JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 75, 287–292 (1980)

Strong Convergence Theorems for Resolvents of Accretive Operators in Banach Spaces*

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Submitted by Ky Fan

Let E be a real Banach space, and let I denote the identity. Recall that an an operator $A \subset E \times E$ with domain $D(A)$ and range $R(A)$ is said to be m-accretive if $R(I + rA) = E$ and $|x_1 - x_2| \leqslant |x_1 - x_2 + r(y_1 - y_2)|$ for all $y_i \in Ax_i$, $i = 1, 2$, and $r > 0$. Let $J_t = (I + tA)^{-1}$, $t > 0$, be the resolvent of A, and assume that $0 \in R(A)$. It is known that if E is a Hilbert space, then for each x in E, the strong $\lim_{t\to\infty} J_t x$ exists and belongs to $A^{-1}0$. Although this result was extended to a restricted class of Banach spaces in [13, 15], it has remained an open question whether it is true in, say, the L^p spaces, $1 < p < \infty$, $p \neq 2$. In the present paper, we provide an affirmative answer to this problem (Theorem 1). This positive solution is of special interest because it leads to strong convergence results for several explicit and implicit iterative methods (Corollary 2, Theorems 2 and 3). A similar argument yields a new result on the asymptotic behavior of solutions of a certain evolution equation (Theorem 4). Finally, we solve another open problem by showing that the strong $\lim_{r\to 0} J_r x$ also exists in, say, all L^p spaces, $1 < p < \infty$ (Theorem 5). This has been known so far only in Hilbert space (and in smooth finite-dimensional spaces). In addition, we identify the limits obtained, and point out that our results cannot be extended to all Banach spaces.

THEOREM 1. *Let E be a uniformly smooth Banach space, and let $A \subset E \times E$ be m-accretive. If $0 \in R(A)$, then for each x in E the strong $\lim_{t\to\infty} J_t x$ exists and belongs to $A^{-1}0$.*

Proof. Fix a point x in E and a positive r. Let C be a bounded closed convex separable subset of E that contains x and is invariant under J_r. Let $t_n \to \infty$, $x_n = J_{t_n} x$, and $y_n = (x - x_n)/t_n$. Since $A^{-1}0$ is nonempty, $\{x_n\}$ is bounded. We also have $|J_r x_n - x_n| \leqslant r|y_n| \to 0$. By [22, Lemma 1.1], there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $f(z) = \lim_{k\to\infty} |x_{n_k} - z|$ exists for all z in C. Since f is continuous and convex, it attains its infimum over C. Let

* Partially supported by the National Science Foundation under Grant MCS 78-02305.

0022-247X/80/050287-06$02.00/0

$K = \{u \in C: f(u) = \inf\{f(z): z \in C\}\}$. If $u \in K$, then $f(J_r u) = \lim_{k\to\infty} | x_{n_k} - J_r u | = \lim_{k\to\infty} | J_r x_{n_k} - J_r u | \leqslant \lim_{k\to\infty} | x_{n_k} - u | = f(u)$, so that K is invariant under J_r. It is also bounded, closed, and convex. Since weakly compact convex subsets of E have the fixed-point property for nonexpansive mappings [2], K contains a fixed point of J_r. Denote such a fixed point by v, and let J be the duality map of E. Since $v \in A^{-1}0$, we have, on the one hand, $(x_n - x, J(x_n - v)) \leqslant 0$ for all n. Since $v \in K$, we have, on the other hand,

$$\limsup_{k\to\infty}(x - v, J(x_{n_k} - v)) \leqslant 0$$

[22, Lemma 1.2; 1]. Therefore $\{x_{n_k}\}$ converges strongly to v. Finally, assume that the strong $\lim_{k\to\infty} x_{n_k} = v_1$ and that the strong $\lim_{m\to\infty} x_{p_m} = v_2$. Then $(v_1 - x, J(v_1 - v_2)) \leqslant 0$, $(v_2 - x, J(v_2 - v_1)) \leqslant 0$, and $v_1 = v_2$. This completes the proof.

Remark 1. The assumptions on E can be weakened: instead of assuming that E is uniformly smooth (equivalently, E^* is uniformly convex), we could have assumed that E is reflexive with a uniformly Gâteaux differentiable norm, and that every weakly compact convex subset of E has the fixed-point property for nonexpansive mappings. But the result is not true in all Banach spaces, even if A is linear [18, p. 321]. We can also replace the assumption that A is m-accretive with the assumption that $\mathrm{cl}(D(A))$, the closure of $D(A)$, is convex, and that A satisfies the range condition: $R(I + rA) \supset \mathrm{cl}(D(A))$ for all $r > 0$.

Remark 2. If we denote the strong $\lim_{t\to\infty} J_t x$ by Px, then $P: E \to A^{-1}0$ is the unique sunny nonexpansive retraction of E onto $A^{-1}0$.

Remark 3. If A is linear, then $J_t x = (1/t) \int_0^\infty e^{-r/t} S(r)\, x\, dr$, where S is the semigroup generated by $-A$. Hence the result follows in this case (in all reflexive spaces) from the mean ergodic theorem. In the nonlinear case, this representation is no longer valid. We do know [16, 19], however, that if E is uniformly convex with a Fréchet differentiable norm, then the integral converges weakly as $t \to \infty$ to a zero of A.

