On Schauder's Fixed Point Theorem and Forced Second-Order Nonlinear Oscillations*

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Let \( e(t) \) be continuous, periodic with period \( T > 0 \), and have mean zero. In this note we will show that for any number \( c \) the differential equation

\[
\ddot{x} + c\dot{x} + g(x) = e(t)
\]

has a \( T \)-periodic solution provided that \( g \) is continuous, \( xg(x) \geq 0 \) for \( |x| \) sufficiently large, and \( g(x)/x \to 0 \) as \( |x| \to \infty \). Our proof makes use of what appears to be a new method of applying the Schauder fixed point theorem to establish the existence of periodic solutions of nonlinear differential equations. In a future paper we hope to be able to formulate this method in a general setting and thereby establish the existence of periodic solutions of more general nonlinear differential equations.

For brevity we introduce some notation. \( P \) will denote the set of real-valued continuous functions with period \( T \). \( Q \) will denote the set of function \( f \) with

\[
\int_{0}^{T} f(s) \, ds = 0.
\]

If \( f \in P \), \( \|f\| \) will denote \( \max |f(t)| \).

For ease in proving the first part of our main result, we state three easily established lemmas.

**Lemma 1.** If \( f \in Q \) and \( I(f)(t) = \int_{0}^{t} f(s) \, ds \), then \( I(f) \in P \) and

\[
\|I(f)\| \leq T/2 \|f\|.
\]

**Lemma 2.** If \( F \in P \) and \( G(F)(t) = \left[ e^{T} - 1 \right]^{-1} \int_{t}^{T} e^{-T-s} F(s) \, ds \), then \( G(F) \in P \), \( \|G(F)\| \leq \|F\| \), and \( G(F) \) is a solution of the differential equation

\[
\ddot{x} + c\dot{x} = F(t).
\]

From Lemmas 1 and 2 we obtain:

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LEMMA 3. If \( f \in Q \) and \( H(f) \equiv G(I(f)) \), then \( H(f) \in P \),
\[
\|H(f)\| \leq \|I(f)\| \leq (T/2)\|f\|.
\]

THEOREM. Let \( e \in Q \). If \( g \) is continuous, if
\[
g(x)/x \rightarrow 0 \text{ as } |x| \rightarrow \infty,
\]
and if there exists a number \( b \) such that
\[
xg(x) > 0 \text{ for } |x| > b,
\]
then for any number \( c \) the differential equation
\[
\dot{x} + cx + g(x) = e(t)
\]
has at least one \( T \)-periodic solution.

PROOF: Case I. \( c \neq 0 \).

In this case it suffices to consider \( c = 1 \) for under the change of independent variable \( s = ct \), the equation takes the form \( x'' + x' + h(x) = E(s) \),
\[
' = (d/ds) \quad \text{where} \quad h(x) = g(x)/c^2, \quad E(s) = e(s/c)/c^2,
\]
so that \( h(x)x > 0 \) for \( |x| > b \), \( h(x)/x \rightarrow 0 \) as \( |x| \rightarrow \infty \), and \( E(s + S) = E(s) \), \( \int_S^N E(v) \, dv = 0 \), if \( S = |c| \cdot T \).

We note that the condition (2) implies that for any \( \epsilon > 0 \) there exists a number \( L(\epsilon) \) such that
\[
|g(x)| \leq \epsilon D \quad \text{if} \quad D \geq L(\epsilon) \quad \text{and} \quad |x| \leq D. \tag{4}
\]

Indeed, if \( r(\epsilon) \) is such that \( |g(x)| \leq \epsilon |x| \) for \( |x| \geq r(\epsilon) \), if
\[
M = \max\{|g(x)| \mid |x| \leq r(\epsilon)\}
\]
and \( L(\epsilon) = \max(r(\epsilon), M/\epsilon) \), \( L(\epsilon) \) satisfies (4).

