Invariant subspaces of operator Lie algebras and Lie algebras with compact adjoint action

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To the memory of Irina Fedorovna Turovskaya (June 16, 1925–July 5, 2001)

Abstract

It is proved that a Lie algebra of compact operators with a non-zero Volterra ideal is reducible (has a nontrivial invariant subspace). A number of other criteria of reducibility for collections of operators is obtained. The results are applied to the structure theory of Lie algebras of compact operators and normed Lie algebras with compact adjoint action.

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1. Introduction

The interplay between Lie algebras and the theory of invariant subspaces begins with a search of algebraic relations which provide (under some analytic restrictions) the possibility to obtain triangularization of a family of operators. We may think of such relations as the forms of weakened commutativity that are described naturally and...
traditionally in terms of the Lie algebra generated by the family. The classical examples are the famous Engel and Lie theorems which state that nilpotent and, respectively, solvable Lie algebras of operators on finite-dimensional spaces are triangularizable. To compact operators they were extended in [41] for Engel Lie algebras and in [40] for solvable Lie algebras.

Recall that a normed Lie algebra \( \mathfrak{L} \) is Engel if all operators \( \text{ad}_L a : x \mapsto [a, x] \) \( (a \in \mathfrak{L}) \) on \( \mathfrak{L} \) are quasi-nilpotent. This notion seems to be a right functional-analytic extension of the nilpotence. In a contrast, the definition of solvability in [40] repeats the classical one (with the only distinction that \( \mathfrak{L} \) can be infinite-dimensional). In particular Engel Lie algebras need not be solvable. In the present work, we radically extend the definition and call a normed Lie algebra \( \mathfrak{E} \)-solvable (\( E \)-solvable for brevity) if every its nonzero quotient by a closed ideal has a nonzero Engel ideal. One of the central results (Corollary 4.25) states that any \( E \)-solvable Lie algebra of compact operators is triangularizable.

The main step in the proof is the solution of the Volterra Ideal Problem (VIP) posed in [40]. Recall that an operator on a Banach space is Volterra if it is compact and quasi-nilpotent; a set of operators is Volterra if its elements are Volterra. The result (Theorem 4.14) states that if a Lie algebra of compact operators has a nonzero Volterra ideal then it has a nontrivial invariant subspace (is reducible in other words).

The work contains many criteria of reducibility or of the existence of hyperinvariant or superinvariant subspaces.

Let \( \mathcal{B}(\mathfrak{X}) \) denote the algebra of all bounded linear operators on a Banach space \( \mathfrak{X} \), \( \mathcal{K}(\mathfrak{X}) \) its ideal of compact operators. A subspace \( Y \subset \mathfrak{X} \) is hyperinvariant for \( M \subset \mathcal{B}(\mathfrak{X}) \) if it is invariant for \( M \) and for its commutant \( M' = \{ T \in \mathcal{B}(\mathfrak{X}) : [T, M] = 0 \} \). Setting \( \text{Nor} M = \{ S \in \mathcal{B}(\mathfrak{X}) : [S, M] \subset M \} \), we say that \( Y \) is superinvariant (resp., \( \mathcal{K} \)-superinvariant) for \( M \) if it is invariant for \( M \cup \text{Nor} M \) (resp., \( M \cup (\mathcal{K}(\mathfrak{X}) \cap \text{Nor} M) \)). In these terms Theorem 4.14 asserts that any Volterra Lie algebra has a \( \mathcal{K} \)-superinvariant subspace. To prove the existence of a superinvariant subspace is more difficult: even for commutative sets of compact operators we can do it only under some additional restrictions (Theorem 4.17).

The results on invariant subspaces of Lie algebras of compact operators may be used for the classification of such algebras. In this approach the building blocks of the classification are the topologically irreducible Lie algebras, and the important first step is their description. It is proved in Theorem 4.26 that any irreducible Lie algebra of compact operators is ‘\( E \)-semisimple’ (this means that it does not have Engel ideals). The notions of \( E \)-simplicity and \( E \)-solvability are united by the following result (Theorems 5.9 and 5.14): any Lie algebra \( \mathfrak{L} \) of compact operators has the largest \( E \)-solvable ideal \( \mathcal{R}(\mathfrak{L}) \) and the normed Lie algebra \( \mathfrak{L}/ \mathcal{R}(\mathfrak{L}) \) is \( E \)-semisimple. The map \( \mathfrak{L} \mapsto \mathcal{R}(\mathfrak{L}) \) shares many properties of the radical in the theory of finite-dimensional Lie algebras. We prove also the existence of the largest Engel ideal (the analog of the nil-radical) and of the largest Volterra ideal (= the intersection of \( \mathfrak{L} \) with the Jacobson radical of the norm-closed subalgebra of \( \mathcal{B}(\mathfrak{X}) \) generated by \( \mathfrak{L} \)) and characterize all three versions of radicality in several ways.

However, the operator setting is not completely appropriate for the theory of general Lie radicals because the quotients of operator Lie algebras are not operator Lie algebras.
So, we develop the theory of $E$-solvable radical in the frame of normed Lie algebras with compact adjoint action (ad-compact Lie algebras for brevity). They are defined by the condition: all operators $\text{ad}_L a$, $a \in L$, are compact. We show that in this class of Lie algebras the $E$-radical (i.e. the largest $E$-solvable ideal) always exists and has all required radical properties.

The study of ad-compact algebras was initiated by Vaksman and Gurarij [46] and Wojtyński [51]. Our results answer some basic questions left open in these works.

The classification of Banach Lie algebras with compactness conditions seems to be a natural direction for the extension of the classical finite-dimensional theory in the general theory of infinite-dimensional topological Lie algebras. Comparing two types of such conditions considered here, note that the class of Lie algebras of compact operators, a more special subject than the class of ad-compact Lie algebras, is far from being contained in the latter. On the contrary the adjoint representation often allows to reduce the study of ad-compact Lie algebras to the Lie algebras of compact operators. Such a possibility is permanently used in our work.

The paper is organized in the following way. The required definitions and preliminary results are gathered in Section 2.

In Section 3 we work out our main technical tool, namely a theory that relates elementary spectral subspaces of operators in an operator Lie algebra to spectral subspaces of operators of the adjoint representation on the operator Lie algebra. It has a close link with the classical theory of finite-dimensional Lie algebras and local spectral theory [24]. The important examples of such relations are given by the formulae

\[
\mathcal{E}_{\lambda,r}^\mu S, q (\text{ad} S) \subset \mathcal{E}_{\lambda + \mu, r + q} (\text{ad} S)
\]

and

\[
\mathcal{E}_{\lambda,r}^\mu S (\text{ad} S) \mathcal{E}_{\mu, q}^\mu (S) \subset \mathcal{E}_{\lambda + \mu, r + q} (S),
\]

where

\[
\mathcal{E}_{\lambda,r}^\mu (S) = \{x \in X : \lim sup \| (S - \lambda)^n x \|^{1/n} \leq r \}
\]

denotes an elementary spectral manifold of an operator $S \in \mathcal{B}(X)$.

The main results of the section express some commutativity properties of the set of Riesz projections of $S$ with spectral manifolds of $\text{ad} S$ which are necessary for solving VIP.

Speaking about the technical tools, note that in this work we do not use the machinery of the joint spectral radius theory which was the main instrument in [41]. But of course the results of [41] are heavily used.

In Section 4, a number of reducibility criteria for Lie algebras containing compact operators are given. For the convenience of the reader, we summarize these results (in their simplest forms) in the following theorem.
Theorem 1.1. Each of the following conditions implies reducibility of a Lie algebra \( \mathfrak{L} \) of compact operators on an infinite-dimensional Banach space \( X \).

- \( \mathfrak{L} \) has a nonzero E-solvable (more specially, Engel, Volterra, commutative) ideal.
- \( \mathfrak{L} \) has a nonzero ideal without finite rank operators (more specially, \( \mathfrak{L} \) does not contain nonzero finite rank operators).
- \( \mathfrak{L} \) is closed and has a nonzero ideal without finite rank operators (more specially, \( \mathfrak{L} \) does not contain nonzero finite rank operators).
- \( \mathfrak{L} \) has a nonzero ad-compact ideal (for example finite-dimensional ideal).
- \( \mathfrak{L} \) has two nonzero ideals with zero intersection.
- \( \mathfrak{L} \) has a nonzero ideal that commutes with a nonzero compact operator.
- \( \mathfrak{L}^{qc} \) is nonscalar.
- There is a nonzero finite rank operator \( T \) such that \( \rho(S + T) = \rho(S) \) for all \( S \in \mathfrak{L} \).
- There are a nonzero operator \( T \in \mathfrak{L} \) and a constant \( \beta > 0 \) such that \( \rho(S + T) \leq \rho(S) + \beta \) (resp., \( \rho(\text{ad}_L(S + T)) \leq \rho(\text{ad}_L S) + \beta \)) for all \( S \in \mathfrak{L} \).

Here, by \( \mathfrak{L}^{qc} \) we denote the quasi-commutant of an operator Lie algebra \( \mathfrak{L} \subset B(X) \); it consists of all elements \( T \in B(X) \) for which \( \| (\text{ad}_n S)T \|^{1/n} \to 0 \) as \( n \to \infty \) for any \( S \in \mathfrak{L} \). Also, \( \rho(S) \) means the spectral radius of an operator \( S \).

In particular, every triangularizable set of compact operators has a nontrivial \( \mathfrak{K} \)-superinvariant subspace. Note that \( \text{Nor}(\mathfrak{L}) \) is an operator Lie algebra and \( \mathfrak{L} \) is a Lie ideal of \( \text{Nor}(\mathfrak{L}) \). Properties of the triplet \( (\mathfrak{L}, \text{Nor}(\mathfrak{L}), \text{Lat} \mathfrak{L}) \) for some operator algebras \( \mathfrak{L} \) were studied in [20–22]. It was shown in [22] that some hyperinvariant subspaces of operator algebras are in fact superinvariant. Other invariant subspace results (under the assumption on the existence of finite-dimensional solvable ideals) were obtained in [13,33–35], see also [45].

In Section 5, spectral and root ideals of a normed Lie algebra with respect to a Lie homomorphism into a Banach algebra are introduced. Using these ideals, we show that every Lie algebra of compact operators has the largest Volterra ideal, the largest Engel ideal and the largest \( E \)-solvable ideal. Moreover, it is proved that each Engel Lie algebra of compact operators generates the associative algebra which is Engel. We also show that if \( J \) is the largest \( E \)-solvable ideal of a Lie algebra \( \mathfrak{L} \) of compact operators then \( [J, A] \subset \text{rad} \ A \), where \( A \) is the closed associative hull of \( \mathfrak{L} \). This is related to the main result of [42].

The proofs in this section are based on the results of Section 4 as well as on some spectral theory for elementary operators on algebras of compact operators which can be of the independent interest.

In Section 6, the structure theory of ad-compact Lie algebras is developed. We define and investigate \( E \)-radical and \( E \)-semisimple ad-compact Lie algebras. Examples of ad-compact Lie algebras demonstrating various possible phenomena in the structure of such algebras are given. The \( \mathfrak{K} \)-superinvariant subspace results of Section 4 are applied to obtain the important ideal structure theorems of ad-compact Lie algebras. We prove that any irreducible representation of an ad-compact Lie algebra acts on a finite-dimensional space, and characterize \( E \)-radical and the largest Engel ideal in terms of the representations. We also show that a complete \( E \)-semisimple ad-compact Lie algebra has the unique complete norm topology.
The concluding part of the section is devoted to the characterization of \( E \)-semisimple and \( E \)-radical complete ad-compact Lie algebras by the properties of their Killing forms.

2. Main definitions and preliminary results

2.1. Notation

Let \( \mathfrak{X} \) be a Banach space, \( B(\mathfrak{X}), K(\mathfrak{X}) \) and \( \mathcal{F}(\mathfrak{X}) \) the sets of all bounded linear operators, all compact operators and all finite rank operators on \( \mathfrak{X} \), respectively. Quasi-nilpotent compact operators and sets of such operators are called Volterra operators and Volterra sets, respectively. Recall that \( T \in B(\mathfrak{X}) \) is a Riesz operator if its image in the Calkin algebra \( B(\mathfrak{X})/K(\mathfrak{X}) \) is quasi-nilpotent. Let us say that a set of operators is scalar if it consists of scalar multiples of the identity operator. If \( A \) is a unital algebra and \( M \subset A \) then \( M + \mathbb{C} \) denotes the set \( \{ a + \lambda : a \in M, \lambda \in \mathbb{C} \} \), where a complex number \( \lambda \) is identified with the element \( \lambda 1 \) of \( A \). Let \( \text{rad} A \) denote the Jacobson radical of an algebra \( A \).

If \( T \in B(\mathfrak{X}) \) then \( \sigma(T) \) denotes the spectrum of \( T \), and if \( T \in \mathcal{F}(\mathfrak{X}) \) then \( \text{tr}(T) \) denotes the trace of \( T \). Given an element \( a \) of a normed algebra \( A \), \( \rho(a) \) is the topological spectral radius of \( a \), i.e. \( \lim \| a^n \|^{1/n} \) (so quasi-nilpotents are always the elements \( a \) such that \( \rho(a) = 0 \)). Let \( \sigma_e(T) \) denote the essential spectrum of \( T \in B(\mathfrak{X}) \), i.e. the spectrum of the image of \( T \) in the Calkin algebra; \( \rho_e(T) \) the essential spectral radius of \( T \), i.e. \( \lim \| | T_n | |^{1/n} \), where

\[
||| T ||| = \inf_{S \in K(\mathfrak{X})} \| T + S \|
\]

is the essential norm of \( T \). If \( Y \) is a normed linear space, \( \tilde{Y} \) denotes the norm-completion of \( Y \) and \( Y_{(1)} \) denotes the closed unit ball of \( Y \); for \( T \in B(Y) \), let \( T|\tilde{Y} \) denote the extension of \( T \) to \( \tilde{Y} \) by continuity. If \( Z \subset Y \) is a linear manifold, \( \tilde{Z} \) is identified with the norm-closure of \( Z \) in \( \tilde{Y} \). For an operator \( T \in B(Y) \), let \( T^* \) stand for the adjoint operator of \( T \) on the dual space \( Y^* \). An operator (more generally, an element of an algebra) is called scattered if its spectrum is (finite or) countable. We call a collection of operators (or elements of an algebra) scattered if it consists of scattered operators (resp., elements).

As a rule, we use the term operator algebra (or algebra of operators) only for associative subalgebras of \( B(\mathfrak{X}) \). Lie algebras of operators are always Lie subalgebras of the Lie algebra \( (B(\mathfrak{X}), [\cdot, \cdot]) \), where

\[
[T, S] = TS - ST
\]

for all \( S, T \in B(\mathfrak{X}) \). The same is assumed when we speak about Lie subalgebras of a normed algebra \( A \). If \( M \subset A \), \( \mathcal{A}(M) \) denotes the associative subalgebra generated by
As usual, let \( M \) be a subset of a normed space \( Y \). \( M \) denotes the norm-closure of \( M \) in \( Y \). If \( M \subset B(\mathfrak{X}) \), \( M^{\text{wot}} \) denotes the closure of \( M \) in the weak operator topology.

For any Lie algebra \( \mathfrak{L} \) we denote by \( \text{ad}_\mathfrak{L} \) the adjoint representation of \( \mathfrak{L} \) on \( \mathfrak{L} \) defined by the formula

\[
(\text{ad}_\mathfrak{L} a)b = [a, b].
\]

If \( M \subset \mathfrak{L} \) then the normalizer \( \text{Nor}_\mathfrak{L}(M) \) of \( M \) in \( \mathfrak{L} \) is defined [10, 1.1.9] as \( \{a \in \mathfrak{L} : [a, M] \subset M \} \). If \( M \) is a Lie subalgebra of \( \mathfrak{L} \) then \( \text{Nor}_\mathfrak{L}(M) \) is a Lie algebra and \( M \) is a Lie ideal of \( \text{Nor}_\mathfrak{L}(M) \). We write \( \text{Nor}(M) \) for \( \text{Nor}_{B(\mathfrak{X})}(M) \) if \( M \subset B(\mathfrak{X}) \).

Recall that a normed (resp., Banach) Lie algebra is a Lie algebra equipped with a norm (resp., complete norm) such that the Lie multiplication is jointly continuous. Any normed algebra \( A \) is simultaneously a normed Lie algebra with respect to the Lie product \( [a, b] = ab - ba \) for all \( a, b \in A \). As it was already defined in the introduction, a normed Lie algebra \( \mathfrak{L} \) is called Engel [41] if \( \text{ad}_\mathfrak{L} a \) is quasi-nilpotent for every \( a \in \mathfrak{L} \) (Engel Lie algebras are also called quasi-nilpotent [53] or ad-quasi-nilpotent [27]).

A subspace means a closed linear manifold. If at least one of subsets \( M, N \) of an algebra (resp., a Lie algebra) is a linear manifold then \( MN \) (resp., \( [M, N] \)) denotes the linear manifold generated by all \( ab \) (resp., \( [a, b] \)) with \( a \in M \) and \( b \in N \). The definition of \( MN \) is similar for the case when \( M \subset B(\mathfrak{X}) \) and \( N \subset \mathfrak{X} \) if at least one of \( M, N \) is a linear manifold. Otherwise (i.e. if both of them are not linear manifolds) \( MN \) denotes the set \( \{ab : a \in M, b \in N\} \), and \( N + M = \{a + b : a \in N, b \in M\} \). If \( M \) is a bounded subset of a normed algebra, \( \|M\| \) denotes \( \sup \{\|x\| : x \in M\} \).

A linear manifold \( Y \subset \mathfrak{X} \) is invariant for a set \( M \) of operators if \( TY \subset Y \) for all \( T \in M \); \( T|Y \) denotes the restriction of \( T \) to \( Y \) (if \( Y \) is not invariant then \( T|Y \in B(Y, \mathfrak{X}) \)). As usual, \( \text{Lat} M \) denotes the lattice of all invariant subspaces for \( M \). A set \( M \subset B(\mathfrak{X}) \) is called reducible if \( \text{Lat} M \) contains nontrivial (different from \( 0 \) and \( \mathfrak{X} \)) elements. Given a collection \( G \) of subspaces of \( \mathfrak{X} \), let \( \text{Alg} G \) denote the set of all \( S \in B(\mathfrak{X}) \) for which each subspace in \( G \) is invariant. If \( Y \) is an invariant subspace for \( M \cup M' \) (\( M' \) is the commutant of \( M \)), \( Y \) is called a hyperinvariant subspace for \( M \). We will often use the following result.

**Theorem 2.1** ([39, Theorem 2]). Any operator algebra containing a nonzero Volterra ideal has a nontrivial hyperinvariant subspace.

We consider also subspaces that are invariant for \( \text{Nor}(M) \cup M \). In the case when \( M \) is a Lie subalgebra of \( B(\mathfrak{X}) \), \( \text{Nor}(M) \) contains \( M \) and the commutant \( M' \) of \( M \). Thus invariant subspaces for \( \text{Nor}(M) \cup M \) are “more invariant” for \( M \) than hyperinvariant ones; we call them superinvariant for \( M \). If \( M \subset \text{Nor}(M) \) then we have that \( (\text{Nor}(M))' \subset M' \subset \text{Nor}(M) \), whence superinvariant subspaces for \( M \) are automatically hyperinvariant for \( \text{Nor}(M) \).

More generally, let \( \mathcal{R} \) be a Lie subalgebra of \( B(\mathfrak{X}) \). A subspace \( Y \) of \( \mathfrak{X} \) is called superinvariant with respect to \( \mathcal{R} \) (or, simply, \( \mathcal{R}\text{-superinvariant} \)) for \( M \subset B(\mathfrak{X}) \) if \( Y \) is simultaneously invariant for both \( \text{Nor}_{\mathcal{R}}(M) \) := \( \mathcal{R} \cap \text{Nor}(M) \) and \( M \). In these terms
an $M'$-superinvariant subspace for $M$ is in fact hyperinvariant. Clearly $\mathcal{K}(\mathfrak{X})$ and $\mathbb{C}$ are Lie ideals of $\mathcal{B}(\mathfrak{X})$, so is $\mathcal{K}(\mathfrak{X}) + \mathbb{C}$ which is denoted by $\mathcal{K}^1(\mathfrak{X})$ for brevity. So, given a Lie subalgebra $\mathfrak{L} \subset \mathcal{B}(\mathfrak{X})$, the set $\mathfrak{L} \cap \mathcal{K}^1(\mathfrak{X})$ is a Lie ideal of $\mathfrak{L}$. We write ‘$\mathcal{K}^1(\mathfrak{X})$-superinvariant’ for ‘$\mathcal{K}^1(\mathfrak{X})$-superinvariant’.

A simple example of a superinvariant subspace of a set $M$ of operators is its kernel $\ker M = \{x \in \mathfrak{X} : Tx = 0 \text{ for all } T \in M\}$. This is a consequence of the following simple lemma which is a slight modification of Kaplansky [19, Theorem 19].

**Lemma 2.2.** Let $\mathfrak{L} \subset \mathcal{B}(\mathfrak{X})$ be a Lie algebra and let $J$ be a linear manifold of operators such that $[\mathfrak{L}, J] \subset J$. Then $\ker J$ and $J\mathfrak{X}$ are invariant for $\mathfrak{L}$; they are hyperinvariant for $\mathfrak{L}$ if $J \subset \mathfrak{L}$.

**Proof.** Since $[T, S] \in J$ for all $T \in \mathfrak{L}$, $S \in J$, we have

$$STx = T Sv - [T, S]v = 0$$

for every $v \in \ker J$. Hence $\ker J$ is invariant for $\mathfrak{L}$. Similarly,

$$TSy = STy + [T, S]y \in J\mathfrak{X}$$

for any $y \in \mathfrak{X}$, whence $J\mathfrak{X}$ is invariant for $\mathfrak{L}$. In particular, $\ker J$ and $J\mathfrak{X}$ are invariant for $J'$. If $J \subset \mathfrak{L}$ then $\mathfrak{L}' \subset J'$, whence $\ker J$ and $J\mathfrak{X}$ are also invariant for $\mathfrak{L}'$. □

### 2.2. The case of finite-dimensional $\mathfrak{X}$

In the following theorem, we discuss conditions under which a set of operators on a finite-dimensional space has a superinvariant subspace.

**Theorem 2.3.** Let $\dim \mathfrak{X} < \infty$.

(i) If a subalgebra $A$ of $\mathcal{B}(\mathfrak{X})$ is not isomorphic to a full matrix algebra, then it has a nontrivial superinvariant subspace.

(ii) A set $M \subset \mathcal{B}(\mathfrak{X})$ has a nontrivial superinvariant subspace iff it has a nontrivial hyperinvariant subspace.

**Proof.** (i) Suppose firstly that $\text{rad} A$ is nonzero. Since $\text{rad} A$ is a finite-dimensional algebra of nilpotent matrices, $\ker \text{rad} A \neq 0$. It follows from Lemma 2.2 that $\text{rad} A$ has a nontrivial superinvariant subspace. On the other hand,

$$\text{Nor}(A) \subset \text{Nor}(\text{rad} A)$$

because the Jacobson radical is a characteristic ideal (i.e. an ideal preserved by all derivations of an algebra). Hence, $A$ has a nontrivial superinvariant subspace.

So one can suppose that $A$ is semisimple. If $A$ is commutative, there exists a nonzero eigenspace $Y$ of $A$ corresponding to a weight $w$, that is $Y = \{x \in \mathfrak{X} : Tx = w(T)x$
for all $T \in A$. If $S \in \text{Nor}(A)$ then, by Kleinecke–Shirokov theorem [38,23], $[S, T]$ is nilpotent whence $w([S, T]) = 0$. It follows that

$$TSx = STx = w(T)Sx$$

for all $x \in Y, T \in A$. This means that $SY \subset Y$, whence $Y$ is superinvariant ($Y$ is not trivial because $A$ is nonscalar).

Suppose now that $A$ is not commutative, but has the nonscalar center $C$. It is easy to see from Lie identities that

$$\text{Nor}(A) \subset \text{Nor}(C).$$

Since $\text{Nor}(C)$ has an invariant subspace (by the above argument), the same is true for $\text{Nor}(A)$.

If the center of $A$ is scalar then, being semisimple, $A$ is isomorphic to a full matrix algebra by the Wedderburn theorem. This is impossible by the assumption.

(ii) Let $M$ have a hyperinvariant subspace. It will be sufficient to prove that the algebra $A = \mathcal{A}(M)$ has a superinvariant subspace. In virtue of (i) it suffices to consider the case that $A$ is isomorphic to a full matrix algebra. In this case, it is easy to see that any derivation of $A$ is inner and therefore

$$\text{Nor} A = A + A',$$

whence any hyperinvariant subspace of $A$ is superinvariant. □

**Corollary 2.4.** Let $\dim \mathfrak{x} < \infty$. Then $M \subset \mathcal{B}(\mathfrak{x})$ has a nontrivial superinvariant subspace iff $\mathcal{A}(M \cup M') \neq \mathcal{B}(\mathfrak{x})$.

**Proof.** Follows from Theorem 2.3 and the Burnside’s theorem [17, p. 276]. □

Formally, the last two results solve the question on the existence of superinvariant and hyperinvariant subspaces, but sometimes it is not more easy to check if $\mathcal{A}(M)$ is isomorphic (or even equal) to $\mathcal{B}(\mathfrak{x})$ than to consider invariant subspaces of $M$. On the other hand, the part (ii) of Theorem 2.3 gives a universal result for finite-dimensional spaces. It is not clear if it holds for sets of compact operators on an infinite-dimensional Banach space.

The following example shows that even for algebras of compact operators the existence of $\mathcal{K}$-superinvariant subspaces does not imply the existence of hyperinvariant ones.

**Example 2.5.** Let $\mathfrak{x} = H \oplus H$ be the direct sum of infinite-dimensional Hilbert spaces, and let $A$ be $\{S \oplus S \in \mathcal{B}(\mathfrak{x}) : S \in \mathcal{K}(H)\}$. Then $A$ has nontrivial $\mathcal{K}$-superinvariant and only trivial hyperinvariant subspaces.
This means that $M$ of the chain. Given a Lie algebra $X$ chain (that is a nonextendable linearly ordered set of closed subspaces of $g$), and $A$ has only trivial hyperinvariant subspaces. On the other hand, it is easy to check that

$$\text{Nor}_{K(\mathcal{X})}(A) = A,$$

whence $A$ has a nontrivial $K$-superinvariant subspace (for example, $\{x \oplus x : x \in H\}$). \hfill \Box

2.3. Triangularization, hypertriangularization, and supertriangularization

A subset $M \subset B(\mathcal{X})$ is called triangularizable if $\text{Lat} M$ contains a maximal subspace chain (that is a nonextendable linearly ordered set of closed subspaces of $\mathcal{X}$). If $\Gamma$ is a complete chain of closed subspaces of $\mathcal{X}$ (see [29]) and $Y \in \Gamma$ then $Y_\leftarrow$ is defined as the closed span of all $Z \in \Gamma$ such that $Z \subset Y$ and $Z \neq Y$. If $Y \neq Y_\leftarrow$ then the pair $(Y_\leftarrow, Y)$ forms a gap of $\Gamma$.

Let $\Pi \subset \text{Lat} M$. We say that $V$ is a gap-quotient of $\Pi$ if $V = Y/Z$, where $Y, Z \in \Pi$, $Z \subset Y$ and there exist no subspaces in $\Pi$ which are intermediate between $Y$ and $Z$. If $\Gamma$ is a maximal chain in $\text{Lat} M$ then all gap-quotients of $\Gamma$ are clearly gap-quotients of $\text{Lat} M$. If $Y, Z \in \text{Lat} M$, $Z \subset Y$ and $V = Y/Z$ then $T|V$ for every $T \in M$ denotes the operator induced by $T$ on $V$, and $M|V = \{T|V : T \in M\}$.

We need the following simple lemma (it should be noted that the equivalence (i) \iff (ii) was proved in [46, Lemma 4.2]).

**Lemma 2.6.** Let $M \subset K(\mathcal{X})$, $N \subset B(\mathcal{X})$ and $M \subset N$. The following conditions are equivalent.

(i) $M|V = 0$ for any gap-quotient $V$ of $\text{Lat} N$.
(ii) $M \subset \text{rad} \overline{\mathcal{A}(N)}$.
(iii) $M \subset \text{rad} \overline{\mathcal{A}(N)}^\text{wot}$.

**Proof.** (ii), (iii) \implies (i) follows from Lomonosov’s results ([26], see also [29]).

(i) \implies (ii), (iii) Note first that $\text{Lat} N = \text{Lat} \overline{\mathcal{A}(N)}^\text{wot}$, so $M\overline{\mathcal{A}(N)}^\text{wot}|V = 0$ for any gap-quotient $V$ of $\text{Lat} \overline{\mathcal{A}(N)}^\text{wot}$. Since every operator $T$ in $M\overline{\mathcal{A}(N)}^\text{wot}$ is compact and, roughly speaking, equals 0 on gap-quotients of a maximal chain in $\text{Lat} \overline{\mathcal{A}(N)}^\text{wot}$, it follows from the Ringrose’s theorem (see [29, Theorem 5.12]) that $T$ is quasi-nilpotent. This means that $M \subset \text{rad} \overline{\mathcal{A}(N)}^\text{wot}$ and, of course, $M \subset \text{rad} \overline{\mathcal{A}(N)}$. \hfill \Box

A subset $M \subset B(\mathcal{X})$ is called hypertriangularizable if there exists a complete chain of hyperinvariant subspaces for $M$ such that $M|V$ is scalar for every gap-quotient $V$ of the chain. Given a Lie algebra $\mathcal{R} \subset B(\mathcal{X})$, if we replace in this definition hyperinvariant subspaces by $\mathcal{R}$-superinvariant subspaces for $M$ then we obtain the definition of $\mathcal{R}$-supertriangularization of $M$. If $M$ is a Lie algebra of operators then hypertriangularization of $M$ coincides with $M'$-supertriangularization of $M$. On the other hand, if $M \subset B(\mathcal{X})$ is hypertriangularizable then $M$ also is triangularizable.
The following theorem easily follows from [54, Theorem 5 and Corollary 6].

**Theorem 2.7.** Any triangularizable set in $K^1(\mathcal{X})$ is hypertriangularizable.

We will show that a triangularizable set in $K^1(\mathcal{X})$ is $K^1$-supertriangularizable (see Corollary 4.19). So, for collections of compact operators, the three notions of ‘triangularization’ coincide.

We will use the following result of [41, Theorem 11.4 and Corollary 11.5].

**Theorem 2.8.** Suppose that a nonscalar Lie algebra $\mathfrak{L} \subset B(\mathcal{X})$ is the image of an Engel Banach Lie algebra under a bounded representation. If $\mathfrak{L}$ contains a nonzero compact operator then $\mathfrak{L}$ has a nontrivial hyperinvariant subspace. As a consequence, any Engel Lie algebra of compact operators is triangularizable.

It is clear that any Volterra Lie algebra is Engel. It follows easily from Theorem 2.8 that the associative hull of a Volterra Lie algebra is Volterra.

3. Elementary spectral manifolds

We gather necessary lemmas on spectral manifolds. Some of them admit wide generalizations, but we choose the formulations convenient for applications to scattered operators. One of the main aims here is to show that $QW = WQ$ for any closed operator subspace $W \subset B(\mathcal{X})$ and any scattered operator $S$ with $[S, W] \subset W$, where $Q$ is the set of all Riesz projections of $S$. The other aim (inspired by results of Wojtyński) is to find conditions on $S$ which imply $W \cap \mathcal{F}(\mathcal{X}) \neq 0$.

3.1. Elementary spectral manifolds and glocal spectral spaces

Let $\mathcal{X}$ be a Banach space, $S \in B(\mathcal{X})$, $\lambda \in \mathbb{C}$ and $r \geq 0$. Put

$$E_{\lambda, r}(S) := \{x \in \mathcal{X} : \lim \sup \|(S - \lambda)^n x\|^{1/n} \leq r\}$$

and

$$E_{\lambda}(S) := E_{\lambda, 0}(S).$$

It is clear that $E_{\lambda, r}(S)$ is a hyperinvariant linear manifold for $S$. We call $E_{\lambda, r}(S)$ an elementary spectral manifold of $S$. The definition of elementary spectral manifolds is appropriate for operators on incomplete spaces as well.

Let $\Omega(\lambda, r) \subset \mathbb{C}$ be the circle of radius $r > 0$ centered at $\lambda \in \mathbb{C}$. Put $\hat{\Omega}(\lambda, r) = \{\mu \in \mathbb{C} : |\mu - \lambda| < r\}$, and let $\overline{\Omega}(\lambda, r)$ be the closure of $\hat{\Omega}(\lambda, r)$. Let $\text{res}(S)$ denote the
resolvent set of an operator $S$, i.e.

$$\text{res}(S) = \mathbb{C} \setminus \sigma(S).$$

If $\Omega = \Omega(\lambda, r) \subset \text{res}(S)$ then one can define the Riesz projection $P_{\lambda,r}(S)$ by

$$P_{\lambda,r}(S) = \frac{1}{2\pi i} \int_{\Omega} (\zeta - S)^{-1} d\zeta.$$  \hspace{1cm} (3.1)

If $\Omega(\lambda, q) \subset \text{res}(S)$ for all $q$, $0 < q \leq r$, then

$$P_{\lambda,q}(S) = P_{\lambda,r}(S)$$

and we write $P_{\lambda}(S)$ instead of $P_{\lambda,r}(S)$.

More generally, if $G$ is an open subset of $\mathbb{C}$ with the boundary $\partial G \subset \text{res}(S)$ then $P_{G}(S)$ denotes the Riesz projection of $S$ which corresponds to the part $\sigma(S) \cap G$ of spectrum of $S$. Recall that $P_{G}(S)$ is given by formula (3.1), where $\Omega$ is some admissible contour that is the union of a finite number of closed continuous curves (it coincides with $\partial G$ in many simple cases, for instance if $G = \Omega(\lambda, r)$).

Now let $S$ be scattered, and let $r > 0$ be arbitrary. It is clear that $\sigma(S) \cap G$ can be covered by a finite set of $\hat{\Omega}(\lambda_i, q_i)$ with $q_i < r$ such that all $\Omega(\lambda_i, q_i) \subset \text{res}(S)$, $\cup \Omega(\lambda_i, q_i) \subset G$ and $\hat{\Omega}(\lambda_i, q_i) \cap \hat{\Omega}(\lambda_j, q_j) = \emptyset$ for all $i \neq j$. It follows from the theory of Riesz projections [32] that

$$P_{G}(S) = P_{\lambda_1,q_1}(S) + \ldots + P_{\lambda_n,q_n}(S).$$  \hspace{1cm} (3.2)

So, dealing with scattered operators, we may restrict our attention to projections given by formula (3.1).

It is known [32, Section 149] that if $\Omega(\lambda, r) \subset \text{res}(S)$ then

$$P_{\lambda,r}(S)\mathcal{X} = \{ x \in \mathcal{X} : \lim \sup \left\| (S - \lambda)^n x \right\|^{1/n} < r \}.$$  

Hence it easy to see that

$$P_{\lambda,r}(S)\mathcal{X} = \mathcal{E}_{\lambda,r}(S)$$  \hspace{1cm} (3.3)

and so $\mathcal{E}_{\lambda,r}(S)$ is closed. In particular, if $\lambda$ is an isolated point of $\sigma(S)$ (or $\lambda \in \text{res}(S)$) then $P_{\lambda}(S)\mathcal{X} = \mathcal{E}_{\lambda}(S)$ and $\mathcal{E}_{\lambda}(S)$ is closed.

The following useful lemma seems to belong to the folklore. The theory of decomposable operators is presented in [8,24]; note that every scattered operator is decomposable.
Lemma 3.1. Let $S \in \mathcal{B}(\mathfrak{X})$ be decomposable, and let $F \subset \mathbb{C}$ be closed. Then, given a positive function $r$ on $\mathbb{C}$, one can find $\hat{\lambda}_1, \ldots, \hat{\lambda}_n \in \sigma(S)$ and positive numbers $q_1, \ldots, q_n$ such that \( \mathfrak{X} = \sum_{i=1}^n \mathcal{E}_{\hat{\lambda}_i, q_i}(S) \), $q_i < r(\hat{\lambda}_i)$, and $q_i < \text{dist}(\hat{\lambda}_i, F)$ if $\hat{\lambda}_i \notin F$. If $S$ is scattered, one can require in addition that $\mathcal{E}(\hat{\lambda}_i, q_i) \subset \text{res}(S)$ for every $\hat{\lambda}_i$ and \( 1 = \sum_{i=1}^n P_{\hat{\lambda}_i, q_i}(S) \).

