{12} = A_4JA_4^* - C_4 + P^*K_4C_1A_4^* + P^*K_5C_2^*A_4^* + P^*K_7C_4.$$

Multiplying (3.44), (3.45) by $-P^*M^{\dagger}$ from left side and adding them to (3.46) and (3.47), respectively, we can get

$$\begin{aligned}&(NK_6 + P^*M^{\dagger})(C_3 - A_3J^*A_3^* - A_3C_1^*K_4^*M^* - A_3C_2K_5^*M^* - C_3K_6^*M^*)\\ &\quad + (A_4JA_4^* - C_4 - P^*M^{\dagger}A_3JA_4^* + NK_4C_1A_4^* + NK_5C_2^*A_4^* + NK_7C_4)K_7^*M^* = 0,\end{aligned} \tag{3.48}$$
$$\begin{aligned}&(NK_6 + P^*M^{\dagger})(-A_3J^*A_4^* - A_3C_1^*K_4^*P - A_3C_2K_5^*P - C_3K_6^*P)\\ &\quad + (A_4JA_4^* - C_4 - P^*M^{\dagger}A_3JA_4^* + NK_4C_1A_4^* + NK_5C_2^*A_4^* + NK_7C_4)(K_7^*P - I_{\mathcal{K}_4}) = 0.\end{aligned} \tag{3.49}$$

Multiplying (3.48) by $-(M^{\dagger})^*P$ from right side and adding it to (3.49) gives

$$\begin{aligned}&(NK_6 + P^*M^{\dagger})[-A_3J^*A_4^* - (C_3 - A_3J^*A_3^*)(M^{\dagger})^*P - A_3C_1^*K_4^*N^* - A_3C_2K_5^*N^* - C_3K_6^*N^*]\\ &\quad + [A_4JA_4^* - C_4 - P^*M^{\dagger}A_3JA_4^* + NK_4C_1A_4^* + NK_5C_2^*A_4^* + NK_7C_4](K_7^*N^* - I_{\mathcal{K}_4}) = 0.\end{aligned} \tag{3.50}$$

Note $R_NN = 0$, then multiplying (3.50) by R_N from two sides yields

$$R_N[C_4 - A_4JA_4^* + P^*M^{\dagger}A_3(J - J^*)A_4^* - P^*M^{\dagger}(C_3 - A_3J^*A_3^*)(M^{\dagger})^*P]R_N = 0. \tag{3.51}$$

In view of (3.2), (3.17) and $M^{\dagger}MPR_N = PR_N$,

$$\begin{aligned}
&-R_NP^*M^{\dagger}(C_3 - A_3J^*A_3^*)(M^{\dagger})^*PR_N \\
&= -R_NP^*M^{\dagger}[G + A_3(J + L_{A_1}L_FJ^* - J^*)A_3^*](M^{\dagger})^*PR_N \\
&= -R_NP^*M^{\dagger}[G(M^{\dagger})^*P + A_3JM^{\dagger}MP]R_N \\
&= -R_NP^*M^{\dagger}[G(M^{\dagger})^*P + A_3JP]R_N \\
&= -R_NP^*M^{\dagger}[G(M^{\dagger})^*P + A_3JL_FL_{A_1}A_4^*]R_N \\
&= -R_NP^*M^{\dagger}[G(M^{\dagger})^*P + A_3(J - J^*)A_4^* + MJ^*A_4^*]R_N \\
&= -R_NP^*M^{\dagger}[G(M^{\dagger})^*P + A_3(J - J^*)A_4^*]R_N - R_NP^*J^*A_4^*R_N. \qquad (3.52)
\end{aligned}$$

Substituting (3.52) into (3.51) gives

$$R_N[C_4 - A_4JA_4^* - P^*M^{\dagger}G(M^{\dagger})^*P - P^*J^*A_4^*]R_N = 0,$$

implying $R_NQR_N = 0$ by $P^*M^{\dagger} = A_4M^{\dagger}$.

Now we show that if Eqs. (1.4) have a common Hermitian solution, i.e., the equalities in (3.4)–(3.6) hold, then its general Hermitian solution can be expressed by (3.12), where U can be expressed as (3.13).

Assume $X_0 \in \mathfrak{B}(\mathcal{H})sa$ is any Hermitian solution to (1.4). Then

$$\begin{aligned}
L_{A_1}L_FX_0L_FL_{A_1} &= (I_{\mathcal{H}} - A_1^{\dagger}A_1 - F^{\dagger}F)X_0L_FL_{A_1} = (X_0 - J)L_FL_{A_1} \\
&= X_0(I_{\mathcal{H}} - A_1^{\dagger}A_1 - F^{\dagger}F) - J + J^* - L_{A_1}L_FJ^* \\
&= X_0 - J - L_{A_1}L_FJ^*. \qquad (3.53)
\end{aligned}$$

