JOURNAL OF APPROXIMATION THEORY **66**, 53–57 (1991)

The Linear Selections of Metric Projections in the L_p Spaces

SONG WENHUA

Institute of Applied Mathematics, Dalian University of Technology, Dalian 116024, People's Republic of China

Communicated by Frank Deutsch

Received April 25, 1990; revised August 2, 1990

A characterization is given of those subspaces of L_p space whose metric projection is linear, and of L_1, which is finitely codimensional, whose metric projection admits a linear selection.

1. INTRODUCTION

Let X be a normed linear space and Y a non-empty subset of X. The metric projection (or nearest point mapping $P_Y: X \to 2^Y$ is a set mapping to be defined by $P_Y(x) = \{y \in Y;\ \|x-y\| = d(x, Y)\}$ for any $x \in X$, where $d(x, Y) = \inf\{\|x-g\|;\ g \in Y\}$. The subset Y is called proximinal if $P_Y(x) \neq \varnothing$ for each $x \in X$. It is well known that it is proximinal for any closed convex subset of a uniformly convex Banach space. A selection for P_Y is a mapping $s: X \to Y$ such that $s(x) \in P_Y(x)$ for each $x \in X$. If Y is a subspace, a linear selection for P_Y is a selection with the additional property of being linear. The kernel of a metric projection P_Y onto a proximinal subspace Y is the set $\ker P_Y = \{x \in X; 0 \in P_Y(x)\}$.

F. Deutsch [3] has shown that, for a proximinal subspace Y, P_Y has a linear selection if and only if $\ker P_Y$ contains a closed subspace N such that $X = Y + N$.

Pei-Kee Lin [6] has proved that, for a finite dimensional subspace Y of L_p $(1 < p < \infty$ and $p \neq 2)$, P_Y admits a linear selection if and only if there exist k disjoint subsets $B_1, B_2, ..., B_k$, every one of which is the union of some atoms of T, such that $Y = (\oplus Y_i)_p$, where Y_i is either $L_p(B_i)$ or a hyperplane of $L_p(B_i)$.

In Section 2, we study the linear metric projection on $L_p(T)$. For any closed subspace of $L_p(T, \Sigma, \mu)$, which μ is a purely atomic measure, we prove that P_Y is linear if and only if there exists a disjoint subset collection $\{A_\lambda\}_{\lambda \in \Lambda}$ of T such that $Y = (\bigoplus_{\lambda \in \Lambda} Y_\lambda)_p$, where Y_λ is either $L_p(A_\lambda)$ or a hyperplane of $L_p(A_\lambda)$.

0021-9045/91 $3.00

In Section 3, we consider the space $L_1(T, \Sigma, \mu)$ of integrable functions on the finite measure space (T, Σ, μ). For an n-codimensional subspace Y of L_1, we prove that P_Y has a linear selection P such that $P1_T=0$, where $1_T(t)=1$, if and only if there exist n disjoint subsets $B_i \in \Sigma$, $i=1, 2, ..., n$, such that $Y=\{x \in L_1; \int_{A_i} x\, d\mu=0,\ i=1, 2, ..., n\}$.

2. Linear Selections in L_p ($1<p<\infty$ and $p \neq 2$)

Let (T, Σ, μ) be a measure space. An atom A is a measurable set such that $\mu(A)<\infty$ and, if B is a measurable subset of A, then it has either $\mu(A)=\mu(B)$ or $\mu(B)=0$. Hence, any measurable function is constant a.e. (μ) on an atom, and we can assume that every atom contains only one point. For $x \in L_p$, the supported subset of x is defined (up to a set of measure zero) by $\text{supp}(x)=\{t \in T; x(t) \neq 0\}$.

We shall use the following theorem. The proof is similar to that in [2].

THEOREM 2.1. *Let (T, Σ, μ) be a measure space, μ a purely atomic measure, and P a contractive projection on X. Then there exists a vector family $\{y_\lambda\}_{\lambda \in \Lambda}$ of norm 1 with the disjoint supported subsets in X such that: For each $x \in X$, $P(x)=\sum_{\lambda \in \Lambda} y_\lambda^*(y_\lambda)$, where y_λ^* is the peak functional of y_λ for each $\lambda \in \Lambda$.*

Using this theorem, we can show the following theorem.