**Proof.** For every $\hat{\lambda}$, take a positive $q(\hat{\lambda}) < \min\{ \text{dist}(\hat{\lambda}, F), r(\hat{\lambda}) \}$ if $\hat{\lambda} \notin F$; otherwise take a positive $q(\hat{\lambda}) < r(\hat{\lambda})$. Since $\{ \mathcal{E}(\hat{\lambda}, q(\hat{\lambda})) : \hat{\lambda} \in \sigma(S) \}$ is an open covering of $\sigma(S)$, there exist $\hat{\lambda}_1, \ldots, \hat{\lambda}_n \in \sigma(S)$ such that $\sigma(S) \subset \mathcal{E}(\hat{\lambda}_1, q(\hat{\lambda}_1)) \cup \ldots \cup \mathcal{E}(\hat{\lambda}_n, q(\hat{\lambda}_n))$.

Set $q_i = q(\hat{\lambda}_i)$ for any $i \leq n$. Since $S$ is decomposable, there exist spectral maximal subspaces $Y_i$ such that $\mathfrak{X} = \sum_{i=1}^n Y_i$ with $\rho((S - \hat{\lambda}_i)|Y_i) \leq q_i$ for every $i$. Therefore $\mathfrak{X} = \sum_{i=1}^n \mathcal{E}_{\hat{\lambda}_i, q_i}(S)$. The refinement in the case of scattered $S$ is obvious. □

There is a close link between manifolds $\mathcal{E}_{\lambda, r}(S)$ and glocal spectral spaces $\mathcal{X}_S(F)$ of an operator $S \in \mathcal{B}(\mathfrak{X})$. Recall from [24] that, given a closed set $F \subset \mathbb{C}$, $\mathcal{X}_S(F)$ is defined as the set of all $x \in \mathfrak{X}$ for which there exists an analytic function $f : \mathbb{C} \setminus F \rightarrow \mathfrak{X}$ with the property

$$ (S - \hat{\lambda})f(\hat{\lambda}) = x $$

for all $\hat{\lambda} \in \mathbb{C} \setminus F$. It follows from [24, Proposition 3.3.20] that

$$ \mathcal{E}_{\lambda, r}(S) = \mathcal{X}_S(\mathfrak{Q}(\lambda, r)). $$

(3.4)

If $S$ is decomposable then all $\mathcal{E}_{\lambda, r}(S)$ are closed (see [8, Theorem 1.5], [24, Proposition 3.3.2]).

Let $F \subset \mathbb{C}$, and let $\mathcal{O}(F)$ be the set of all disks $\mathfrak{Q}(\lambda, r)$ satisfying $F \subset \mathfrak{Q}(\lambda, r)$. If $F$ is a convex compact set in $\mathbb{C}$, it is well known that

$$ F = \cap \{ \mathfrak{Q}(\lambda, r) \in \mathcal{O}(F) \}. $$

Since

$$ \mathcal{X}_S(\cap_{\lambda \in \Lambda} F_\lambda) = \cap_{\lambda \in \Lambda} \mathcal{X}_S(F_\lambda) $$

(3.5)

for every family $(F_\lambda)_{\lambda \in \Lambda}$ of convex closed sets in $\mathbb{C}$ [24, Proposition 3.3.1(g)], we have

$$ \mathcal{X}_S(F) = \cap \{ \mathcal{E}_{\lambda, r}(S) : \mathfrak{Q}(\lambda, r) \in \mathcal{O}(F) \}. $$

(3.6)
Lemma 3.2. Let $S \in B(\mathcal{X})$.

(i) $\mathcal{E}_{\lambda, r}(S|Y) = \mathcal{E}_{\lambda, r}(S) \cap Y$ for any linear manifold $Y$ invariant for $S$.
(ii) $\mathcal{E}_{\mu, q}(S) \subset \mathcal{E}_{\lambda, r}(S)$ whenever $|\lambda - \mu| + q \leq r$.

Proof. (i) is clear, and (ii) follows from (3.5). □

Some statements in the following proposition are well known.

Proposition 3.3. Let $S \in B(\mathcal{X})$.

(i) If $S$ is scattered then $\mathcal{E}_{\lambda, r}(S)$ is closed for every $r \geq 0$, $\lambda \in \mathbb{C}$.
(ii) If $\mathcal{E}_{\lambda, r}(S)$ is closed then $\rho((S - \lambda)|\mathcal{E}_{\lambda, r}(S)) \leq r$.
(iii) If $\mathcal{E}_{\lambda, q}(S)$ is closed for any $q \leq r$, and if $G$ is a dense subset of $\mathcal{E}_{\lambda, r}(S)$ then

$$
\sup_{x \in G} \limsup_{n \to \infty} \frac{\| (S - \lambda)^n x \|}{n} = \rho((S - \lambda)|\mathcal{E}_{\lambda, r}(S)).
$$

Proof. (i) is known for decomposable operators [24], but there is the following easy argument for scattered operators. If $\sigma(S)$ is countable then for arbitrary $\lambda \in \mathbb{C}$, $r > 0$ and $\varepsilon > 0$, there exists $q > 0$ such that $\Omega(\lambda, q) \subset \text{res}(S)$ and $|q - r| < \varepsilon$. In particular each $\mathcal{E}_{\lambda, r}(S)$ is closed because $\mathcal{E}_{\lambda, r}(S)$ is the intersection of all closed $\mathcal{E}_{\lambda, q}(S)$ with $\Omega(\lambda, q) \subset \text{res}(S)$ and $q > r$.

(ii) is evident (see for instance, [24, Proposition 3.3.20]).

(iii) Suppose, to the contrary, that

$$
\sup_{x \in G} \limsup_{n \to \infty} \frac{\| T^n x \|}{n} < \rho(T|\mathcal{E}_{0, r}(T)),
$$

where $T = S - \lambda$. Then $G \subset \mathcal{E}_{0, q}(T) \subset \mathcal{E}_{0, r}(T)$ for some $q < \rho(T|\mathcal{E}_{0, r}(T))$. Since $\mathcal{E}_{0, q}(T)$ is closed, $\mathcal{E}_{0, q}(T) = \mathcal{E}_{0, r}(T)$ and

$$
\rho(T|\mathcal{E}_{0, r}(T)) = \rho(T|\mathcal{E}_{0, q}(T)) \leq q
$$

by (ii), a contradiction. □

Proposition 3.4. Let a sequence $(S_n)$ of operators in $B(\mathcal{X})$ converge to $S \in B(\mathcal{X})$.

(i) If an open subset $G \subset \mathbb{C}$ is enclosed by an admissible contour $\Omega \subset \text{res}(S)$ then there exists $m \in \mathbb{N}$ such that $\Omega \subset \text{res}(S_n)$ for every $n > m$ and the sequence $(P_G(S_n))_{n > m}$ converges to $P_G(S)$.
(ii) If $S$ is scattered then $\cap_{n \geq k} \mathcal{E}_{\lambda, r}(S_n) \subset \mathcal{E}_{\lambda, r}(S)$ for every $k \in \mathbb{N}$.

Proof. (i) Since the set of all invertible operators is open, for arbitrary $\lambda \in \Omega$ there exist a number $k = k(\lambda) \in \mathbb{N}$ and $\varepsilon = \varepsilon(\lambda) > 0$ such that $\Omega(\lambda, \varepsilon) \subset \text{res}(S_n)$ for all $n > k$. Since $\Omega$ is a compact subset of $\mathbb{C}$, there exists a number $m$ such that $\Omega \subset \text{res}(S_n)$ for
every $n > m$. Since the sequence $((\zeta - S_n)^{-1})_{n>m}$ converges to $(\zeta - S)^{-1} \text{ uniformly on } \Omega$, it follows from (3.1) that the sequence $(P_G(S_n))_{n>m}$ converges to $P_G(S)$.

(ii) Let $x \in \cap_{n>k} E_{\lambda,r}(S_n)$ and take $q > r$ such that $\Omega(\lambda, q) \subset \text{res}(S)$. By (i), there exists $m \in \mathbb{N}$ such that the sequence $(P_{\lambda,q}(S_n))_{n>m}$ converges to $P_{\lambda,q}(S)$. It follows from (3.3) that $x = P_{\lambda,q}(S_n)x$ for all $n > \max\{m, k\}$, whence

$$x = \lim P_{\lambda,q}(S_n)x = P_{\lambda,q}(S)x \in E_{\lambda,q}(S).$$

Since $S$ is scattered, $E_{\lambda,r}(S)$ is the intersection of all $E_{\lambda,q}(S)$ with $\Omega(\lambda, q) \subset \text{res}(S)$ and $q > r$. Therefore $x \in E_{\lambda,r}(S)$. □

Our final remark here concerns two convex compact sets $F_1$ and $F_2$ in $\mathbb{C}$. It is easy to check that $F_1 + F_2$ is also a convex compact set, and

$$F_1 + F_2 = \cap \left[ \Omega(\lambda, r) + \Omega(\mu, q) : \Omega(\lambda, r) \in \mathcal{O}(F_1), \Omega(\mu, q) \in \mathcal{O}(F_2) \right]. \quad (3.7)$$

We will use this formula in the following subsections.

### 3.2. The algebraic formulae for spectral manifolds

Recall that a linear operator $D$ on an algebra (resp., a Lie algebra) $A$ is a derivation (resp., a Lie derivation) if

$$D(ab) = D(a)b + aD(b)$$

(resp., $D[a, b] = [Da, b] + [a, Db]$) for all $a, b \in A$.

Finite-dimensional analogs of the following lemma under $r = q = 0$ can be found in the textbooks on Lie algebras (e.g. [19, Theorem 14]).

**Lemma 3.5.** If $D$ is a derivation (resp., a Lie derivation) of a normed (resp., a normed Lie) algebra $A$ then

$$E_{\lambda,r}(D)E_{\mu,q}(D) \subset E_{\lambda+\mu,r+q}(D)$$

(resp., $[E_{\lambda,r}(D), E_{\mu,q}(D)] \subset E_{\lambda+\mu,r+q}(D)$).

**Proof.** Set $\Delta = D - (\lambda + \mu)$, $\Delta_1 = D - \lambda$, and $\Delta_2 = D - \mu$. Then for any $S_1, S_2 \in B(\mathbb{C})$ one has

$$\Delta(S_1S_2) = \Delta_1(S_1)S_2 + S_1\Delta_2(S_2).$$
The subsequent application of this equality gives

\[ \Delta^n(S_1S_2) = \sum_{k=0}^{n} \binom{n}{k} \Delta^k_1(S_1) \Delta^{n-k}_2(S_2). \tag{3.8} \]

If \( S_1 \in \mathcal{E}_{\lambda, r}(D) \) and \( S_2 \in \mathcal{E}_{\mu, q}(D) \) then for any \( \varepsilon > 0 \) there exists \( \beta > 0 \) with \( \|\Delta^1_1(S_1)\| \leq \beta(r + \varepsilon)^n \) and \( \|\Delta^2_2(S_2)\| \leq \beta(q + \varepsilon)^n \) for all integers \( n > 0 \). Using (3.8), we obtain

\[ \|\Delta^n(S_1S_2)\| \leq \sum_{k=0}^{n} \binom{n}{k} \|\Delta^k_1(S_1)\| \|\Delta^{n-k}_2(S_2)\| \]

\[ \leq \sum_{k=0}^{n} \binom{n}{k} \beta(r + \varepsilon)^k \beta(q + \varepsilon)^{n-k} \]

\[ = \beta^2 (r + q + 2\varepsilon)^n. \]

This shows that \( S_1S_2 \in \mathcal{E}_{\lambda+\mu, r+q}(D) \).

The proof for Lie algebras and derivations is similar. \( \square \)

To our regret we could not avoid multiple variations of the above argument with binomial estimation.

It is easy to receive a glocal version of Lemma 3.5 in the following way: if \( F_1 \) and \( F_2 \) are convex compact sets in \( \mathbb{C} \) and \( A \) is complete then

\[ \mathcal{X}_D(F_1)\mathcal{X}_D(F_2) \subset \mathcal{X}_D(F_1 + F_2). \]

Indeed, if \( S \in \mathcal{X}_D(F_1) \) and \( T \in \mathcal{X}_D(F_2) \) then, by Lemma 3.5 and (3.4), \( ST \in \mathcal{X}_D(G_1 + G_2) \) for all \( G_1 \in \mathcal{O}(F_1) \) and \( G_2 \in \mathcal{O}(F_2) \), whence the required result follows from (3.5) and (3.7).

**Corollary 3.6.** Let \( A \) be a normed algebra (resp., a normed Lie algebra) and \( D \) a bounded derivation (resp., a bounded Lie derivation) of \( A \). If \( |\lambda| > r \) then \( \mathcal{E}_{\lambda, r}(D) \) generates a nilpotent subalgebra (resp., a nilpotent Lie subalgebra) of \( A \).

**Proof.** One can assume that \( A \) is complete. If \( S_i \in \mathcal{E}_{\lambda, r}(D) \) for \( i = 1, 2, \ldots \) then \( S_1 \cdots S_n \in \mathcal{E}_{n\lambda, nr}(D) \) by Lemma 3.5. If \( n(|\lambda| - r) > \rho(D) \) then \( \Omega(n\lambda, q) \subset \text{res}(D) \) for all \( q, 0 \leq q \leq nr \), and therefore

\[ \mathcal{E}_{n\lambda, nr}(D) \subset P_{n\lambda}(D)A = 0. \]

Hence \( S_1 \cdots S_n = 0. \)
The proof for Lie algebras and derivations is similar. □

Given a family \((E_x)\) of linear manifolds in a normed space, \(\sum_x E_x\) denotes the set of all finite sums \(\sum x_{x_i} \) with \(x_{x_i} \in E_{x_i}\).

**Corollary 3.7.** Let \(D\) be a scattered bounded derivation of a Banach algebra \(A\) and \(W \in \text{Lat} D\). Then

(i) \(\rho(D|W^n) \leq n \rho(D|W)\) for every \(n \in \mathbb{N}\).

(ii) If \(\rho(D|W) = 0\) then \(\rho(D|A(W)) = 0\).

**Proof.** (i) Let \(r = \rho(D|W)\) and \(V = \mathcal{E}_{0,r}(D)\). Then \(W^n \in \text{Lat} D, W \subset \mathcal{E}_{0,r}(D)\), and \(W^n \subset V\) by Lemma 3.5. Since \(\rho(D|V) \leq nr\) by Corollary 3.3, \(\rho(D|W^n) \leq nr\).

(ii) Let \(\rho(D|W) = 0\). Since \(A(W) = \sum_{n=1}^\infty W^n\),

\[ A(W) \subset \mathcal{E}_0(D) \]

by (i). Since \(D\) is quasi-nilpotent on \(\mathcal{E}_0(D)\) (because \(\mathcal{E}_0(D)\) is closed), \(D\) is quasi-nilpotent on \(A(W)\). □

The following construction of a Lie ideal was used in a more special case by Vaksman and Gurarij [46].

**Proposition 3.8.** Let \(A\) be a Banach algebra (resp., a Banach Lie algebra) and \(D\) a bounded derivation (resp., a bounded Lie derivation) of \(A\). Suppose that \(\Lambda := \sigma(D) \setminus \{0\}\) consists of isolated points. Then

\[ J := \sum_{\lambda \in \Lambda} \mathcal{E}_\lambda(D) + \sum_{\lambda \in \Lambda} \mathcal{E}_\lambda(D)\mathcal{E}_{-\lambda}(D) \]  \hspace{1cm} (3.9)

is a two-sided ideal of \(A\) (resp.,

\[ I := \sum_{\lambda \in \Lambda} \mathcal{E}_\lambda(D) + \sum_{\lambda \in \Lambda} [\mathcal{E}_\lambda(D), \mathcal{E}_{-\lambda}(D)] \]

is a Lie ideal of \(A\).

**Proof.** Let \(a \in J\). It is sufficient to prove that \(aA + Aa \subset J\) if \(a \in \mathcal{E}_\lambda(D)\) or \(a \in \mathcal{E}_{\lambda_1}(D)\mathcal{E}_{-\lambda_2}(D)\) for every \(\lambda \in \Lambda\).

Suppose first that \(a \in \mathcal{E}_\lambda(D)\). Let \(0 < r < |\lambda|\) and \(\Omega(0, r) \subset \text{res}(D)\), and let \(\Lambda(r)\) be the set of all \(\mu \in \sigma(D)\) with \(|\mu| > r\). Then \(\Lambda(r)\) is finite and

\[ A = \mathcal{E}_{0,r}(D) + \sum_{\mu \in \Lambda(r)} \mathcal{E}_\mu(D) . \]
It follows from Lemma 3.5 that
\[
\sum_{\mu \in \Lambda(r)} a\mathcal{E}_\mu(D) + \sum_{\mu \in \Lambda(r)} \mathcal{E}_\mu(D)a \subset J.
\]

Also, it follows from the same lemma that
\[
\mathcal{E}_{\lambda}(D)\mathcal{E}_{0,r}(D) + \mathcal{E}_{0,r}(D)\mathcal{E}_{\lambda}(D) \subset \mathcal{E}_{\lambda,r}(D),
\]
but it is easy to see that \(\mathcal{E}_{\lambda,r}(D) \subset \sum_{v \in \Lambda} \mathcal{E}_v(D)\) (since \(r < |\lambda|\)). Thus \(\mathcal{A} + \mathcal{A}a \subset J\).

The proof for \(I\) is similar. □

3.3. The spatial formulae for spectral manifolds

As usually, by \(\text{ad}_S\) we denote the inner derivation on \(\mathcal{B}(\mathfrak{X})\) implemented by \(S \in \mathcal{B}(\mathfrak{X})\):
\[(\text{ad}_S)T = [S, T]\]

for all \(T \in \mathcal{B}(\mathfrak{X})\). The restriction of \(\text{ad}_S\) to an invariant linear manifold \(W \subset \mathcal{B}(\mathfrak{X})\) will be denoted by \((\text{ad}_S)|_W\) or, briefly, \(\text{ad}_W S\). Note that
\[\text{ad}_S = L_S - R_S,\]
where \(L_S\) and \(R_S\) are the multiplication operators: \(L_ST = ST\) and \(R_ST = TS\) for all \(T \in \mathcal{B}(\mathfrak{X})\). We also use the same notation for elements of a normed algebra.

The following proposition, for the case when \(\lambda = 0\) and \(r = q = 0\), was proved in [50, Lemma 13] (see a more strong result in [24, Theorem 3.6.6]).

Lemma 3.9. Let \(S \in \mathcal{B}(\mathfrak{X})\). Then \(\mathcal{E}_{\lambda,r}(\text{ad}_S)\mathcal{E}_{\mu,q}(S) \subset \mathcal{E}_{\lambda+\mu,r+q}(S)\).

Proof. Set \(\Delta = \text{ad}_S - \lambda\) and \(Q = S - \mu\). If \(x \in \mathcal{E}_{\mu,q}(S)\) and \(T \in \mathcal{E}_{\lambda,r}(\text{ad}_S)\) then for any \(\varepsilon > 0\) there exists a constant \(\beta > 0\) such that \(\|Q^n x\| \leq \beta (q + \varepsilon)^n\) and \(\|A^n T\| \leq \beta (r + \varepsilon)^n\) for all \(n > 0\). Therefore
\[
\|(S - \lambda - \mu)^n T x\| = \|(L_Q - \lambda)^n T x\|
\]
\[
= \|(\Delta + R_Q)^n T x\| = \left\| \sum_{i=0}^{n} \binom{n}{i} (R_Q^{n-i} \Delta^i T) x \right\|
\]
\[ \leq \sum_{i=0}^{n} \binom{n}{i} \| (\Delta^i T) Q^{n-i} x \| \leq \sum_{i=0}^{n} \binom{n}{i} \| \Delta^i T \| \| Q^{n-i} x \| \]

\[ \leq \sum_{i=0}^{n} \binom{n}{i} \beta_i (r + \varepsilon)^i \beta (q + \varepsilon)^{n-i} = \beta^2 (r + q + 2\varepsilon)^n. \]

Since \( \varepsilon \) is arbitrarily small, we obtain that \( Tx \in \mathcal{E}_{\lambda + \mu, r+q}(S) \). \( \square \)

Note that setting \( q = 0 \) we get that an arbitrary \( \mathcal{E}_0(\text{ad} S) \) is invariant for \( \mathcal{E}_0(\text{ad} S) \). We call \( \mathcal{E}_0(\text{ad} S) \) the quasi-commutant of \( S \). It follows that the quasi-commutant of a nonquasi-nilpotent decomposable operator is reducible.

The glocal version of Lemma 3.9 is the following: if \( F_1 \) and \( F_2 \) are convex compact sets in \( \mathbb{C} \) then

\[ \mathcal{X}_{\text{ad} S}(F_1) \mathcal{X}_S(F_2) \subset \mathcal{X}_S(F_1 + F_2). \]

The proof is similar to one of the algebraic formula. A more general consideration of this inclusion (for closed \( F_1, F_2 \) and \( LS - RT \) instead of \( \text{ad} S \)) is in [24, Theorem 3.6.6].

We interpret the tensor product \( \mathfrak{X} \otimes \mathfrak{X}^* \) as the set of all finite rank operators on \( \mathfrak{X} \) under the action \( (x \otimes f) y = f(y)x \) for every \( x, y \in \mathfrak{X} \) and \( f \in \mathfrak{X}^* \). With this agreement the following proposition holds.

**Proposition 3.10.** Let \( S \in \mathcal{B}(\mathfrak{X}) \). Then

\[ \mathcal{E}_{\lambda, r}(S) \otimes \mathcal{E}_{\mu, q}(S^*) \subset \mathcal{E}_{\lambda - \mu, r+q}(\text{ad} S), \] (3.10)

\[ \mathcal{E}_{\eta, p}(\text{ad} S)(\mathcal{E}_{\lambda, r}(S) \otimes \mathcal{E}_{\mu, q}(S^*)) \subset \mathcal{E}_{\lambda + \eta, r+p}(S) \otimes \mathcal{E}_{\mu, q}(S^*), \] (3.11)

\[ (\mathcal{E}_{\lambda, r}(S) \otimes \mathcal{E}_{\mu, q}(S^*)) \mathcal{E}_{\eta, p}(\text{ad} S) \subset \mathcal{E}_{\lambda, r}(S) \otimes \mathcal{E}_{\mu - \eta, q+p}(S^*). \] (3.12)

**Proof.** If \( x \in \mathcal{E}_{\lambda, r}(S) \) and \( f \in \mathcal{E}_{\mu, q}(S^*) \) then for any \( \varepsilon > 0 \) there exists a constant \( \beta > 0 \) for which \( \|(S - \lambda)^n x\| < \beta (r + \varepsilon)^n \) and \( \|(S^* - \mu)^n f\| < \beta (q + \varepsilon)^n \) for all \( n \). Then

\[ \| (\text{ad} S - \lambda + \mu)^n (x \otimes f) \| = \| (LS - \lambda - RS - \mu)^n (x \otimes f) \| \]

\[ \leq \sum_{i=0}^{n} \binom{n}{i} \| (S - \lambda)^i x \| \| (S^* - \mu)^{n-i} f \| \]

\[ < \beta^2 (r + q + 2\varepsilon)^n, \]

whence we easily obtain (3.10).
It is easy to check that
\[ \{ T^* : T \in \mathcal{E}_{\eta,p}(\text{ad } S) \} \subset \mathcal{E}_{-\eta,p}(\text{ad } S^*). \]

Then inclusions (3.11) and (3.12) follow from Lemma 3.9. □

Note that if \( S \in \mathcal{B}(\mathfrak{X}) \) has a finite spectrum then
\[ \mathfrak{X} \otimes \mathfrak{X}^* = \sum \{ \mathcal{E}_\lambda(S) \otimes \mathcal{E}_\mu(S^*) : \lambda, \mu \in \sigma(S) \}, \]
because \( \mathfrak{X} \) and \( \mathfrak{X}^* \) are decomposed into finite sums of corresponding spectral manifolds.

It is easy to check the following glocal version of (3.10): if \( F_1 \) and \( F_2 \) are convex compact sets in \( \mathbb{C} \) then
\[ \mathcal{X}_S(F_1) \otimes \mathcal{X}_{S^*}(F_2) \subset \mathcal{X}_{\text{ad } S}(F_1 - F_2). \]

### 3.4. Existence of nilpotent finite rank operators

We need the following lemmas.

**Lemma 3.11.** Let \( S \in \mathcal{B}(\mathfrak{X}) \). Then

(i) \( \mathcal{E}_{\lambda,r}(S) \cap \mathcal{E}_{\eta,q}(S) = 0 \) if \( r + q < |\eta - \lambda| \) and at least one of the linear manifolds \( \mathcal{E}_{\lambda,r}(S) \) or \( \mathcal{E}_{\eta,q}(S) \) is closed.

(ii) \( P_{\lambda,r}(S)\mathcal{E}_{\eta,q}(S) = 0 \) if \( \Omega(\lambda, r) \subset \text{res}(S) \) and \( r + q < |\eta - \lambda| \).

(iii) \( P_{\lambda,r_1}(S)\mathcal{E}_{\eta,q}(\text{ad } S)P_{\mu,r_2}(S) = 0 \) if \( \Omega(\lambda, r_1), \Omega(\mu, r_2) \subset \text{res}(S) \) and \( r_1 + r_2 + q < |\eta + \mu - \lambda| \).

**Proof.**

(i) Let \( r + q < |\eta - \lambda| \), and let \( \mathcal{E}_{\lambda,r}(S) \) be closed. Put \( Y = \mathcal{E}_{\lambda,r}(S) \) and \( T = (S - \lambda)Y \). Suppose, to the contrary, that there exists a vector \( x \in \mathcal{E}_{\lambda,r}(S) \cap \mathcal{E}_{\eta,q}(S) \) with \( \|x\| = 1 \). Note that \( \rho(T) \leq r \) by Proposition 3.3. So there exists \( \beta > 0 \) such that \( \|T^n\| \leq \beta(r + \varepsilon)^n \) and \( \| (S - \eta)^n x \| \leq \beta(q + \varepsilon)^n \) for every integer \( n \geq 0 \). Since \( (S - \eta)^j x \in Y \) for every integer \( j \geq 0 \), we obtain

\[
|\eta - \lambda|^n = \|(\eta - \lambda)^n x\| = \|((S - \lambda) - (S - \eta))^n x\| \\
= \left\| \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} (S - \lambda)^i (S - \eta)^{n-i} x \right\| \\
\leq \sum_{i=0}^{n} \binom{n}{i} \|T^i\| \|(S - \eta)^{n-i} x\| \\
\leq \beta^2 \sum_{i=0}^{n} \binom{n}{i} (r + \varepsilon)^i (q + \varepsilon)^{n-i} = \beta^2 (r_1 + q + \varepsilon)^n,
\]

whence \( |\eta - \lambda| \leq r + q + 2\varepsilon \) and therefore \( |\eta - \lambda| \leq r + q \), a contradiction.
(ii) Follows from (i) because \( P_{\lambda,r}(S)E_{\eta,q}(S) \subset E_{\lambda,r}(S) \cap E_{\eta,q}(S) \) and \( E_{\lambda,r}(S) \) is closed.

(iii) Taking into account Lemma 3.9, we obtain

\[
P_{\lambda,r_1}(S)E_{\eta,q}(\text{ad } S)P_{\mu,r_2}(S)X \subset P_{\lambda,r_1}(S)E_{\eta,q}(\text{ad } S)E_{\mu,r_2}(S)
\]

\[
\subset P_{\lambda,r_1}(S)E_{\eta+\mu,q+r_2}(S) = 0
\]

by (ii) whenever \( r_1 + r_2 + q < |\eta + \mu - \lambda| \). Hence \( P_{\lambda,r_1}(S)E_{\eta,q}(\text{ad } S)P_{\mu,r_2}(S) = 0 \).

\[ \square \]

**Lemma 3.12.** Let \( S \in B(\mathcal{X}) \). Then \( E_{\lambda,r}(\text{ad } S) \) consists of nilpotent finite rank operators if \( r + 2\min\{\rho_\epsilon(S - \mu) : \mu \in \mathbb{C}\} < |\lambda| \).

**Proof.** Note first that the map \( \mu \mapsto \rho_\epsilon(S - \mu) \) is continuous on \( \mathbb{C} \) and \( \rho_\epsilon(S - \mu) \to \infty \) as \( |\mu| \to \infty \) (if \( \dim \mathcal{X} = \infty \)). So there exists \( \min\{\rho_\epsilon(S - \mu) : \mu \in \mathbb{C}\} \). Let \( q > 0 \) such that \( r + 2q < |\lambda| \) and \( q > \rho_\epsilon(S - \mu) \) for some \( \mu \in \mathbb{C} \). Put \( P_2 = P_{\lambda,q}(S - \mu) \) and \( P_1 = 1 - P_2 \). It follows from the theory of essential spectra (see for instance [6]) that \( P_1 \in \mathcal{F}(\mathcal{X}) \). Also \( P_2E_{\lambda,r}(\text{ad } S)P_2 = 0 \) by Lemma 3.11. If \( T \in E_{\lambda,r}(\text{ad } S) \) then

\[
T = (P_1 + P_2)T(P_1 + P_2) = P_1TP_1 + P_1TP_2 + P_2TP_1,
\]

whence \( T \in \mathcal{F}(\mathcal{X}) \). Moreover, \( T \) is nilpotent by Corollary 3.6. \( \square \)

It follows from Lemma 3.12 that if \( S \) is a Riesz operator, \( \lambda \in \sigma(\text{ad } S) \) and \( q < |\lambda| \) then \( E_{\lambda,q}(\text{ad } S) \) consists of nilpotent operators of finite rank (note that \( E_{\lambda,q}(\text{ad } S) \) is not zero). The same was proved in [50, Theorem 3] for \( E_{\lambda}(\text{ad } S) \), if \( \lambda \) is a nonzero isolated point in the spectrum \( \sigma(\text{ad } S) \), where \( S \) is a compact operator.

It follows from Lemma 3.12 and Proposition 3.8 that if \( A = B(\mathcal{X}) \) and \( D \) is the inner derivation implemented by a nonquasi-nilpotent Riesz operator then the ideal \( J \) defined by formula (3.9) is equal to \( \mathcal{F}(\mathcal{X}) \).

If \( S \in B(\mathcal{X}) \) is scattered then the same is true for its restrictions to invariant subspaces and quotients; moreover, if \( Z, Y \in \text{Lat}\{S\} \), \( Z \subset Y \) and \( V = Y/Z \) then

\[
\sigma(S|V) \subset \sigma(S|Y) \subset \sigma(S).
\]

(see [29, Theorems 0.12 and 0.8]). Also \( \text{ad } S \) is scattered (because \( \sigma(\text{ad } S) = \sigma(S) - \sigma(S) \) [24, Theorem 3.5.1]) and therefore \( \text{ad } W S \) is scattered for any \( W \in \text{Lat } \text{ad } S \).

Note that for any \( S \in B(\mathcal{X}) \) and \( Y \in \text{Lat } S \), \( \sigma(S|Y) \) is contained in the polynomially convex hull of \( \sigma(S) \) (e.g. [50, Corollary 6]). Since the polynomially convex hull of a compact set \( K \subset \mathbb{C} \) is the union of \( K \) and bounded components of \( \mathbb{C} \setminus K \), the polynomially convex hull of the union of two compact sets in \( \mathbb{C} \) is the union of their polynomially convex hulls, and the polynomially convex hull of a countable compact set in \( \mathbb{C} \) coincides with the set. We use these facts in the following assertion.
Corollary 3.13. Let \( S \in \mathcal{B}(X) \), \( W \in \text{Lat} \, \text{ad} \, S \) be nonzero. Then \( W \) contains a nonzero nilpotent finite rank operator if one of the following conditions holds:

(i) \( \rho(\text{ad} \, W \, S) > \rho(S) + \rho_e(S) \).
(ii) \( \text{ad} \, W \, S \) is scattered (more generally, is decomposable) and \( \rho(\text{ad} \, W \, S) > 2\rho_e(S) \).

\textbf{Proof.} (i) Let \( \lambda \in \sigma(\text{ad} \, W \, S) \) with \( |\lambda| = \rho(\text{ad} \, W \, S) \). Since \( \sigma(\text{ad} \, W \, S) \) is contained in the polynomially convex hull of \( \sigma(\text{ad} \, S) \), there exists a point \( \zeta \in \sigma(\text{ad} \, S) \) with

\[
|\zeta| > r > \rho(S) + \rho_e(S)
\]

for some \( r < |\lambda| \). Let \( G = \{(\mu, \eta) : \mu, \eta \in \sigma(S), |\mu - \eta| > r \} \). It is clear that

\[
\min\{|\mu|, |\eta|\} > r - \rho(S) > \rho_e(S)
\]

for every \( (\mu, \eta) \in G \). Since the set \( \{\mu \in \sigma(S) : |\mu| > r - \rho(S)\} \) is finite, so is \( G \). Since \( \sigma(\text{ad} \, S) = \sigma(S) - \sigma(S) \), the set \( F := \{\xi \in \sigma(\text{ad} \, S) : |\xi| > r\} \) coincides with \( \{\mu - \eta : (\mu, \eta) \in G\} \), hence is finite and consists of isolated points of \( \sigma(\text{ad} \, S) \). Moreover, \( \mathcal{E}_f(\text{ad} \, W \, S) \) is nonzero because any point \( v \in \sigma(\text{ad} \, W \, S) \) with \( |v| > r \) is clearly in the polynomially convex hull of \( F \) (that is equal to \( F \)) and so \( \lambda \) is an isolated point of \( \sigma(\text{ad} \, W \, S) \). It remains to apply Lemmas 3.2(i) and 3.12.

(ii) If \( \text{ad} \, W \, S \) is decomposable, there exists a nonzero \( \mathcal{E}_{\lambda, r}(\text{ad} \, W \, S) \) with \( r + 2\rho_e(S) < |\lambda| \). Now the result follows as in (i). \( \square \)

3.5. Commutativity of Riesz projections with operator spectral manifolds

One can find the idea of the following lemma in the proof of Frunză [15, Lemma 5].

\textbf{Lemma 3.14.} Let \( S \in \mathcal{B}(X) \). If \( \Omega = \Omega(\lambda, r) \subset \text{res} \, S \) and \( \Omega(\lambda + \mu, r) \subset \text{res} \, S \) then

\[
L_{P_{\lambda, r}}(S) - R_{P_{\lambda, r} + \mu, r}(S) = F(\text{ad} \, S + \mu) = (\text{ad} \, S + \mu)F,
\]

where \( F \) is a bounded linear operator on \( \mathcal{B}(X) \) defined as follows:

\[
F = \frac{1}{2\pi i} \int_{\Omega} L_{(S - \zeta)^{-1}} R_{(S - \mu - \zeta)^{-1}} d\zeta.
\]

\textbf{Proof.} It is clear that \( P_{\lambda + \mu, r}(S) = P_{\lambda, r}(S - \mu) \) and \( \Omega \subset \text{res} \, (S - \mu) \). Using (3.1) for \( L_{P_{\lambda, r}}(S) \) and \( R_{P_{\lambda, r}(S - \mu)} \), we easily obtain that

\[
L_{P_{\lambda, r}}(S) - R_{P_{\lambda, r} + \mu, r}(S) = \frac{1}{2\pi i} \int_{\Omega} (R_{(S - \mu - \zeta)^{-1}} - L_{(S - \zeta)^{-1}}) d\zeta.
\]
But

\[
R(S - \mu - \zeta)^{-1} - L(S - \zeta)^{-1} = (R_S - \mu - \zeta)^{-1} - (L_S - \zeta)^{-1}
= (\text{ad } S + \mu)(L(S - \zeta)^{-1}R(S - \mu - \zeta)^{-1})
= (L(S - \zeta)^{-1}R(S - \mu - \zeta)^{-1})(\text{ad } S + \mu)
\]

for any \( \zeta \in \Omega \), whence, integrating both sides of the last equality, we obtain the result. \( \square \)

A special case of the following lemma is contained in [46, Lemma 1.1].

**Lemma 3.15.** Let \( S \in \mathcal{B}(X) \). If \( \Omega(\lambda, r), \Omega(\mu, r) \subset \text{res}(S) \) then there exists \( q > 0 \) such that

\[
P_{\lambda, r}(S)T = TP_{\mu, r}(S)
\]

for every \( T \in \mathcal{E}_{\lambda - \mu, q}(\text{ad } S) \).