Put $V = \frac{1}{2}(X_0 + X_0M^{\dagger}M)$. In view of (3.2) and (3.53),

$$\begin{aligned}
&L_{A_1}L_FL_MVL_FL_{A_1} + L_{A_1}L_FV^*L_ML_FL_{A_1} \\
&= \frac{1}{2}L_{A_1}L_F[L_M(X_0 + X_0M^{\dagger}M) + (X_0 + M^{\dagger}MX_0)L_M]L_FL_{A_1} \\
&= L_{A_1}L_F[X_0 - M^{\dagger}MX_0M^{\dagger}M)]L_FL_{A_1} \\
&= X_0 - J - L_{A_1}L_FJ^* - M^{\dagger}A_3(X_0 - J - L_{A_1}L_FJ^*)A_3^*(M^{\dagger})^* \\
&= X_0 - J - L_{A_1}L_FJ^* - M^{\dagger}G(M^{\dagger})^*
\end{aligned}$$

and

$$\begin{aligned}
NVP + P^*V^*N^* &= A_4(L_{A_1}L_FL_MVL_FL_{A_1} + L_{A_1}L_FV^*L_ML_FL_{A_1})A_4^* \\
&= C_4 - A_4(J + L_{A_1}L_FJ^* + M^{\dagger}G(M^{\dagger})^*)A_4^* = Q.
\end{aligned}$$

In this case,

$$\begin{aligned}
&-N^{\dagger}NVPP^{\dagger} + \frac{1}{2}N^{\dagger}NVPNN^{\dagger}P^{\dagger} - \frac{1}{2}N^{\dagger}P^*V^*N^*P^{\dagger} \\
&= -\frac{1}{2}N^{\dagger}NVPP^{\dagger} - \frac{1}{2}N^{\dagger}NVPR_NP^{\dagger} - \frac{1}{2}N^{\dagger}P^*V^*N^*P^{\dagger} \\
&= -\frac{1}{2}N^{\dagger}QP^{\dagger} - \frac{1}{2}N^{\dagger}(Q - P^*V^*N^*)R_NP^{\dagger} = -\frac{1}{2}N^{\dagger}Q(R_N + I_{\mathcal{K}_4})P^{\dagger}
\end{aligned}$$

and

$$U + U^* = X_0 - J - L_{A_1}L_FJ^* - M^\dagger G(M^\dagger)^* - \frac{1}{2}[N^\dagger Q(R_N + I_{\mathcal{K}_4})P^\dagger + (P^\dagger)^*(R_N + I_{\mathcal{K}_4})Q(N^\dagger)^*].$$

Then X_0 can be expressed as

$$X_0 = J + L_{A_1}L_FJ^* + M^\dagger G(M^\dagger)^* + \frac{1}{2}[N^\dagger Q(R_N + I_{\mathcal{K}_4})P^\dagger + (P^\dagger)^*(R_N + I_{\mathcal{K}_4})Q(N^\dagger)^*] + U + U^*.$$

This expression implies that (3.13), where U can be expressed as (3.12), is the general Hermitian solution to (1.4). □

Now we consider some special cases of Theorem 3.1.

Corollary 3.2. *Let A_1, C_1, B_1, C_2, A_3, C_3 and F, D, M, J, G be as in Theorem* 3.1. *Suppose that A_1, B_1, A_3 and F, M have closed ranges. Then the following conditions are equivalent*:

(1) *Equations*

$$A_1X = C_1, \quad XB_1 = C_2, \quad A_3XA_3^* = C_3 \tag{3.54}$$

have a solution $X \in \mathfrak{B}(\mathcal{H})sa$.

(2) *$C_3 = C_3^*$, the equalities in (3.4), (3.5) and the first equality of (3.6) hold.*

(3) *$C_3 = C_3^*$, the equalities in (3.4), (3.7) and (3.8) hold.*

In this case, the general Hermitian solution to (3.54) can be expressed as

$$X = J + L_{A_1}L_FJ^* + L_{A_1}L_FM^\dagger G(M^\dagger)^*L_FL_{A_1} + L_{A_1}L_FL_MVL_FL_{A_1} + L_{A_1}L_FV^*L_ML_FL_{A_1},$$

where $V \in \mathfrak{B}(\mathcal{H})$ is arbitrary.

Corollary 3.3. *Let A_3, C_3, A_4, C_4 be as in Theorem* 3.1 *and A_3, A_4 and N have closed ranges, where $N = A_4L_{A_3}$. Put*

$$Q = C_4 - A_4A_3^\dagger C_3(A_3^\dagger)^*A_4^*.$$

Then the following conditions are equivalent:

(1) *Eqs. (1.3) have a solution $X \in \mathfrak{B}(\mathcal{H})sa$.*

(2) *$C_3 = C_3^*$, $C_4 = C_4^*$ and*

$$R_{A_3}C_3 = 0, \quad R_{A_4}C_4 = 0, \quad R_NQR_N = 0.$$

(3) *$C_3 = C_3^*$, $C_4 = C_4^*$ and*

$$\mathcal{R}(C_3) \subseteq \mathcal{R}(A_3), \quad \mathcal{R}(C_4) \subseteq \mathcal{R}(A_4), \quad \mathcal{R}(\phi R_\psi) \subseteq \mathcal{R}(\psi),$$

where

$$\psi = \begin{pmatrix} A_3 \\ A_4 \end{pmatrix}, \quad \phi = \begin{pmatrix} -C_3 & 0 \\ 0 & C_4 \end{pmatrix}.$$