If \( \theta \in P \), let us define
\[
\hat{g}(\theta)(t) = g(\theta(t)) - N(\theta); \quad N(\theta) = \frac{1}{T} \int_0^T g(\theta(s)) \, ds. \tag{5}
\]

Clearly, for all \( \theta \in P \), \( \hat{g}(\theta) \in Q \) and
\[
\|\hat{g}(\theta)\| \leq 2\epsilon D \quad \text{if} \quad D \geq L(\epsilon) \quad \text{and} \quad \|\theta\| \leq D. \tag{6}
\]

Let \( R \) denote the real numbers and let \( B = P \times R \). If \((\theta, a), (\theta_1, a_1), (\theta_2, a_2) \in B, x_1, x_2 \in R, \) let us define
\[
[(\theta, a)] = \|\theta\| + |a|,
\]
\[
x_1(\theta_1, a_1) + x_2(\theta_2, a_2) = (x_1\theta_1 + x_2\theta_2, x_1a_1 + x_2a_2).
\]

With these definitions \((B, \| \|)\) is a complete normed linear space. If for each \((\theta, a) \in B \) we define \( A[(\theta, a)] = (\theta^*, a^*) \) where
\[
\theta^* = a + H[e - \hat{g}(\theta)]
\]
\[
a^* = a - N(\theta^*) \tag{7}
\]
then, by Lemma 3, \( A \) is a continuous mapping of \( B \) into \( B \).
Let

$$0 < \delta < \min\{1/3, 1/3T\},$$

$$D = \max\{b/(1 - 3\delta), (b + (3/2) T \| e \|)(1 - 3T\delta), L(\delta)\}$$

(8)

where $L(\delta)$ is as in (4), and let

$$m = \max\{\delta D, (T/2)\| e \| + T\delta D\}$$

(9)

so that

$$b + 3m \leq D.$$  

(10)

Let

$$K = \{(\theta, a) \in B \| \theta \| \leq D, \| a \| \leq b + 2m\}$$

(11)

so that $K$ is a closed and convex subset of $B$. We assert:

(i) $A(K) \subset K$,

(ii) $A(K)$ is conditionally compact ($A(K)$ closure compact).

To prove (i) consider $(\theta, a) \in K$, from (6)-(11) and Lemma 3,

$$\| \theta^* \| \leq \| a \| + \| H[e - \hat{g}(\theta)] \|$$

$$\leq (b + 2m) + (T/2)\| e - \hat{g}(\theta) \|$$

$$\leq (b + 2m) + T/2(\| e \| + 2\delta D) \leq b + 3m \leq D.$$  

(12)

If $-(b + m) \leq a \leq b + m$, then since

$$D \geq L(\delta) \quad \text{and} \quad \| \theta^* \| \leq D,$$

$$\| N(\theta^*) \| = \frac{1}{T} \int_0^T g(\theta^*(s)) \, ds \leq \delta D \leq m,$$

so that $-(b + 2m) \leq a - N(\theta^*) \leq b + 2m$, and so

$$a \in [-(-b + m), (b + m)] \implies a^* \in [-(b + 2m), b + 2m].$$

(13)

By (6), (7), and (9),

$$\| \theta^* - a \| = \| H[e - \hat{g}(\theta)] \| \leq (T/2)(\| e \| + 2\delta D) \leq m,$$

so that $a \geq b + m$ implies $\theta^*(t) \geq b$ and $a \leq -(b + m)$ implies $\theta^*(t) \leq -b$ for all $t$. Hence, by (3) $a \geq b + m$ implies $g(\theta^*(t)) \geq 0$, and $a \leq -(b + m)$ implies $g(\theta^*(t)) \leq 0$ for all $t$. Hence, by (3) and (5) $(b + m) \leq a \leq (b + 2m)$ implies $b \leq a - N(\theta^*) \leq a \leq b + 2m$ and $-(b + 2m) \leq a \leq -(b + m)$ implies $-(b + 2m) \leq a \leq a - N(\theta^*) \leq b$. Hence,

$$a \in [b + m, b + 2m] \quad \implies \quad a^* \in [b, b + 2m],$$

$$a \in [-(b + 2m), -(b + m)] \quad \implies \quad a^* \in [-(b + 2m), -b].$$

(14)

Assertion (i) follows from (11)-(14).
To prove assertion (ii) we must show that if \( \{(\theta_n^*, a_n^*)\} = \{A(\theta_n, a_n)\} \) is a sequence in \( A(K) \), then there exists a subsequence \( \{(\theta^*_n, a^*_n)\} \) of \( \{(\theta_n^*, a_n^*)\} \) and an element \((\bar{\theta}, \bar{a}) \in B \) such that
\[
\lim_{n_k \to \infty} [(\theta^*_n, a^*_n) - (\bar{\theta}, \bar{a})] = 0.
\]