**Proof.** Set \( \Delta = L_{P_{\lambda, r}(S)} - R_{P_{\mu, r}(S)} \). By Lemma 3.14, there exists a bounded operator \( F \) on \( \mathcal{B}(X) \) such that

\[
L_{P_{\lambda, r}(S)} - R_{P_{\mu, r}(S)} = F(\text{ad } S + (\mu - \lambda)) = (\text{ad } S + (\mu - \lambda))F.
\]

Take \( q > 0 \) such that \( q \rho(F) < 1 \). If \( T \in \mathcal{E}_{\lambda - \mu, q}(\text{ad } S) \) then we obtain

\[
\limsup \| \Delta^n T \|^{1/n} = \limsup \| F^n(\text{ad } S + (\mu - \lambda))^n T \|^{1/n}
\leq (\lim \| F^n \|^{1/n})(\limsup \| (\text{ad } S + (\mu - \lambda))^n T \|^{1/n})
\leq q \rho(F) < 1.
\]

(3.13)

However, it is easy to check that \( \Delta^3 = \Delta \) since \( P_{\lambda, r}(S) \) and \( P_{\mu, r}(S) \) are projections. So \( \Delta^{2n+1} = \Delta \) for every \( n \in \mathbb{N} \) and, by (3.13),

\[
\limsup \| \Delta T \|^{1/(2n+1)} < 1,
\]

whence \( \Delta T = 0 \). \( \square \)

For the case of sufficiently small \( |\lambda - \mu| \), the statement of Lemma 3.15 can be deduced from the results of Beltiţa and Şabac [3].

It follows from Lemma 3.15 that \( P_{\lambda, r}(S) \) commutes with all \( T \in \mathcal{E}_{0, q}(\text{ad } S) \) for sufficiently small \( q > 0 \). The following result is a slight modification of this fact.
Proposition 3.16. Given a Riesz projection $P$ of an operator $S \in B(\mathcal{X})$, there exists $q > 0$ such that $P$ commutes with all operators in $\mathcal{E}_{0,q}(\text{ad } S)$.

Proof. Follows as in Lemmas 3.14 and 3.5 (see also the proof of [41, Lemma 11.3]) under changing the circle $\Omega$ by an admissible contour in (3.1). □

Now, we can prove the main result of the section.

Theorem 3.17. Let $S \in B(\mathcal{X})$ be scattered, $W \in \text{Lat } \text{ad } S$, $Q$ the set of all Riesz projections of $S$, $P$ a Riesz projection of $S$ corresponding to an isolated point $\zeta$ of $\sigma(S)$. Then

(i) $QW = WQ$.

(ii) There is $r > 0$ such that $PW = \mathcal{E}_{0,r}(\text{ad } W S)P$ and $P$ commutes with every operator in $\mathcal{E}_{0,r}(\text{ad } W S)$.

(iii) If $W$ is an operator algebra then so is $QW$.

Proof. (i) It suffices to show that $P_{\lambda,r}(S)W \subset WQ$ for every Riesz projection $P_{\lambda,r}(S)$. It is obvious that there exists a number $\varepsilon > 0$ such that $P_{\lambda,r}(S) = P_{\lambda,q}(S)$ for all $q > 0$ satisfying $|q - r| < \varepsilon$. It follows from Lemmas 3.1 and 3.15 that $W$ is a finite sum of $\mathcal{E}_{\eta_i,q_i}(\text{ad } W S)$ satisfying the following conditions:

• There exist $r_i > 0$ such that $|r_i - r| < \varepsilon$, $\Omega(\mu_i, r_i) \subset \text{res}(S)$ and

$$P_{\lambda,r_i}(S)T = TP_{\mu_i,r_i}(S)$$

for any $T \in \mathcal{E}_{\eta_i,q_i}(\text{ad } W S)$, where $\mu_i = -\eta_i + \lambda$.

This shows that

$$P_{\lambda,r}(S)W \subset WQ.$$ 

A similar argument proves $WP_{\lambda,r}(S) \subset QW$.

(ii) One may suppose that $\zeta = 0$ and $P = P_0(S) = P_{0,\varepsilon}(S)$ for some $\varepsilon > 0$. It follows from Proposition 3.16, Lemmas 3.1 and 3.15 that $W$ is a finite sum of $\mathcal{E}_{\eta_i,q_i}(\text{ad } W S)$ satisfying the following conditions:

• $\eta_0 = 0$, and $q_0 > 0$ is sufficiently small so that $P$ commutes with every operator in $\mathcal{E}_{0,q_0}(\text{ad } W S)$; all $\eta_i \neq 0$ if $i > 0$;

• there exist $r_i > 0$ (for $i > 0$) such that $r_i < \text{min}\{\varepsilon, |\eta_i|\}$, $\Omega(\mu_i, r_i) \subset \text{res}(S)$ and

$$PT = P_{0,r_i}(S)T = TP_{\mu_i,r_i}(S)$$

for every $T \in \mathcal{E}_{\eta_i,q_i}(\text{ad } W S)$, where $\mu_i = -\eta_i$.

Since $PP_{\mu_i,r_i}(S) = P_{\mu_i,r_i}(S)P = 0$ for every nonzero $\eta_i$, we obtain that

$$PWP = P(\mathcal{E}_{0,q_0}(\text{ad } W S))P = (\mathcal{E}_{0,q_0}(\text{ad } W S))P.$$ 

Now it suffices to put \( r = q_0 \).

(iii) If \( W \) is an operator algebra then, by (i),

\[
(QW)(QW) = (QQ)(WW) \subset QW,
\]

whence \( QW \) is also an operator algebra. \( \square \)

3.6. Riesz projections of inner derivations

We first show that any operator in \( \mathcal{B}(\mathfrak{X}) \) may be decomposed into the sum of elements of different \( \mathcal{E}_{\lambda, \mu}(\text{ad} S) \)-spaces for a fixed operator \( S \in \mathcal{B}(\mathfrak{X}) \) if \( S \) has a “good” set of Riesz projections (in particular if \( S \) is scattered).

Lemma 3.18. Let \( S \in \mathcal{B}(\mathfrak{X}) \), and let \( \Omega(\lambda, r), \Omega(\mu, q) \subset \text{res}(S) \). Then

\[
P_{\lambda, r}(S)\mathcal{B}(\mathfrak{X})P_{\mu, q}(S) \subset \mathcal{E}_{\lambda - \mu, r + q}(\text{ad} S).
\]

Proof. Let \( T \in P_{\lambda, r}(S)\mathcal{B}(\mathfrak{X})P_{\mu, q}(S) \). It is easy to see that \( \rho((S - \lambda)P_{\lambda, r}(S)) < r \) and \( \rho((S - \mu)P_{\mu, q}(S)) < q \). Therefore, there exists a constant \( \beta > 0 \) such that \( \| (S - \lambda)^n P_{\lambda, r}(S) \| \leq \beta r^n \) and \( \| (S - \mu)^n P_{\mu, q}(S) \| \leq \beta q^n \) for any integer \( n \geq 0 \). So we obtain

\[
\| (\text{ad} S - \lambda + \mu)^n T \| = \| (L_{S - \lambda} - RS - \mu)^n L_{P_{\lambda, r}(S)} R_{P_{\mu, q}(S)} T \|
\]

\[
\leq \| T \| \sum_{k=0}^{n} \binom{n}{k} \| (S - \lambda)^{n-k} P_{\lambda, r}(S) \| \| (S - \mu)^k P_{\mu, q}(S) \|
\]

\[
\leq \beta^2 \| T \| \sum_{k=0}^{n} \binom{n}{k} r^{n-k} q^k = \beta^2 \| T \| (r + q)^n
\]

whence we immediately have \( T \in E_{\lambda - \mu, r + q}(\text{ad} S) \). \( \square \)

Proposition 3.19. Let \( S \in \mathcal{B}(\mathfrak{X}) \) be scattered, and let \( G \subset \mathbb{C} \) be open with \( \partial G \subset \text{res}(S) \). Then \( P_G(\text{ad} S) \) is a finite sum of some \( L_{P_V(S)} R_{P_U(S)} \), where \( V, U \subset \mathbb{C} \) are open with \( \partial V, \partial U \subset \text{res}(S) \) and \( V - U \subset G \). In particular, if \( G \cap \sigma(\text{ad} S) = \{ \lambda \} \) then

\[
P_{\lambda}(\text{ad} S) = \sum_{\mu - \eta = \lambda} L_{P_{\mu}(S)} R_{P_{\eta}(S)}
\]

(the sum contains only a finite number of nonzero summands).

Proof. Let \( \Theta = \sigma(S) \times \sigma(S) \). Take an open cover \( \{ V(\mu) \times U(\eta) \}_{(\mu, \eta) \in \Theta} \) of \( \Theta \), where \( V(\mu), U(\eta) \) are neighborhoods of \( \mu \) and \( \eta \), respectively, such that \( \partial V(\mu), \partial U(\eta) \subset \)}
res(S) and \((V(\mu) - U(\eta)) \cap \partial G = \emptyset\). Then clearly there exists a finite subcovering
\[
\{ V(\mu_1) \times U(\eta_1), \ldots, V(\mu_n) \times U(\eta_n) \}
\]
of \(\Theta\). Set \(P_1(i) = P_{\mu_i}(S)\) and \(P_2(i) = P_{\eta_i}(S)\) for \(i = 1, \ldots, n\), and let
\[
\Lambda = \{(i, j) : \mu_i - \eta_j \in G\}.
\]

Let \(Q(i, j) = L_{P_1(i)} R_{P_2(j)}\) for \(i, j = 1, \ldots, n\) and let
\[
Q = \sum_{(i, j) \in \Lambda} Q(i, j).
\]

Clearly \(1 = \sum_{i=1}^n P_1(i) = \sum_{i=1}^n P_2(i)\), whence \(\sum_{i,j=1}^n Q(i, j)\) is the identity operator on \(B(\mathcal{X})\). Taking into account (3.2), we obtain that
\[
P_G(\text{ad} S) = P_G(\text{ad} S) \sum_{i,j=1}^n Q(i, j) = P_G(\text{ad} S) Q
\]
by Lemmas 3.18 and 3.11(ii). On the other hand, if \((i, j) \in \Lambda\) then
\[
Q(i, j) = P_G(\text{ad} S) Q(i, j)
\]
by Lemma 3.18, whence we obtain that \(Q = P_G(\text{ad} S) Q\). Therefore \(P_G(\text{ad} S) = Q\).

If \(G \cap \sigma(\text{ad} S) = \{\lambda\}\), use \(\sigma(\text{ad} S) = \sigma(S) - \sigma(\text{ad} S)\) to obtain that the set \(\{ (\mu, \eta) \in \Theta : \mu - \eta = \lambda \}\) consists of isolated points and hence is finite. \(\square\)

Note that the spectrum of a scattered operator does not change if it is calculated in an arbitrary unital closed subalgebra of \(B(\mathcal{X})\) containing the operator (see for instance [9, Theorem 2.3.21]). So, taking a scattered \(S \in B(\mathcal{X})\) and the closed subalgebra generated by \(S\), one obtains that every \((\lambda - S)^{-1}\) for \(\lambda \in \text{res}(S)\) is norm-approximated by polynomials in \(S\).

The following proposition is a generalization of Cartan’s lemma [4, Lemma1.5.3]. Let \(\overline{\lambda}\) denote the complex-conjugate number to \(\lambda \in \mathbb{C}\).

**Proposition 3.20.** (i) Let \(S \in B(\mathcal{X})\) have a finite spectrum and
\[
T = \sum_{\lambda \in \sigma(S)} \overline{\lambda} P_{\lambda}(S).
\]
Then \(\text{ad} T\) is a limit of some polynomials \(q_n\) in \(\text{ad} S\) with \(q_n(0) = 0\). If \(S - \sum_{\lambda \in \sigma(S)} \overline{\lambda} P_{\lambda}(S)\) is nilpotent then \(\text{ad} T = q(\text{ad} S)\), where \(q\) is a polynomial with \(q(0) = 0\).
(ii) Let $M, N \subset B(\mathcal{X})$ be linear manifolds, $M \subset N$, and let $\mathfrak{L} \subset B(\mathcal{X})$ be a Lie algebra containing every $T \in \mathcal{F}(\mathcal{X})$ with $[T, N] \subset M$. Let $J$ be the set of all $T \in \mathfrak{L} \cap \mathcal{F}(\mathcal{X})$ such that $\text{tr}(ST) = 0$ for every $S \in \mathfrak{L} \cap \mathcal{F}(\mathcal{X})$. Then $J$ is a Lie ideal of $\mathfrak{L}$ consisting of nilpotent operators.

**Proof.** (i) Since $\text{ad}S$ has a finite spectrum, it is well known that each $P_\lambda(\text{ad}S)$ corresponding to $\lambda \neq 0$ is a norm-limit of some polynomials $p_n(\text{ad}S)$ with $p_n(0) = 0$. It is easy to check that $\sigma(T) = \{\tilde{\lambda} : \lambda \in \sigma(S)\}$ and

$$\text{ad} T = \sum_{\lambda \in \sigma(\text{ad}T)} \lambda P_\lambda(\text{ad}T).$$

Since $P_\lambda(T) = P_\lambda(S)$ for every $\lambda$, we have

$$P_\lambda(\text{ad}T) = P_\lambda(\text{ad}S)$$

for every $\lambda$, by Proposition 3.19, whence

$$\text{ad} T = \sum_{\lambda \in \sigma(\text{ad}S)} \tilde{\lambda} P_\lambda(\text{ad}S)$$

and $\text{ad} T$ is a limit of some polynomials $q_n(\text{ad}S)$ with $q_n(0) = 0$.

Now, suppose that $S - \sum_{\lambda \in \sigma(S)} \lambda P_\lambda(S)$ is nilpotent. Let $A$ be the subalgebra generated by $S$ and its Riesz projections. Since $A$ is a unital finite-dimensional algebra, so is $B$, the subalgebra of $B(B(\mathcal{X}))$ generated by $L_A R_A$. Note that $\text{ad}S$, $\text{ad}T$ and all their Riesz projections lie in $B$ by Proposition 3.19. Since $B$ is finite-dimensional, every $P_\lambda(\text{ad}S)$ corresponding to $\lambda \neq 0$ is some polynomial $p$ in $\text{ad}S$ with $p(0) = 0$. Thus $\text{ad} T$ is some polynomial $q$ in $\text{ad} S$ with $q(0) = 0$.

(ii) Since

$$\text{tr}([S, K]T) = \text{tr}(SK)$$

for every $S \in \mathcal{F}(\mathcal{X})$ and $K, T \in B(\mathcal{X})$, $J$ is a Lie ideal of $\mathfrak{L}$. Let $S \in J$ be arbitrary, and let $T = \sum_{\lambda \in \sigma(S)} \tilde{\lambda} P_\lambda(S)$. Then $[T, N] \subset M$ by (i), so $T \in \mathfrak{L} \cap \mathcal{F}(\mathcal{X})$ and then

$$0 = \text{tr}(ST) = \sum_{\lambda \in \sigma(S)} |\tilde{\lambda}|^2,$$

whence $S$ is nilpotent. □
3.7. Concluding remarks

Given an operator \( S \in \mathcal{B}(\mathfrak{X}) \), let \( \text{res}_\infty(S) \) stand for the unbounded component of \( \text{res}(S) \). The following lemma is standard, but we did not find appropriate references.

**Lemma 3.21.** Let \( S \in \mathcal{B}(\mathfrak{X}) \) and \( V = Y/Z \), where \( Y, Z \in \text{Lat} S \) with \( Z \subset Y \).

(i) If \( \lambda \in \text{res}_\infty(S) \) then \( (S|V - \lambda)^{-1} = (S - \lambda)^{-1}|V \).

(ii) If \( G \subset \mathbb{C} \) is open with the boundary \( \partial G \subset \text{res}_\infty(S) \) then \( P_G(S|V) = P_G(S)|V \).

(iii) If \( S \) is a Riesz operator then so is \( S|V \).

**Proof.** (i) It is well known that \( \sigma(S|V) \) lies in the polynomially convex hull of \( \sigma(S) \), whence \( \text{res}_\infty(S) \subset \text{res}(S|V) \). Let \( H \) be the set of all \( \lambda \in \text{res}_\infty(S) \) such that \( (S|V - \lambda)^{-1} = (S - \lambda)^{-1}|V \). Then \( H \) is open. Indeed, if \( \lambda \in H \) and \( |\mu - \lambda| < \rho((S - \lambda)^{-1})^{-1} \) then

\[
(S - \mu)^{-1} = \sum_{n=1}^{\infty} (\mu - \lambda)^{n-1}(S - \lambda)^{-n},
\]

whence clearly \( \mu \in H \). Since \( \eta \in H \) for \( \eta \in \mathbb{C} \) with \( |\eta| > \rho(S) \) and \( \text{res}_\infty(S) \) is connected, we easily obtain that \( H = \text{res}_\infty(S) \).

(ii) Immediately follows from (i).

(iii) If \( S \) is a Riesz operator then \( \sigma(S|V) \subset \sigma(S) \) and all Riesz projections \( P_\lambda(S|V) \) corresponding to nonzero \( \lambda \in \sigma(S|V) \) are of finite rank by (ii). Therefore \( S|V \) is a Riesz operator. \( \square \)

If \( S \) is a Riesz operator on a Banach space \( \mathfrak{X} \) then \( \sum_{\lambda \in \sigma(T)} \mathcal{E}_\lambda(S) \) need not be dense in \( \mathfrak{X} \) whenever \( \sigma(S) \) is not finite. However, the following proposition holds:

**Corollary 3.22.** Let \( S \) be a Riesz operator on \( \mathfrak{X} \).

(i) If \( Y = \sum_{\lambda \in \sigma(T) \setminus \{0\}} \mathcal{E}_\lambda(S) \) then \( S|(\mathfrak{X}/Y) \) is quasi-nilpotent.

(ii) \( S \) is algebraic if and only if \( S^m \in \mathcal{F}(\mathfrak{X}) \) for some \( m \in \mathbb{N} \).

**Proof.** (i) It follows from Lemma 3.21 that \( P_\lambda(S|(\mathfrak{X}/Y)) = P_\lambda(S)|(\mathfrak{X}/Y) \) for each nonzero \( \lambda \in \sigma(S|(\mathfrak{X}/Y)) \). But \( P_\lambda(S)\mathfrak{X} \subset Y \), so \( P_\lambda(S)|(\mathfrak{X}/Y)) = 0 \).

(ii) If \( S^m \in \mathcal{F}(\mathfrak{X}) \) then \( S^m \) is algebraic and so is \( S \).

Now let \( S \) be algebraic. Then \( \sigma(S) \) is a finite set and \( \mathfrak{X} \) is a finite direct sum of \( \mathcal{E}_0(S), \mathcal{E}_{\lambda_1}(S), \ldots, \mathcal{E}_{\lambda_n}(S) \). Since \( S|\mathcal{E}_0(T) \) is algebraic and quasi-nilpotent, it is a nilpotent. If \( (S|\mathcal{E}_0(S))^m = 0 \) for some \( m \in \mathbb{N} \) then \( S^m \in \mathcal{F}(\mathfrak{X}) \) because all \( \mathcal{E}_{\lambda_i}(S) \) with nonzero \( \lambda_i \in \sigma(S) \) are finite-dimensional. \( \square \)

The following proposition is an easy consequence of the Fredholm theory of linear operators. Since we did not find a suitable reference, we give the proof.
Proposition 3.23. Let $S \in \mathcal{B}(\mathcal{X})$. If $S + \mathcal{K}(\mathcal{X})$ in the Calkin algebra $\mathcal{B}(\mathcal{X})/\mathcal{K}(\mathcal{X})$ is scattered then so is $S$.

Proof. Note that $\lambda - S$ is a Fredholm operator for every $\lambda \in \sigma(S) \setminus \sigma_e(S)$, i.e., $\sigma(S) \setminus \sigma_e(S)$ consists of Fredholm points of $\sigma(S)$. Since every boundary Fredholm point of $\sigma(S)$ is isolated in $\sigma(S)$ by the Punctured Neighborhood theorem [2, Theorem R.2.4], every point of the boundary $\partial \sigma(S)$ belongs to $\sigma_e(S)$ or is isolated. Since $\sigma_e(S)$ and the set of all isolated points of $\sigma(S)$ are countable, so is $\partial \sigma(S)$. Therefore $\sigma(S)$ is countable. □

It follows from Proposition 3.23 that every perturbation of a scattered operator by a compact operator is scattered.

4. Invariant subspaces for operator Lie algebras

In this section, we prove several statements on reducibility of operator Lie algebras. In particular, we give the affirmative answer to Volterra Ideal Problem and extend the statement to much more wide class of ideals than Volterra ones.

4.1. A reducibility criterion

To obtain a convenient technical criterion of reducibility of an operator Lie algebra we need two lemmas.

Lemma 4.1 (Wojtyński [49, Lemma 1]). If $\mathcal{L}$ is a Lie subalgebra of an algebra $A$ then $A(\mathcal{L})$ consists of sums of powers of elements of $\mathcal{L}$.

Lemma 4.2. Let $T, S \in \mathcal{B}(\mathcal{X})$. Then $\rho(S + \zeta T) = \rho(S)$ for all $\zeta \in \mathbb{C}$ if and only if $\rho((S - \mu)^{-1} T) = 0$ for all $\mu \in \mathbb{C}$ with $|\mu| > \rho(S)$. If this condition holds and $T \in \mathcal{F}(\mathcal{X})$ then $\text{tr}(S^n T) = 0$ for all $n \geq 0$.

Proof. Let $\lambda \in \mathbb{C}$ be nonzero. If $\rho((S - \mu)^{-1} T) = 0$ for every $\mu \in \mathbb{C}$ with $|\mu| > \rho(S)$, then $\lambda^{-1} (S + \lambda T - \mu)$ is invertible for every $\lambda \neq 0$, as the product of invertible operators $S - \mu$ and $(S - \mu)^{-1} T + \lambda^{-1}$. Hence

$$\rho(S + \lambda T) \leq \rho(S)$$

for all $\lambda \in \mathbb{C}$. Therefore the map $\lambda \mapsto \rho(S + \lambda T)$ is bounded on $\mathbb{C}$. Being subharmonic [48], it is constant, i.e.,

$$\rho(S + \lambda T) = \rho(S)$$

for all $\lambda \in \mathbb{C}$. 
Suppose now that $\rho(S + \lambda T) = \rho(S)$ for all $\lambda \in \mathbb{C}$. Let $\lambda, \mu \in \mathbb{C}$ be such that $\lambda \neq 0$ and $|\mu| > \rho(S)$. Since $S - \mu$ and $\lambda^{-1}(S + \lambda T - \mu)$ are invertible, so is their product $(S - \mu)^{-1}T + \lambda^{-1}$, whence

$$\rho((S - \mu)^{-1}T) = 0$$

(4.1)

for every $\mu \in \mathbb{C}$ with $|\mu| > \rho(S)$.

Now let $T \in \mathcal{F}(\mathfrak{X})$. Put $v = \mu^{-1}$ in (4.1). Then we have

$$\sigma((1 - vS)^{-1}T) = 0$$

and therefore $\text{tr}((1 - vS)^{-1}T) = 0$ for all $v \in \mathbb{C}$ such that $|v| \rho(S) < 1$. Thus the analytic function $v \mapsto \text{tr}((1 - vS)^{-1}T)$ vanishes on some neighborhood of the origin. Since

$$(1 - vS)^{-1} = 1 + vS + v^2S^2 + \ldots$$

for all such $v$, we obtain that

$$\text{tr}((1 - vS)^{-1}T) = \text{tr}(T) + v\text{tr}(ST) + v^2\text{tr}(S^2T) + \ldots,$$

whence $\text{tr}(S^nT) = 0$ for every $n \geq 0$. \qed

**Theorem 4.3.** Let $\mathfrak{L} \subset \mathcal{B}(\mathfrak{X})$ be a Lie algebra, and let $T \in \mathcal{F}(\mathfrak{X})$ be a nonzero operator such that $\text{tr}(S^nT) = 0$ for all $S \in \mathfrak{L}$ and $n \in \mathbb{N}$.

(i) If $\overline{\mathcal{A}(\mathfrak{L})}^{\text{wot}}$ contains a nonzero compact operator then $\mathfrak{L}$ is reducible.

(ii) If $T \in \overline{\mathcal{A}(\mathfrak{L})}^{\text{wot}}$, then $T \in \text{rad} \overline{\mathcal{A}(\mathfrak{L})}^{\text{wot}}$, and therefore $\mathfrak{L}$ has a nontrivial hyperinvariant subspace.

**Proof.** (i) It follows from Lemma 4.1 that $\text{tr}(ST) = 0$ for all $S \in \mathcal{A}(\mathfrak{L})$. It follows from the continuity of the map $S \mapsto \text{tr}(ST)$ on $\mathcal{B}(\mathfrak{X})$ in the weak operator topology that

$$\text{tr}(ST) = 0$$

for all $S \in \overline{\mathcal{A}(\mathfrak{L})}^{\text{wot}}$. Then $\overline{\mathcal{A}(\mathfrak{L})}^{\text{wot}} \neq \mathcal{B}(\mathfrak{X})$ and, when $\overline{\mathcal{A}(\mathfrak{L})}^{\text{wot}}$ contains a nonzero compact operator, Lat $\overline{\mathcal{A}(\mathfrak{L})}^{\text{wot}}$ is nontrivial by Lomonosov’s results ([26], see also [29,31]).

(ii) If $T \in \overline{\mathcal{A}(\mathfrak{L})}^{\text{wot}}$, it follows from the argument above that $\text{tr}((ST)^n) = 0$ for each $n \in \mathbb{N}$ and therefore

$$\rho(ST) = 0$$
for all \( S \in \mathcal{A}(\mathfrak{U})^{\text{wot}} \). Thus, \( T \in \text{rad} \mathcal{A}(\mathfrak{U})^{\text{wot}} \) and \( \mathcal{A}(\mathfrak{U}) \) has a nontrivial hyperinvariant subspace by Theorem 2.1. \( \square \)

**Corollary 4.4.** A Lie algebra \( \mathfrak{U} \subset \mathcal{B}(\mathfrak{X}) \) having a nonzero compact operator in \( \mathcal{A}(\mathfrak{U})^{\text{wot}} \) is reducible if and only if there exists a nonzero \( T \in \mathcal{F}(\mathfrak{X}) \) such that \( \rho(S + T) = \rho(S) \) for all \( S \in \mathfrak{U} \).

**Proof.** If \( \mathfrak{U} \) has a nontrivial invariant subspace, say \( Y \), then one can pick \( T = x \otimes f \), where \( x \in Y \) and \( f \in \mathfrak{X}^* \) are nonzero and \( f(Y) = 0 \). Clearly \( T \) is in \( \text{rad Alg}\{Y\} \), whence \( \rho(S + T) = \rho(S) \) for all \( S \in \mathfrak{U} \).

The converse follows from Lemma 4.2 and Theorem 4.3. \( \square \)

### 4.2. Quasi-commutant and quasi-center

Let \( M \) be a subset of \( \mathcal{B}(\mathfrak{X}) \). The set \( \cap_{T \in M} \mathcal{E}_0(\text{ad} T) \) is called the quasi-commutant of \( M \) and denoted by \( M^{qc} \). It follows easily from Lemma 3.5 that \( M^{qc} \) is a unital operator algebra containing the commutant \( M' \) of \( M \). It should be underlined that if \( M \) is scattered then \( M^{qc} \) is closed by Proposition 3.3. We will prove that Lomonosov Theorem extends in some way from \( M' \) to \( M^{qc} \), namely if a Lie algebra of compact operators has a nonscalar quasi-commutant then it is reducible (see Corollary 4.9 below).

The following lemma shows that the quasi-commutant of a set of operators has a property of weakened commutativity with the set.

**Lemma 4.5.** Let \( M \subset \mathcal{B}(\mathfrak{X}) \). Then each Riesz projection of every operator in \( M \) commutes with every operator in \( M^{qc} \).

**Proof.** Let \( S \in M \), and let \( P \) be its Riesz projection. Since \( M^{qc} \subset \mathcal{E}_0(\text{ad} S) \), \( P \) commutes with every operator in \( M^{qc} \) and hence in \( M^{qc} \), by Proposition 3.16. \( \square \)

It follows from the lemma that the quasi-commutant of an operator Lie algebra containing an operator with nonconnected spectrum is reducible. On the other hand, the quasi-commutant of a Volterra Lie algebra is \( \mathcal{B}(\mathfrak{X}) \), and so is irreducible.

**Proposition 4.6.** Let \( M \subset \mathcal{B}(\mathfrak{X}) \). If \( \overline{M} \) is scattered then \( M^{qc} = (\overline{M})^{qc} \).

**Proof.** It suffices to show that \( M^{qc} \subset (\overline{M})^{qc} \). Since \( \overline{M} \) is scattered, \( \text{ad}\overline{M} \) is scattered. Then \( M^{qc} \subset \mathcal{E}_0(\text{ad} S) \) by Proposition 3.4 for every \( S \in \overline{M} \). Therefore \( M^{qc} \subset (\overline{M})^{qc} \). \( \square \)

If \( \mathfrak{U} \) is an Engel Lie subalgebra of \( \mathcal{B}(\mathfrak{X}) \) then clearly \( \mathfrak{U} \subset \mathfrak{U}^{qc} \). On the other hand, if \( \mathfrak{U} \) is a closed operator Lie algebra and \( \mathfrak{U} \subset \mathfrak{U}^{qc} \) then \( \mathfrak{U} \) is Engel by Proposition 3.3. The set \( \mathfrak{U} \cap \mathfrak{U}^{qc} \) is called the quasi-center of \( \mathfrak{U} \) and coincides with \( \cap_{T \in \mathfrak{U}} \mathcal{E}_0(\text{ad} \mathfrak{U} T) \).

So the quasi-center of \( \mathfrak{U} \) can be defined for an abstract normed Lie algebra \( \mathfrak{U} \). It is
denoted by $Z_q(\mathfrak{L})$. Again, it should be underlined that $Z_q(\mathfrak{L})$ is closed if all operators $\text{ad}_L T$ are scattered.

A normed Lie algebra $\mathfrak{L}$ is quasi-central if $\mathfrak{L} = Z_q(\mathfrak{L})$. Thus the class of all quasi-central Lie algebras contains the class of all Engel ones. For Banach Lie algebras the classes coincide.

The following proposition establishes some properties of a quasi-commutant and a quasi-center. Recall that an ideal of a normed Lie algebra is called topologically characteristic if it is invariant for all bounded derivations of the Lie algebra.

**Proposition 4.7.** (i) Let $\mathfrak{L} \subset B(\mathfrak{X})$ be a closed Lie algebra. Then

$$\text{Nor}(\mathfrak{L}) \subset \text{Nor}(\overline{\mathfrak{L}}^{qc}).$$

(ii) Let $\mathfrak{L}$ be a Banach Lie algebra. Then $\overline{Z_q(\mathfrak{L})}$ is a topologically characteristic ideal of $\mathfrak{L}$.

**Proof.** (ii) Let $D$ be a bounded derivation of $\mathfrak{L}$. Then $\exp(D)$ is a bounded automorphism of $\mathfrak{L}$, in particular

$$\exp(D)[a, b] = [\exp(D)a, \exp(D)b]$$

for all $a, b \in \mathfrak{L}$. Let $x \in \overline{Z_q(\mathfrak{L})}$ and $a \in \mathfrak{L}$, and let $(x_k)$ be a sequence of elements of $Z_q(\mathfrak{L})$ convergent to $x$. For brevity, put $\psi_D = \exp(D)$. Then $\psi_D^{-1} = \psi_{-D}$ and

$$\limsup_{n \to \infty} \|(\text{ad}_L a)^n \psi_D x_k\|^{1/n} = \limsup_{n \to \infty} \|\psi_D ((\text{ad}_L \psi_{-D}(a))^n x_k)\|^{1/n}$$

$$\leq \lim_{n \to \infty} \|\psi_D\|^{1/n} \limsup_{n \to \infty} \|(\text{ad}_L \psi_{-D}(a))^n x_k\|^{1/n} = 0$$

for every $k$. Therefore

$$\exp(D)x_k = \psi_D(x_k) \in Z_q(\mathfrak{L})$$

for every $k$, whence

$$\exp(D)x = \lim \exp(D)x_k \in \overline{Z_q(\mathfrak{L})}.$$

Replacing $D$ by $\lambda D$, we have that

$$\exp(\lambda D)x = \psi_{\lambda D}(x) \in \overline{Z_q(\mathfrak{L})},$$

(4.2)

for every $\lambda \in \mathbb{C}$. Since $x \in \overline{Z_q(\mathfrak{L})}$, we obtain, differentiating by $\lambda$, that $Dx \in \overline{Z_q(\mathfrak{L})}$. In particular, $\overline{Z_q(\mathfrak{L})}$ is an ideal of $\mathfrak{L}$. 
(i) Let $T \in \mathfrak{L}^{qc}$ and $S \in \text{Nor}(\mathfrak{L})$ be arbitrary. Then $\text{ad} \ S$ is a bounded derivation of $\mathfrak{L}$, and we obtain (in the same way as (4.2)) that
\[
\exp(\lambda \text{ad} \ S)T \in \bar{\mathfrak{L}}^{qc}
\]
for every $\lambda \in \mathbb{C}$, whence it follows that $[S, T] \in \bar{\mathfrak{L}}^{qc}$, i.e. $S \in \text{Nor}(\bar{\mathfrak{L}}^{qc})$. □

In the following two propositions, we obtain invariant subspace results, imposing some conditions on the quasi-center or quasi-commutant of an operator Lie algebra. Recall [41] that an operator $S \in B(X)$ is called principal if $\rho_e(S) = \rho(S)$. We say that $S$ is strictly principal if $S$ has no isolated eigenvalues $\lambda \in \sigma(S)$ of finite multiplicity (i.e. such that $P_\lambda(S)$ is a nonzero finite rank projection). It is clear that if $S$ is strictly principal then $S$ is principal because all $\lambda \in \sigma(S)$ with $|\lambda| > \rho_e(S)$ are isolated of finite multiplicity.

**Theorem 4.8.** A closed operator Lie algebra $\mathfrak{L}$ is reducible if at least one of the following conditions holds.

(i) the closure of the quasi-center of $\mathfrak{L}$ contains an operator $K$ such that the polynomially convex hull of $\sigma(K)$ is not connected.

(ii) $\mathfrak{L}^{qc}$ is nonscalar and $\mathfrak{L}$ contains a nonzero compact operator and an operator which is not strictly principal.

(iii) $\mathfrak{L}^{qc}$ is nonscalar and $\mathfrak{L}$ contains a nonzero finite rank operator.

**Proof.** (i) Let $W = \bar{\mathfrak{L}}^{qc}$. Then $W$ is a unital closed operator algebra and $K \in W$. Let $\sigma_W(K)$ be the spectrum of $K$ with respect to $W$. Since polynomially convex hulls of $\sigma_W(K)$ and $\sigma(K)$ coincide, $\sigma_W(K)$ is not connected and hence there exists a nontrivial Riesz projection $P \in W$ of the operator $K$. By Lemma 4.5, $P$ commutes with every operator in $W$. Put
\[
I = \{T \in W : TP = 0\}. \tag{4.3}
\]
Then $I \in \text{Lat ad } \mathfrak{L}$. Indeed, for any $S \in \mathfrak{L}$, $T \in I$ one has $[S, T] \in W$ by Proposition 4.7(i) and
\[
[S, T]P = P[S, T]P = PSTP - PTPS = P(S(TP) - (TP)SP) = 0, \tag{4.4}
\]
whence $[S, T] \in I$. Since $I \neq 0$ (because $1 - P \in I$) and $\ker I \neq 0$, ker $I$ is a nontrivial invariant subspace for $\mathfrak{L}$ by Lemma 2.2.

(ii) Let $W$ be defined as in (i). It follows from Proposition 3.16 that there exists a nonzero finite rank projection $P$ commuting with every operator in $W$ (one may take for $P$ a Riesz projection of an operator in $\mathfrak{L}$ which is not strictly principal). Let $I$ be defined by (4.3).

If $I \neq 0$, the result follows as in (i).

Suppose that $I = 0$. Then the linear map $SP \mapsto S$ from $WP$ onto $W$ is well defined. Since $WP = PWP$ is finite-dimensional, $W$ is finite-dimensional and $\text{ad}_W \mathfrak{L}$ is a Lie
algebra of nilpotent operators on a finite-dimensional space. Hence, \( \text{ad}_W \mathcal{L} \) is a nilpotent Lie algebra. Since \( W \) is nonscalar, it is clear that there is a nonscalar operator \( T \in W \) commuting with every operator in \( \mathcal{L} \). Since \( T \) commutes with a nonzero compact operator from \( \mathcal{L} \), the commutant of \( T \) is reducible by Lomonosov’s results \([26]\) (see also \([31]\)), and so is \( \mathcal{L} \).