THEOREM 2.2. *Let (T, Σ, μ) be a purely atomic measure space and Y a closed subspace of L_p $(1<p<\infty,\ p \neq 2)$. Then the following statements are equivalent,*

(a) *P_Y is linear.*

(b) *There exists a disjoint subset family $\{A_\lambda\}_{\lambda \in \Lambda}$ of T such that $Y=[\bigoplus_{\lambda \in \Lambda} M_\lambda]_p$, where M_λ is either $L_p(A_\lambda)$ or a hyperplane of $L_p(A_\lambda)$ for any $\lambda \in \Lambda$.*

Proof. (a) $\Rightarrow$ (b). Let $P=P_Y$ and $Q=id-P$. It is obvious that Q is a contractive projection operator. By Theorem 2.1, there exists a vector family $\{y_\lambda\}_{\lambda \in \Lambda_0}$ of norm 1 in L_p in which the supported subsets $\text{supp}(y_\lambda)$ of y_λ are disjoint such that, for each $x \in L_p$,

$$Qx=\sum_{\lambda \in \Lambda_0} y_\lambda^*(x) \cdot y_\lambda, \tag{2.1}$$

where y_λ^* is the peak functional of y_λ for each $\lambda \in \Lambda_0$. We can assume that $0 \notin \Lambda_0$. Let $\Lambda=\{0\} \cup \{\lambda \in \Lambda_0;\ \text{card}[\text{supp}(y_\lambda)] \geqslant 2\}$. Since for any $x \in L_p$

and $\|x\| = 1$ the peak functional f of x is $|x|^{p-1}\operatorname{sgn}(x)$, we get $\operatorname{supp}(f) = \operatorname{supp}(x)$. So $y_\lambda^* \in L_q(\operatorname{supp}(y_\lambda))$ for any $\lambda \in \Lambda$. Let $A_0 = T \setminus \bigcup_{\lambda \in \Lambda_0} \operatorname{supp}(y_\lambda)$ and $A_\lambda = \operatorname{supp}(y_\lambda)$ $\lambda \in \Lambda \setminus \{0\}$. Let $M_0 = L_p(A_0)$ and $M_\lambda = \{x \in L_p(A_\lambda);\ y_\lambda^*(x) = 0\}$ $\lambda \in \Lambda \setminus \{0\}$. Then M_λ is a hyperplane of $L_p(A_\lambda)$ for each $\lambda \in \Lambda \setminus \{0\}$. For any $x \in L_p$, let $x_\lambda = x|_{A_\lambda}$ ($= x(t)$, $t \in A_\lambda$; and $= 0$, $t \notin A_\lambda$) for each $\lambda \in \Lambda$. Then $x = \sum_{\lambda \in \Lambda} x_\lambda$. By (2.1),

$$Qx = \sum_{\lambda \in \Lambda_0} y_\lambda^*(x_\lambda) \cdot y_\lambda. \tag{2.2}$$

If $x \in Y$, then $Qx = 0$. By (2.2), we get $y_\lambda^*(x_\lambda) = 0$. If $\operatorname{supp}(y_\lambda)$ is a singleton, let $\operatorname{supp}(y_\lambda) = \{t_0\}$. Since $y_\lambda^*(t) = x_\lambda(t) = 0$ when $t \neq t_0$, we have $x_\lambda(t_0) = 0$ by $0 = y_\lambda^*(x_\lambda) = \mu(t_0) \cdot y_\lambda^*(t_0) \cdot x_\lambda(t_0)$ and $\mu(t_0) \neq 0$, $y_\lambda^* \neq 0$. Hence $x = \sum_{\lambda \in \Lambda} x_\lambda$. By $y_\lambda^*(x_\lambda) = 0$, we get $x \in [\bigoplus_{\lambda \in \Lambda} M_\lambda]_p$, that is, $Y \subseteq [\bigoplus_{\lambda \in \Lambda} M_\lambda]_p$. If $x = \sum_{\lambda \in \Lambda} x_\lambda$ and $x_\lambda \in M_\lambda$, by (2.2) we get $Qx = 0$. Hence $x \in Y$, i.e., $[\bigoplus_{\lambda \in \Lambda} M_\lambda]_p \subseteq Y$.

The (b) $\Rightarrow$ (a) is the following theorems. ∎

THEOREM 2.3 (F. Deutsch [3]). *Let Y be a proximinal hyperplane of a Banach space X. Then P_Y admits a linear selection.*

THEOREM 2.4 (Pei-Kee Lin [6]). *Suppose M_i is a proximinal subspace of X_i, P_{M_i} has a linear selection s_i. Then $M = (\bigoplus M_i)_p$ $(1 \leqslant p < \infty)$ is a proximinal subspace of $X = (\bigoplus X_i)_p$. Moreover, P_M has a linear selection $\bigoplus s_i$.*

3. THE LINEAR SELECTION IN L_1

In this section, we consider the linear selections in L_1 space. We will need to use the following theorems.

THEOREM 3.1 (R. G. Douglas [5]). *Let (T, Σ, μ) be a finite measure space and P a contractive projection on $L_1(T)$ and $P1_T = 1_T$, where $1_T(t) = 1$ for any $t \in T$. Let $\Sigma_0 = \{\operatorname{supp} f;\ f \in R(P)\}$. Then Σ_0 is a σ-subring of Σ and $Pf = 0$ if and only if $\int_A f\, d\mu = 0$ for each $A \in \Sigma_0$.*

Using this theorem, we can get the following theorem.