Remark 4. If A is zero free, then $\lim_{t\to\infty} | J_t x | = \infty$ for each x in E [23].

Remark 5. If A is m-accretive and A^{-1} is bounded, then $I - J_r$ is unbounded on unbounded subsets of E. Therefore J_r has a fixed point and $0 \in R(A)$.

COROLLARY 1. *Let C be a closed convex subset of a uniformly smooth Banach space E, and let $T: C \to C$ be a nonexpansive mapping with a fixed point. Let x belong to C. Define for each $0 < k < 1$ a point x_k in C by $x_k = kTx_k + (1 - k)x$. Then the strong $\lim_{k\to 1} x_k$ exists and is a fixed point of T.*

Consider now the iteration

$$x_{n+1} = (1 - k_n) x_0 + k_n T x_n . \tag{1}$$

Corollary 1 implies that [11, Theorem 3.1] is valid in all uniformly smooth Banach spaces:

COROLLARY 2. *In the setting of Corollary* 1, *let* $\{x_n\}$ *be defined by* (1) *with* $k_n = 1 - (n + 2)^{-a}$, *where* $0 < a < 1$. *Then* $\{x_n\}$ *converges strongly to a fixed point of* T.

This provides a partial positive answer to Problem 6 of [12]. The Hilbert space case is due to Halpern [8] (see also [9]).

Returning to accretive operators A, we consider the implicit scheme

$$x_n \in x_{n+1} + h_{n+1}(A x_{n+1} + p_{n+1} x_{n+1}), \tag{2}$$

where $\{h_n\}$ and $\{p_n\}$ are positive sequences such that $\{p_n\}$ decreases to 0, $\sum_{n=1}^{\infty} p_n h_n = \infty$, and $\lim_{n\to\infty}(p_{n-1}/p_n - 1)/(p_n h_n) = 0$, and the explicit scheme

$$x_{n+1} \in x_n - h_n(A x_n + p_n x_n), \tag{3}$$

where, in addition, $\lim_{n\to\infty} \beta(h_n)/p_n = 0$ (β is defined on p. 89 of [17]). We denote $((1 - h_n p_n) x_n - x_{n+1})/h_n$ by $y_n \in A x_n$. In the Hilbert space case, (3) was studied in [3, 5].

Theorem 1 implies that [18, Theorem 1] and [20, Theorem 4] are valid in all uniformly smooth spaces:

THEOREM 2. *Let* E *be a uniformly smooth Banach space, and let* $A \subset E \times E$ *be m-accretive. If* $0 \in R(A)$, *then the sequence* $\{x_n\}$ *defined by* (2) *converges strongly to a zero of* A.

THEOREM 3. *Let* E *be a uniformly smooth Banach space, and let* $A \subset E \times E$ *be m-accretive with* $0 \in R(A)$. *If* $\{x_n\}$ *can be defined by* (3), *and* $\{x_n\}$ *and* $\{y_n\}$ *remain bounded, then* $\{x_n\}$ *converges strongly to a zero of* A.

It is clear that other results (e.g., those of [7, 21, 10]) can also be improved.

Let $A \subset E \times E$ be an m-accretive operator, g: $[0, \infty) \to [0, \infty)$ a non-increasing function of class C^1 such that $\lim_{t\to\infty} g(t) = 0$ and

$$\int_0^\infty g(t)\, dt = \infty, \qquad x \in E, \quad \text{and} \quad x_0 \in D(A).$$

Consider the following initial-value problem:

$$\begin{aligned} u'(t) + Au(t) + g(t)\,u(t) &\ni g(t)\,x, \\ u(0) &= x_0\,. \end{aligned} \tag{4}$$

The method of proof of Theorem 1, when combined with that of [17, Theorem 1.1], yields the following new result.

THEOREM 4. *Let E be uniformly smooth, and let u: $[0, \infty) \to E$ be the strong solution of* (4). *If $0 \in R(A)$, then the strong* $\lim_{t\to\infty} u(t)$ *exists and belongs to* $A^{-1}0$.

Proof. Let $r > 0$, and let C be a bounded closed convex separable subset of E that contains x and is invariant under J_r. Let $t_n \to \infty$ and $x_n = u(t_n)$. There is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $f(z) = \lim_{k\to\infty} | x_{n_k} - z |$ exists for all z in C. Since $\lim_{t\to\infty} \| Au(t)\| = 0$ and $| x_n - J_r x_n | \leqslant r \| Ax_n \|$, the argument used in the proof of Theorem 1 shows f attains its infimum over C at a point v in $A^{-1}0$. By the proof of [17, Theorem 1.1], $\limsup_{n\to\infty}(x_n - x, J(x_n - y)) \leqslant 0$ for all y in $A^{-1}0$. The result now follows because we also have $\limsup_{k\to\infty}(x - v, J(x_{n_k} - v)) \leqslant 0$.