(iii) Let \( T \in \mathcal{L} \) be a nonzero finite rank operator. Suppose, to the contrary, that \( \mathcal{L} \) is irreducible. Then all operators in \( \mathcal{L} \) must be strictly principal by (ii). Hence

\[
\rho(S + T) = \rho(S)
\]

for each \( S \in \mathcal{L} \). It follows from Corollary 4.4 that \( \mathcal{L} \) is reducible, a contradiction. □

**Corollary 4.9.** (i) Let \( \mathcal{L} \subset \mathcal{B}(\mathcal{X}) \) be a closed nonscalar Lie algebra of operators with one-point essential spectra. If \( \mathcal{L}^{qc} \) is nonscalar and \( \mathcal{L} \) contains a nonzero compact operator then \( \mathcal{L} \) is reducible.

(ii) A nonscalar Lie algebra \( \mathcal{L} \subset \mathcal{K}^1(\mathcal{X}) \) is reducible if \( \mathcal{L}^{qc} \) is nonscalar.

**Proof.** (i) Suppose, to the contrary, that \( \mathcal{L} \) is irreducible. Then all operators in \( \mathcal{L} \) are strictly principal by Theorem 4.8 (ii) and hence have one-point spectra. So \( \mathcal{L} \) is Engel. It follows from Theorem 2.8 that \( \mathcal{L} \) is reducible, a contradiction.

(ii) If \( \mathcal{L} \subset \mathcal{K}^1(\mathcal{X}) \) then \( \mathcal{L} \subset \mathcal{K}^1(\mathcal{X}) \) and \( \mathcal{L}^{qc} = (\mathcal{L})^{qc} \) by Proposition 4.6. The result follows by (i). □

**Corollary 4.10.** Let \( \mathcal{L} \subset \mathcal{K}^1(\mathcal{X}) \) be a Lie algebra, \( W = \overline{A(\mathcal{L})}^{\text{wot}} \cap \mathcal{L}^{qc} \) and \( Z = \overline{A(\mathcal{L})} \cap \mathcal{L}^{qc} \). Then

\[
[W, \overline{A(\mathcal{L})}] \cup [Z, \overline{A(\mathcal{L})}^{\text{wot}}] \subset \text{rad} \overline{A(\mathcal{L})}^{\text{wot}}.
\]

(4.5)

In particular,

\[
\rho(T + S) \leq \rho(T) + \rho(S), \quad \rho(TS) \leq \rho(T)\rho(S)
\]

(4.6)

if \( T \in W \) and \( S \in \overline{A(\mathcal{L})} \), or \( T \in Z \) and \( S \in \overline{A(\mathcal{L})}^{\text{wot}} \).

**Proof.** Let \( V \) be a gap-quotient of \( \text{Lat} \overline{A(\mathcal{L})}^{\text{wot}} \). If \( T \in W \) then

\[
\| (\text{ad} S)^n(T|V) \|^n \leq \| (\text{ad} S)^n T \|^{1/n} \to 0
\]

as \( n \to \infty \) for all \( S \in \mathcal{L} \). Hence \( W|V \subset (\mathcal{L}|V)^{qc} \). By Corollary 4.9(ii), \( W|V \) is scalar, whence

\[
[W, \overline{A(\mathcal{L})}^{\text{wot}}]|V = 0.
\]

Since \([W, \overline{A(\mathcal{L})}]\) and \([Z, \overline{A(\mathcal{L})}^{\text{wot}}]\) are subsets of \([W, \overline{A(\mathcal{L})}^{\text{wot}}]\) and consist of compact operators, (4.5) holds by Lemma 2.6.
We see that, given an operator \( T \in W \) (resp., \( T \in Z \)) and an operator \( S \in \mathcal{A}(\mathfrak{L}) \) (resp., \( S \in \overline{\mathcal{A}(\mathfrak{L})} \)), \( T \) and \( S \) commute modulo the Jacobson radical \( \text{rad} \mathcal{A}(\mathfrak{L}) \). Hence (4.6) holds. □

**Corollary 4.11.** Let \( \mathfrak{L} \subset \mathcal{K}^1(\mathfrak{X}) \) be a nonscalar Lie algebra and \( W \in \text{Lat ad} \mathfrak{L} \) be nonscalar. If \( W \) does not contain nonzero nilpotent finite rank operators then \( \mathfrak{L} \) is reducible.

**Proof.** It is clear that \( W \subset \mathfrak{L}^\text{qc} \) by Corollary 3.13. Then \( \mathfrak{L} \) is reducible by Corollary 4.9(ii). □

The corollary implies in particular the following result:

*If a closed ideal \( J \) of a closed Lie algebra \( \mathfrak{L} \) of compact operators has no (nonzero) nilpotent finite rank operators then \( \mathfrak{L} \) is reducible.*

This is quite surprising even if \( J = \mathfrak{L} \). We will see in Corollary 4.20 that this extends to nonclosed ideals.

4.3. The solution of Volterra Ideal Problem

We start to solve the Volterra Ideal Problem posed in [40]. The following lemma is a key one.

**Lemma 4.12.** Let \( W \subset B(\mathfrak{X}) \) be a nonzero Volterra Lie algebra, \( S \) a scattered operator and \([S, W] \subset W\). Then \( W \subset \text{rad} \overline{\mathcal{A}([S] \cup W)} \) wot and so \( [S] \cup W \) has a nontrivial hyperinvariant subspace.

**Proof.** Set \( A = \overline{\mathcal{A}([S] \cup W)} \). Since \( \overline{\mathcal{A}(W)} \) is a Volterra algebra [41, Theorem 11.6] which is invariant for \( \text{ad} S \), one may suppose that \( W \) is a Volterra (associative!) subalgebra of \( A \). Set \( I = \sum_{n \geq 0} S^n W \). It follows easily from our assumptions that \( I \) is a nonzero two-sided ideal of \( A \) (namely, the closed ideal of \( A \) generated by \( W \)). We consider two cases.

**Case 1:** \( S \) is a Riesz operator.

Note that \( \overline{CS + W} \) consists of Riesz operators because its image in the Calkin algebra is commutative, whence consists of quasi-nilpotents.

**Claim 1.** If \( S + T \) is quasi-nilpotent for every \( T \in W \), then \( [S] \cup W \) is reducible.

Indeed, in this case \( \overline{CS + W} \) consists of quasi-nilpotents (because the spectral radius is continuous on Riesz operators) and is an Engel Lie algebra containing a nonscalar compact operator. So Claim 1 follows from Theorem 2.8.

(!) Thus we assume (changing \( S \) by \( S + T \), if necessary) that \( S \) is not quasi-nilpotent.

Let \( P = P_\lambda(S) \) be the Riesz projection of \( S \) corresponding to a nonzero point \( \lambda \in \sigma(S) \).
Claim 2. For every $T \in W$ and every $n \in \mathbb{N}$

$$\text{tr}(S^n PT) = 0.$$ \hfill (4.7)

Since the left-hand side in (4.7) is linear in $T$, Lemma 3.1 shows that it will suffice to check Claim 2 locally with respect to decomposition of $W$ into a finite sum of spectral subspaces $\mathcal{E}_{\mu,r}(\text{ad}_W S)$ with $r$ sufficiently small. In other words, we have to prove that for any $\mu$ there exists $r > 0$ such that (4.7) holds for any $T \in \mathcal{E}_{\mu,r}(\text{ad}_W S)$.

If $\mu \neq 0$ and $r < |\mu|$, (4.7) follows from the fact that $S^n PT \in \mathcal{E}_{\mu,r}(\text{ad}_S)$, whence $S^n PT$ is nilpotent by Corollary 3.6.

Assume now that $T \in \mathcal{E}_{0,r}(\text{ad}_W S)$. One can suppose using Lemma 3.15 that $r$ is so small that $P$ commutes with $\mathcal{E}_{0,r}(\text{ad}_W S)$. Let $E_0$ be the Lie algebra generated by $\mathcal{E}_{0,r}(\text{ad}_W S)$, and $Y = P\mathfrak{X}$. Since $P$ commutes with $S$ and with every operator from $E_0$, we obtain that $[S|Y, E_0|Y] \subset E_0|Y$ and

$$\text{tr}(S^n PT) = \text{tr}((S|Y)^n (T|Y)).$$

Since $E_0|Y$ is an algebra of nilpotent operators on the finite-dimensional space $Y$, the Engel theorem implies that $\ker E_0|Y \neq 0$, and $\ker E_0|Y \in \text{Lat}([S|Y] \cup E_0|Y)$. Taking a maximal chain $\Gamma \subset \text{Lat}([S|Y] \cup E_0|Y)$, we have that

$$(E_0|Y)|V = 0$$

for any gap-quotient $V$ of $\Gamma$, whence

$$E_0|Y \subset \text{rad} \mathcal{A}([S|Y] \cup E_0|Y)$$

and (4.7) clearly follows.

Claim 3. $P\overline{T} \cup \overline{T} P$ consists of nilpotent operators of finite rank.

For any $T_0 \in PI$, a simple calculation shows that every power of $T_0$ is a linear combination of operators of the form $S^n PT$ with $T \in W$. By Claim 2,

$$\text{tr}(T_0^m) = 0$$

for every $m \in \mathbb{N}$. Since $T_0 \in \mathcal{F}(\mathfrak{X})$, $T_0$ is nilpotent. Since $P\overline{T} \subset \overline{P}T$, we have that $P\overline{T}$ is Volterra. The same is true for $\overline{T} P$ because operators $AB$ and $BA$ can be quasi-nilpotent only simultaneously. Since $P \in \mathcal{F}(\mathfrak{X})$, Claim 3 follows.

Claim 4. $\{S\} \cup W$ is reducible.
Note that $P \overline{T}$ (resp., $\overline{TP}$) is a right (resp., left) ideal of $\overline{T}$ consisting of nilpotents by Claim 3. So

$$P \overline{T} \cup \overline{TP} \subset \text{rad } \overline{T}$$

by well-known properties of the Jacobson radical. If $P \overline{T} \cup \overline{TP}$ is nonzero, $\overline{T}$ is reducible by Lomonosov Lemma; otherwise ker $\overline{T}$ is a nontrivial invariant subspace for $\overline{T}$. Since $\overline{T}$ is a nonzero two-sided ideal of $A$, $A$ is reducible in any case.

**Claim 5.** $W \subset \text{rad } A$.

Indeed, for a gap-quotient $V$ of Lat$(\{S\} \cup W)$, we have

$$W|V = 0$$

by Claim 4 applied to $S|V$ and $W|V$. So Claim 5 follows from Lemma 2.6.

**Case 2.** $S$ is an arbitrary scattered operator.

Note that $\sigma(S)$ has an isolated point $\lambda$. Let, as above, $P = P_\lambda(S)$. One may suppose that $\lambda = 0$ and $W$ is closed. Since $P \in A$, $P\overline{T}$ is a right ideal of $A$ and $P\overline{TP}$ is a left ideal of the algebra $P\overline{T}$ by Theorem 3.17. Let $E$ be the Lie algebra generated by $E_{0,r}(\text{ad}_W S)$, where $r$ is chosen (see Theorem 3.17 (ii)) in such a way that $P$ commutes with every operator in $E_{0,r}(\text{ad}_W S)$ and $P WP = E_{0,r}(\text{ad}_W S)P$. Then

$$P WP = \sum_{n \geq 0} (SP)^n (PWP) = \sum_{n \geq 0} (SP)^n (E_{0,r}(\text{ad}_W S)P) \subset \sum_{n \geq 0} (SP)^n (PE). \quad (4.8)$$

It is easy to see that $SP$ is a quasi-nilpotent operator (whence $SP$ is a Riesz operator) and

$$[SP, PE] \subset PE.$$ 

Since $P$ commutes with every operator in $E \subset W$, $PE$ is a Volterra Lie algebra. So we can apply the case 1 to $SP$ and $PE$. Let $A_P = \AA((SP) \cup PE)$. By Claim 5 applied to $SP$ and to $PE$, we have that

$$PE \subset \text{rad } A_P,$$

whence $PIP$ consists of Volterra operators by (4.8). Then $P\overline{TP}$ is a (closed) left Volterra ideal of the closed algebra $P\overline{T}$, whence

$$P\overline{TP} \subset \text{rad } P\overline{T}.$$
Hence \((P\overline{T})^2 \subseteq \text{rad} \; P\overline{T}, \; P\overline{T} = \text{rad} \; P\overline{T}\). Since \(P\overline{T}\) is a right ideal of \(A\),

\[ P\overline{T} \subseteq \text{rad} \; A. \]

A similar argument shows that \(TP \subseteq \text{rad} \; A\). Arguing as in the proof of Claim 4, we see that \(A\) is reducible. At last, reducing to gap-quotients of \(\text{Lat} \; A\) in the same way as in Claim 5, we obtain that \(W \subseteq \text{rad} \; A\). In particular,

\[ W \subseteq \text{rad} \; \overline{A}\wot\]

by Lemma 2.6, and \(A\) has a hyperinvariant subspace by Theorem 2.1. \(\square\)

**Theorem 4.13.** Let a scattered Lie algebra \(\mathcal{L} \subset B(\mathfrak{X})\) have a nonzero Volterra ideal \(W\). If \(\overline{\mathcal{L}}\) contains a nonzero finite rank operator then it has a nontrivial hyperinvariant subspace.

**Proof.** Since a perturbation of a scattered operator by a compact operator is a scattered operator by Proposition 3.23, one may suppose, replacing \(W\) by \(\mathcal{A}(W)\) and then \(\mathcal{L}\) by \(\mathcal{L} + W\), that \(W\) is a closed Volterra algebra which is a Lie ideal of \(\mathcal{L}\). Clearly, \(W\) is also a Lie ideal of \(\overline{\mathcal{L}}\).

Let \(I = \mathcal{L} \cap \mathcal{F}(\mathfrak{X})\). Then \(I\) is a nonzero Lie ideal of \(\overline{\mathcal{L}}\), whence \(I \cap W\) is a Lie ideal of \(\overline{\mathcal{L}}\), too. If \(T \in I \cap W\) then

\[ \text{tr}(S^nT) = 0 \]

for all \(S \subset \mathcal{L}\), by Lemma 4.12. So, if \(I \cap W \neq 0\), \(\mathcal{L}\) has a nontrivial hyperinvariant subspace by Theorem 4.3.

Suppose that \(I \cap W = 0\). Since \([I, W] \subset I \cap W\), we have that

\[ [I, W] = 0, \]

whence \(IW = WI\). If \(IW = 0\) then \(\ker I\) is a nontrivial hyperinvariant subspace for \(\mathcal{L}\). So one may suppose that \(IW\) is nonzero. Note that

\[ \mathcal{A}(I)W = W\mathcal{A}(I), \]

so \(\mathcal{A}(I)W\) is an operator algebra.

Let \(J = \mathcal{A}(I)W\). Clearly \(J\) consists of nilpotent finite rank operators. Note that \([\mathcal{L}, J] \subset J\). Hence, \(J\) is a nonzero Lie ideal of \(\mathcal{L} + J\) consisting of nilpotent finite rank operators, while \(\mathcal{L} + J\) is a scattered Lie algebra. Applying again Lemma 4.12, we see that

\[ \text{tr}(S^nT) = 0 \]
for every $T \in J$, $S \in \mathfrak{L} + J$ and $n \in \mathbb{N}$. By Theorem 4.3, $\mathfrak{L} + J$ has a nontrivial hyperinvariant subspace and so has $\mathfrak{L}$ because $\mathfrak{L}' = (\mathfrak{L} + J)'$. □

**Theorem 4.14.** (i) If $\mathfrak{L} \subset \mathcal{B}(\mathfrak{X})$ is a closed Lie algebra of operators with one-point essential spectra and $W$ is a nonzero Volterra Lie ideal of $\mathfrak{L}$ then $\mathfrak{L}$ is reducible.

(ii) Any Lie algebra $W$ of Volterra operators is $\mathcal{K}^1$-supertriangularizable. As a consequence, 

$$W \subset \text{rad} \overline{\mathcal{A}(\text{Nor}(W) \cap \mathcal{K}^1(\mathfrak{X}))}.$$ 

**Proof.** (i) Suppose, to the contrary, that $\mathfrak{L}$ is irreducible. Then $W \cap \mathcal{F}(\mathfrak{X}) = 0$ by Theorem 4.13, and therefore 

$$\rho(\text{ad}_W S) = 0$$ 

for all $S \in \mathfrak{L}$, by Corollary 3.13. We see that the quasi-commutant of $\mathfrak{L}$ is not scalar because it contains $W$. Then $\mathfrak{L}$ is reducible by Theorem 4.9(i), a contradiction.

(ii) Follows from (i). □

The theorem solves VIP with some exceeding: for Lie algebras of Riesz operators. It would be interesting to know if it can be extended to all Lie algebras of scattered operators.

**Corollary 4.15.** Let $\mathfrak{L}$ be a scattered Lie algebra and $W = \overline{\mathcal{A}(\mathfrak{L} \cap \mathcal{K}(\mathfrak{X}))}$. If $\text{rad} W$ differs from $(0)$ and $W$ then $\mathfrak{L}$ has a nontrivial hyperinvariant subspace.

**Proof.** It is easy to check that $[\mathfrak{L}, W] \subset W$. Since $\text{rad} W$ is invariant for bounded derivations of $W$, 

$$[\mathfrak{L} + W, \text{rad} W] \subset \text{rad} W.$$ 

Note that $\mathfrak{L} + W$ consists of scattered operators by Proposition 3.23 and contains a nonzero finite rank operator (for instance, a Riesz projection of a nonquasi-nilpotent compact operator in $W$). By Theorem 4.13, $\mathfrak{L} + W$ has a nontrivial hyperinvariant subspace, whence so has $\mathfrak{L}$. □

If $\text{rad} V \neq 0$ for $V = \overline{\mathcal{A}(\mathfrak{L}) \cap \mathcal{K}(\mathfrak{X})}$, the statement of the corollary would trivially follow from Theorem 2.1 even for an arbitrary Lie algebra $\mathfrak{L} \subset \mathcal{B}(\mathfrak{X})$. However, in general $V \neq W$ and $W$ is not an ideal of $\overline{\mathcal{A}(\mathfrak{L})}$.

**4.4. Superinvariant subspaces for commuting operators**

Here, we find some conditions which guarantee the existence of superinvariant subspaces for commutative families of operators. By using them we prove a result on triangularization promised after Theorem 2.7.
Lemma 4.16. Let $L \subset K^1(\mathfrak{x})$ be a Lie algebra, and let $J \in \text{Lat \, ad} \, L$ be a commutative subspace. Then $[L, J] \subset \text{rad} \, \mathcal{A}(L \cup J)^{wot}$, so $L \cup J$ is reducible whenever $L$ and $J$ are not scalar.

Proof. Note that $J$ is a solvable ideal of the Lie algebra $L + J$. Therefore for a finite-dimensional $\mathfrak{x}$ the statement follows immediately from [42, Theorem 1]. So we assume that $\dim \mathfrak{x} = \infty$.

One may suppose that $[L, J]$ is nonzero. Let $W = \mathcal{A}(J)$, and let $M$ be the set of all nilpotent finite rank operators in $W$. It is clear that $M$ is a commutative operator algebra. It is easy to check that 

$$(\text{ad} \, T)^n S^n = n![T, S]^n$$

for every $S \in L, T \in W$ and $n \in \mathbb{N}$. So, if $T \in M$ then $\text{ad} \, T$ is nilpotent and we obtain that $[S, T]$ is a nilpotent operator of finite rank. Hence, we conclude that 

$$[L, M] \subset M.$$ 

If $[S, T]$ is nonzero for some $S \in L$ and $T \in M$, then the image $Y$ of $[S, T]$ is a finite-dimensional invariant subspace for $M$, and clearly $\ker(M|Y) \neq 0$. Hence $\ker M$ is nonzero and is a nontrivial invariant subspace for $L \cup W$. If $[L, M] = 0$ and $M$ contains a nonscalar operator $T$ then $\ker T$ is a nontrivial invariant subspace for $L \cup J$.

It remains to consider the case that $W$ contains no nilpotent finite rank operators. In this case

$$W \subset \mathfrak{L}^{qc}$$

by Corollary 3.13. Since $\text{ad}_W \, W = 0$, it is easy to see that

$$W \subset (L + W)^{qc}.$$ 

Let $I$ be the set of all Volterra operators in $W$. Since $[\mathfrak{L}, W] \subset I$ by Kleinecke–Shirokov theorem [38,23], $I$ is a closed ideal of the Lie algebra $L + W$. We have that

$$I \subset (\mathfrak{L} + W)^{qc} \subset (\mathfrak{L} + I)^{qc}$$

and hence

$$[I, \mathcal{A}(\mathfrak{L} \cup I)] \subset \text{rad} \, \mathcal{A}(\mathfrak{L} \cup I)$$

by Corollary 4.10. Since $I$ consists of quasi-nilpotents,

$$I \subset \text{rad} \, \mathcal{A}(\mathfrak{L} \cup I).$$
by Le Page’s theorem [25]. Let $B = I + \mathcal{L}I + \mathcal{L}^2I + \ldots$. Clearly,

$$I \subset \text{rad } B.$$ 

On the other hand, it is easy to check that $B$ is a two-sided ideal of $\mathcal{A}(\mathcal{L} \cup W)$ (because $IW = WI \subset I$). Therefore

$$I \subset \text{rad } B \subset \text{rad } \mathcal{A}(\mathcal{L} \cup W).$$

Since $I \neq 0$, $\mathcal{L} \cup W$ is reducible.

We have proved that if $[\mathcal{L}, J] \neq 0$ then $\mathcal{L} \cup J$ is reducible. Then $[\mathcal{L}, J]|V = 0$ for any gap-quotient $V$ of $\text{Lat}(\mathcal{L} \cup J)$. Hence

$$[\mathcal{L}, J] \subset \text{rad } \mathcal{A}(\mathcal{L} \cup J)^{wot}$$

by Lemma 2.6.

It remains to note that if $[\mathcal{L}, J] = 0$, $\mathcal{L}$ and $J$ are nonscalar, then $\mathcal{L} \cup J$ is in the commutant of some nonscalar $S \in J$ commuting with a nonzero compact operator in $\mathcal{L} + C$. So $\mathcal{L} \cup J$ is reducible by Lomonosov’s results [26].

We do not know if the assertion “any commutative set of compact operators has a superinvariant subspace” holds. The problem reduces easily to the case of commutative algebras. The next result gathers some sufficient conditions. The easy condition (iii) is included because of its contrast with (iv).

**Theorem 4.17.** A nonscalar commutative algebra $W$ of compact operators has a superinvariant subspace if at least one of the following conditions holds:

(i) $W$ is not Volterra.
(ii) $W$ contains a nonzero operator of finite rank.
(iii) $W^k = 0$ for some integer $k > 1$.
(iv) $\cap W^n^{wot} \neq 0$.

**Proof.** One may suppose that $\dim \mathfrak{X} = \infty$ and $W$ is closed. Let $\mathcal{L} = \text{Nor}(W)$.

(i) If $W \neq \text{rad } W$ then there exists a nontrivial Riesz projection $P$ corresponding to a nonzero point of the spectrum of some operator in $W$. It is obvious that $P \in W$, whence

$$(\text{ad } P)^2 \mathcal{L} = 0.$$ 

Since $\text{ad } P = (\text{ad } P)^3$, we obtain that $P$ commutes with $\mathcal{L} + W$. Therefore, $\mathcal{L} \cup W$ is reducible.
(ii) Let $W = \text{rad } W$, and let $W$ contain a nonzero operator of finite rank. As in the proof of Lemma 4.16, we obtain that $\ker M$ is a nontrivial invariant subspace for $L + W$, where $M$ is the set of all finite rank operators in $W$.

(iii) If $W^{k-1} \neq 0$ then $\ker W^{k-1}$ is clearly a nontrivial invariant subspace for $L \cup W$.

(iv) In virtue of (i) one may suppose that $W$ is a Volterra algebra. Put $I = W^{\text{wot}}$ and $J = \cap W^n^{\text{wot}}$. Then $J$ is a two-sided ideal of $I$. Let $H$ be a two-sided ideal of $A(L \cup I)$ generated by $J$. It is clear that

$$A(L \cup I) = \sum_{i=0}^{\infty} \hat{W}^i I, \quad H = \sum_{i=0}^{\infty} \hat{W}^i J$$

and

$$A(L \cup I)^{\text{wot}} = A(\hat{W} \cup W)^{\text{wot}}.$$

We claim that $H \subset A(\hat{W} \cup W)^{\text{wot}}$. Indeed, let $T \in J$ be arbitrary. It follows from Lemma 4.1 that it is sufficient to show that $S^n T \in A(\hat{W} \cup W)^{\text{wot}}$ for every $S \in L$ and integer $n \geq 0$. But it follows from a simple induction argument that $S^n W^n \subset A(\hat{W} \cup W)$, whence

$$S^n W^n^{\text{wot}} \subset A(\hat{W} \cup W)^{\text{wot}}$$

and so $S^n T \in A(\hat{W} \cup W)^{\text{wot}}$.

It is easy to see that $\hat{W} + W$ is an operator Lie algebra, and

$$[\hat{W} + W, W] \subset W.$$

Moreover, $\hat{W} + W$ consists of Volterra operators. Indeed, if $V$ is a gap-quotient of $\text{Lat}(\hat{W} \cup W)$ then $W|V = 0$ by Lemma 4.16. Hence

$$\hat{W} + W \subset \text{rad } A(\hat{W} \cup W)$$

by Lemma 2.6. Therefore $A(\hat{W} \cup W)$ is a Volterra algebra. Since $\text{Lat } A(\hat{W} \cup W) = \text{Lat } A(\hat{W} \cup W)^{\text{wot}}$, $H$ and $H^{\text{wot}}$ are reducible. Since $H^{\text{wot}}$ is a nonzero two-sided ideal of $A(\hat{W} \cup W)^{\text{wot}}$, $\hat{W} \cup W$ is reducible by Wojtyński [52, Lemma 5].

Clearly the condition (iv) of Theorem 4.17 holds if the identity operator belongs to $W^{\text{wot}}$. More generally, we have the following result:

**Corollary 4.18.** A commutative linear manifold $W \subset K(\mathfrak{X})$ is supertriangularizable if $W \cap F(\mathfrak{X})$ is dense in $W$ or $A(W) \subset (A(W))^2$.
Proof. Let $\mathcal{L} = \text{Nor} W$, and let $V$ be a gap-quotient of $\text{Lat} \mathcal{L}$.
If $W \cap \mathcal{F}(\mathfrak{X})$ is dense in $W$ then so is $W|V$, whence $W|V$ is scalar by Theorem 4.17(ii).
If $\mathcal{A}(W) \subset (\mathcal{A}(W))^{2 \text{wot}}$ then it follows from $\text{Lat} W = \text{Lat} \mathcal{A}(W)^{\text{wot}}$ that
$$\mathcal{A}(W|V) = \mathcal{A}(W)|V \subset (\mathcal{A}(W))^{2 \text{wot}}|V \subset (\mathcal{A}(W|V))^{2 \text{wot}},$$
whence $\mathcal{A}(W|V)^{\text{wot}} = (\mathcal{A}(W|V))^{2 \text{wot}}$. It follows from Theorem 4.17(iv) that $W|V$ is scalar. □

As an application we obtain

Corollary 4.19. A triangularizable set $M$ in $\mathcal{K}^1(\mathfrak{X})$ is $\mathcal{K}^1$-supertriangularizable.

Proof. Let $N = (\text{Nor} \mathcal{A}(M)) \cap \mathcal{K}^1(\mathfrak{X})$. Since $\text{Nor}(M) \subset \text{Nor} \mathcal{A}(M)$, it suffices to prove that there exists a complete chain $\Gamma \subset \text{Lat} N$ such that $M$ is scalar on every gap-quotient of $\Gamma$.
Let $\Gamma$ be a maximal chain in $\text{Lat} N$. Suppose, to the contrary, that $M|V$ is not scalar for some gap-quotient $V$ of $\Gamma$. Since $\mathcal{A}(M)$ is commutative modulo the Jacobson radical, so is $\mathcal{A}(M|V)$. Let $J = \text{rad} \mathcal{A}(M|V)$. Since $N|V \subset \text{Nor} \mathcal{A}(M|V)$, we obtain that
$$[N|V, J] \subset J.$$ If $J \neq 0$ then $N|V$ is reducible by Theorem 4.14(i), a contradiction. So $J = 0$, $M|V$ is commutative and generates a semisimple closed algebra. Since $M|V$ is nonscalar, $N|V$ is reducible by Theorem 4.17, again a contradiction. □

Corollary 4.20. If a closed Lie algebra $\mathcal{L} \subset \mathcal{K}^1(\mathfrak{X})$ has a nonscalar ideal $J$ with $J \cap \mathcal{F}(\mathfrak{X}) = 0$ then $\mathcal{L}$ is reducible.

Proof. Suppose that $\mathcal{L}$ is irreducible. Then $I := \overline{J} \cap \mathcal{F}(\mathfrak{X})$ is nonzero by Corollary 4.11. By our assumption, $[J, I] = 0$, whence $[J, I] = 0$. Thus $I$ is a nonzero commutative ideal of $\mathcal{L}$. So $\mathcal{L}$ is reducible by Theorem 4.17, a contradiction. □

As a consequence, we conclude that every nonzero ideal of an irreducible closed Lie algebra of compact operators contains a nonzero finite rank operator. It would be interesting to know whether the set of all finite rank operators in an irreducible closed Lie algebra of compact operators is irreducible. We only know that this set is not triangularizable.

4.5. Engel and E-solvable ideals

An ideal $J$ of a normed Lie algebra is called quasi-central (resp., Engel) if it is a quasi-central (resp., Engel) Lie algebra. Recall that the first definition means that $\lim \| (\text{ad} J a)^n b \|^{1/n} = 0$ for all $a, b \in J$, and the second one means that
\[ \lim \| (\text{ad}_J a)^n \|^{1/n} = 0 \text{ for all } a \in J. \] Clearly, these classes contain all commutative ideals and all Volterra ideals of operator algebras.

It is standard that a Banach Lie algebra is quasi-central iff it is Engel. The following result implies in particular that, dealing with Lie algebras of compact operators, one need not distinguish Engel and quasi-central ideals.

Recall that \( \tilde{Y} \) means the completion of a normed linear space \( Y \).

**Lemma 4.21.** Let \( \mathcal{L}, M \) be normed Lie algebras, \( J \) a closed Lie ideal of \( \mathcal{L} \), and let \( \pi : \mathcal{L} \to M \) be a bounded homomorphism. Suppose that \( \mathcal{L}/J \) is a quasi-central Lie algebra, and \( f \) is the composition of \( \pi : \mathcal{L} \to \overline{\pi\mathcal{L}} \) and the quotient map \( \overline{\pi\mathcal{L}} \to \overline{\pi\mathcal{L}}/\overline{\pi J} \), where \( \overline{\pi\mathcal{L}}, \overline{\pi J} \) are the corresponding norm-closures in \( M \). If \( \text{ad}_{\overline{\pi J}} \pi \mathcal{L} \) (resp., \( \text{ad}_{\overline{\pi J}} \pi \mathcal{L} \)) is scattered then \( f(\mathcal{L}) \) (resp., \( f(\mathcal{L}) \)) is Engel.

**Proof.** First we claim that \( f(\mathcal{L}) \) is a quasi-central Lie algebra. Indeed,

\[
\| (\text{ad}(\pi a + \overline{\pi J}))^n (\pi b + \overline{\pi J}) \|^{1/n} = \text{inf}_{x \in J} \left\| (\text{ad} (\pi a)^n \pi b + \pi x \right\|^{1/n} \\
\leq \| \pi \|^{1/n} \cdot \text{inf}_{x \in J} \| (\text{ad} (a)^n b + x \right\|^{1/n} \\
= \| \pi \|^{1/n} \cdot \| (\text{ad} (a)^n (b + J) \right\|^{1/n} \to 0
\]

as \( n \to \infty \), for every \( a, b \in \mathcal{L} \). Also, if \( \text{ad}_{\overline{\pi J}} \pi \mathcal{L} \) is scattered then so is \( \text{ad}_{\overline{\pi J}} \pi \mathcal{L} \). So, without loss of generality, one may assume that \( M = \mathcal{L} \) and \( \pi \) is the identity homomorphism.

Suppose now that \( \text{ad}_{\overline{\pi J}} \mathcal{L} \) is scattered; we have to prove that \( \mathcal{L}/J \) is Engel. Note that \( \mathcal{L}/J \) is isometrically isomorphic to \( \overline{\mathcal{L}}/\overline{J} \). One can identify \( \overline{J} \) with the corresponding ideal of \( \overline{\mathcal{L}} \). Given an element \( a \in \mathcal{L} \) in an equivalence class \( b \in \overline{\mathcal{L}}/\overline{J} \), the operator \( \text{ad}_{\overline{\pi J}}\overline{b} \) coincides with the operator induced on \( \overline{\mathcal{L}}/\overline{J} \) by \( \text{ad}_{\overline{\mathcal{L}}/\overline{J}} a \), and \( \text{ad}_{\overline{\mathcal{L}}/\overline{J}} a \) is clearly scattered. Hence \( \text{ad}_{\overline{\pi J}} \mathcal{L}/J \) is scattered. Since \( \mathcal{L}/J \) is quasi-central and \( \text{ad}_{\overline{\pi J}} b \) is scattered for every \( b \in \mathcal{L}/J \), we obtain that \( \text{ad}_{\overline{\pi J}} b \) is quasi-nilpotent for every \( b \in \mathcal{L}/J \), by Proposition 3.3(iii). In particular

\[
\lim \| (\text{ad}_{\overline{\pi J}} b)^n \|^{1/n} = 0
\]

for every \( b \in \mathcal{L}/J \).

If now \( \text{ad}_{\overline{\pi J}} \mathcal{L} \) is scattered then, as we saw, so is \( \text{ad}_{\overline{\pi J}} \mathcal{L}/J \). Since \( \text{ad}_{\overline{\pi J}} \mathcal{L}/J \) consists of quasi-nilpotents, by the above argument, and \( \rho \) is continuous on scattered operators, \( \text{ad}_{\overline{\pi J}} \mathcal{L}/J \) consists of quasi-nilpotents. \( \square \)

It follows from Lemma 4.21 that quasi-central quotients of a Lie algebra \( \mathcal{L} \subset \mathcal{K}^1(X) \) by closed ideals are in fact Engel.

Note that a Lie algebra \( \mathcal{L} \subset \mathcal{K}^1(X) \) is Engel iff \( (\mathcal{L} + \mathbb{C}) \cap \mathcal{K}(X) \) is Engel and iff \( (\mathcal{L} + \mathbb{C})/\mathbb{C} \) is Engel. The first statement is evident, and, for the second one, suppose
that \( \mathfrak{L} \) is not Engel while \( (\mathfrak{L} + \mathbb{C})/\mathbb{C} \) is Engel. Then there exist non scalar \( S, T \in \mathfrak{L} + \mathbb{C} \) such that \( T \in \mathcal{E}_\lambda(ad S) \) for some nonzero \( \lambda \in \mathbb{C} \). On the other hand, \( \mathfrak{L} + \mathbb{C}/\mathbb{C} \) is Engel by Lemma 4.21, whence

\[
T + \mathbb{C} \in \mathcal{E}_\lambda(ad_{\mathfrak{L} + \mathbb{C}/\mathbb{C}}(S + \mathbb{C})) = 0
\]

and \( T \in \mathbb{C} \), a contradiction.

**Lemma 4.22.** If \( J \) is an Engel ideal of a Lie algebra \( \mathfrak{L} \subset K^1(\mathfrak{X}) \) then \( J \) is in the center of \( \mathcal{A}(\mathfrak{L}) \) modulo the Jacobson radical.

**Proof.** It follows from Lemma 4.21 that the norm-closure of \( J \) is Engel. So \( J \) is triangularizable by Theorem 2.8. Hence \([J,J]\) is a Lie ideal of \( \mathfrak{L} \) consisting of Volterra operators, and

\[
[J,J] \subset \text{rad} \mathcal{A}(\mathfrak{L})
\]

by Theorem 4.14. Let \( V \) be a gap-quotient of \( \text{Lat} \mathfrak{L} \). Then \( J|V \) is commutative. It follows from Lemma 4.16 that \( J|V \) is in the center of \( \mathcal{A}(\mathfrak{L}|V) \) modulo the Jacobson radical. Since \( \mathcal{A}(\mathfrak{L}|V) \) is semisimple, \( J|V \) is scalar, whence \([\mathfrak{L},J]|V = 0 \). Hence

\[
[\mathcal{A}(\mathfrak{L}),J] \subset \text{rad} \mathcal{A}(\mathfrak{L})
\]

by Lemma 2.6. \( \Box \)

In particular, a Lie algebra of compact operators with a non scalar Engel ideal is reducible. It will be convenient for further references to formulate this result for Lie subalgebras of \( K^1(\mathfrak{X}) \).