In this case, the general Hermitian solution to (1.3) can be expressed as

$$X = A_3^\dagger C_3(A_3^\dagger)^* + \frac{1}{2}[N^\dagger Q(R_N + I_{\mathcal{K}_4})(A_4^*)^\dagger + A_4^\dagger(R_N + I_{\mathcal{K}_4})Q(N^\dagger)^*] + U + U^*,$$

where

$$U = L_{A_3}V - N^\dagger NVA_4^\dagger A_4 + \frac{1}{2}N^\dagger NVA_4^*NN^\dagger(A_4^*)^\dagger - \frac{1}{2}N^\dagger A_4V^*N^*(A_4^*)^\dagger$$

and $V \in \mathfrak{B}(\mathcal{H})$ is arbitrary.

Remark 3.1. The finite-dimensional case of the above corollary was considered in [11,1] by rank and the singular-value decomposition, respectively.

Corollary 3.4. *Let $A_1, C_1, B_1, C_2, F, D, J$ be as in Theorem* 3.1. *Suppose that A_1, B_1 and F have closed ranges. Then the following conditions are equivalent*:

(1) *Eqs.* (1.1) *have a solution $X \in \mathfrak{B}(\mathcal{H})sa$.*
(2) *The equalities in* (3.4) *and* (3.5) *hold.*
(3) *The equalities in* (3.4) *and* (3.7) *hold.*
In this case, the general Hermitian solution to (1.1) *can be expressed as*

$$X = J + L_{A_1}L_FJ^* + L_{A_1}L_FYL_FL_{A_1}, \tag{3.55}$$

where $Y \in \mathfrak{B}(\mathcal{H})sa$ is arbitrary.

Remark 3.2. Corollary 3.4 is one of the main results of [4,13].

Remark 3.3. For matrices, we revisit Khatri and Mitra's solvable conditions for the existence of the common Hermitian solution to (1.1) over $\mathbb{C}$ in [8]. The following counterexample shows that these conditions given in [8] are not sufficient for the existence of a common Hermitian solution to (1.1). Take, for example,

$$A_1 = \begin{pmatrix} 1 & 1 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 1-i & 1+i \end{pmatrix}, \quad B_1 = \begin{pmatrix} i \\ i \end{pmatrix}, \quad C_2 = \begin{pmatrix} i \\ i \end{pmatrix}.$$

Then it is easy to verify that the conditions in [8] are all satisfied, and $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is one of the inner inverses of $\begin{pmatrix} A_1 \\ B_1^* \end{pmatrix}$. According to the expression for the Hermitian solution to (1.1), given in [8], we have that

$$X = \begin{pmatrix} 0 & 1+i \\ 1-i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} U \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix},$$

where U is an arbitrary Hermitian matrix with suitable size. However,

$$XB_1 = \begin{pmatrix} i-1 \\ i+1 \end{pmatrix} \neq C_2.$$

The correction is as follows. Eqs. (1.1) have a common Hermitian solution if and only if (3.4) and (3.5) hold.

Remark 3.4. For matrices, we revisit the expression for general symmetric solution to (1.1)

$$X = A_1^{\dagger}C_1 + L_{A_1}(A_1^{\dagger}C_1)^* + F^{\dagger}DL_{A_1} + L_FL_{A_1}(F^{\dagger}D)^* + L_FXL_F$$

in [16]. By simply computing, we can show that the solvable conditions for (1.1) to have a symmetric solution in [16] are equivalent to (3.4) and (3.5). However, under the conditions (3.4) and (3.5),

$$XB_1 = C_2 + L_FXL_FB_1 \neq C_2.$$

The correct version of the general symmetric solution should be

$$X = A_1^{\dagger}C_1 + L_{A_1}(A_1^{\dagger}C_1)^* + F^{\dagger}DL_{A_1} + L_FL_{A_1}(F^{\dagger}D)^* + L_{A_1}L_FXL_FL_{A_1}.$$

By (3.16), the expression mentioned above is the same as (3.55).

4. Conclusion

In this paper, we derive necessary and sufficient conditions for the existence of the common Hermitian solution to (1.4) for Hilbert C^*-modules operators, and give an expression for the general common Hermitian solution to (1.4) when the solvability conditions are satisfied. Some corresponding results on special cases are also given. Some known results can be viewed as special cases of this paper.

It is worthy to say that the approach and results in this paper are also true to the bounded operators between quaternionic Hilbert spaces, which plays an important role in certain physical problems (see, for example [5]).

Motivated by the work in this paper, it would be of interest to investigate the common nonnegative and positive solutions to equations (1.4) for Hilbert C^*-modules operators. Moreover, two challenging tasks are to derive the extremal ranks and inertias of the common general Hermitian solution to (1.4) in matrix equation version.

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