For the Hilbert space case see Browder [4, p. 174]. The strong $\lim_{t\to\infty} u(t) = Px$, where P: $E \to A^{-1}0$ is again the unique sunny nonexpansive retraction of E onto $A^{-1}0$.

It follows that the doubly iterative procedure presented in [17] works in all uniformly smooth spaces.

We now turn to the behavior of the resolvent J_r when $r \to 0$.

THEOREM 5. *Let E be a Banach space that is both uniformly convex and uniformly smooth. If $A \subset E \times E$ is m-accretive, then for each x in E the strong* $\lim_{r\to 0} J_r x$ *exists.*

Proof. Fix a point x in E. Since E is uniformly convex, $\mathrm{cl}(D(A))$ is a nonexpansive retract of E [15, p. 382]. Let R: $E \to \mathrm{cl}(D(A))$ be a nonexpansive retraction. Let C be a bounded closed convex separable subset of E that contains x and is invariant under R. Let $r_n \to 0$ and $x_n = J_{r_n} x$. Since $\{x_n\}$ remains bounded as $r_n \to 0$, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $f(z) = \lim_{k\to\infty} | x_{n_k} - z |$ exists for all z in C. Let f attain its infimum over C at $u \in C$. We have $f(Ru) = \lim_{k\to\infty} | x_{n_k} - Ru | = \lim_{k\to\infty} | Rx_{n_k} - Ru | \leqslant \lim_{k\to\infty} | x_{n_k} - u | = f(u)$. Therefore f attains its infimum over C at $v = Ru$. Hence $\limsup_{k\to\infty}(x - v, J(x_{n_k} - v)) \leqslant 0$. On the other hand, $(x - x_n - (y - J_{r_n} y), J(x_n - J_{r_n} y)) \geqslant 0$ for all n, so that $\limsup_{n\to\infty}(x_n - x, J(x_n - y)) \leqslant 0$ for all y in $\mathrm{cl}(D(A))$. It follows that $\{x_{n_k}\}$ converges strongly to v. If $x_{n_k} \to v_1$ and $x_{p_m} \to v_2$, then $(x - v_1, J(v_1 - v_2)) \geqslant 0$, $(x - v_2, J(v_2 - v_1)) \geqslant 0$, and $v_1 = v_2$. The proof is complete.

This is the first result of its kind for non-Hilbert infinite-dimensional Banach spaces. The Hilbert space case in well known [6, p. 388]. For weak convergence results outside Hilbert space see [14, p. 288; 15, p. 383].

Remark 6. If we denote the strong $\lim_{r\to 0} J_r x$ by Qx, then $Q: E \to \mathrm{cl}(D(A))$ is the unique sunny nonexpansive retraction of E onto $\mathrm{cl}(D(A))$.

Remark 7. The assumptions on E can be relaxed: it suffices to assume that E has a uniformly Gâteaux differentiable norm and that its dual E^* has a Fréchet differentiable norm. But the result is not true in all Banach spaces (see the last example of [22]).

Remark 8. We emphasize that the known Hilbert space proofs of Theorems 1, 4, and 5 do not extend to L^p, $p \neq 2$.

We conclude with a new result on the exponential formula.

COROLLARY 3. *In the setting of Theorem 5, let S be the semigroup generated by $-A$, and let Q: $E \to \mathrm{cl}(D(A))$ be the unique sunny nonexpansive retraction of E onto* $\mathrm{cl}(D(A))$. *Then* $\lim_{n\to\infty} J_{t/n}^n x = S(t)Qx$ *for all x in E and $t > 0$.*

Note added in proof. 1. Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm. Theorem 1 is also true if, in addition, E is strictly convex, or if the metric projection on every closed convex subset of E is upper semicontinuous.

2. The idea of the present paper has led to a "dual" nonlinear ergodic theorem in Banach spaces and to a new short proof of the nonlinear mean ergodic theorem in Hilbert space. See the paper by R. E. Bruck and the author entitled "Accretive Operators, Banach Limits, and Dual Ergodic Theorems."

3. Combining Corollary 1 of the present paper with Theorem 2 of the paper by R. Haydon, E. Odell, and Y. Sternfeld entitled "A Fixed Point Theorem for a Class of Star-Shaped Sets in c_0," we obtain the following result: Let C_1 be a closed subset of a Banach space E_1, and let C_2 be a bounded closed convex subset of a uniformly smooth Banach space E_2. If C_1 has the fixed point property for nonexpansive mappings, then so does the subset $C_1 \oplus C_2$ of $(E_1 \oplus E_2)_\infty$.

4. Let E be a Banach space, and let $A \subset E \times E$ be an accretive operator that satisfies the range condition. Assume either that E is (UG) and E^* is (F), or that E is uniformly convex. Then for each x in $\mathrm{cl}(D(A))$, $\lim_{t\to\infty} J_t x/t = -v$, where v is the point of least norm in $\mathrm{cl}(R(A))$. For this and related results see our papers entitled "A Solution to a Problem on the Asymptotic Behavior of Nonexpansive Mappings and Semigroups," *Proc. Japan Acad.*, in press, and "On the Asymptotic Behavior of Nonlinear Semigroups and the Range of Accretive Operators."

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Printed in Belgium