**Corollary 4.23.** A Lie algebra \( \mathfrak{L} \subset K^1(\mathfrak{X}) \) having a non scalar Engel ideal is reducible.

This complements the positive answer to VIP. Now, it will be extended to a more wide class of ideals.

Let \( \mathfrak{L} \) be a normed Lie algebra. We say that \( \mathfrak{L} \) is \( E \)-solvable if any nonzero quotient of \( \mathfrak{L} \) by a closed ideal contains a nonzero closed Engel ideal. An ideal of \( \mathfrak{L} \) is called \( E \)-solvable if it itself is an \( E \)-solvable Lie algebra.

**Theorem 4.24.** The image of a bounded representation of an \( E \)-solvable Lie algebra \( \mathfrak{L} \) by compact operators on a Banach space is triangularizable.

**Proof.** It suffices to prove that if \( \pi : \mathfrak{L} \to K(\mathfrak{X}) \) is a bounded irreducible representation of \( \mathfrak{L} \) then \( \dim \mathfrak{X} = 1 \). Suppose, to the contrary, that \( \dim \mathfrak{X} > 1 \). If \( \dim \mathfrak{X} < \infty \) then clearly \( \pi \mathfrak{L} \) has a finite complete chain of its Lie ideals with commutative gap-quotients
by Lemma 4.21. So \( \pi \mathcal{U} \) is a solvable Lie algebra of operators; by the Lie theorem, \( \dim \mathfrak{X} = 1 \), a contradiction.

Now let \( \dim \mathfrak{X} = \infty \). Let \( J \) be a nonzero Engel ideal of \( \mathcal{U}/\ker \pi \). Then \( \pi J \) is a nonzero Engel ideal of \( \pi \mathcal{U} \) by Lemma 4.21. By Corollary 4.23, \( \pi J \) is scalar. Since \( \pi J \) consists of compact operators, \( \pi J = 0 \), a contradiction. \( \square \)

**Corollary 4.25.** For a Lie algebra \( \mathcal{U} \subset \mathcal{K}^1(\mathfrak{X}) \), the following conditions are equivalent.

(i) \( \mathcal{U} \) is \( E \)-solvable.
(ii) \( \mathcal{U} \) is triangularizable.
(iii) \( \mathcal{U} \) is \( \mathcal{K}^1 \)-supertriangularizable.

**Proof.** (ii) \( \iff \) (iii) was in fact established in Corollary 4.19.

(i) \( \implies \) (ii) The case of \( \dim \mathfrak{X} < \infty \) is obvious. Otherwise define a bounded representation \( \pi : \mathcal{U} \to \mathcal{K}(\mathfrak{X}) \) as follows: for every \( S = \lambda + T \in \mathcal{U} \) with \( \lambda \in \mathbb{C} \) and \( T \in \mathcal{K}(\mathfrak{X}) \), let \( \pi S = T \). Clearly the definition is correct, \( \mathcal{U} \) and \( \pi \mathcal{U} \) are simultaneously triangularizable or not, and the implication follows by Theorem 4.24.

(ii) \( \implies \) (i) Suppose that \( \mathcal{U} \) is triangularizable. Then \( \mathcal{A}(\mathcal{U}) \) is commutative modulo the Jacobson radical, whence \( I := [\mathcal{U}, \mathcal{U}] \) is Engel. Let \( J \) be a closed ideal of \( \mathcal{U} \). If \( I \subset J \) then \( \mathcal{U}/J \) is commutative (hence Engel). Otherwise the image of \( I \) in \( \mathcal{U}/J \) (under the standard map \( \mathcal{U} \to \mathcal{U}/J \)) is a nonzero Engel ideal of \( \mathcal{U}/J \). Thus, \( \mathcal{U} \) is \( E \)-solvable. \( \square \)

As a consequence, \([\mathcal{U}, \mathcal{U}]\) is Volterra for each \( E \)-solvable Lie algebra \( \mathcal{U} \subset \mathcal{K}^1(\mathfrak{X}) \). The converse also follows from the equivalence of (i) and (ii). The other consequence is that any subalgebra of an \( E \)-solvable Lie algebra of compact operators is also \( E \)-solvable.

**Theorem 4.26.** Let \( \mathcal{U} \subset \mathcal{K}^1(\mathfrak{X}) \) be a Lie algebra. If \( W \) is a nonscalar \( E \)-solvable Lie ideal of \( \mathcal{U} \) then \([\mathcal{U}, W] \subset \text{rad} \, \mathcal{A}(\mathcal{U})^{\text{wot}} \) and \( \mathcal{U} \) is reducible.

**Proof.** It suffices to show that \( W|V \) is scalar for any gap-quotient \( V \) in \( \text{Lat} \, \mathcal{U} \). If \( W|V \) is commutative then it is Engel and the statement follows from Corollary 4.23. Otherwise \([W, W]|V \) is a nonzero Volterra ideal of \( \mathcal{U}|V \), in contradiction to Theorem 4.14. \( \square \)

### 4.6. ad-compact elements in operator Lie algebras

**Lemma 4.27.** Let \( \mathcal{U} \subset \mathcal{K}^1(\mathfrak{X}) \) be a nonscalar Lie algebra, and let \( W \in \text{Lat} \, \text{ad} \, \mathcal{U} \) be nonscalar. If \( \dim W < \dim \mathfrak{X} = \infty \) then \( \mathcal{U} \) is reducible.

**Proof.** Suppose, to the contrary, that \( \mathcal{U} \) is irreducible. It follows from Corollary 4.11 that \( I := W \cap \mathcal{F}(\mathfrak{X}) \neq 0 \). Then \( I \in \text{Lat} \, \text{ad} \, \mathcal{U} \) and clearly \( I \mathfrak{X} \) is a nonzero finite-dimensional subspace of \( \mathfrak{X} \). By Lemma 2.2, \( I \mathfrak{X} \) is invariant for \( \mathcal{U} \). \( \square \)

Let \( \mathcal{U} \) be a normed Lie algebra. We say that an element \( a \in \mathcal{U} \) is an \( \text{ad-compact element} \) of \( \mathcal{U} \) if \( \text{ad}_{\mathcal{U}} a \) is a compact operator. It is clear that the center of \( \mathcal{U} \) consists of ad-compact elements of \( \mathcal{U} \).
Lemma 4.28. Let $\mathfrak{L} \subset \mathcal{K}^1(\mathfrak{X})$ be a Lie algebra and $\dim \mathfrak{X} = \infty$. Then $\mathfrak{L}$ is reducible if one of the following conditions holds.

(i) $\mathfrak{L}$ has a nonscalar $\text{ad}$-compact element.
(ii) There exists a compact Lie derivation $D$ of $\mathfrak{L}$ with nonscalar range.

Proof. (i) Clearly $\mathfrak{L}$ can be assumed to be closed. Suppose, to the contrary, that $\mathfrak{L}$ is irreducible. Let $J$ be the set of all $\text{ad}$-compact elements of $\mathfrak{L}$. Since $\text{ad}_\mathfrak{L}[T, S] = [\text{ad}_\mathfrak{L} T, \text{ad}_\mathfrak{L} S]$ for all $T, S \in \mathfrak{L}$, $J$ is a closed Lie ideal of $\mathfrak{L}$. Since $J$ is not scalar, $J$ contains a nonzero finite rank operator by Corollary 4.11. Let $I = J \cap \mathcal{F}(\mathfrak{X})$. If $I$ consists of nilpotent operators then $\mathfrak{L}$ is reducible by Theorem 4.14. So one can suppose that there exists a non-nilpotent operator $S \in I$. Let $\sigma(\text{ad}_\mathfrak{L} S) = \{0, \lambda_1, \ldots, \lambda_n\}$. Then

$$M := \sum_{i=1}^{n} \mathcal{E}_{\lambda_i}(\text{ad}_\mathfrak{L} S) + \sum_{i=1}^{n} [\mathcal{E}_{\lambda_i}(\text{ad}_\mathfrak{L} S), \mathcal{E}_{-\lambda_i}(\text{ad}_\mathfrak{L} S)]$$

is an ideal of $\mathfrak{L}$ by Proposition 3.8. Since $\text{ad}_\mathfrak{L} S$ is a compact operator, all $\mathcal{E}_{\lambda_i}(\text{ad}_\mathfrak{L} S)$ are finite-dimensional. Hence $M$ is finite-dimensional, clearly $M$ is nonscalar. Therefore $\mathfrak{L}$ is reducible by Lemma 4.27, a contradiction.

(ii) Let $S = DT$ be nonscalar for some $T \in \mathfrak{L}$. Since $\text{ad}_\mathfrak{L} S = [D, \text{ad}_\mathfrak{L} T]$, we obtain that $\text{ad}_\mathfrak{L} S$ is compact. Therefore, $\mathfrak{L}$ is reducible by (i). □

Theorem 4.29. If a Lie algebra $\mathfrak{L} \subset \mathcal{B}(\mathfrak{X})$ has a compact derivation $D$ with nonscalar range then $\mathfrak{L} \cap \mathcal{K}(\mathfrak{X})$ is reducible.

Proof. Suppose that $\mathfrak{L}_1 := \mathfrak{L} \cap \mathcal{K}(\mathfrak{X})$ is irreducible. Let $K \in \mathfrak{L}_1$ and $S = D(K)$. Then $\text{ad}_\mathfrak{L} S = [D, \text{ad}_\mathfrak{L} K]$ is a compact derivation of $\mathfrak{L}$ leaving invariant $\mathfrak{L}_1$. By Lemma 4.28(ii) $\text{ad}_\mathfrak{L} S$ must have scalar range. But this means that $\text{ad}_\mathfrak{L}_1 S = 0$ because if a commutator is a scalar operator then it is zero. Hence $S$ commutes with an irreducible set of compact operators; by Lomonosov’s theorem, $S$ must be scalar.

We proved that $D(\mathfrak{L}_1)$ consists of scalar operators. Hence, for any $T \in \mathfrak{L}$ and $K \in \mathfrak{L}_1$, one has that

$$[D(T), K] = D([T, K]) - [T, D(K)] = D([T, K])$$

is a scalar operator. Being a commutator it is zero. Again by Lomonosov’s theorem, $D(T)$ is scalar, $D$ has a scalar range, a contradiction. □

Since a nonzero bounded derivation of an associative normed algebra is nonscalar, and the reducibility of a nonzero ideal implies the reducibility of the algebra itself, we obtain
**Corollary 4.30.** Let $A \subset B(\mathcal{X})$ be an algebra containing a nonzero compact operator and \(\dim \mathcal{X} = \infty\). If there exists a nonzero compact derivation $D$ on $A$ then $A$ is reducible.

For $A = B(\mathcal{X})$ the result (that is the triviality of compact derivations) follows from [14, Example 1] and the fact that every bounded derivation on $B(\mathcal{X})$ is inner [7]; earlier for $A = B(\mathcal{H})$ (where $\mathcal{H}$ is a Hilbert space) it was proved in [16].

A normed Lie algebra $\mathfrak{L}$ is called an ad-compact Lie algebra (a $\mathcal{K}$-algebra, a Lie algebra with compact adjoint action in the terminology of Vaksman and Gurarij [46]) if it consists of ad-compact elements. Clearly, commutative and finite-dimensional algebras are ad-compact. An ideal $J$ of $\mathfrak{L}$ is called an ad-compact ideal if it is an ad-compact Lie algebra.

**Lemma 4.31.** If $J$ is an ad-compact ideal of a normed Lie algebra $\mathfrak{L}$ then $[J, J]$ consists of ad-compact elements of $\mathfrak{L}$.

**Proof.** For any $S, T \in J$, we obtain that

$$\text{ad}_\mathfrak{L} [S, T] \mathfrak{L}(1) \subset (\text{ad}_\mathfrak{L} S)(\text{ad}_\mathfrak{L} T) \mathfrak{L}(1) - (\text{ad}_\mathfrak{L} T)(\text{ad}_\mathfrak{L} S) \mathfrak{L}(1)$$

$$\subset \|\text{ad}_\mathfrak{L} T\| (\text{ad}_J S) J(1) + \|\text{ad}_\mathfrak{L} S\| (\text{ad}_J T) J(1)$$

(recall that $\mathfrak{L}(1)$ and $J(1)$ are closed unit balls), whence $\text{ad}_\mathfrak{L} [S, T]$ is compact and then all operators in $\text{ad}_\mathfrak{L} [J, J]$ are compact. $\square$

**Theorem 4.32.** Let $\mathfrak{L} \subset \mathcal{K}^1(\mathcal{X})$ be a Lie algebra and \(\dim \mathcal{X} = \infty\). Then $\mathfrak{L}$ is reducible if one of the following conditions holds.

(i) $\mathfrak{L}$ has a nonscalar ideal $J$ such that $J' \cap \mathcal{K}(\mathcal{X})$ is nonzero.
(ii) $\mathfrak{L}$ has two nonscalar ideals $J_1$ and $J_2$ with $J_1 \cap J_2 \subset \mathbb{C}$.
(iii) $\mathfrak{L}$ has a nonscalar ad-compact ideal $J$.

**Proof.** (i) Set $W = J' \cap \mathcal{K}(\mathcal{X})$. Then $\mathfrak{L} + W$ is an operator Lie algebra and $W$ is an ideal of $\mathfrak{L} + W$. If $W$ is Volterra then the result follows from Theorem 4.14. Let $W$ be not Volterra. Then there exists an operator $S \in W$ with nonzero spectrum. Therefore, $J$ has a finite-dimensional invariant subspace $Y$, namely the range of a Riesz projection $P$ of $S$, and $P$ commutes with $J$. If $J$ is finite-dimensional, the result follows from Lemma 4.27. Let $J$ be infinite-dimensional. Then the set $I := \{T \in J : TP = 0\}$ is not zero (because $TP = 0$ if $T|Y = 0$). Note that $I$ is an ideal of $\mathfrak{L}$ (see (4.4) in the proof of Theorem 4.8(i)), and ker $I$ is a nontrivial invariant subspace for $\mathfrak{L}$.

(ii) Since $[J_1, J_2] \subset J_1 \cap J_2 \subset \mathbb{C}$, $[J_1, J_2] = 0$ and the result follows from (i).

(iii) If $[J, J]$ is not zero then it contains a nonscalar ad-compact element of $\mathfrak{L}$ by Lemma 4.31. Hence $\mathfrak{L}$ is reducible by Lemma 4.28 (i). If $[J, J] = 0$ then $\mathfrak{L}$ is reducible by Lemma 4.16. $\square$
Theorem 4.33. Let \( \mathfrak{U} \) be a nonzero finite-dimensional Lie algebra of compact operators on an infinite-dimensional Banach space. Then \( \mathfrak{U} \) has a nontrivial superinvariant subspace.

Proof. If \( \mathfrak{U} \) has a nonzero finite rank operator then the proof follows as in Lemma 4.27. Otherwise \( \text{ad}_\mathfrak{U} \text{ } \mathfrak{U} \) is Engel by Corollary 3.13(ii) and then consists of nilpotent operators, whence \( \mathfrak{U} \) is a nilpotent Lie algebra. By Turovskii [42, Theorem 1],

\[
[\mathfrak{U}, \text{Nor}(\mathfrak{U})] \subset \text{rad} \overline{A(\text{Nor}(\mathfrak{U}))}
\]

and \( \text{Nor}(\mathfrak{U}) \) is reducible by Lomonosov’s results [26]. □

Corollary 4.34. Let \( \mathfrak{U} \subset B(\mathfrak{X}) \) be a Lie algebra containing a nonzero compact operator. If \( \dim[\mathfrak{U}, \mathfrak{U}] < \dim \mathfrak{X} = \infty \) then \( \mathfrak{U} \) is reducible.

Proof. Let \( J = \mathcal{K}(\mathfrak{X}) \cap \mathfrak{U} \). Then \( \dim[\mathfrak{U}, J] < \infty \). If \( [\mathfrak{U}, J] \neq 0 \) then the result follows by Theorem 4.33. Otherwise every operator in \( \mathfrak{U} \) commutes with a nonzero compact operator, so \( \mathfrak{U} \) is reducible. □

Corollary 4.35. Any nonscalar commutative finite-dimensional subspace \( W \subset \mathcal{K}(\mathfrak{X}) \) has a nontrivial superinvariant subspace.

Proof. It follows from Theorem 4.33 in the case when \( \dim \mathfrak{X} = \infty \) and from Theorem 2.3 in the case when \( \dim \mathfrak{X} < \infty \). □

The results of the present subsection as well as of the previous one will be used in Section 6.

5. Radical-like ideals of operator Lie algebras

In this section, we show that every operator Lie algebra \( \mathfrak{U} \) has the largest \( E \)-solvable ideal and the largest Engel ideal among ideals consisting of compact operators. We also prove that \( \mathfrak{U} \) has the largest Volterra ideal. The key to these results lies in the consideration of spectral and root ideals of normed Lie algebras with respect to their homomorphisms into Banach algebras. Moreover, this approach allows to characterize the largest Engel and \( E \)-solvable ideals in spectral terms.

5.1. Spectral and root ideals of normed Lie algebras

Here, we describe the structure of all operators \( T \in \mathfrak{U} \) satisfying the spectral condition of Corollary 4.4 in the context of normed Lie algebras.
Let $\mathfrak{L}$ be a normed Lie algebra, $A$ a Banach algebra and $\pi : \mathfrak{L} \to A$ a bounded Lie homomorphism. Sometimes we denote the continuous extension of $\pi$ to the completion $\tilde{\mathfrak{L}}$ of $\mathfrak{L}$ by the same symbol, $\pi$. We define $\mathcal{R}_\pi(\mathfrak{L})$ as the set of all $a \in \mathfrak{L}$ such that

$$\rho(\pi(a + b)) \leq \beta_a + \rho(\pi b)$$

for every $b \in \tilde{\mathfrak{L}}$ and some constant $\beta_a$ depending of $a$.

Let $\mathcal{R}_\pi^0(\mathfrak{L}) = \{a \in \mathfrak{L} : \rho(\pi(a + b)) = \rho(\pi b) \text{ for every } b \in \tilde{\mathfrak{L}}\}$. It is clear that

$$\mathcal{R}_\pi^0(\mathfrak{L}) = \mathfrak{L} \cap \mathcal{R}_\pi^0(\tilde{\mathfrak{L}}) \subset \mathcal{R}_\pi(\mathfrak{L}) = \mathfrak{L} \cap \mathcal{R}_\pi(\tilde{\mathfrak{L}}). \quad (5.1)$$

The following proposition gives a more precise information about $\mathcal{R}_\pi^0(\mathfrak{L})$.

**Proposition 5.1.** Let $\mathfrak{L}$ be a normed Lie algebra, $A$ a Banach algebra and $\pi : \mathfrak{L} \to A$ a bounded Lie homomorphism. If $a \in \mathfrak{L}$ is such that

$$\rho(\pi(a + b)) \leq \beta \rho(\pi b)$$

for every $b \in \tilde{\mathfrak{L}}$ and some constant $\beta > 0$, then $a \in \mathcal{R}_\pi^0(\mathfrak{L})$.

**Proof.** Indeed, the map $\psi : \lambda \mapsto \rho(\lambda \pi a + \pi b)$ is a bounded subharmonic function on $\mathbb{C}$ [48]. Therefore it is constant and the equality $\psi(0) = \psi(1)$ gives the required result. □

**Theorem 5.2.** Let $\pi$ be a bounded Lie homomorphism from a normed Lie algebra $\mathfrak{L}$ into a Banach algebra $A$. Then $\mathcal{R}_\pi(\mathfrak{L})$ and $\mathcal{R}_\pi^0(\mathfrak{L})$ are ideals of $\mathfrak{L}$. Moreover $[\mathfrak{L}, \mathcal{R}_\pi(\mathfrak{L})] \subset \mathcal{R}_\pi^0(\mathfrak{L})$.

**Proof.** It is clear that $\mathcal{R}_\pi(\mathfrak{L})$ and $\mathcal{R}_\pi^0(\mathfrak{L})$ are linear manifolds of $\mathfrak{L}$. Let $a, b \in \tilde{\mathfrak{L}}$ and $c \in \mathcal{R}_\pi(\mathfrak{L})$. Then $\exp(\lambda \ad \pi b)$ is clearly a bounded automorphism of $A$ for each $\lambda \in \mathbb{C}$, and

$$\exp(\lambda \ad \pi b)\pi \tilde{\mathfrak{L}} \subset \pi(\exp(\lambda \ad b)\tilde{\mathfrak{L}}) \subset \pi \tilde{\mathfrak{L}}.$$

We obtain that

$$\rho([\exp(\lambda \ad \pi b)\pi c - \pi c] + \lambda \pi a) = \rho(\pi c + \exp(-\lambda \ad \pi b)(\lambda \pi a - \pi c))$$

$$\leq \beta_c + \rho(\lambda \pi a - \pi c) \leq 2\beta_c + |\lambda| \rho(\pi a), \quad (5.2)$$

where $\beta_c$ is a constant depending on $c$. It is clear that the map

$$\phi : \lambda \mapsto \pi a + (\ad \pi b)\pi c + \lambda(\ad \pi b)^2\pi c/2 + \ldots + \lambda^{n-1}(\ad \pi b)^n\pi c/n! + \ldots$$
is analytic. Then \( \lambda \mapsto \rho(\varphi(\lambda)) \) is a subharmonic function on \( \mathbb{C} \) by Vesentini [48]. It follows from (5.2) that

\[
\rho(\varphi(\lambda)) \leq 2\beta_{\lambda}/|\lambda| + \rho(\pi a)
\]

whenever \( \lambda \neq 0 \). Hence, the map \( \lambda \mapsto \rho(\varphi(\lambda)) \) is bounded and therefore is constant. Then

\[
\rho(\pi a + [\pi b, \pi c]) = \rho(\varphi(0)) = \lim_{|\lambda| \to \infty} \rho(\varphi(\lambda)) \leq \rho(\pi a).
\]

Since \( a, b, c \) are arbitrary, we obtain

\[
[\mathfrak{U}, \mathcal{R}_\pi(\mathfrak{U})] \subset \mathcal{R}_\pi(\mathfrak{U}),
\]

whence \([\mathfrak{U}, \mathcal{R}_\pi(\mathfrak{U})] \subset \mathcal{R}^\circ(\mathfrak{U}). \) So \( \mathcal{R}_\pi(\mathfrak{U}) \) and \( \mathcal{R}^\circ(\mathfrak{U}) \) are ideals of \( \mathfrak{U} \). \( \square \)

We call \( \mathcal{R}_\pi(\mathfrak{U}) \) the \( \pi \)-spectral ideal of \( \mathfrak{U} \) and \( \mathcal{R}^\circ(\mathfrak{U}) \) the \( \pi \)-root ideal of \( \mathfrak{U} \). For every \( a \in \mathcal{R}_\pi(\mathfrak{U}) \), put

\[
\vartheta_\pi(a) = \sup \{ \rho(\pi(a + b)) - \rho(\pi b) : b \in \tilde{\mathfrak{U}} \}.
\]

It is easy to check that \( \vartheta_\pi(a) \) is a seminorm on \( \mathcal{R}_\pi(\mathfrak{U}) \) and \( \vartheta_\pi([a, b]) = 0 \) for all \( b \in \tilde{\mathfrak{U}} \).

Clearly \( \pi \mathcal{R}^\circ(\mathfrak{U}) \) consists of quasi-nilpotent elements. Among the open questions on spectral and root ideals (for instance: is \( \mathcal{R}_\pi(\mathfrak{U}) \) closed in \( \mathfrak{U} \)? Is \( \vartheta_\pi(a) \) equal to \( \rho(\pi a) \) for \( a \in \mathcal{R}_\pi(\mathfrak{U}) \)?) the following ones seem to be the most interesting:

Let \( \mathfrak{U} \) be a normed Lie algebra. Is \( \mathcal{R}^\circ(\mathfrak{U}) \) the largest of ideals of \( \mathfrak{U} \) whose \( \pi \)-images consist of quasi-nilpotent operators? Is this true if \( \overline{\pi \mathfrak{U}} \) is scattered?

We will obtain the positive answer for representations by compact operators (Corollary 5.6) and for the adjoint representation of a Lie algebra of compact operators (Theorem 5.19).

To simplify notations we remove the subscript \( \pi \) when \( \pi \) is the adjoint representation \( \text{ad}_{\mathfrak{U}} \) of a normed Lie algebra \( \mathfrak{U} \) on its completion \( \tilde{\mathfrak{U}} \); we call \( \mathcal{R}(\mathfrak{U}) \) and \( \mathcal{R}^\circ(\mathfrak{U}) \) the spectral and root ideals of \( \mathfrak{U} \), respectively.

If \( \mathfrak{U} \) is a Lie subalgebra of \( B(\mathfrak{X}) \) (or, more generally, of an obviously fixed Banach algebra \( A \)), we consider also the identity homomorphism \( \text{id} \) from \( \mathfrak{U} \) into \( B(\mathfrak{X}) \) (resp., \( A \)) and the adjoint representation \( \text{ad} \) of \( \mathfrak{U} \) on \( B(\mathfrak{X}) \) (resp., \( A \)).

Let us denote by \( \mathcal{Z}_{\text{rad}}(A) \) the center modulo the Jacobson radical of a normed algebra \( A \), that is the preimage of the center of \( A/\text{rad} A \) under the standard epimorphism \( A \to A/\text{rad} A \). If \( A \) is complete, it is evident that

\[
\mathfrak{U} \cap \mathcal{Z}_{\text{rad}}(A) \subset \mathcal{R}_{\text{id}}(\mathfrak{U})
\]
for a Lie subalgebra \( \mathfrak{L} \subset A \). We will see in Theorem 5.9 that this inclusion is in fact an equality if \( \mathfrak{L} \subset \mathcal{K}^1(\mathfrak{X}) \) and \( A = \mathcal{A}(\mathfrak{L}) \).

**Proposition 5.3.** Let \( \mathfrak{L} \subset B(\mathfrak{X}) \) be a Lie algebra. Then

\[
\mathcal{R}_{id}^0(\mathfrak{L}) \subset \mathcal{R}_{ad}^0(\mathfrak{L}) \subset \mathcal{R}^0(\mathfrak{L}).
\]

If \( \mathfrak{L} \) is contained in a proper ideal of \( B(\mathfrak{X}) \) (whence \( \dim \mathfrak{X} = \infty \)) then \( \mathcal{R}_{id}^0(\mathfrak{L}) = \mathcal{R}_{ad}^0(\mathfrak{L}) \).

**Proof.** If \( S \in \mathcal{R}_{id}^0(\mathfrak{L}) \), \( T \in \mathcal{R}_{ad}^0(\mathfrak{L}) \) and \( K \in \tilde{\mathfrak{L}} \) then the functions \( \phi : \lambda \mapsto \rho(\text{ad}(\lambda S + K)) \) and \( \psi : \lambda \mapsto \rho(\text{ad}_{\mathfrak{L}}(\lambda T + K)) \) are bounded on \( \mathbb{C} \) because

\[
\rho(\text{ad}(\lambda S + K)) \leq 2\rho(\lambda S + K) = 2\rho(K)
\]

and

\[
\rho(\text{ad}_{\mathfrak{L}}(\lambda T + K)) \leq \rho(\text{ad}(\lambda T + K)) = \rho(\text{ad}K).
\]

Being subharmonic, \( \phi \) and \( \psi \) are constant. So the equalities \( \phi(0) = \phi(1) \) and \( \psi(0) = \psi(1) \) imply the first statement.

If \( \mathfrak{L} \) lies in a proper ideal of \( B(\mathfrak{X}) \) then \( \rho(F) \leq \rho(\text{ad} F) \) for every \( F \in \mathfrak{L} \), because \( 0 \in \sigma(F) \). Hence \( \mathcal{R}_{ad}^0(\mathfrak{L}) \subset \mathcal{R}_{id}^0(\mathfrak{L}) \). \( \square \)

### 5.2. The largest Volterra ideal

Let us firstly note that if a Lie algebra \( \mathfrak{L} \subset B(\mathfrak{X}) \) is scattered and closed then \( \mathcal{R}_{id}^0(\mathfrak{L}) \) contains all its Volterra ideals:

**Proposition 5.4.** Let \( J \) be a Volterra ideal of a scattered closed Lie algebra \( \mathfrak{L} \subset B(\mathfrak{X}) \) then \( J \subset \mathcal{R}_{id}^0(\mathfrak{L}) \).

**Proof.** Follows from Lemma 4.12. \( \square \)

**Theorem 5.5.** Let \( \mathfrak{L} \subset \mathcal{K}^1(\mathfrak{X}) \) be a Lie algebra. Then

(i) \( \mathcal{R}_{id}^0(\mathfrak{L}) \) is the largest Volterra ideal of \( \mathfrak{L} \).

(ii) \( \mathcal{R}_{id}^0(\mathfrak{L}) = \mathfrak{L} \cap \text{rad} \mathcal{A}(\mathfrak{L}) \).

**Proof.** (i) By Theorem 5.2, \( \mathcal{R}_{id}^0(\mathfrak{L}) \) is a Volterra ideal; by Proposition 5.4, it contains all Volterra ideals.

(ii) The inclusion \( \mathfrak{L} \cap \text{rad} \mathcal{A}(\mathfrak{L}) \subset \mathcal{R}_{id}^0(\mathfrak{L}) \) follows from (i). The converse inclusion was established in Theorem 4.14. \( \square \)
The next result answers the question posed in the previous subsection (p. 39) for the case of representations by compact operators.

**Corollary 5.6.** Let $\mathcal{L}$ be a normed Lie algebra and $\pi : \mathcal{L} \to \mathcal{K}(\mathcal{X})$ its bounded representation. Then $\mathcal{R}^0(\mathcal{L})$ is the largest of ideals $J \subset \mathcal{L}$ such that $\pi J$ consists of quasi-nilpotents.

**Proof.** If $\pi J$ consists of quasi-nilpotents then $\pi J$ is a Volterra ideal of $\pi \mathcal{L}$. By the previous theorem, $\pi J \subset \mathcal{R}^0(\pi \mathcal{L})$, whence $J \subset \mathcal{R}^0(\mathcal{L})$. Since $\pi \mathcal{R}^0(\mathcal{L})$ consists of quasi-nilpotents, we are done. □

Now, we extend Theorem 5.5 to arbitrary operator algebras. Recall that an ideal $J$ of an operator Lie algebra $\mathcal{L}$ is *inner-characteristic* if $J$ is invariant for all $\text{ad} S$ with $S \in \text{Nor}(\mathcal{L})$.

**Corollary 5.7.** Every Lie algebra $\mathcal{L} \subset \mathcal{B}(\mathcal{X})$ has the largest Volterra ideal $\mathcal{V}(\mathcal{L})$ which is equal to $\mathcal{R}^0(\mathcal{L} \cap \mathcal{K}(\mathcal{X}))$. This ideal is closed in $\mathcal{L}$ and is inner-characteristic.

**Proof.** Put $\mathcal{V}(\mathcal{L}) = \mathcal{L} \cap \text{rad} \mathcal{A}(\mathcal{L} \cap \mathcal{K}(\mathcal{X}))$. If $I$ is a Volterra ideal of $\mathcal{L}$ then $I$ is an ideal of $\mathcal{L} \cap \mathcal{K}(\mathcal{X})$. Hence

$$I \subset \text{rad} \mathcal{A}(\mathcal{L} \cap \mathcal{K}(\mathcal{X}))$$

by Theorem 4.14 and therefore $I \subset \mathcal{V}(\mathcal{L})$. On the other hand, $\mathcal{V}(\mathcal{L})$ is Volterra. It follows from Corollary 5.6 that $\mathcal{V}(\mathcal{L}) = \mathcal{R}^0(\mathcal{L} \cap \mathcal{K}(\mathcal{X}))$. In particular, $\mathcal{V}(\mathcal{L})$ is closed in $\mathcal{L}$.

Let $S \in \text{Nor}(\mathcal{L})$. Then $S \in \text{Nor}(\mathcal{L} \cap \mathcal{K}(\mathcal{X}))$. Moreover, $\text{ad} S$ being a bounded derivation of $\mathcal{A}(\mathcal{L} \cap \mathcal{K}(\mathcal{X}))$ preserves $\text{rad} \mathcal{A}(\mathcal{L} \cap \mathcal{K}(\mathcal{X}))$, whence $[S, \mathcal{V}(\mathcal{L})] \subset \mathcal{V}(\mathcal{L})$. □

### 5.3. The largest $E$-solvable ideal consisting of compact operators

Here we prove that $\mathcal{R}_{id}(\mathcal{L})$ is the largest $E$-solvable ideal in a Lie algebra $\mathcal{L}$ of compact operators and thus establish the existence of such an ideal.

**Lemma 5.8.** Let $\mathcal{L} \subset \mathcal{K}^1(\mathcal{X})$ be a Lie algebra. Then

(i) $\mathcal{R}_{id}(\mathcal{L}) \subset \mathcal{Z}_{rad}(\mathcal{A}(\mathcal{L}))$.

(ii) $\sigma(T + S) \subset \sigma(T) + \sigma(S)$ and $\sigma(T S) \subset \sigma(T)\sigma(S)$ for all $S \in \mathcal{R}_{id}(\mathcal{L}), T \in \mathcal{A}(\mathcal{L})$.

**Proof.** By Theorems 5.2 and 5.5,

$$[\mathcal{L}, \mathcal{R}_{id}(\mathcal{L})] \subset \mathcal{R}^0(\mathcal{L}) \subset \text{rad} \mathcal{A}(\mathcal{L}).$$

Hence $[\mathcal{A}(\mathcal{L}), \mathcal{R}_{id}(\mathcal{L})] \subset \text{rad} \mathcal{A}(\mathcal{L})$ which proves (i). Now (ii) follows from (i) and the fact that for a scattered operator $S$, $\sigma(S)$ coincides with the spectrum $\sigma_A(S)$ of $S$ with respect to any closed subalgebra $A \subset \mathcal{B}(\mathcal{X})$ containing $S$. □
Theorem 5.9. Let $\mathfrak{L} \subset \mathcal{K}^1(\mathfrak{X})$ be a Lie algebra. Then $\mathcal{R}_{id}(\mathfrak{L})$ is the largest $E$-solvable ideal of $\mathfrak{L}$ and coincides with $\mathfrak{L} \cap Z_{rad}(\mathcal{A}(\mathfrak{L}))$.

Proof. The inclusion $\mathcal{R}_{id}(\mathfrak{L}) \subset \mathfrak{L} \cap Z_{rad}(\mathcal{A}(\mathfrak{L}))$ was proved in Lemma 5.8. The converse inclusion is evident.

Since $Z_{rad}(\mathcal{A}(\mathfrak{L})) \subset \mathcal{K}^1(\mathfrak{X})$ is commutative modulo the Jacobson radical, it is triangularizable [28], hence $\mathcal{R}_{id}(\mathfrak{L})$ is also triangularizable. By Corollary 4.25, $\mathcal{R}_{id}(\mathfrak{L})$ is $E$-solvable. By Theorem 4.26, it contains all $E$-solvable ideals of $\mathfrak{L}$. □

5.4. Lemmas on spectra of elementary operators

Our next aim is to obtain an intrinsic description of the largest $E$-solvable ideal. Namely, we are going to prove that it coincides with $\mathcal{R}(\mathfrak{L})$. For this, we need to consider general spectral problems for elementary operators on a Banach algebra $\mathcal{B}$ of compact operators (by definition, elementary operators are polynomials in operators of the left and right multiplication). Actually, we can consider elementary operators with coefficients from $\mathcal{B}$ not only on $\mathcal{B}$ but also on $\mathcal{B}(\mathfrak{X})$ or on any closed subspace of $\mathcal{B}(\mathfrak{X})$ invariant under left and right multiplications by elements of $\mathcal{B}$. In all cases, we preserve the notations $L_T, R_T$ for the operators of left and right multiplication by $T \in \mathcal{B}$.

Our calculations in this subsection are quite long and complicated because we could not overcome the following obstacle:

Let $\mathcal{B}$ be a closed algebra of compact operators. Is it true that

$$L_{rad} B \cup R_{rad} B \subset \text{rad} \mathcal{A}(L_B \cup R_B)?$$

In what follows, for an algebra $B \subset \mathcal{B}(\mathfrak{X})$ we set $B_K = B \cap \mathcal{K}(\mathfrak{X})$.