THEOREM 3.2. *Let (T, Σ, μ) be a finite measure space and Y an n-codimensional proximinal subspace of $L_1(T)$. Then the following statements are equivalent*

(1) *P_Y admits a linear selection P such that $P1_T = 0$.*

(2) *There exist measurable subsets* $A_1, A_2, ..., A_n$ *such that*

(a) $A_i \cap A_j = \emptyset$ $(i \neq j)$ *and* $\bigcup_{k=1}^{n} A_k = T$.

(b) $Y = \{f \in L_1(T); \int_{A_i} f\, d\mu = 0,\ i = 1, 2, ..., n\}$.

Proof. It is evident that if Y has the form in (b), then codim $Y = n$.

(1)$\Rightarrow$(2). Let $Q = id - P$. Then Q is a contractive projection on L_1 and $Q1_T = 1_T$. Let $\Sigma_0 = \{\text{supp}\, x;\ x \in R(Q)\}$. By Theorem 3.1,

$$Y = \left\{f \in L_1;\ \int_A f\, d\mu = 0 \text{ for each } A \in \Sigma_0\right\}. \tag{3.1}$$

Let $A_1, A_2, ..., A_m$ be all atoms in Σ_0. Since $R(Q)$ is separable (finite dimension), the subset $T_0 = \cup\{\text{supp}\, x;\ x \in R(Q)\}$ is measurable. Let $D = T_0 \setminus \bigcup_{k=1}^{m} A_k$. Suppose $\mu(D) > 0$. Since D does not contain any atoms, there exist disjoint $B_1, B_2, ..., B_{n+1}$ such that $0 < \mu(B_i) \leqslant \mu(D)/(n+1)$ and $B_i \in \Sigma_0$. Hence there exist $y_i \in R(Q)$ such that $\text{supp}(y_i) = B_i$. It is obvious that $y_1, y_2, ..., y_{n+1}$ are linear independent. So $\dim R(Q) > n$. It is in contradiction with the codim $Y = n$. So we get $T_0 = \bigcup_{k=1}^{m} A_i$ and $\Sigma_0 = \{A_1, A_2, ..., A_m\}$. By (3.1), we get that Y has the form of (b) and $m = n$. If $\mu(T \setminus T_0) > 0$, let f be the characteristic function. Then $f \neq 0$. But

$$\|1_T - f\| = \int_{T_0} f\, d\mu = \mu(T_0) < \mu(T) = \|1_T\|.$$

This is in contradiction with the $P1_T = 0$. So we can assume that $T = T_0$ (up to a set of measure zero). Hence (a) holds.

(2)$\Rightarrow$(1). Let x_i be the characteristic function of A_i and $y_i = x_i/\mu(A_i)$. For any $x \in L_1$, let $f_i(x) = \int_{A_i} x\, d\mu$. Then $f_i \in L_1^*$, $\|f_i\| = 1$, and $f_i(y_j) = \delta_{ij}$, where δ_{ij} is the Kronecker symbol. Since $A_i \cap A_j = \emptyset$ and $T = \bigcup_{k=1}^{n} A_k$, for any $x \in L_1$, $x = \sum_{k=1}^{n} (x_i x)$. Let $x_0 = \sum_{k=1}^{n} [x_i x - f_i(x_i x)\, y_i]$. It is obvious that $f_i(x_0) = 0$. So $x_0 \in Y$. For any $y \in Y$, by $f_i(x_i y) = f_i(y) = 0$,

$$\begin{aligned} \|x - x_0\| &= \sum_{k=1}^{n} \|f_i(x_i x)\, y_i\| = \sum_{k=1}^{n} |f_i(x_i x)| \\ &= \sum_{k=1}^{n} |f_i(x_i x - x_i y)| \leqslant \sum_{k=1}^{n} \|x_i x - x_i y\| = \|x - y\|. \end{aligned}$$

So $x_0 \in P_Y x$. Let $Px = x_0$. It is evident that P is a linear selection of P_Y. We need only prove $P1_T = 0$. By definition, $P1_T = \sum_{k=1}^{n} [x_i - f_i(x_i)\, y_i]$. Since $f_i(x_i)\, y_i = \mu(A_i)\, y_i = x_i$, $P1_T = 0$. ∎

References

1. J. Lindenstrauss and L. Tzafriri, "Classical Banach Spaces," Springer-Verlag, New York, 1977.
2. H. E. Lacey, "The Isometric Theory of Classical Banach Spaces," Springer-Verlag, Berlin Heidelberg New York, 1974.
3. F. Deutsch, Linear selections for the metric projections, *J. Funct. Anal.* **49** (1982), 269–292.
4. F. Deutsch, When does the metric projection admit a linear selection? *Canad. Math. Soc. Confer. Proc.* **3** (1983), 135–141.
5. R. G. Douglas, Contractive projections on an L_1-space, *Pacific J. Math.* **15** (1965), 443–462.
6. Pei-Kee Lin, Remarks on linear selections for the metric projections, *J. Approx. Theory* **43** (1985), 64–74.