Suppose then \( \{(\theta_n^*, a_n^*)\} = \{A(\theta_n, a_n)\} \) is such a sequence. We consider the functions \( v_n = H[e - \dot{g}(\theta_n)] = G[I(e - \dot{g}(\theta_n))] \). By (9) and Lemmas 1-3,
\[
\|v_n\| \leq (T/2)\|e - \dot{g}(\theta_n)\| \leq (T/2)(\|e\| + 2\delta D) \leq m
\]
and
\[
\left\| \frac{dv_n}{dt} \right\| = \|v_n - I(e - \dot{g}(\theta_n))\| \leq \|v_n\| + (T/2)(\|e - \dot{g}(\theta_n)\| \leq 2m.
\]

Hence, the sequence \( \{v_n\} \) is equicontinuous and uniformly bounded, and thus, since \( \{v_n\} \subset P \), by Ascoli's Lemma, there exists a subsequence \( \{v_{n_k}\} \) of \( \{v_n\} \) and a \( w \in P \) such that
\[
\lim_{n_k \to \infty} \|v_{n_k} - w\| = 0.
\]

From the condition \( a_{n_k} \in [- (b + 2m) \), we may assume by again taking subsequences that
\[
\lim_{n_k \to \infty} a_{n_k} = \alpha
\]
exists and so by (7),
\[
\lim_{n_k \to \infty} \theta_n^*(t) = \lim_{n_k \to \infty} (a_{n_k} + v_{n_k}(t)) = \alpha + w = \tilde{\theta}(t)
\]
uniformly in \( t \). Obviously,
\[
\lim_{n_k \to \infty} a_{n_k}^* = \lim_{n_k \to \infty} (a_{n_k} - N(\theta_n^*)) = \alpha - N(\tilde{\theta}) = \tilde{a};
\]
and hence,
\[
\lim_{n_k \to \infty} [(\theta_n^*, a_n^*) - (\tilde{\theta}, \tilde{a})] = \lim_{n_k \to \infty} (\|\theta_n^* - \theta_n\| + |a_n^* - \tilde{a}|) = 0,
\]
which proves assertion (ii).

To conclude the proof of Theorem 1, Case I, we note that (i), (ii), the fact that \( K \) is closed and convex, and the Schauder Fixed Point Theorem as given in [1, p. 131] imply the existence of \( a(\phi, \tilde{a}) \in K \) such that \( (\phi, \tilde{a}) = A[\phi, \tilde{a}] = (\phi^*, \tilde{a}^*) \).

Therefore by (7),
\[
N(\phi) = 0, \quad \dot{g}(\phi) = g(\phi),
\]
\[
\phi = \tilde{a} + H[e - g(\phi)],
\]
and so by Lemma 3, $\phi \in P$ and

$$\dot{\phi} + \phi + g(\phi) = e(t).$$

**Case II.** $c = 0$.

To take care of this case we need a substitute for Lemma 3. If $f \in P$, we define $f^\#(t) = f(t) - (1/T) \int_0^T f(s) \, ds$. It follows immediately that

$$f^\# \in Q, \|f^\#\| \leq 2 \|f\|. \quad (15)$$

**Lemma 4.** If $f \in Q$ and $S(f) \equiv I(I(f^\#))$, then

$$s(f) \in P, \quad S(f) \equiv T^2/2 \|f\|$$

and $S(f)$ is a solution of

$$\ddot{x} = f(t). \quad (16)$$

**Proof.** By Lemma 1 and (15) $I(f) \in P$ and $I(f^\#) \in Q$, another application of Lemma 1 implies that $S(f) = I(I(f^\#)) \in P$. Moreover, by Lemma 1 and (15),

$$\|I(I(f^\#))\| \leq (T/2)\|I(f^\#)\| \leq T\|I(f)\| \leq T^2/2 \|f\|.$$

If $x = I(I(f^\#))$, then

$$\ddot{x} = I(f^\#) = \int_0^T f(s) \, ds - \frac{1}{T} \int_0^T \left( \int_0^s f(u) \, du \right) \, ds.$$

Another differentiation gives (16).

**Conclusion of Proof.** For $(\theta, a) \in P \times R$ define $E(\theta, a) = (\theta^*, a^*)$ where

$$\theta^* = a + S(e - \dot{\theta}(\theta)),
\quad a^* = a - N(\dot{\theta}^*).$$

By mimicking the proof for Case I we show that $E$ has a fixed point $(u, b)$ and $u$ is $T$-periodic solution of

$$\ddot{x} + g(x) = e(t).$$

**Reference**