Lemma 5.10. Let $B \subset \mathcal{K}^1(\mathfrak{X})$ be a closed algebra, $Q$ the set of all quasi-nilpotents in $B$. Then

$$\mathcal{A}(L_{Z_{rad}(B)} \cup R_{Z_{rad}(B)}), \mathcal{A}(L_B \cup R_B) \subset \mathcal{A}(L_{rad} B R_B \cup L_B R_{rad} B),$$

$$\mathcal{A}(L_{rad} B R_{B_K} \cup L_B R_{rad} B) \subset \text{rad} \mathcal{A}(L_B \cup R_B)$$

and $L_Q + R_Q + \text{rad} \mathcal{A}(L_B \cup R_B)$ consists of quasi-nilpotents.

Proof. If $F, G \in Z_{rad}(B)$ and $S, T \in B$ then


$$= L_{FS} R_{[T, G]} + L_{[S, F]} R_{TG} \in \mathcal{A}(L_{rad} B R_B \cup L_B R_{rad} B).$$

Now, the first inclusion is immediate.
It is clear that \( \text{rad } B \subset K(\mathfrak{X}) \). Then \( L_{\text{rad } B} R_{B \mathcal{K}} \cup L_{B \mathcal{K}} R_{\text{rad } B} \) consists of compact operators by Vala’s theorem \([47]\). It is obvious that \( L_{\text{rad } B} R_{B \mathcal{K}} \cup L_{B \mathcal{K}} R_{\text{rad } B} \) is a multiplicative semigroup consisting of Volterra operators. By \([43, \text{ Theorem 4}]\), \( \mathcal{A}(L_{\text{rad } B} R_{B \mathcal{K}} \cup L_{B \mathcal{K}} R_{\text{rad } B}) \) is a Volterra algebra. Since \( \overline{\mathcal{A}(L_{\text{rad } B} R_{B \mathcal{K}} \cup L_{B \mathcal{K}} R_{\text{rad } B})} \) is a two-sided ideal of \( \mathcal{A}(L_B \cup R_B) \),

\[
\mathcal{A}(L_{\text{rad } B} R_{B \mathcal{K}} \cup L_{B \mathcal{K}} R_{\text{rad } B}) \subset \text{rad } \mathcal{A}(L_B \cup R_B).
\]

Further, for \( S = L_{T_1} + R_{T_2} + S_0 \), where \( S_0 \in \text{rad } \mathcal{A}(L_B \cup R_B) \) and \( T_1, T_2 \in Q \), we have

\[
\rho(S) = \rho(L_{T_1} + R_{T_2} + S_0) = \rho(L_{T_1} + R_{T_2}) \leq \rho(L_{T_1}) + \rho(R_{T_2}) = \rho(T_1) + \rho(T_2) = 0. \quad \square
\]

Let \( B \subset \mathcal{B}(\mathfrak{X}) \) be a closed algebra. For an admissible contour \( \Gamma \subset \mathbb{C} \), we consider the algebra \( \mathcal{A}_\Gamma(B) \) of mappings from \( \Gamma \) into \( \mathcal{B}(B) \) generated by elements of \( \mathcal{A}(L_B \cup R_B) \) considered as constant maps and the maps of the form \( \mu \mapsto (F - \mu)^{-1} \), where \( F \in \mathcal{A}(L_B \cup R_B) \) and \( \Gamma \subset \text{res}(F) \).

**Lemma 5.11.** Let \( B \subset \mathcal{K}^1(\mathfrak{X}) \) be a closed algebra, \( \Gamma \subset \mathbb{C} \) an admissible contour, \( G_1(\mu) \) and \( G_2(\mu) \) belong to \( \mathcal{A}_\Gamma(B) \). Then, for every \( S \in \text{rad } B \), there exist \( S_1, S_2 \in \text{rad } B \) such that

\[
\left( \int_{\Gamma} G_1(\mu) L_S G_2(\mu) d\mu - L_{S_1} \right), \left( \int_{\Gamma} G_1(\mu) R_S G_2(\mu) d\mu - R_{S_2} \right) \in \text{rad } \overline{\mathcal{A}(L_B \cup R_B)}.
\]

**Proof.** For \( i = 1, 2 \), one has \( G_i(\mu) = p_i((F_1 - \mu)^{-1}, \ldots, (F_m - \mu)^{-1}, F_{m+1}, \ldots, F_n) \), where \( p_i \) is a noncommutative polynomial and \( F_1, \ldots, F_n \in \mathcal{A}(L_B \cup R_B) \). It follows from the definition that \( F_j = \lambda_j + L_{T_j} + R_{K_j} + H_j \), where \( \lambda_j \in \mathbb{C}, T_j, K_j \in B_\mathcal{K} \) and \( H_j \in \mathcal{A}(L_{B \mathcal{K}} R_{B \mathcal{K}}) \). Since all \( F_j, \lambda_j + L_{T_j} \) and \( \lambda_j + R_{K_j} \) are scattered, there exists an admissible contour \( \Gamma' \) in the intersection of resolvent sets of operators mentioned above such that

\[
\int_{\Gamma'} G_1(\mu)L_S G_2(\mu) d\mu = \int_{\Gamma'} G_1(\mu)L_S G_2(\mu) d\mu.
\]

Let \( S_1 = \int_{\Gamma'} p_1((\lambda_1 + T_1 - \mu)^{-1}, \ldots, (\lambda_n + T_n)^{-1}, (\lambda_n + T_n)^{-1}, \ldots, (\lambda_n + T_n)^{-1}) d\mu \). Then \( S_1 \in \text{rad } B \). Since

\[
(F_j - \mu)^{-1} = -(F_j - \mu)^{-1}(R_{K_j} + H_j)(\lambda_j + L_{T_j} - \mu)^{-1} + (\lambda_j + L_{T_j} - \mu)^{-1}
\]

for \( j \leq m \), and \( F_j = (R_{K_j} + H_j) + (\lambda_j + L_{T_j}) \) for \( j > m \), a simple calculation shows that \( \int_{\Gamma'} G_1(\mu)L_S G_2(\mu) d\mu - L_{S_1} \) is a finite sum of elements of the closed ideal of
\( A(L_B \cup R_B) \) generated by \( L_SR_{B_K} \). By Lemma 5.10, they are in \( \text{rad } A(L_B \cup R_B) \). The case of \( S_2 \) is similar. \( \square \)

**Theorem 5.12.** Let \( B \subset K^1(\mathfrak{N}) \) be a unital closed algebra. Then

\[
\sigma((S + T)|Y) \subset \sigma(S|Y) + \sigma(T|Y)
\]

and

\[
\sigma((TS)|Y) \subset \sigma(T|Y)\sigma(S|Y)
\]

for every \( S \in A(L_{Z_{\text{rad}}(B)} \cup R_{Z_{\text{rad}}(B)}) \), \( T \in A(L_B \cup R_B) \) and \( Y \in \text{Lat}\{S, T\} \).

**Proof.** Let \( \lambda \notin \sigma(S|Y) + \sigma(T|Y) \) be arbitrary. Then clearly

\[
\sigma((S - \lambda)|Y) \cap \sigma(-T|Y) = \emptyset.
\]

Take admissible contour \( \Gamma \subset \mathbb{C} \) which surrounds \( \sigma((S - \lambda)|Y) \) and lies outside of \( \sigma(-T|Y) \) (this means that \( \sigma(-T|Y) \) is contained in the unbounded component of the complement to \( \Gamma \)). One can assume that \( \Gamma \) lies in resolvent sets of \( S - \lambda \) and \( -T \). Since \( S \) and \( T \) are scattered operators, one can write that

\[
(\mu + \lambda - S)^{-1}|Y = (\mu + \lambda - S|Y)^{-1}
\]

for \( \mu \in \text{res}(S - \lambda) \), and

\[
(\mu + T)^{-1}|Y = (\mu + T|Y)^{-1}
\]

for \( \mu \in \text{res}(-T) \), by Lemma 3.21. Let \( g(\mu) = (\mu + \lambda - S|Y)^{-1}(\mu + T|Y)^{-1} \) for \( \mu \in \Gamma \), and \( K = (2\pi i)^{-1} \int_{\Gamma} g(\mu) d\mu \). Then

\[
(\lambda - S|Y - T|Y)K = \frac{1}{2\pi i} \int_{\Gamma} (\lambda - S - T)g(\mu) d\mu
\]

\[
= \frac{1}{2\pi i} \left( \int_{\Gamma} f(\mu) d\mu + \int_{\Gamma} h(\mu) d\mu \right),
\]

where

\[
f(\mu) = (\mu + \lambda - S)^{-1}(\lambda - S - T)(\mu + T)^{-1}|Y
\]

and

\[
h(\mu) = [(\mu + \lambda - S)^{-1}, S + T](\mu + T)^{-1}|Y
\]
for \( \mu \in \Gamma \). Using the resolvent identity, we have
\[
\frac{1}{2\pi i} \int_{\Gamma} f(\mu) d\mu = \frac{1}{2\pi i} \int_{\Gamma} (\mu + \lambda - S|Y)^{-1} d\mu - \frac{1}{2\pi i} \int_{\Gamma} (\mu + T|Y)^{-1} d\mu
\]
\[
= 1 - 0 = 1
\]
by the choice of \( \Gamma \). Therefore
\[
(\lambda - S|Y - T|Y)K = 1 + \frac{1}{2\pi i} \int_{\Gamma} h(\mu) d\mu.
\]
To show that \((\lambda - S|Y - T|Y)\) is right invertible it suffices to prove that \(\int_{\Gamma} h(\mu) d\mu\) is a quasi-nilpotent operator. Let us consider
\[
h_0(\mu) = (\mu + \lambda - S)^{-1}[S, T](\mu + \lambda - S)^{-1}(\mu + T)^{-1}
\]
for \( \mu \in \Gamma \). Then \(\int_{\Gamma} h_0(\mu) d\mu\) is quasi-nilpotent by Lemmas 5.10 and 5.11. Since \(\int_{\Gamma} h(\mu) d\mu = (\int_{\Gamma} h_0(\mu) d\mu)|Y\), we obtain that \(\int_{\Gamma} h(\mu) d\mu\) is quasi-nilpotent. One can show similarly that \(\lambda - S|Y - T|Y\) is left invertible. Therefore \(\lambda \notin \sigma(S|Y + T|Y)\). This shows that
\[
\sigma((S + T)|Y) \subset \sigma(S|Y) + \sigma(T|Y).
\]
Now, let \(\lambda \notin \sigma(T|Y)\sigma(S|Y)\) be arbitrary. If \(\lambda = 0\) then \(TS|Y\) is invertible, so we assume that \(\lambda \neq 0\). Note that \(\mu T|Y - \lambda\) is invertible for \(\mu \in \sigma(S|Y)\). So there exists a contour \(\Gamma \subset \mathbb{C}\setminus \{0\}\) which surrounds \(\sigma(S|Y)\) outside of \(\{\mu \in \mathbb{C} : \mu T|Y - \lambda\) is not invertible\}. Let \(g(\mu) = (\mu - S|Y)^{-1}(\lambda - \mu T|Y)^{-1}\) for \(\mu \in \Gamma\), and \(K = (2\pi i)^{-1} \int_{\Gamma} g(\mu) d\mu\). Note that
\[
\int_{\Gamma} T(\lambda - \mu T)^{-1} d\mu = 0
\]
and
\[
\frac{1}{2\pi i} \int_{\Gamma} (\mu - S)^{-1} d\mu = 1
\]
by the choice of \(\Gamma\). Then
\[
(\lambda - TS|Y)K = \frac{1}{2\pi i} \int_{\Gamma} (\lambda - \mu T + T(\mu - S))(\mu - S|Y)^{-1}(\lambda - \mu T|Y)^{-1} d\mu
\]
\[
= \frac{1}{2\pi i} \int_{\Gamma} ((\lambda - \mu T)(\mu - S|Y)^{-1}(\lambda - \mu T|Y)^{-1} + T(\lambda - \mu T|Y)^{-1}) d\mu
\]
\[
\frac{1}{2\pi i} \int \int_{\Gamma} ((\mu - S|Y)^{-1} + [\lambda - \mu T, (\mu - S|Y)^{-1}](\lambda - \mu T|Y)^{-1}) d\mu
\]
\[= 1 + \frac{1}{2\pi i} \int \int_{\Gamma} ((\mu - S|Y)^{-1}[S|Y, T|Y](\mu - S|Y)^{-1}(\lambda - \mu T|Y)^{-1}) d\mu.\]

The last integral is quasi-nilpotent by Lemmas 5.10 and 5.11 (and the argument above). Therefore \(TS\) is right invertible, and one can show similarly that \(TS\) is left invertible. Hence \(\lambda \not\in \sigma(TS|Y).\) This means that
\[
\sigma((TS)|Y) \subset \sigma(T|Y)\sigma(S|Y).
\]

5.5. The largest E-solvable ideal (an intrinsic description)

As a consequence of Theorem 5.12 we obtain the following result.

**Corollary 5.13.** Let \(L \subset K^1(\mathfrak{X})\) be a Lie algebra. Then the largest E-solvable ideal \(L \cap Z_{\text{rad}}(A(L))\) of \(L\) coincides with \(R(L).\) Moreover,

\[
\sigma(\text{ad}_{L}(S + T)) \subset \sigma(\text{ad}_{L} S) + \sigma(\text{ad}_{L} T)
\]

for all \(S \in R(L), T \in L.\)

**Proof.** The inclusion for spectra follows from Theorem 5.12 with \(Y = \overline{L}.\) Hence \(L \cap Z_{\text{rad}}(A(L)) \subset R(L).\) Note that \(R^o(L)\) is an Engel ideal of \(L\) and \(R(L)/R^o(L)\) is commutative. So we easily conclude that \(R(L)\) is an E-solvable ideal of \(L.\) It follows from Theorem 4.26 that \(R(L) \subset Z_{\text{rad}}(A(L)).\)

The following theorem underlines the radical-like nature of the largest E-solvable ideal in Lie algebras of compact operators.

**Theorem 5.14.** Let \(L \subset K^1(\mathfrak{X})\) be a Lie algebra. Then \(L/R(L)\) does not contain nonzero Engel ideals.

**Proof.** Suppose, to the contrary, that \(L/R(L)\) has a nonzero Engel ideal \(I_0.\) Let \(M\) be the preimage of \(I_0\) in \(L\) under the quotient map \(L \to L/R(L).\) It is clear that \(M\) is an ideal of \(L\) and \(M/R(L)\) is Engel.

We claim that \(M\) is triangularizable. Indeed, let \(V\) be a gap-quotient of \(\text{Lat} M.\) Then \(L/R(L)\) is an Engel Lie algebra. Since \(R(L)/R^o(L)\) is triangularizable, it is an E-solvable ideal of \((M|V)\) by Corollary 4.25 and is scalar by Theorem 4.26. Therefore \((M|V)\) is Engel (see the remark after Lemma 4.21). By Corollary 4.23, \(M|V\) is scalar. Therefore \(\dim V = 1.\) This implies that \(M\) is triangularizable.
By Corollary 4.25, \( M \) is an \( E \)-solvable Lie algebra. Since \( M \) is an ideal of \( \mathfrak{L} \), \( M \subset R(\mathfrak{L}) \), a contradiction. □

We will finish this subsection in the same way as the previous one: by extending the main result to general operator Lie algebras.

**Corollary 5.15.** Every Lie algebra \( \mathfrak{L} \subset \mathcal{B}(\mathfrak{X}) \) has the largest \( E \)-solvable ideal among ideals of \( \mathfrak{L} \) contained in \( K^1(\mathfrak{X}) \). This ideal is equal to \( R(\mathfrak{L} \cap K^1(\mathfrak{X})) \), is closed in \( \mathfrak{L} \) and inner-characteristic.

**Proof.** It suffices to note that any ideal \( J \) of \( \mathfrak{L} \) contained in \( K^1(\mathfrak{X}) \) is an ideal of \( \mathfrak{L} \cap K^1(\mathfrak{X}) \), and apply Theorem 5.9. By Theorem 5.9, the required ideal is equal to \( \mathfrak{L} \cap Z_{rad}(A(\mathfrak{L} \cap K^1(\mathfrak{X}))) \); it is equal to \( R(\mathfrak{L} \cap K^1(\mathfrak{X})) \) by Corollary 5.13. This ideal is clearly closed in \( \mathfrak{L} \) and is inner-characteristic because \( Nor(\mathfrak{L}) \subset Nor(Z_{rad}(A(\mathfrak{L} \cap K^1(\mathfrak{X})))) \). □

One can similarly show that every Lie algebra \( \mathfrak{L} \subset \mathcal{B}(\mathfrak{X}) \) has the largest \( E \)-solvable ideal among ideals contained in \( K(\mathfrak{X}) \).

We know that a Lie algebra of compact operators is triangularizable iff it is \( E \)-solvable. The following result relates this condition to the spectral and root ideals.

**Corollary 5.16.** Let \( \mathfrak{L} \subset K^1(\mathfrak{X}) \) be a Lie algebra. Then the following assertions are equivalent.

(i) \( \mathfrak{L} \) is triangularizable.

(ii) \( \mathfrak{L} = R_{id}(\mathfrak{L}) \).

(iii) \( [\mathfrak{L}, \mathfrak{L}] \subset R_{id}(\mathfrak{L}) \).

(iv) \( \mathfrak{L} = R(\mathfrak{L}) \).

(v) \( [\mathfrak{L}, \mathfrak{L}] \subset R^\circ(\mathfrak{L}) \).

**Proof.** (i) \( \iff \) (ii) \( \iff \) (iv) follows immediately from Corollaries 4.25, 5.13 and Theorem 5.9. The implications (ii) \( \implies \) (iii) and (iv) \( \implies \) (v) follow from Theorem 5.2.

(iii) \( \implies \) (i) If (iii) holds then \( [\mathfrak{L}, \mathfrak{L}] \subset radA(\mathfrak{L}) \) by Theorem 5.5. Then \( A(\mathfrak{L}) \) is commutative modulo the Jacobson radical and hence triangularizable [28].

(v) \( \implies \) (i) If (v) holds then \( [\mathfrak{L}, \mathfrak{L}] \) is triangularizable by Theorem 2.8, so is \( \mathfrak{L} \). □

Now, we prove that for Lie algebras of compact operators the local triangularization implies the global one (see a similar result for operator semigroups in [30]).

**Corollary 5.17.** Let \( \mathfrak{L} \subset K^1(\mathfrak{X}) \) be a Lie algebra. Then \( \mathfrak{L} \) is triangularizable if and only if \( \{T, S\} \) is triangularizable for all \( T, S \in \mathfrak{L} \).

**Proof.** Suppose that every pair \( \{S, T\} \) of elements of \( \mathfrak{L} \) is triangularizable. Then \( A(\{S, T\}) \) is commutative modulo the Jacobson radical. Hence

\[
\rho(S + T) \leq \rho(S) + \rho(T)
\]
for all $S, T \in \mathfrak{L}$, and the result follows from equivalence of (i) and (ii) in Corollary 5.16. □

5.6. The largest Engel ideal consisting of compact operators

We need the following lemma.

Lemma 5.18. Let $\mathfrak{L} \subset \mathcal{B}(\mathfrak{X})$ be a Lie algebra, $J \subset \mathcal{K}^1(\mathfrak{X})$ its Engel ideal, and let $S \in J$ be arbitrary. Then $\rho(\text{ad}_{\overline{\mathcal{A}(\mathfrak{L})}} S) = 0$ and each Riesz projection of $S$ commutes with each operator in $\overline{\mathcal{A}(\mathfrak{L})}$.

Proof. Suppose, to the contrary, that $\rho(\text{ad}_{\overline{\mathcal{A}(\mathfrak{L})}} S) > 0$. Then there exists a nonzero isolated point $\lambda \in \sigma(\text{ad}_{\overline{\mathcal{A}(\mathfrak{L})}} S)$ such that $\|(\text{ad}_{\overline{\mathcal{A}(\mathfrak{L})}} S - \lambda)^n T\|^{1/n} \to 0$ as $n \to \infty$ for some nonzero $T \in \mathfrak{L}$. Let $I$ be the closure of $J$ in $\mathfrak{L}$. Then

$$\|(\text{ad}_{\overline{\mathcal{A}(\mathfrak{L})}} S - \lambda)^n (T + I)\|^{1/n} \to 0$$

as $n \to \infty$. Since $S \in J$, we obtain

$$\|\lambda^n (T + I)\|^{1/n} \to 0$$

as $n \to \infty$, whence $T \in I$. But quasi-nilpotence of $\text{ad}_{\overline{\mathcal{A}(\mathfrak{L})}} S$ implies $T = 0$, a contradiction.

Now, it follows from Corollary 3.7 that $\rho(\text{ad}_{\overline{\mathcal{A}(\mathfrak{L})}} S) = 0$. Then, every Riesz projection of $S$ commutes with all operators in $\overline{\mathcal{A}(\mathfrak{L})}$ by Proposition 3.16. □

Theorem 5.19. Let $\mathfrak{L} \subset \mathcal{K}^1(\mathfrak{X})$ be a Lie algebra. Then

(i) $\mathcal{R}^\circ(\mathfrak{L})$ is the largest Engel ideal in $\mathfrak{L}$.
(ii) $\mathcal{R}^\circ(\mathfrak{L}) = \{S \in \mathfrak{L} \cap \mathcal{Z}_{\text{rad}}(\overline{\mathcal{A}(\mathfrak{L})}) : \rho(\text{ad}_{\overline{\mathcal{A}(\mathfrak{L})}} S) = 0\}$.

Proof. One can suppose that $\mathfrak{L}$ is closed. Let $W$ be the right-hand side of (ii). It follows from Lemmas 4.22 and 5.18 that any Engel ideal of $\mathfrak{L}$ is contained in $W$. Since $\mathcal{R}^\circ(\mathfrak{L})$ is also an Engel ideal, it suffices to prove that $W \subset \mathcal{R}^\circ(\mathfrak{L})$. But this inclusion is immediate by Corollary 5.13. □

Theorem 5.20. Every Lie algebra $\mathfrak{L} \subset \mathcal{B}(\mathfrak{X})$ has the largest Engel ideal among ideals of $\mathfrak{L}$ contained in $\mathcal{K}^1(\mathfrak{X})$. This ideal is closed in $\mathfrak{L}$ and inner-characteristic.

Proof. Let $J$ be the sum of all Engel ideals of $\mathfrak{L}$ contained in $\mathcal{K}^1(\mathfrak{X})$. Since every such ideal is an ideal of $\mathfrak{L} \cap \mathcal{K}^1(\mathfrak{X})$,

$$J = \mathcal{R}^\circ(\mathfrak{L} \cap \mathcal{K}^1(\mathfrak{X}))$$

by Theorem 5.19. Clearly $J$ is an Engel ideal of $\mathfrak{L}$ closed in $\mathfrak{L}$.

Now let $S \in \text{Nor}(\mathfrak{L})$ and $T \in J$ be arbitrary, and let $I$ be the largest Engel ideal of $\mathfrak{L}$ among all ideals of $\mathfrak{L}$ contained in $\mathcal{K}^1(\mathfrak{X})$. Using Lemma 4.21, we obtain that
\[ I = \tilde{J}. \] Then \( \exp(\lambda \text{ad} S) T \in \overline{\mathfrak{U}}. \) For every \( \lambda \in \mathbb{C} \)

\[ \left\| (\text{ad}_T \exp(\lambda \text{ad} S) T)^n \right\|^{1/n} \leq \left\| \exp(\lambda \text{ad} S) \right\|^{1/n} \left\| (\text{ad}_T T)^n \right\|^{1/n} \to 0 \]
as \( n \to \infty \) by Lemma 5.18. So the closure of \( \exp(\lambda \text{ad} S) J \), being an ideal of \( \overline{\mathfrak{U}} \), is Engel. Then

\[ \exp(\lambda \text{ad} S) J \subseteq I \]

for every \( \lambda \in \mathbb{C} \), whence \( [S, J] \subseteq I \) and therefore

\[ [S, J] \subseteq I \cap \mathfrak{U} = J, \]
i.e., \( J \) is inner-characteristic. \( \Box \)

5.7. Transference of the Engel property

There are several results that state that some properties of a Lie algebra \( \mathfrak{U} \) of compact operators transfer to the Banach algebra \( \overline{\mathfrak{U}(\mathfrak{U})} \) generated by \( \mathfrak{U} \). The first one is Theorem 2.8 that states that if \( \mathfrak{U} \subseteq \mathcal{K}^1(X) \) consists of quasi-nilpotent operators then the same is true for \( \overline{\mathfrak{U}(\mathfrak{U})} \). Furthermore if \( \mathfrak{U} \subseteq \mathcal{K}^1(X) \) is \( E \)-solvable then, by Corollary 4.25, \( \overline{\mathfrak{U}(\mathfrak{U})} \) is \( E \)-solvable (and also is commutative modulo the Jacobson radical). Now, we will prove that if \( \mathfrak{U} \) is Engel then \( \overline{\mathfrak{U}(\mathfrak{U})} \) is Engel. Recall that a normed algebra \( A \) is called Engel if it is Engel as a normed Lie algebra, that is if all operators \( x \mapsto ax - xa \) on \( A \) are quasi-nilpotent.

Recall some known related facts. For a Banach algebra \( A \), Zemanek [55, Theorem 1] showed that \( R_{\text{id}}(A) = Z_{\text{rad}}(A) \) and \( R_{\text{id}}^0(A) = \text{rad} A \); also Aupetit and Mathieu [1, Proposition] proved that \( a \in Z_{\text{rad}}(A) \) iff \( \rho(\text{ad}[a, b]) = 0 \) for all \( b \in A \) (the last result was also announced in [44, Theorem 10]). Using these results, we formulate the following proposition.

**Proposition 5.21.** Let \( A \) be a normed algebra which is an ideal of its completion \( \tilde{A} \).

Then

(i) \( R(A) \subseteq R_{\text{id}}(A) = Z_{\text{rad}}(A) = A \cap Z_{\text{rad}}(\tilde{A}) \) and \( R_{\text{id}}^0(A) = \text{rad} A \).

(ii) \( a \in Z_{\text{rad}}(A) \) iff \( \rho(\text{ad}[a, b]) = 0 \) for all \( b \in A \).

(iii) If \( A \) is Engel then \( A \) is commutative modulo the Jacobson radical.

**Proof.** (i) Note that \( \text{rad} A = A \cap \text{rad} \tilde{A} \) (because the Jacobson radical is hereditary). By Zemanek [55, Theorem 1] and (5.1), \( R_{\text{id}}^0(A) = \text{rad} A \) and for the other equalities in (i), it suffices to show that \( Z_{\text{rad}}(A) \subseteq A \cap Z_{\text{rad}}(\tilde{A}) \). For this, if \( a \in Z_{\text{rad}}(A) \) and \( b \in \tilde{A} \), we have \( [a, b] \in A \), whence

\[ [a, [a, b]] \in \text{rad} A \subseteq \text{rad} \tilde{A}. \]
By the Kleinecke–Shirokov theorem [38,23], \( \rho([a,b] + \operatorname{rad} \tilde{A}) = 0 \), whence
\[
\rho([a,b]) = 0
\]
for every \( b \in \tilde{A} \), and, by the Le Page’s theorem [25], \( a \in \mathcal{Z}_{\operatorname{rad}}(\tilde{A}) \).

It remains to prove that \( \mathcal{R}(A) \subset \mathcal{Z}_{\operatorname{rad}}(A) \). If \( a \in \mathcal{R}(A) \) then, by Theorem 5.2 and (5.1),
\[
[a, b] \in \mathcal{R}(\tilde{A})
\]
and then \( \rho(\operatorname{ad}[a,b]) = 0 \) for all \( b \in \tilde{A} \). By [1, Proposition], \( a \in \mathcal{Z}_{\operatorname{rad}}(\tilde{A}) \), whence
\[
a \in A \cap \mathcal{Z}_{\operatorname{rad}}(\tilde{A}) = \mathcal{Z}_{\operatorname{rad}}(A).
\]

(ii) It suffices to prove \( \iff \). The proof of this implication in Aupetit and Mathieu [1. Proposition] used only the Jacobson density of images of strictly irreducible representations of \( \tilde{A} \) so that the proof for \( A \) is really the same. So, we refer to the proof of Aupetit and Mathieu [1, Proposition].

(iii) Follows from (ii). \( \square \)

Easy examples (e.g. uppertriangular matrices) show that the converse implication for Proposition 5.21(iii) fails. Therefore, the Engel property is more strong than commutativity modulo the Jacobson radical.

**Theorem 5.22.** Let \( \mathcal{V} \subset \mathcal{K}(X) \) be an Engel Lie algebra. Then \( \overline{\mathcal{A}(\mathcal{V})} \) is an Engel algebra.

**Proof.** Let \( S \in \mathcal{A}(\mathcal{V}) \). By Lemma 4.1, \( S = S_1^{m_1} + \ldots + S_n^{m_n} \) for some \( S_1, \ldots, S_n \in \mathcal{V} \). By Lemma 5.18, \( \rho(\operatorname{ad}_{\overline{\mathcal{A}(\mathcal{V})}} S_j) = 0 \) for every \( j \). Clearly, \( \mathcal{V} \subset \mathcal{Z}_{\operatorname{rad}}(\overline{\mathcal{A}(\mathcal{V})}) \). Hence, by Theorem 5.12,
\[
\rho(\operatorname{ad}_{\overline{\mathcal{A}(\mathcal{V})}} S) = \rho \left( \sum_{j=1}^{n} \operatorname{ad}_{\overline{\mathcal{A}(\mathcal{V})}} S_j^{m_j} \right) = \rho \left( \sum_{j=1}^{n} \sum_{k=0}^{m_j-1} L_{S_j}^{k} \operatorname{ad}_{\overline{\mathcal{A}(\mathcal{V})}} S_j R_{S_j}^{m_j-1-k} | \overline{\mathcal{A}(\mathcal{V})} \right)
\]
\[
\leq \sum_{j=1}^{n} \sum_{k=0}^{m_j-1} \rho(L_{S_j}^{k} \operatorname{ad}_{\overline{\mathcal{A}(\mathcal{V})}} S_j R_{S_j}^{m_j-1-k} | \overline{\mathcal{A}(\mathcal{V})})
\]
\[
\leq \sum_{j=1}^{n} \sum_{k=0}^{m_j-1} \rho(L_{S_j}^{k}) \rho(\operatorname{ad}_{\overline{\mathcal{A}(\mathcal{V})}} S_j) \rho(R_{S_j}^{m_j-1-k}) = 0.
\]
Therefore \( \mathcal{A}(\mathcal{V}) \) is Engel. Then \( \overline{\mathcal{A}(\mathcal{V})} \) is Engel by Lemma 4.21. \( \square \)
Corollary 5.23. Let $\mathfrak{L} \subset \mathcal{K}^1(\mathfrak{X})$ be an Engel Lie algebra, and let $M$ be a Lie subalgebra of $\mathcal{A}(\mathfrak{L})$. Then $M$ is an Engel Lie algebra. In particular, if $M$ is finite-dimensional then $M$ is a nilpotent Lie algebra.

Proof. Follows from Theorem 5.22 since any Lie subalgebra of an Engel algebra is Engel. □

This corollary has applications to a spectral mapping theorem for representations of nilpotent Lie algebras (see for instance [12, Theorem 5.1]).

6. ad-Compact Lie algebras

Recall that a normed Lie algebra $\mathfrak{L}$ is called ad-compact if all operators $\text{ad} \, a$ on $\mathfrak{L}$ are compact ($a \in \mathfrak{L}$). The theory of ad-compact Lie algebras was initiated by Vaksman and Gurarij [46] with a very important contribution by Wojtyński [51], who proved that any Engel ad-compact Lie algebra (of dimension $> 1$) has a nontrivial closed ideal. Note that the original proof of Wojtyński theorem was based on a deep analysis of linear operator equations; now it can be immediately deduced from Theorem 2.8 applied to the adjoint representation. In [46], a strong theory of radicality and semisimplicity for complete ad-compact Lie algebras was produced, that led to a quite satisfactory description of semisimple complete ad-compact Lie algebras. In particular, the results of Vaksman and Gurarij [46] (if one takes into account the Wojtyński theorem) imply that all (topologically) simple complete ad-compact Lie algebras are finite-dimensional. Nevertheless, many important questions remained unanswered. Applying the results of previous sections to the algebra $\text{ad} \, \mathfrak{L} \subset \mathcal{K}(\mathfrak{L})$, we will answer most of them and obtain a new information about the structure of ideals and representations of ad-compact Lie algebras.

The class of ad-compact Lie algebras is quite wide and stable under standard constructions. Namely, it contains completions, subalgebras, quotients by closed ideals and restricted direct sums of ad-compact Lie algebras.

6.1. The largest Engel ideal

As before, $\tilde{\mathfrak{L}}$ means the norm-completion of a normed Lie algebra $\mathfrak{L}$. If $J$ is an ideal of $\mathfrak{L}$, $\tilde{J}$ is identified with the norm-closure of $J$ in $\tilde{\mathfrak{L}}$, i.e. $J$ is a closed ideal of $\mathfrak{L}$; on the other hand, $\bar{J}$ means usually the closure of $J$ in $\mathfrak{L}$, i.e. $\bar{J}$ is a closed ideal of $\mathfrak{L}$. As a rule, we identify $\text{ad}_\mathfrak{L} \, \mathfrak{L}$ and $\text{ad}_{\tilde{\mathfrak{L}}} \, \tilde{\mathfrak{L}}$.

Recall that an ideal $J$ of a normed Lie algebra is called an Engel ideal if it is Engel as a normed Lie algebra, that is if $\text{ad}_J \, J$ consists of quasi-nilpotent operators. The following result shows that in ad-compact Lie algebras Engel ideals can be defined by a formally more weak property (namely, quasi-centrality) and also by a formally more strong property.

Lemma 6.1. Let $\mathfrak{L}$ be an ad-compact Lie algebra and $J$ a (nonnecessarily closed) ideal of $\mathfrak{L}$. The following conditions are equivalent.

(i) $\tilde{J}$ is an Engel Lie algebra.

(ii) $\bar{J}$ is an Engel Lie algebra.
(iii) \( \| (\text{ad} a)^n b \|^{1/n} \to 0 \) for any \( a, b \in J \), as \( n \to \infty \).
(iv) \( \text{ad}_J \) consists of quasi-nilpotents.

**Proof.** The implications (ii) \(\Rightarrow\) (i) \(\Rightarrow\) (iii) are evident.

(iii) \(\Rightarrow\) (iv) Since \( J \subset \mathcal{E}_0(\text{ad}_J a) \) for any \( a \in J \), and \( \mathcal{E}_0(\text{ad}_J a) \) is closed by Lemma 3.3(i), we obtain that

\[
\| (\text{ad} a)^n b \|^{1/n} \to 0
\]

for all \( a \in J \) and \( b \in \hat{J} \), as \( n \to \infty \). If now the operator \( \text{ad}_J a \) for some \( a \in J \), is not quasi-nilpotent then being compact it has a nonzero eigenvalue \( \lambda \), i.e. \([a, x] = \lambda x\) for some nonzero \( x \in \hat{U} \). But then \( x \in \hat{J} \) and

\[
\| (\text{ad} a)^n x \|^{1/n} \to |\lambda|
\]

as \( n \to \infty \), a contradiction.

(iv) \(\Rightarrow\) (ii) It follows from the continuity of spectral radius on compact operators that \( \text{ad}_J \) consists of quasi-nilpotents, so does \( \text{ad}_J \). \(\square\)

It follows from Lemma 6.1 that the closure of an Engel ideal in an ad-compact Lie algebra is again an Engel ideal.

**Corollary 6.2.** Let \( \mathfrak{U} \) and \( M \) be ad-compact Lie algebras.

(i) If \( \phi : \mathfrak{U} \to M \) a bounded homomorphism with dense image then the closure of \( \phi(J) \) in \( M \) is an Engel ideal of \( M \) for every Engel ideal \( J \) of \( \mathfrak{U} \).

(ii) If \( \mathfrak{U} \) is Engel then a subalgebra and a quotient of \( \mathfrak{U} \) by a closed ideal are Engel.

**Proof.** (i) It is clear from Lemma 6.1(iii) that \( \phi(J) \) is an Engel ideal of \( \phi(M) \). Therefore, the closure of \( \phi(J) \) in \( M \) is also Engel; clearly it is an ideal of \( M \).

(ii) Follows from Lemma 6.1. \(\square\)

**Theorem 6.3.** Let \( \mathfrak{U} \) be an ad-compact Lie algebra.

(i) \( \mathfrak{U} \) has the largest Engel ideal \( E(\mathfrak{U}) \) which is equal to \( \{ a \in \mathfrak{U} : \text{ad}_\mathfrak{U} a \in \text{rad } \mathcal{A}(\text{ad}_\mathfrak{U} \mathfrak{U}) \} \) and also to \( \mathcal{R}_c(\mathfrak{U}) \).

(ii) The ideal \( E(\mathfrak{U}) \) is topologically characteristic and closed.

(iii) \( E(\mathfrak{U}) \) is invariant for every bounded Lie endomorphism of \( \mathfrak{U} \) with dense image.

(iv) If \( J \) is an ideal of \( \mathfrak{U} \) then \( E(J) = E(\mathfrak{U}) \cap J \).

**Proof.** (i)–(ii) Let \( \mathfrak{U}_0 = \text{ad}_\mathfrak{U} \mathfrak{U} \). Then \( \mathfrak{U}_0 \) is a Lie algebra of compact operators on \( \mathfrak{U} \). By Corollary 5.7, \( \mathfrak{U}_0 \) has the largest Volterra ideal

\[
\mathcal{V}(\mathfrak{U}_0) := \{ S \in \mathfrak{U}_0 : S \in \text{rad } \mathcal{A}(\mathfrak{U}_0) \}.
\]
Let \( E(\mathcal{U}) \) be the preimage of \( \mathcal{V}(\mathcal{U}_0) \) under the adjoint representation \( \mathcal{U} \to \mathcal{U}_0 \). It is clear that \( E(\mathcal{U}) \) is a closed Engel ideal, and any Engel ideal of \( \mathcal{U} \) is contained in \( E(\mathcal{U}) \).

Since \( \mathcal{V}(\mathcal{U}_0) \) is inner-characteristic, \( E(\mathcal{U}) \) is topologically characteristic.

Since \( \text{ad} \tilde{\mathcal{U}} E(\mathcal{U}) \subset \text{rad} A(\mathcal{U}_0) \),

\[
\rho(\text{ad}_\mathcal{U}(a + b)) = \rho(\text{ad}_\mathcal{U} a)
\]

for all \( a \in \mathcal{U}, b \in E(\mathcal{U}) \), whence \( E(\mathcal{U}) \subset \mathcal{R}(\mathcal{U}) \).

On the other hand, \( \mathcal{R}(\mathcal{U}) \) is an ideal of \( \mathcal{U} \) by Theorem 5.2 and \( \text{ad}_\mathcal{U} b \) is quasinilpotent for each \( b \in \mathcal{R}(\mathcal{U}) \). Therefore \( \mathcal{R}(\mathcal{U}) \) is an Engel ideal of \( \mathcal{U} \), whence \( \mathcal{R}(\mathcal{U}) \subset E(\mathcal{U}) \). Thus

\[
E(\mathcal{U}) = \mathcal{R}(\mathcal{U}).
\]

(iii) If \( \phi : \mathcal{U} \to \mathcal{U} \) is an endomorphism with dense image then the closure of \( \phi E(\mathcal{U}) \) in \( \mathcal{U} \) is Engel by Corollary 6.2, so

\[
\phi E(\mathcal{U}) \subset E(\mathcal{U}).
\]

(iv) It follows from Lemma 6.1(iv) that \( \text{ad}_\mathcal{U} J \) consists of Volterra operators, whence

\[
E(J) \subset E(\mathcal{U}) \cap J.
\]

The reverse inclusion is obvious. □

Recall that a representation of a Lie algebra is a Lie homomorphism from this algebra into the Lie algebra of all linear operators on a linear space. Given a normed Lie algebra \( \mathcal{U} \) and a representation \( \pi : \mathcal{U} \to B(\mathcal{X}) \), \( \pi \) is called a Banach representation if \( \mathcal{X} \) is a nonzero Banach space and \( \pi \) is bounded. A Banach representation \( \pi \) is called irreducible if \( \text{Lat} \pi \mathcal{U} \) is trivial; \( \pi \) is called a representation by compact operators if \( \pi \mathcal{U} \) consists of compact operators. Let \( \text{Irr}_K \mathcal{U} \) be the set of all irreducible Banach representations of \( \mathcal{U} \) by compact operators. It is clearly nonempty because \( \mathcal{U} \) admits a zero irreducible representation by compact operators on an one-dimensional space.

**Lemma 6.4.** Let \( \mathcal{U} \) be an ad-compact Lie algebra. If \( a \in \mathcal{U} \) then

\[
\sigma(\text{ad}_\mathcal{U} a) \setminus \{0\} = \bigcup_V \sigma(\text{ad}_V a) \setminus \{0\},
\]

where \( V \) runs over all gap-quotients of the lattice of all closed ideals of \( \tilde{\mathcal{U}} \) (i.e., of \( \text{Lat} \text{ad}_\mathcal{U} \mathcal{U} \)). If \( \text{ad}_\mathcal{U} a \) is of finite rank then

\[
\text{tr}(\text{ad}_\mathcal{U} a) = \sum_Z \text{tr}(\text{ad}_Z a),
\]

where \( Z \) runs over all gap-quotients of a maximal nest in \( \text{Lat} \text{ad}_\mathcal{U} \mathcal{U} \).
**Proof.** Let $a \in \mathfrak{L}$ and $S = \text{ad}_\mathfrak{L} a$. Since $S$ is a compact operator, $S$ is scattered and then

$$\sigma(S|V) \subset \sigma(S)$$

for every gap-quotient $V$ of $\text{Lat} \text{ad}_\mathfrak{L} \mathfrak{L}$. The equality (6.1) follows from [31, Theorem 7.2.7]. If $S$ is of finite rank, [31, Theorem 7.2.9] shows (6.2). □

Let $\Lambda(\mathfrak{L}) = \cap \{\ker \pi : \pi \in \text{Irr}_K \mathfrak{L}\}$ for a normed Lie algebra $\mathfrak{L}$.

**Theorem 6.5.** Let $\mathfrak{L}$ be an ad-compact Lie algebra. Then $\Lambda(\mathfrak{L})$ is a closed Engel ideal of $\mathfrak{L}$.

**Proof.** One may identify $\text{Irr}_K \mathfrak{L}$ and $\text{Irr}_K \tilde{\mathfrak{L}}$ by continuous extension of the representations to $\tilde{\mathfrak{L}}$ (or by restriction to $\mathfrak{L}$). Clearly, $\Lambda(\mathfrak{L})$ is a closed ideal of $\mathfrak{L}$. Let $\Gamma = \{J_\mathfrak{L} : \mathfrak{L} \in \text{Irr}_K \mathfrak{L}\}$ be a maximal nest of closed ideals of $\tilde{\mathfrak{L}}$. Any gap in $\Gamma$ determines (by “restriction” of the representation $\text{ad}_\mathfrak{L}$ to the gap) an irreducible representation of $\tilde{\mathfrak{L}}$ by compact operators. Hence $\text{ad}_\mathfrak{L} a$ for any $a \in \Lambda(\mathfrak{L})$, has zero “gap parts” on any gap-quotient of $\Gamma$. Then $\Lambda(\mathfrak{L})$ is an Engel Lie algebra by Lemma 6.4. If $\Gamma$ has no gaps, $\Gamma$ is a continuous nest and then, being a Lie algebra of compact operators, $\text{ad}_\mathfrak{L}$ is a Volterra Lie algebra by Radjavi and Rosenthal [29, Corollary 5.13], whence $\Lambda(\mathfrak{L}) = \mathfrak{L}$ is Engel. □

6.2. Irreducible representations and ad-finite elements

An element $a$ of a normed Lie algebra $\mathfrak{L}$ is called ad-finite if $\text{ad}_\mathfrak{L} a$ is of finite rank. We denote the set of all ad-finite elements of $\mathfrak{L}$ by $\mathfrak{I}_\mathfrak{L}(\mathfrak{L})$. Since $\text{ad}_\mathfrak{L} \{a, b\} = [\text{ad}_\mathfrak{L} a, \text{ad}_\mathfrak{L} b]$ for all $a, b \in \mathfrak{L}$, $\mathfrak{I}_\mathfrak{L}(\mathfrak{L})$ is an ideal of $\mathfrak{L}$.

**Lemma 6.6.** Any non-Engel ideal $J$ of a complete ad-compact Lie algebra $\mathfrak{L}$ contains a nonzero ad-finite element of $\mathfrak{L}$. Moreover, for an element $a \in \mathfrak{L}$, every eigenvector of $\text{ad}_\mathfrak{L} a$ corresponding to a nonzero eigenvalue belongs to $\mathfrak{I}_\mathfrak{L}(\mathfrak{L})$.

**Proof.** Since $\text{ad} J$ contains nonquasi-nilpotent operators, there are $a, b \in J$ with $[a, b] = b$. Hence

$$[\text{ad}_\mathfrak{L} a, \text{ad}_\mathfrak{L} b] = \text{ad}_\mathfrak{L} b \in \mathfrak{E}(\text{ad}_\mathfrak{L} a)$$

and $\text{ad}_\mathfrak{L} b$ is of finite rank by Lemma 3.12. □

For brevity we will write FDS instead of ‘finite-dimensional semisimple’. For finite-dimensional Lie algebras, the classical semisimplicity means the absence of nonzero solvable ideals. Recall that a Lie ideal (in general, a Lie algebra) $J$ is called (classically) solvable if $J^{(n)} = 0$ for some $n$, where $J^{(n)}$ is defined recurrently by $J^{(n)} = [J^{(n-1)}, J^{(n-1)}]$ with $J^{(0)} = J$. It is well known that every finite-dimensional Lie al-
algebra \( \mathfrak{L} \) has the largest solvable ideal (the \textit{radical} of \( \mathfrak{L} \)) and the quotient of \( \mathfrak{L} \) by the radical is semisimple.

The following result is a refinement of Vaksman and Gurarij [46, Theorem 1.1].

**Theorem 6.7.** Let \( \mathfrak{L} \) be a complete ad-compact Lie algebra and \( J \) a nonzero ideal of \( \mathfrak{L} \). Then \( J \) contains either a nonzero FDS or a nonzero Engel ideal of \( \mathfrak{L} \).

**Proof.** Let \( J_0 = J \cap \mathcal{Z}(\mathfrak{L}) \). If \( J_0 = 0 \) then, by Lemma 3.12, \( J \) is Engel and we are done. Let \( J_0 \neq 0 \). If \( J_0 \) is Engel then again there is nothing to prove. Suppose that \( J_0 \) is not Engel then it contains an element \( a \) with \( \sigma(\text{ad}_a) \neq \{0\} \). Clearly \( \sigma(\text{ad}_a) \) is finite and all \( E_{\lambda}(\text{ad}_a) \) with \( \lambda \in \Lambda \) are finite-dimensional. Hence

\[
I := \sum_{\lambda \in \Lambda} E_{\lambda}(\text{ad}_a) + \sum_{\lambda \in \Lambda} [E_{\lambda}(\text{ad}_a), E_{-\lambda}(\text{ad}_a)]
\]

is finite-dimensional. By Proposition 3.8, \( I \) is an ideal of \( \mathfrak{L} \). If \( I \) is not FDS then the radical \( I_0 \) of \( I \) is a solvable ideal of \( \mathfrak{L} \) because \( I_0 \) is characteristic. For some \( n \), \( I_0^{(n)} \) is a nonzero commutative (hence Engel) ideal of \( \mathfrak{L} \). \( \square \)

**Theorem 6.8.** Any representation \( \pi \in \text{Irr}_K(\mathfrak{L}) \) of an ad-compact Lie algebra \( \mathfrak{L} \) is finite-dimensional.

**Proof.** One can assume that \( \mathfrak{L} \) is complete; let \( \pi \) maps \( \mathfrak{L} \) into \( \mathcal{B}(\mathfrak{X}) \). Suppose, to the contrary, that \( \mathfrak{X} \) is infinite-dimensional. Changing \( \mathfrak{L} \) by \( \mathfrak{L}/\ker \pi \), we can assume that \( \pi \) is faithful. If \( \mathfrak{L} \) has a finite-dimensional ideal then the same is true for \( \pi \mathfrak{L} \); this implies that \( \pi \mathfrak{L} \) has an invariant subspace, by Lemma 4.27. So \( \mathfrak{L} \) has no finite-dimensional ideals and, by Theorem 6.7, has a nonzero Engel closed ideal \( J \). Since

\[
\| (\pi a)^n \pi b \| \leq \| \pi \| \| (\text{ad}a)^n b \|
\]

for all \( a, b \in J \), as \( n \to 0 \). It follows from Proposition 3.3 that \( \text{ad}_{\pi J} \pi a \) is quasi-nilpotent for every \( a \in J \). Then \( \pi \mathfrak{L} \) is a Lie algebra of compact operators containing a nonscalar Engel ideal \( \pi J \). It follows from Theorem 4.26 that \( \pi \mathfrak{L} \) is reducible, a contradiction. \( \square \)

**Corollary 6.9.** If \( J_1, J_2 \) are closed ideals of a complete ad-compact Lie algebra \( \mathfrak{L} \), \( J_1 \subseteq J_2 \) and \( \dim(J_2/J_1) = \infty \) then there is a closed ideal \( J \) of \( \mathfrak{L} \) such that \( J_1 \subset J \subset J_2 \) and \( J_1 \neq J \neq J_2 \).

**Proof.** Let \( \mathfrak{X} = J_2/J_1 \), and \( \pi \) the representation of \( \mathfrak{L} \) on \( \mathfrak{X} \) induced by \( \text{ad}_{J_2} \). By Theorem 6.8, \( \pi \mathfrak{L} \) has a nontrivial invariant subspace \( Y \). Setting \( J = \{ x \in J_2 : x + J_1 \in Y \} \), we obtain an intermediate closed ideal. \( \square \)
Corollary 6.10. Every minimal closed ideal \( J \) of a complete ad-compact Lie algebra \( \mathfrak{L} \) is finite-dimensional and either simple or commutative.

Proof. Applying Corollary 6.9 to the pair \((0, J)\), we see that \( \dim J < \infty \). If \( J \) is not FDS then the radical \( R \) of \( J \) is also a nonzero ideal of \( \mathfrak{L} \), since \( R \) is characteristic. Hence \( R = J \) and \( J \) is solvable. Since \([J, J] \subset J\), \( J \) is commutative.

If \( J \) is FDS then \( J \) is a finite sum of its FDS simple ideals. Since a FDS simple ideal of \( J \) is characteristic, it is an ideal of \( \mathfrak{L} \), whence \( J \) is FDS simple. \( \square \)

Corollary 6.11. An infinite-dimensional complete ad-compact Lie algebra contains an infinite number of closed ideals. The gap-quotients of the lattice of closed ideals are finite-dimensional.

Proof. Follows from Corollary 6.9. \( \square \)

Example 6.12. (i) There exists an infinite-dimensional incomplete ad-compact Lie algebra \( \mathfrak{L} \) having only a finite set of closed ideals.

(ii) There exists a complete ad-compact Lie algebra \( \mathfrak{L} \) without nonzero finite-dimensional ideals but with the following properties:
\[ \dim \mathfrak{L}/\mathcal{I}_\mathcal{F}(\mathfrak{L}) = 1. \]
\[ \text{ad}_{\mathfrak{L}} a \text{ is of rank one for every nonzero element } a \in \mathcal{I}_\mathcal{F}(\mathfrak{L}). \]

Proof. Let \( S \) be a compact operator on a Banach space \( \mathfrak{X} \), and let \( Y \) be a nonzero linear manifold invariant for \( S \). Supply the direct sum \( \mathfrak{L} = \mathbb{C} \oplus Y \) with norm \( \| \lambda \oplus y \| = |\lambda| + \| y \| \) and Lie product
\[ [\lambda \oplus x, \mu \oplus y] = 0 \oplus (\lambda Sy - \mu Sx). \]

Then \( \mathfrak{L} \) is an ad-compact Lie algebra, and it is easy to check that each its ideal is of the form \( \mathbb{C} \oplus Z \) or \( 0 \oplus Z \), where \( Z \subset Y \) is an invariant linear manifold for \( S \).

Let \( \mathfrak{X} \) be the Hilbert space \( L^2_{[0,1]} \), and let \( S \) be the Volterra integration operator defined by
\[ Sf(x) = \int_0^x f(t) \, dt \]
for every \( f \in L^2_{[0,1]} \).

(i) Let \( Y \) be the set of all polynomials (restricted to \([0,1]\)). If \( Z \) is a linear manifold closed in \( Y \) and invariant for \( S \), then \( \widetilde{Z} \) is an invariant subspace for \( S \) and coincides with \( \{ f \in H : f = 0 \text{ a.e. on } [0, \alpha] \} \) for some \( \alpha \in [0,1] \) (see [29, Theorem 4.14]). Hence, since \( Z \) consists of polynomials, \( Z = 0 \). Therefore \( S|Y \) has no nontrivial invariant linear manifolds closed in \( Y \). Then \( \mathfrak{L} \) has only one nontrivial closed ideal \( 0 \oplus Y \).

(ii) Let \( Y = \mathfrak{X} \). Then \( \mathfrak{L} \) is complete, \( \mathcal{I}_\mathcal{F}(\mathfrak{L}) = \{ 0 \oplus x : x \in \mathfrak{X} \} \) and
\[ \dim(\text{ad}_{\mathfrak{L}}(0 \oplus x)\mathfrak{L}) = 1 \]
for each nonzero \( x \in X \). Since \( S \) has no nonzero finite-dimensional invariant subspaces, \( \mathfrak{L} \) has no nonzero finite-dimensional ideals. □

Let \( \mathfrak{sl}(2) \) be the Lie algebra of all complex \( 2 \times 2 \)-matrices with zero trace. For brevity, we say that a Lie algebra \( \mathfrak{L} \) contains \( \mathfrak{sl}(2) \) if it contains a Lie subalgebra isomorphic to \( \mathfrak{sl}(2) \). This means actually that \( \mathfrak{L} \) contains elements \( h, u, v \) such that

\[
[h, u] = u, \quad [h, v] = -v, \quad [u, v] = h.
\] (6.4)

It is known (see, for instance, [5]) that a complex finite-dimensional Lie algebra contains \( \mathfrak{sl}(2) \) if and only if it is not solvable.

The following construction will be useful. Let \( A \) be a normed commutative algebra, \( \mathfrak{N} \) a normed Lie algebra, and let \( A \otimes \mathfrak{N} \) be their tensor product. Then \( A \otimes \mathfrak{N} \) is a Lie algebra with Lie multiplication defined on elementary tensors by

\[
[a \otimes f, b \otimes g] = ab \otimes [f, g]
\] (6.5)

for all \( a, b \in A \) and \( f, g \in \mathfrak{N} \). Supply \( A \otimes \mathfrak{N} \) with the projective tensor norm. If \( A \) is completely continuous (i.e. \( L_a = R_a \) is compact for every \( a \in A \)) and \( \mathfrak{N} \) is ad-compact, then every \( \text{ad}_{A \otimes \mathfrak{N}}(a \otimes f) = L_a \otimes \text{ad}_{\mathfrak{N}} f \) is compact, whence \( A \otimes \mathfrak{N} \) is an ad-compact Lie algebra. If \( \mathfrak{N} \) is isomorphic to a Lie algebra of complex matrices, it is convenient to represent \( A \otimes \mathfrak{N} \) by matrices with components in \( A \).

The following example shows that a complete ad-compact Lie algebra need not have finite-dimensional ideals and commutative ideals.

**Example 6.13.** Let \( A \) be a commutative completely continuous radical Banach algebra which is an integral domain. For example, \( A \) may be realized as the weighted semigroup algebra consisting of all power series \( \sum_{n=1}^{\infty} \lambda_n x^n \) with \( \lambda_n \in \mathbb{C} \) and (the norm) \( \sum |\lambda_n| \exp(-n^2) < \infty \). Let \( \mathfrak{L} = A \otimes \mathfrak{sl}(2) \). Then \( \mathfrak{L} \) is a complete Engel ad-compact Lie algebra without minimal closed ideals, nonzero commutative ideals and finite-dimensional ideals and, moreover, without nonzero ad-finite elements.

**Proof.** Note that \( \mathfrak{L} \) is a Lie subalgebra of the Banach algebra \( A \otimes M_2(\mathbb{C}) \), where \( M_2(\mathbb{C}) \) denotes the algebra of all complex \( 2 \times 2 \)-matrices. Since \( A \) is radical, \( A \otimes M_2(\mathbb{C}) \) is radical by the Jacobson’s Theorem [18, Theorem 5.14.1]. Therefore \( \mathfrak{L} \) consists of quas-nilpotent elements of \( A \otimes M_2(\mathbb{C}) \), whence \( \mathfrak{L} \) is Engel.

Let \( J \) be an ideal of \( \mathfrak{L} \), and let \( S = a_1 \otimes h + a_2 \otimes u + a_3 \otimes v \) be an element of \( J \). Let \( S_1 = b \otimes h \), \( S_2 = c \otimes u \), \( S_3 = d \otimes v \) be elements of \( \mathfrak{L} \) with nonzero \( b, c, d \in A \). A simple calculation shows that

\[
[[S_1, S], S_2] = a_3 bc \otimes h \in J,
\] (6.6)

\[
[[S_1, S], S_3] = a_2 bd \otimes h \in J.
\] (6.7)
\[
\begin{align*}
[[S_1, [S_2, S]], [[S, S_3], S_1]] &= a_1^{2b}cd \otimes h \in [J, J], \quad (6.8) \\
[[[S_1, S], S_3], [[S, S_3], S_3]] &= a_2^{2b}d^3 \otimes v \in [J, J], \quad (6.9) \\
[[[S_1, S], S_2], [S_2, [S, S_2]]] &= a_3^{2bc} \otimes u \in [J, J]. \quad (6.10)
\end{align*}
\]

Now assume that \(J\) is commutative. Since \(A\) is an integral domain, it follows from (6.8)–(6.10) that \(a_1 = a_2 = a_3 = 0\), whence \(\mathfrak{L}\) has no nonzero commutative ideals.

Note that \(\mathfrak{L}\) has no nonzero finite-dimensional ideals because in any Engel Lie algebra finite-dimensional ideals are nilpotent and contain commutative ideals of the Lie algebra. Also, \(\mathfrak{L}\) has no minimal closed ideals by Corollary 6.10.

Suppose that \(S\) is a nonzero element of \(I_{\mathcal{F}}(\mathfrak{L})\). It follows from (6.6)–(6.8) that \(a \otimes h \in I_{\mathcal{F}}(\mathfrak{L})\) for some nonzero \(a \in A\). But \([a \otimes h, A \otimes u] = aA \otimes u\) is infinite-dimensional, a contradiction. So \(I_{\mathcal{F}}(\mathfrak{L}) = 0\). □

The following lemma gives a sufficient condition of the existence of nonzero finite-dimensional ideals.

**Lemma 6.14.** Let \(\mathfrak{L}\) be an ad-compact Lie algebra. If there exists an element \(a \in I_{\mathcal{F}}(\mathfrak{L})\) such that \(ad\mathfrak{L}a\) is not nilpotent then there exists a nonzero finite-dimensional ideal of \(\mathfrak{L}\) which is contained in the ideal of \(\mathfrak{L}\) generated by \(a\).

**Proof.** It is clear that \(a \in I_{\mathcal{F}}(\tilde{\mathfrak{L}})\). There exists a finite-dimensional ideal

\[
I := \sum_{\lambda \in \Lambda} E_\lambda(ad\tilde{\mathfrak{L}}a) + \sum_{\lambda \in \Lambda} E_\lambda(ad\tilde{\mathfrak{L}}a)E_{-\lambda}(ad\tilde{\mathfrak{L}}a)
\]

of \(\tilde{\mathfrak{L}}\), where \(\Lambda = \sigma(ad\tilde{\mathfrak{L}}a) \backslash \{0\}\). Clearly \(J := I \cap \mathfrak{L}\) is an ideal of \(\mathfrak{L}\). Since each \(E_\lambda(ad\tilde{\mathfrak{L}}a)\) is equal to \(ker(ad\tilde{\mathfrak{L}}a - \lambda^n)\) for \(n\) sufficiently large, \(J\) lies in the ideal of \(\tilde{\mathfrak{L}}\) generated by \(a\), so \(J\) lies in the ideal of \(\mathfrak{L}\) generated by \(a\).

We show that \(J\) is nonzero. Let \(W = [a, \mathfrak{L}]\). Then \(W\) is a finite-dimensional invariant subspace for \(ad\mathfrak{L}a\). It is clear that \(ad\mathfrak{L}a\) is not nilpotent, so there exists a nonzero \(\lambda\) such that \(E_\lambda(ad\mathfrak{L}a)\) is nonzero. Since \(E_\lambda(ad\mathfrak{L}a) \subset J\), \(J\) is nonzero. □

### 6.3. E-solvable algebras and E-radical

Recall that a normed Lie algebra is called E-solvable if all its nonzero quotients by closed ideals have nonzero closed Engel ideals.

**Theorem 6.15.** An ad-compact Lie algebra \(\mathfrak{L}\) is E-solvable iff \([\mathfrak{L}, \mathfrak{L}]\) is Engel.

**Proof.** If \(\mathfrak{L}\) is E-solvable then the image of the representation \(ad\tilde{\mathfrak{L}}\) is triangularizable by Corollary 4.24. Hence \(ad\tilde{\mathfrak{L}}(\mathfrak{L}, \mathfrak{L})\) consists of Volterra operators. It follows that \([\mathfrak{L}, \mathfrak{L}]\) is Engel.
Conversely, if $[\mathfrak{L}, \mathfrak{L}]$ is Engel then the closure $J_0$ of $[\mathfrak{L}, \mathfrak{L}]$ is Engel. Let $J$ be a closed ideal of $\mathfrak{L}$ such that $J \neq \mathfrak{L}$. If $J_0 \subset J$ then $\mathfrak{L}/J$ is commutative and hence Engel. So we may suppose that $J_1 := J \cap J_0 \neq J_0$. Then the image of $J_0/J_1$ in $\mathfrak{L}/J$ under the standard map $\mathfrak{L}/J_1 \to \mathfrak{L}/J$ is Engel by Corollary 6.2. Therefore, $\mathfrak{L}$ is $E$-solvable. □

It follows directly from the definition that the quotient of an $E$-solvable Lie algebra by a closed ideal is $E$-solvable.

**Corollary 6.16.** Let $\mathfrak{L}$ be an ad-compact Lie algebra.

(i) $\mathfrak{L}$ is $E$-solvable iff $\widetilde{\mathfrak{L}}$ is $E$-solvable.

(ii) If $\mathfrak{L}$ is $E$-solvable then so are its subalgebras (and quotients by closed ideals).

**Proof.** Apply Theorem 6.15 and Lemma 6.1. For (ii), apply also Corollary 6.2(ii).

Given a subset $N$ of a Lie algebra $\mathfrak{L}$, $N^\sharp = \{a \in \mathfrak{L} : [a, N] = 0\}$ is called the commutant (or centralizer) of $N$ in $\mathfrak{L}$. It is easy to check that $N^\sharp$ is a Lie subalgebra, and if $N$ is an ideal of $\mathfrak{L}$ then so is $N^\sharp$. Furthermore if $\mathfrak{L}$ is normed then $N^\sharp$ is closed in $\mathfrak{L}$. The following lemma is well known in the case of finite-dimensional Lie algebras.

**Lemma 6.17.** If $J$ is a FDS ideal of a normed Lie algebra $\mathfrak{L}$ then $\mathfrak{L}$ is the direct sum of the ideals $J$ and $J^\sharp$.

**Proof.** For any $a \in \mathfrak{L}$, the derivation $\text{ad}_J a$ of $J$ is inner by the Weyl theorem [36, Corollary of Theorem 5.2]. Hence there is $b \in J$ such that $\text{ad}_J a = \text{ad}_J b$. This means that $a - b \in J^\sharp$, whence $a \in J + J^\sharp$. Clearly $J \cap J^\sharp = 0$ since a FDS ideal has a trivial center. □

**Lemma 6.18.** If $\mathfrak{L}$ is an ad-compact Lie algebra and $M$ is its subalgebra isomorphic to $\mathfrak{sl}(2)$ then there is a finite-dimensional ideal $J$ of $\mathfrak{L}$ containing $M$.

**Proof.** Let $h, u, v \in M$ satisfy (6.4). Since $\text{ad}_\mathfrak{L} h$ is a compact operator and $\text{ad}_\mathfrak{L} u \in \mathcal{E}_1(\text{ad} \text{ad}_\mathfrak{L} h)$,

$$\text{ad}_\mathfrak{L} u \in \mathcal{I}_\mathcal{F}(\mathfrak{L})$$

by Lemma 3.12. Since $\text{ad}_\mathfrak{L} h = [\text{ad}_\mathfrak{L} u, \text{ad}_\mathfrak{L} v]$,

$$\text{ad}_\mathfrak{L} h \in \mathcal{I}_\mathcal{F}(\mathfrak{L}).$$

Let $I$ be defined as in (6.3) with $a = h$ and $\mathfrak{L} = \widetilde{\mathfrak{L}}$. It is clear that $I$ contains $M$. The argument in the proof of Theorem 6.7 shows that $I$ is a finite-dimensional ideal of $\mathfrak{L}$. Then $I \cap \mathfrak{L}$ is a finite-dimensional ideal of $\mathfrak{L}$ that contains $M$. □
Theorem 6.19. Let $\mathfrak{L}$ be a complete ad-compact Lie algebra. The following conditions are equivalent.

(i) $\mathfrak{L}$ is $E$-solvable.
(ii) Every $\pi \in \text{Irr}_{\mathcal{K}} \mathfrak{L}$ is at most one-dimensional.
(iii) $\mathfrak{L}$ has no FDS quotients.
(iv) Every finite-dimensional ideal of $\mathfrak{L}$ is solvable.
(v) $\mathfrak{L}$ does not contain $\text{sl}(2)$.

Proof. (i) $\implies$ (ii) If $\pi \in \text{Irr}_{\mathcal{K}} \mathfrak{L}$ is a representation of $\mathfrak{L}$ on a (finite-dimensional by Theorem 6.8) space $X$ and $\dim X > 1$ then $\mathfrak{L}/\ker \pi$ is isomorphic to the irreducible Lie algebra $M := \pi(\mathfrak{L})$ of operators on $X$. It follows from the Lie theorem that $M$ is not solvable. Let $I$ be a solvable radical of $M$ and $J$ its preimage in $\mathfrak{L}$ under $\pi$. We have that $\mathfrak{L}/J$ is isomorphic to the nonzero Lie algebra $M/I$ which has no Engel ideals. Hence $\mathfrak{L}$ is not $E$-solvable.

(ii) $\implies$ (iii) is evident (because FDS Lie algebras have none one-dimensional irreducible representations).

(iii) $\implies$ (iv) Suppose that (iv) fails. Then $\mathfrak{L}$ has a finite-dimensional nonsolvable ideal $J$. Let $I$ be a solvable radical of $J$. Then $I$ is also an ideal of $\mathfrak{L}$ (since $I$ is characteristic). The ad-compact Lie algebra $\mathfrak{L}_1 := \mathfrak{L}/I$ has a FDS ideal $J_1 = J/I$. By Lemma 6.17, $\mathfrak{L}_1 = J_1 + I_1$, whence $\mathfrak{L}_1/I_1$ is FDS. Since any quotient of $\mathfrak{L}_1$ is isomorphic to a quotient of $\mathfrak{L}$, $\mathfrak{L}$ has a FDS quotient and (iii) fails.

(iv) $\implies$ (v) Follows from Lemma 6.18.

(v) $\implies$ (i) Suppose that (i) fails. Then $\mathfrak{L}/J$ has no Engel ideals, for some ideal $J \subset \mathfrak{L}$. By Theorem 6.7, $\mathfrak{L}/J$ has a FDS ideal. It follows that $\mathfrak{L}/J$ contains $\text{sl}(2)$. In other words, there are elements $h, u, v \in \mathfrak{L}$ whose images $h_1, u_1, v_1 \in \mathfrak{L}/J$ satisfy the conditions

$$[h_1, u_1] = u_1, \quad [h_1, v_1] = -v_1, \quad [u_1, v_1] = h_1.$$ 

Let $P$ be the Riesz projection $P_1(\text{ad}_h)$ of $\text{ad}_h$ corresponding to the eigenvalue 1, and $Q$ the Riesz projection $P_1(\text{ad}_{h_1})$ of $\text{ad}_{h_1}$ corresponding to the eigenvalue 1. It follows from Lemma 3.21(ii) that

$$Q = P_1(\text{ad}_{h_1}) = P_1(\text{ad}_h)(\mathfrak{L}/J) = (P_1(\text{ad}_h))(\mathfrak{L}/J) = P(\mathfrak{L}/J).$$

Then $Q(x + J) = Px + J$ for any $x \in \mathfrak{L}$. Since $Qu_1 = u_1$ we get that $Pu + J = u + J$ whence

$$Pu - u \in J.$$
Changing $u$ by $Pu$ we may suppose that $u \in E_1(\text{ad}_\mathfrak{g} h)$. Similarly we suppose that $v \in E_{-1}(\text{ad}_\mathfrak{g} h)$. Let $\Lambda$ be the set of all nonzero integers in $\sigma(\text{ad}_\mathfrak{g} h)$, and let

$$M = \sum_{n \in \Lambda} E_n(\text{ad}_\mathfrak{g} h) + \sum_{n \in \Lambda} E_n(\text{ad}_\mathfrak{g} h)E_{-n}(\text{ad}_\mathfrak{g} h).$$

It follows from Lemma 3.5 that $M$ is a subalgebra of $\mathfrak{g}$. Since $\Lambda$ is finite and all $E_n(\text{ad}_\mathfrak{g} h)$ are finite-dimensional (because $\text{ad}_\mathfrak{g} h$ is compact), $M$ is a finite-dimensional subalgebra of $\mathfrak{g}$. Set as in [46, Section 2] $L_0 = \mathbb{C}h + M$. Since $\mathfrak{g}$ is invariant for $\text{ad}_\mathfrak{g} h$, $L_0$ is a subalgebra of $\mathfrak{g}$, which is finite-dimensional. By our choice $u, v, h \in L_0$. This means that the quotient $L_0/(J \cap L_0)$ contains $\mathfrak{sl}(2)$. Hence, $\mathfrak{g} = L_0$ is not solvable and therefore it itself contains $\mathfrak{sl}(2)$. Thus $\mathfrak{g}$ contains $\mathfrak{sl}(2)$. This contradicts to (v).

It follows from Theorem 6.16 that ideals and quotients of $E$-solvable ad-compact Lie algebras are again $E$-solvable. The following result is of the opposite kind: it states that the class of such algebras is stable under extensions and sums.

**Corollary 6.20.** Let $\mathfrak{g}$ be a complete ad-compact Lie algebra.

(i) If $J$ is a closed ideal of $\mathfrak{g}$, and if $J$ and $\mathfrak{g}/J$ are $E$-solvable then $\mathfrak{g}$ is $E$-solvable.

(ii) If $(J_\lambda)$ is a family of $E$-solvable ideals of $\mathfrak{g}$ then the closure $J$ of the sum of all $J_\lambda$ is $E$-solvable.

**Proof.** (i) If $\mathfrak{g}$ is not $E$-solvable then it contains a Lie subalgebra $M$ isomorphic to $\mathfrak{sl}(2)$. Since $M$ is simple, $M \cap J$ is either $M$ or $0$. This first possibility contradicts the assumption that $J$ is $E$-solvable. Hence the image of $M$ in $\mathfrak{g}/J$ is isomorphic to $M$, a contradiction to $E$-solvability of $\mathfrak{g}/J$.

(ii) Suppose, to the contrary, that $J$ is not $E$-solvable. Since $J$ is an ad-compact Lie algebra, by Theorem 6.19, there exists a closed ideal $I$ of $J$ such that $J/I$ is a nonzero FDS Lie algebra. Let $\pi : J \to J/I$ be the standard epimorphism. Then $\pi J_\lambda$ is an ideal of $J/I$, hence is a FDS Lie algebra for each $\lambda$. Since $\pi J_\lambda$ is also Engel by Corollary 6.2, $\pi J_\lambda = 0$. Thus $J_\lambda \subset I$ for each $\lambda$ and $J = I$, a contradiction.

**Corollary 6.21.** An ad-compact Lie algebra $\mathfrak{g}$ has the largest $E$-solvable ideal. This ideal is topologically characteristic and closed.

**Proof.** If $\mathfrak{g}$ is complete, it follows from part (ii) of Corollary 6.20 that the closure of the sum $J$ of all $E$-solvable ideals of $\mathfrak{g}$ is the largest solvable ideal. Clearly it is closed and invariant under all bounded automorphisms of $\mathfrak{g}$, hence topologically characteristic.

In general, let $J$ be the largest $E$-solvable ideal of $\mathfrak{g}$. If $I$ is an $E$-solvable ideal of $\mathfrak{g}$, then $\tilde{I}$ is an $E$-solvable ideal of $\mathfrak{g}$ by Corollary 6.16. Hence

$$I \subset \mathfrak{g} \cap \tilde{I} \subset \mathfrak{g} \cap J.$$
But $\mathfrak{L} \cap J$ is an $E$-solvable ideal of $\mathfrak{L}$ by the same corollary. Therefore $\mathfrak{L} \cap J$ is the largest $E$-solvable ideal of $\mathfrak{L}$. It is clear that $\mathfrak{L} \cap J$ is closed in $\mathfrak{L}$ and topologically characteristic. □

The existence of the largest $E$-solvable ideal (more precisely, the largest ideal satisfying condition (v) or (iii) of Theorem 6.19) in a complete ad-compact Lie algebra was stated in [46, Section 2]. We included the full proof because we could not overcome some gap in arguments of Vaksman and Gurarij [46].

Given an ad-compact Lie algebra $\mathfrak{L}$, we denote the largest $E$-solvable ideal of $\mathfrak{L}$ by $\text{rad}_E \mathfrak{L}$ and call it the $E$-radical of $\mathfrak{L}$. We say that $\mathfrak{L}$ is $E$-semisimple if $\text{rad}_E \mathfrak{L} = 0$.

**Theorem 6.22.** Let $\mathfrak{L}$ be an ad-compact Lie algebra.

(i) $\text{rad}_E \mathfrak{L}$ consists of all elements $a \in \mathfrak{L}$ such that $\pi a \in \mathfrak{C}$ for any Banach irreducible representation $\pi$ of $\mathfrak{L}$ by compact operators.

(ii) The following relations are true:

$$[\text{rad}_E \mathfrak{L}, \mathfrak{L}] \subset \Lambda(\mathfrak{L}) = \text{rad}_E \mathfrak{L} \cap [\mathfrak{L}, \mathfrak{L}] \subset E(\mathfrak{L}) = \mathcal{R}_0(\mathfrak{L}) \subset \mathcal{R}(\mathfrak{L}) = \text{rad}_E \mathfrak{L} = \mathfrak{L} \cap \text{rad}_E \tilde{\mathfrak{L}}.$$

**Proof.** Let $R$ be the set of all elements $a \in \mathfrak{L}$ such that $\pi a \in \mathfrak{C}$ for every $\pi \in \text{Irr}_K \mathfrak{L}$. Then $R$ contains $\Lambda(\mathfrak{L})$ and satisfies the condition $[R, \mathfrak{L}] \subset \Lambda(\mathfrak{L})$. If $\pi = \text{ad}_V$ for a gap-quotient $V$ of $\text{Lat ad}_V \mathfrak{L}$, and $\pi a = \lambda_a \in \mathfrak{C}$ for $a \in R$, then

$$\sigma(\text{ad}_V (a + b)) = \lambda_a + \sigma(\text{ad}_V b)$$

for every $a \in R$, $b \in \mathfrak{L}$. Since $\lambda_a \in \sigma(\text{ad}_V a)$, it follows from Lemma 6.4 that

$$\rho(\text{ad}_\mathfrak{L} (a + b)) \leq \rho(\text{ad}_\mathfrak{L} a) + \rho(\text{ad}_\mathfrak{L} b),$$

whence $R \subset \mathcal{R}(\mathfrak{L})$.

Since $[\mathcal{R}(\mathfrak{L}), \mathfrak{L}] \subset \mathcal{R}_0(\mathfrak{L})$ and $\mathcal{R}_0(\mathfrak{L})$ is Engel, we conclude that $[\mathcal{R}(\mathfrak{L}), \mathcal{R}(\mathfrak{L})]$ is Engel and hence $E$-solvable. Then $\mathcal{R}(\mathfrak{L})$ is an $E$-solvable closed ideal of $\mathfrak{L}$ by Theorem 5.2 and Corollary 6.20, whence

$$R \subset \mathcal{R}(\mathfrak{L}) \subset \text{rad}_E \mathfrak{L}. $$

Conversely, if $\pi \in \text{Irr}_K \mathfrak{L}$ then the image of $\text{rad}_E \mathfrak{L}$ (being isomorphic to a quotient of $\text{rad}_E \mathfrak{L}$) is a solvable ideal of the finite-dimensional irreducible algebra $\pi(\mathfrak{L})$. By the Lie theorem, it consists of scalar multiples of the identity. This shows that $\text{rad}_E \mathfrak{L} \subset R$. Therefore

$$R = \mathcal{R}(\mathfrak{L}) = \text{rad}_E \mathfrak{L}. $$
We have proved that
\[
[r_{\mathcal{E}L}, \mathcal{L}] \subset \Lambda(\mathcal{L}) \subset \mathcal{R}(\mathcal{L}) = r_{\mathcal{E}L} \mathcal{L}.
\]
It follows from Theorems 6.3 and 6.5 that \( \Lambda(\mathcal{L}) \subset E(\mathcal{L}) = \mathcal{R}^\circ(\mathcal{L}) \). The equality \( r_{\mathcal{E}L} \mathcal{L} = \mathcal{L} \cap r_{\mathcal{E}L} \mathcal{L} \) follows from the proof of Corollary 6.21. It remains to prove that \( \Lambda(\mathcal{L}) = r_{\mathcal{E}L} \mathcal{L} \cap [\mathcal{L}, \mathcal{L}] \).

Since every \( f \in \mathcal{L}^* \) with \( f([\mathcal{L}, \mathcal{L}]) = 0 \) can be considered as an one-dimensional representation of \( \mathcal{L} \),
\[
\Lambda(\mathcal{L}) \subset [\mathcal{L}, \mathcal{L}].
\]
On the other hand, if \( a \in r_{\mathcal{E}L} \mathcal{L} \cap [\mathcal{L}, \mathcal{L}] \) is arbitrary then, for every \( \pi \in \text{Irr}_K \mathcal{L} \), \( \pi a \in \mathbb{C} \) and simultaneously
\[
\text{tr}(\pi a) = 0
\]
since a sum of commutators of finite rank operators has a zero trace. Hence
\[
\pi a = 0.
\]
Thus \( r_{\mathcal{E}L} \mathcal{L} \cap [\mathcal{L}, \mathcal{L}] \subset \Lambda(\mathcal{L}) \). We have finally that
\[
\Lambda(\mathcal{L}) = r_{\mathcal{E}L} \mathcal{L} \cap [\mathcal{L}, \mathcal{L}].
\]
The proof is complete. \( \square \)

**Corollary 6.23.** Let \( \mathcal{L} \) be an ad-compact Lie algebra and \( N = \text{ad}_{\mathcal{L}} \mathcal{L} \). Then the closed algebra generated by \( \text{ad}_{\mathcal{L}}(r_{\mathcal{E}L} \mathcal{L}) \) is commutative modulo the Jacobson radical and \( r_{\mathcal{E}L} \mathcal{L} = \{ a \in \mathcal{L} : \text{ad}_{\mathcal{L}} a \in \mathcal{R}(N) \} \). Moreover,
\[
\sigma(\text{ad}_{\mathcal{N}} \text{ad}_{\mathcal{L}} (a + b)) \subset \sigma(\text{ad}_{\mathcal{N}} \text{ad}_{\mathcal{L}} a) + \sigma(\text{ad}_{\mathcal{N}} \text{ad}_{\mathcal{L}} b)
\]
for all \( a \in r_{\mathcal{E}L} \mathcal{L}, b \in \mathcal{L} \).

**Proof.** Let \( M = \text{ad}_{\mathcal{L}}(r_{\mathcal{E}L} \mathcal{L}) \). Then \( M \subset \mathcal{K}(\mathcal{L}) \). For every gap-quotient of \( \text{Lat} M \), we have \( [M, M]|V = 0 \) by Theorem 6.22. Then
\[
[M, M] \subset r_{\mathcal{A}(M)}
\]
by Lemma 2.6. Therefore \( \mathcal{A}(M) \) is commutative modulo the Jacobson radical.

The other statements follows easily from Corollary 5.13. \( \square \)
Corollary 6.24. Let $\mathcal{L}$ be an ad-compact Lie algebra. Then

(i) $\text{rad}_E \mathcal{L} = \{ a \in \mathcal{L} : \sigma(\text{ad}_V (a + b)) \subset \sigma(\text{ad}_V a) + \sigma(\text{ad}_V b) \text{ for all } b \in \mathcal{L} \}$.

(ii) If $\phi : \mathcal{L} \to M$ is a bounded homomorphism of ad-compact Lie algebras with dense image then $\phi(\text{rad}_E \mathcal{L}) \subset \text{rad}_E M$.

(iii) $\mathcal{L}$ is $E$-semisimple iff $\mathcal{L}$ contains no nonzero Engel ideals.

(iv) Every bounded Lie derivation $D$ of $\mathcal{L}$ maps $\text{rad}_E \mathcal{L}$ into $E(\mathcal{L})$.

Proof. (i) It follows from (i) of Theorem 6.22 that

$$\sigma(\text{ad}_V (a + b)) \subset \sigma(\text{ad}_V a) + \sigma(\text{ad}_V b)$$

for every gap-quotient $V$ of $\text{Lat} \text{ad}_V \mathcal{L}$ and all $a \in \text{rad}_E \mathcal{L}$, $b \in \mathcal{L}$. Using Lemma 6.4, we obtain

$$\sigma(\text{ad}_{\mathcal{L}} (a + b)) \setminus \{0\} \subset \sigma(\text{ad}_{\mathcal{L}} a) + \sigma(\text{ad}_{\mathcal{L}} b).$$

If $\mathcal{L}$ is infinite-dimensional then 0 belongs to the spectrum of each compact operator on $\mathcal{L}$. If $\mathcal{L}$ is finite-dimensional then $\sigma(\text{ad}_V b)$ for each $b \in \mathcal{L}$, is clearly equal to the union of $\sigma(\text{ad}_V b)$ for a finite number of gaps $V$. Thus (i) is true in any case.

(ii) Since $[\phi(\text{rad}_E \mathcal{L}), \phi(\text{rad}_E \mathcal{L})]$ is Engel, $\phi(\text{rad}_E \mathcal{L})$ is $E$-solvable. Then the closure of $\phi(\text{rad}_E \mathcal{L})$ in $M$ is also $E$-solvable ideal of $M$. So

$$\phi(\text{rad}_E \mathcal{L}) \subset \text{rad}_E M.$$

(iii) If $\mathcal{L}$ has no nonzero Engel ideals then, since $[\text{rad}_E \mathcal{L}, \text{rad}_E \mathcal{L}]$ is Engel, $\text{rad}_E \mathcal{L}$ is commutative, hence Engel. Therefore $\text{rad}_E \mathcal{L} = 0$, i.e. $\mathcal{L}$ is $E$-semisimple. The converse is evident.

(iv) Since $\text{rad}_E \mathcal{L}$ is topologically characteristic, $D$ is a Lie derivation of $\text{rad}_E \mathcal{L}$. Let $M = \text{rad}_E \mathcal{L}$ and $A = \overline{\text{Ad}(\text{ad}_M M)}$. Then $\text{ad}D$ is a bounded derivation of $A$. Since $A/\text{rad} A$ is commutative, by Corollary 6.23, one has

$$(\text{ad}D)A \subset \text{rad} A$$

by the Singer–Wermer theorem (see [37]). But $(\text{ad}D)\text{ad}_M a = \text{ad}_M Da$ for every $a \in \mathcal{L}$. So we obtain that

$$DM \subset E(M)$$

by Theorem 6.3(i). Since $E(M) = E(\mathcal{L}) \cap M$, by Theorem 6.3(iv), and $E(\mathcal{L}) \subset M$, we have $E(M) = E(\mathcal{L})$, whence $DM \subset E(\mathcal{L})$. □

Let $\mathcal{U}$ be some class of normed Lie algebras closed under topological isomorphisms, passing to ideals and quotients by closed ideals, and let $\mathcal{R} : \mathcal{U} \to \mathcal{U}$ be a map such
that \( \mathfrak{R}(\mathfrak{L}) \) is a closed ideal of \( \mathfrak{L} \) for every \( \mathfrak{L} \in \mathfrak{U} \). Similarly to [11] where the case of normed associative algebras was considered, we call \( \mathfrak{R} \) a strong hereditary topological radical on \( \mathfrak{U} \) if the following conditions hold for arbitrary \( \mathfrak{L}, \mathfrak{M} \in \mathfrak{U} \), an ideal \( J \) of \( \mathfrak{L} \) and a bounded Lie epimorphism \( \phi : \mathfrak{L} \rightarrow \mathfrak{M} \).

\[
R(L/R(L)) = 0.
\]

\[
R(L) \subset R(M).
\]

\[
R(J) = J \cap R(L).
\]

Note that (\( \gamma \)) implies immediately that \( R(R(L)) = R(L) \).

**Theorem 6.25.** The map \( \text{rad}_E : \mathfrak{L} \rightarrow \text{rad}_E \mathfrak{L} \) is a strong hereditary topological radical on the class of all ad-compact Lie algebras.

**Proof.** The property (\( \beta \)) follows from Corollary 6.24(ii).

(\( \alpha \)) Suppose, to the contrary, that \( \mathfrak{L}/\text{rad}_E \mathfrak{L} \) has a nonzero Engel ideal. Since the standard map \( \mathfrak{L}/\text{rad}_E \mathfrak{L} \rightarrow \tilde{\mathfrak{L}}/\text{rad}_E \tilde{\mathfrak{L}} \) is a bounded monomorphism with dense image, one may suppose in virtue of Corollary 6.2(i) that \( \mathfrak{L}/\text{rad}_E \mathfrak{L} \) has a nonzero closed Engel ideal \( I \). Let \( J \) be the preimage of \( I \) under the standard map \( \tilde{\mathfrak{L}} \rightarrow \tilde{\mathfrak{L}}/\text{rad}_E \tilde{\mathfrak{L}} \). Then \( J \) is a complete ad-compact algebra and \( J/\text{rad}_E \tilde{\mathfrak{L}} \) is Engel. It follows from Corollary 6.20(i) that \( J \) is \( E \)-solvable. Since \( J \) is an ideal of \( \tilde{\mathfrak{L}} \), \( J \subset \text{rad}_E \tilde{\mathfrak{L}} \), a contradiction. Now it follows from Corollary 6.24 that \( \mathfrak{L}/\text{rad}_E \mathfrak{L} \) is \( E \)-semisimple.

(\( \gamma \)) Since \( \text{rad}_E J \) is topologically characteristic, \( \text{rad}_E J \) is an ideal of \( \mathfrak{L} \). Since \( \text{rad}_E J \) is \( E \)-solvable, \( \text{rad}_E J \subset \text{rad}_E \mathfrak{L} \). On the other hand, \( J \cap \text{rad}_E \mathfrak{L} \) is \( E \)-solvable ideal of \( J \), whence \( J \cap \text{rad}_E \mathfrak{L} \subset \text{rad}_E J \). \( \square \)

**Corollary 6.26.** Let \( \mathfrak{L} \) be an ad-compact Lie algebra. Then

(i) \( \text{rad}_E \mathfrak{L} \) is the smallest among all closed ideals \( J \) of \( \mathfrak{L} \) having the property that \( \mathfrak{L}/J \) is \( E \)-semisimple.

(ii) If \( J \) is an ideal of \( \mathfrak{L} \) and \( J \cap \text{rad}_E \mathfrak{L} = 0 \) then \( J \) is \( E \)-semisimple. In particular, if \( \mathfrak{L} \) is \( E \)-semisimple then each ideal of \( \mathfrak{L} \) is \( E \)-semisimple.

**Proof.** (i) Let \( \mathfrak{L}/J \) be \( E \)-semisimple. Let \( \phi \) be the standard map \( \mathfrak{L} \rightarrow \mathfrak{L}/J \). It follows from the property (\( \beta \)) of the \( E \)-radical (see Theorem 6.25) that

\[
\phi(\text{rad}_E \mathfrak{L}) \subset \text{rad}_E (\mathfrak{L}/J) = 0,
\]

whence \( \text{rad}_E \mathfrak{L} \subset J \). On the other hand, \( \mathfrak{L}/\text{rad}_E \mathfrak{L} \) is \( E \)-semisimple by property (\( \alpha \)).

(ii) Follows from the property (\( \gamma \)) of the \( E \)-radical. \( \square \)

6.4. \( E \)-semisimple Lie algebras

**Theorem 6.27.** Let \( \mathfrak{L} \) be an \( E \)-semisimple ad-compact Lie algebra. Then

(i) There exist a complete semisimple ad-compact Lie algebra \( \mathfrak{M} \) and a bounded monomorphism \( \phi : \mathfrak{L} \rightarrow \mathfrak{M} \) with dense image.
(ii) There exists a norm $\|\cdot\|'$ on $\mathfrak{L}$ satisfying the condition $\|\cdot\|' \leq \beta \|\cdot\|$ for some $\beta > 0$, such that $(\mathfrak{L}, \|\cdot\|')$ is an $\text{ad}$-compact Lie algebra and $(\mathfrak{L}, \|\cdot\|')$ is $E$-semisimple.

**Proof.** Since $\mathfrak{L} \cap \text{rad}_E \tilde{\mathfrak{L}}$ is an $E$-solvable ideal of $\mathfrak{L}$,

$$\mathfrak{L} \cap \text{rad}_E \tilde{\mathfrak{L}} = 0.$$ 

Therefore the standard map $\phi : \mathfrak{L} \to \tilde{\mathfrak{L}}/\text{rad}_E \tilde{\mathfrak{L}}$ is a bounded monomorphism with dense image. This proves (i). Setting

$$\|a\|' = \|\phi(a)\|$$

for all $a \in \mathfrak{L}$, we obtain (ii). □

Note in connection with (ii) of Theorem 6.27 that the completion of an $E$-semisimple ad-compact Lie algebra can be not $E$-semisimple. Such an example is due to Vaksman and Gurarij [46].

We need the following simple lemma.

**Lemma 6.28.** Let $\mathfrak{X}, \mathfrak{Y}$ be Banach spaces, $S \in B(\mathfrak{Y})$ and $T \in B(\mathfrak{X})$. Suppose that $K : \mathfrak{X} \to \mathfrak{Y}$ is a linear operator with dense range such that $KT = SK$. If $\rho_e(S) < \rho(S)$ or $K$ is surjective then $\rho(S) \leq \rho(T)$.

**Proof.** Let $\lambda \in \sigma(S)$ with $|\lambda| = \rho(S)$. Suppose, to the contrary, that $\rho(S) > \rho(T)$. Then $T - \lambda$ is invertible and

$$(T - \lambda)K\mathfrak{X} = K(T - \lambda)\mathfrak{X} = K\mathfrak{X}.$$ 

If $K\mathfrak{X} = \mathfrak{Y}$ then $(\lambda) = \mathfrak{Y}$; on the other hand, since $\lambda$ is a boundary point of $\sigma(S)$, $(\lambda) \neq \mathfrak{Y}$, a contradiction.

If $\rho_e(S) < \rho(S)$ then

$$\rho_e(S^*) \leq \rho_e(S) < \rho(S) = \rho(S^*),$$

whence $\lambda$ is an eigenvalue of $S^*$ and therefore $\mathfrak{Y} \neq (\lambda)\mathfrak{Y}$. Then

$$\mathfrak{Y} \neq (\lambda)\mathfrak{Y} = (T - \lambda)K\mathfrak{X} = K(T - \lambda)\mathfrak{X} = K\mathfrak{X} = \mathfrak{Y},$$

a contradiction. □

For surjective $K$, the assertion was in fact proved in [1, Lemma B].
Theorem 6.29. Let $M, \mathfrak{L}$ be Banach Lie algebras. If $\mathfrak{L}$ is E-semisimple and ad-compact then every homomorphism $\phi : M \to \mathfrak{L}$ with dense range is continuous.

Proof. Let $S(\phi)$ be the separating space of $\phi$, that is the set of all $b \in \mathfrak{L}$ for which there exists a sequence $(a_n)$ of elements of $M$ such that $a_n \to 0$ and $\phi(a_n) \to b$ as $n \to \infty$. If $c = \phi(a)$ is an arbitrary element of $\phi(M)$ then $[a, a_n] \to 0$ and

$$\phi([a, a_n]) = [c, \phi(a_n)] \to [c, b]$$

as $n \to \infty$, whence we see that

$$[S(\phi), \phi(\mathfrak{L})] \subset S(\phi).$$

Since $S(\phi)$ is closed in $\mathfrak{L}$, $S(\phi)$ is an ideal of $\mathfrak{L}$. Since $\phi \text{ad}_M a = (\text{ad}_\mathfrak{L} \phi(a))\phi$ and $\text{ad}_\mathfrak{L} \phi(a)$ is a compact operator,

$$\rho(\text{ad}_\mathfrak{L} \phi(a)) \leq \rho(\text{ad}_M a))$$

by Lemma 6.28, for every $a \in M$. Therefore

$$\rho(\text{ad}_\mathfrak{L} \phi(a_n)) \leq \rho(\text{ad}_M a_n) \leq \|\text{ad}_M a_n\| \to 0$$

as $n \to \infty$. Since $\text{ad}_\mathfrak{L} b$ is a compact operator and $\text{ad}_\mathfrak{L} \phi(a_n) \to \text{ad}_\mathfrak{L} b$ as $n \to \infty$, we have

$$\rho(\text{ad}_\mathfrak{L} b) = \lim \rho(\text{ad}_\mathfrak{L} \phi(a_n)) = 0.$$

This shows that $S(\phi)$ is an Engel ideal of $\mathfrak{L}$. Since $\mathfrak{L}$ is E-semisimple, $S(\phi) = 0$ and $\phi$ is continuous. □

Corollary 6.30. Let $(\mathfrak{L}, \|\cdot\|)$ be a complete E-semisimple ad-compact Lie algebra. Let $\|\cdot\|$ be a norm on $\mathfrak{L}$ such that $(\mathfrak{L}, \|\cdot\|)$ is a Banach Lie algebra. Then $\|\cdot\|$ is equivalent to $\|\cdot\|$. □

Proof. The identity map $\phi : (\mathfrak{L}, \|\cdot\|') \to (\mathfrak{L}, \|\cdot\|)$ is bounded by Theorem 6.29. The reverse map $\phi^{-1}$ is bounded by the Closed Graph theorem.

Each complete E-semisimple ad-compact Lie algebra has nonzero ad-finite elements by Lemma 6.6. The following example shows that incomplete E-semisimple ad-compact Lie algebras can have no nonzero ad-finite elements (hence no nonzero finite-dimensional ideals).
Example 6.31. Let $c_0$ be a Banach algebra of complex sequences convergent to 0, with sup-norm. Let $x \in c_0$ have a finite number of zero coordinates, $A = A(x)$ a subalgebra of $c_0$ generated by $x$, and let $\mathfrak{U} = A \otimes \mathfrak{sl}(2)$ be a Lie algebra with the Lie product defined by (6.5) and with the projective tensor norm. Then $\mathfrak{U}$ is an incomplete ad-compact Lie algebra which is $E$-semisimple and contains no nonzero ad-finite elements.

Proof. It is clear that $c_0$ is a completely continuous algebra, so $\mathfrak{U}$ is an ad-compact Lie algebra. Let $J$ be a nonzero ideal of $\mathfrak{U}$. Since $A$ is an integral domain, it follows from (6.6)–(6.8) that there exists a nonzero element $\beta \in A$ such that $\beta \otimes h \in J$. Note that

$$(\operatorname{ad}_\mathfrak{U} \beta \otimes h)^{n+1}(x \otimes u) = (\operatorname{ad}_J \beta \otimes h)^n(\beta x \otimes u) = \beta^{n+1} x \otimes u,$$

whence

$$\| (\operatorname{ad}_J \beta \otimes h)^n(\beta x \otimes u) \|^{1/n} = \| \beta^{n+1} x \|^{1/n} \| u \|^{1/n}$$

does not tend to zero. This shows that $\operatorname{ad}_J \beta \otimes h$ is not quasi-nilpotent. So $\mathfrak{U}$ has no Engel ideals, whence is $E$-semisimple.

Suppose, to the contrary, that $\mathcal{I}_\mathcal{F}(\mathfrak{U})$ is not zero. Then it contains a nonzero element of the form $\beta \otimes h$. Since this element is a nonnilpotent ad-finite element of $\mathfrak{U}$, $\mathfrak{U}$ has some nonzero finite-dimensional ideal $I$ by Lemma 6.14. Then $I$ contains $\gamma \otimes h$ with some nonzero $\gamma \in A$. Since $(\operatorname{ad}_\mathfrak{U} \gamma \otimes h)^n(\alpha \otimes u) = \gamma^n \alpha \otimes u \in I$ for every $n \in \mathbb{N}$, the family $(\gamma^n \alpha)_{n=1}^\infty$ is linearly dependent. But it is easy to check that this family must be linearly independent, a contradiction. □

We saw in Example 6.12 that ad-finite elements can generate infinite-dimensional ideals. The following theorem and corollary show that this is impossible for $E$-semisimple ad-compact Lie algebras.

Theorem 6.32. Let $\mathfrak{U}$ be a complete $E$-semisimple ad-compact Lie algebra.

(i) If $J$ is an ideal of $\mathfrak{U}$ and $F = J \cap \mathcal{I}_\mathcal{F}(\mathfrak{U})$ then $F^\perp \cap J = \{0\}$.

(ii) Each ad-finite element is contained in the sum of a finite number of FDS (simple) ideals of $\mathfrak{U}$.

Proof. (i) Let $I = \{a \in J : [a, F] = 0\}$. Since $F$ is an ideal of $\mathfrak{U}$, so is $I$. Suppose that $I \neq 0$. By Theorem 6.7, $I$ contains a nonzero FDS ideal $M$ (because $I$ has no nonzero Engel ideals). But $M \subset F$ and $[M, M] = M$, a contradiction.

(ii) Let $a \in \mathcal{I}_\mathcal{F}(\mathfrak{U})$ be arbitrary, and let $(J_x)$ be a family of all FDS simple ideals of $\mathfrak{U}$. By Lemma 6.17, $a = a_x + b_x$ for each $x$, where $a_x \in J_x$ and $b_x \in J_x^\perp$. For each nonzero $a_x$ there exists an element $c_x \in J_x$ such that $[a, c_x] = [a_x, c_x]$ is a nonzero element of $J_x$. It is clear that the set $G$ of all nonzero $[a, c_x]$ is linearly independent. Since $[a, \mathfrak{U}]$ is finite-dimensional, $G$ is finite. So there exist only a finite number of $x$’s with $a_x \neq 0$. Hence $b = \sum a_x$ is a finite sum and $b \in \mathcal{I}_\mathcal{F}(\mathfrak{U})$. 

We claim that \( a = b \). Suppose, to the contrary, that \( c = a - b \neq 0 \). Let \( J = \cap J^\circ \). Then \( J \) is an ideal of \( \mathfrak{L} \) and \( c \in J \). Since the ideal of \( \mathfrak{L} \) generated by \( c \) is \( E \)-semisimple by Corollary 6.26(ii), it has an element \( d \) with nonnilpotent \( \text{ad}_\mathfrak{L} d \). Since \( d \) is an ad-finite element of \( \mathfrak{L} \) and \( d \in J \), there exists a nonzero finite-dimensional ideal \( I \) of \( \mathfrak{L} \) which is contained in \( J \) by Lemma 6.14. Since \( I \) is \( E \)-semisimple, \( I \) is a FDS ideal which is a finite sum of nonzero FDS simple ideals, say \( J_2 \), \ldots, \( J_n \). But

\[
J_2 = [J_2, J_2] = [J, J_2] \subset [J, J_2] \subset [J^\circ, J_2] = 0,
\]
a contradiction. \( \square \)

**Corollary 6.33.** Let \( \mathfrak{L} \) be an \( E \)-semisimple ad-compact Lie algebra. Each ad-finite element of \( \mathfrak{L} \) generates a finite-dimensional ideal in \( \mathfrak{L} \).

**Proof.** Suppose, to the contrary, that there is an ad-finite element \( a \in \mathfrak{L} \) such that \( a \) generates an infinite-dimensional ideal \( J \) in \( \mathfrak{L} \). By Theorem 6.27, there exist a complete semisimple ad-compact Lie algebra \( M \) and a monomorphism \( \phi : \mathfrak{L} \to M \) with dense image. Then \([\phi(a), \phi(M)]\) is finite-dimensional, so is \([\phi(a), M]\), whence \( \phi(a) \) is an ad-finite element of \( M \). By Theorem 6.32, \( \phi(a) \) generates a finite-dimensional ideal \( I \) of \( M \). Since \( \phi(J) \subset I \) and \( \ker \phi = 0 \), we obtain a contradiction. \( \square \)

### 6.5. Killing forms and the Cartan’s criterion

In this subsection, we characterize \( E \)-radical and \( E \)-semisimple ad-compact Lie algebras in terms of their Killing forms. In distinction to the finite-dimensional theory, these forms are defined only partially. We denote by \( W(\mathfrak{L}) \) the subset \((\mathfrak{L} \times \mathfrak{L}) \cup (\mathfrak{L} \times I_F(\mathfrak{L}))\) of \( \mathfrak{L} \times \mathfrak{L} \) and call pairs \((a, b)\) in \( W(\mathfrak{L})\) admissible. In other words, \((a, b)\) is admissible if at least one of the elements \( a, b \) is ad-finite.

The set \( W(\mathfrak{L}) \) is a bilinear subset of \( \mathfrak{L} \times \mathfrak{L} \): if \((a_1, b), (a_2, b) \in W(\mathfrak{L})\) then

\[
(a_1 + a_2, b) \in W(\mathfrak{L})
\]

and so on. So it is possible to consider bilinear forms on \( W(\mathfrak{L}) \). We define on \( W(\mathfrak{L}) \) the *Killing form* \( f_\mathfrak{L}(a, b) \) of \( \mathfrak{L} \) by

\[
f_\mathfrak{L}(a, b) = \text{tr}(\text{ad}_\mathfrak{L} a \, \text{ad}_\mathfrak{L} b).
\]

Let us say that two ideals \( J_1 \) and \( J_2 \) of \( \mathfrak{L} \) are orthogonal with respect to the Killing form \( f_\mathfrak{L} \) if \( f_\mathfrak{L} \) vanishes on \( W(\mathfrak{L}) \cap (J_1 \times J_2) \). Also, the Killing form \( f_\mathfrak{L} \) is said to be nondegenerate on an ideal \( J \) of \( \mathfrak{L} \) if for any nonzero element \( a \in J \) there is \( b \in J \) such that \((a, b) \in W(\mathfrak{L}) \) and \( f_\mathfrak{L}(a, b) \neq 0 \).

**Theorem 6.34.** Let \( \mathfrak{L} \) be a complete ad-compact Lie algebra and \( J \) its ideal. Then

(i) \( J \) is orthogonal to \( E(J) \) with respect to \( f_\mathfrak{L} \).
(ii) \( J \) is \( E \)-solvable if and only if \( J \) is orthogonal to \( [J, J] \) with respect to \( f_\mathfrak{U} \).

(iii) \( J \) is \( E \)-semisimple if and only if \( f_\mathfrak{U} \) is nondegenerate on \( J \).

**Proof.** 

(i) By Theorem 6.3(iv), \( E(J) \subset E(\mathfrak{U}) \). Since \( \text{ad}_\mathfrak{U} E(\mathfrak{U}) \subset \text{rad} \mathcal{A}(\text{ad}_\mathfrak{U} \mathfrak{U}) \), \( \text{ad}_\mathfrak{U} a \text{ad}_\mathfrak{U} b \) is a Volterra operator for every \( a \in E(J) \) and \( b \in J \). If the pair \((a, b)\) is admissible, we have

\[
\text{tr}(\text{ad}_\mathfrak{U} a \text{ad}_\mathfrak{U} b) = 0.
\]

This means that \( J \) is orthogonal to \( E(J) \) with respect to \( f_\mathfrak{U} \).

(ii) If \( J \) is \( E \)-solvable then \( [J, J] \subset E(J) \), whence \( J \) is orthogonal to \( [J, J] \) by (i).

Assume now that \( J \) is orthogonal to \( [J, J] \) with respect to \( f_\mathfrak{U} \), but is not \( E \)-solvable. Let \( M = \{ T \in \mathcal{F}(\mathfrak{U}) : [T, \text{ad}_\mathfrak{U} J] \subset \text{ad}_\mathfrak{U} [J, J] \} \). For every \( T \in M \) and \( a, b \in J \), we have

\[
\text{tr}(T \text{ad}_\mathfrak{U} [a, b]) = \text{tr}([T, \text{ad}_\mathfrak{U} a] \text{ad}_\mathfrak{U} b) = 0.
\]

(6.11)

Let \( F = J \cap \mathcal{I}_\mathcal{F}(\mathfrak{U}) \). Then clearly \( M \) is a Lie algebra, \( \text{ad}_\mathfrak{U} [F, F] \) is an ideal of \( M \) and

\[
\text{tr}(TS) = 0
\]

for every \( T \in M \) and \( S \in \text{ad}_\mathfrak{U} [F, F] \), by (6.11). It follows from Proposition 3.20(ii) that \( \text{ad}_\mathfrak{U} [F, F] \) consists of nilpotents. Therefore \( F \) is an \( E \)-solvable ideal. Since \( J \) is not \( E \)-solvable by our assumption, \( J \) contains \( \text{sl}(2) \) by Theorem 6.19; moreover, \( F \) contains \( \text{sl}(2) \) by Lemma 6.18. This contradicts to the fact that \( F \) is \( E \)-solvable.

(iii) Let the Killing form \( f_\mathfrak{U} \) be nondegenerate on \( J \). Assume, to the contrary, that there is a nonzero \( a \in E(J) \). It follows from (i) that \( a \notin \mathcal{I}_\mathcal{F}(\mathfrak{U}) \). Therefore there exists an element \( b \in J \cap \mathcal{I}_\mathcal{F}(\mathfrak{U}) \) such that \( f_\mathfrak{U}(a, b) \neq 0 \). Let \( F = J \cap \mathcal{I}_\mathcal{F}(\mathfrak{U}) \). Since, by (i),

\[
f_\mathfrak{U}([F, E(J)], J) \subset f_\mathfrak{U}(F, E(J)) = \{0\},
\]

we have \([F, E(J)] = 0 \). Hence

\[
[\text{ad}_\mathfrak{U} b, \text{ad}_\mathfrak{U} E(J)] = 0;
\]

in particular, since \( \text{ad}_\mathfrak{U} a \) is Volterra, so is \( \text{ad}_\mathfrak{U} a \text{ad}_\mathfrak{U} b \) and then

\[
f_\mathfrak{U}(a, b) = \text{tr}(\text{ad}_\mathfrak{U} a \text{ad}_\mathfrak{U} b) = 0.
\]

This contradiction shows that in fact

\[
E(J) = 0.
\]

By Corollary 6.24(iii), \( \mathfrak{U} \) is \( E \)-semisimple.
Now let \( J \) be \( E \)-semisimple. Assume, to the contrary, that \( f_\mathcal{L} \) is degenerate on \( J \), i.e., there exists a nonzero \( a \in J \) such that \( f_\mathcal{L}(a, b) = 0 \) for every \( b \in \mathcal{L} \) with \((a, b) \in W(\mathcal{L})\). Let \( F = J \cap \mathcal{I}_F(\mathcal{L}) \), and let \( I = \{ b \in F : f_\mathcal{L}(b, J) = \{0\} \} \). Then \( I \) is an ideal of \( \mathcal{L} \) and \( I \) is orthogonal to \([I, I]\) with respect to \( f_\mathcal{L} \). By (ii), \( I \) is \( E \)-solvable, and in the same time \( I \) is \( E \)-semisimple by Corollary 6.26(ii), whence \( I = 0 \). Since \([a, F] \subset I\), we have that

\[
[a, F] = 0.
\]

But \( \{ b \in J : [b, F] = 0 \} = \{0\} \) by Theorem 6.32(i), whence \( a = 0 \), a contradiction.

\[\blacksquare\]

**Corollary 6.35.** Let \( \mathcal{L} \) be a complete ad-compact algebra. Then

(i) \( \mathcal{L} \) is \( E \)-semisimple iff its Killing form is nondegenerate on \( \mathcal{L} \).

(ii) \( \mathcal{L} \) is \( E \)-solvable iff \( \mathcal{L} \) is orthogonal to \([\mathcal{L}, \mathcal{L}]\) with respect to its Killing form.

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**References**


