JOURNAL OF NUMBER THEORY **58**, 1–8 (1996)
ARTICLE NO. 0057

The Number of Embeddings of Quadratic $\mathbb{Z}$-Lattices

DONALD G. JAMES*

Department of Mathematics, Pennsylvania State University, University Park, Pennsylvania 16802

Communicated by J. S. Hsia

Received May 17, 1994; revised January 27, 1995

The number of inequivalent primitive embeddings of a quadratic lattice M into an indefinite even unimodular $\mathbb{Z}$-lattice L, modulo the action of the orthogonal groups $O(L)$, $SO(L)$, and $O'(L)$, are determined.

1. INTRODUCTION

Let L be a unimodular lattice on an S-indefinite quadratic space V of finite dimension $n \geqslant 3$ over an algebraic number field F. Denote by $O(V)$ the orthogonal group of V, and by $O(L)$ the subgroup of those isometries that leave L invariant. Let M be a second S-lattice on a non-degenerate quadratic space with dimension $m < n$. In [1] we studied primitive embeddings of M into L, and the number $N(L, M)$ of inequivalent embedding modulo the action of $O(L)$, $SO(L)$ and the spinorial kernel $O'(L) = O(L) \cap O(V)$. These results were incomplete at dyadic primes and will now be completed when 2 is unramified and L is an even lattice. The notation and terminology in [1] will be continued.

We first determine the number $e_2 = N'(L_2, M_2)$ of local dyadic embeddings modulo the action of $O'(L_2)$ when $m + m_2 = n$ (see Theorems 2.1 and 2.6). By studying the action of the quotient group $O(L)/O'(L)$ on the local embeddings, all situations where there is a unique global embedding modulo the action of $O(L)$ are then determined for even $\mathbb{Z}$-lattices L (see Theorems 3.1 and 3.2). This extends the earlier work of Miranda and Morrison [2].

* This research was partially supported by NSA Grant MDA 904-94-H-2034. It was completed while the author was on sabbatical leave at the University of Auckland.

0022-314X/96 $18.00

2. Unramified Dyadic Embeddings

Let q be the size of the residue class field $\mathcal{O}_2/2\mathcal{O}_2$. Theorems 2.1 and 2.6 evaluate e_2 when $m+m_2=n$. For $q=2$ the data is the same as for e_2 in Table II of [2, 1986]. Let $\wp$ denote the subgroup of integers in $\mathcal{O}_2$ of the form $\alpha(\alpha+1)$. Then $\wp=2\mathcal{O}_2$ when $q=2$, and in general $2\mathcal{O}_2 \subseteq \wp$ with the index $[\mathcal{O}_2 : \wp]=2$. Let $M_2=\perp_{k\geqslant 0} J_k$ be a Jordan splitting with J_k the 2^k-modular component. The component J_k is called *odd* if there exists $z\in J_k$ with $q(z)\in 2^{k-1}\mathcal{U}_2$; otherwise J_k is *even* (including $J_k=0$) and $q(J_k)\subseteq 2^k\mathcal{O}_2$. Then $r_k=\operatorname{rank} J_k$ and the parity of J_k are invariants of M_2. In the notation of [2, p. 31], $r_k=2s(k)+\operatorname{rank} w(k)$, and J_k is even when $w(k)=0$.

There are several cases to consider when J_1 is odd. As in [1], first reduce to $M_2=M_2(1)$ and $n=2m_2$ by cancelling the even unimodular component $M_2(0)=J_0$ from M_2 and L_2. Fix $x\in J_1$ with $q(x)=\lambda\equiv 1 \bmod 2$; λ corresponds to ε in Table II of [2].

Case 1: $r_1\geqslant 3$, or $r_1=2$ with J_2 odd, or $r_1=2$ with J_2 even and the discriminant $dJ_1\in -4(1+4\mathcal{O}_2)\,\mathcal{U}_2^2$.

Case 2: $r_1=2$ with J_2 even and $dJ_1\notin -4(1+4\mathcal{O}_2)\,\mathcal{U}_2^2$. When $q=2$, $\lambda \bmod 4$ is an invariant of m_2.

Case 3: $r_1=1$, J_2 is odd, and $r_2\geqslant 2$ or J_3 odd.

Case 4: $r_1=r_2=0$ and J_3 is even. Choose x with $\lambda\equiv 1 \bmod 4$. Let $J_1=\mathcal{O}_2 x$ and normalize $J_2=\mathcal{O}_2 z$ with $q(z)=2\eta\equiv 2 \bmod 4$. The cosets $\lambda+4\wp$ and $\eta+4\wp$ are then invariants of M_2 (since J_3 is even); η is the same as in Table II of [2].

Case 5: $r_1=1$, and $r_2\geqslant 2$ with J_2 even, or $r_2=0$ with J_3 odd. The coset $\lambda+4\mathcal{O}_2$ is now an invariant of M_2.

Case 6: $r_1=1$, $r_2=0$ and J_3 is even. The coset $\lambda+4\wp$ is an invariant of M_2.

Theorem 2.1. *Assume 2 prime, the even lattice L_2 primitively represents M_2 with $m+m_2=n$, and J_1 odd. Then:*

1. $e_2=1$ *for Cases* 1 *and* 3.
2. $e_2=2$ *for Cases* 2 *and* 4.
3. $e_2=q$ *for Case* 5.
4. $e_2=2q$ *for Case* 6.

If L_2 is not a sum of hyperbolic planes, replace L_2 by $L_2 \perp B$ and M_2 by $M_2 \perp B$, and extend the embedding by the identity on B. Since $B \perp B = H \perp H$ and $B \perp \mathcal{O}_2 x = H \perp \mathcal{O}_2 x'$ with $\lambda = q(x) \equiv q(x') \bmod 4$, after cancelling H, we have $L_2 = H_1 \perp \cdots \perp H_m$, preserving $\lambda \bmod 4$. However, $\lambda \mathcal{U}_2^2$ changes. It was shown in Proposition 3.6 of [1] that any embedding of M_2 into the even lattice L_2, with $m + m_2 = n$, is locally spinor equivalent to a canonical embedding (after cancelling the unimodular component $M_2(0)$). Hence it suffices to concentrate on canonical embeddings. Let $x_1, ..., x_m$ be a basis for M_2, viewed as a primitive sublattice of L_2, with $x_m = x$ and

$$x_i = u_i + v_i + \sum_{j<i} a_{ij} v_j, \qquad 1 \leqslant i \leqslant m,$$

where $a_{ij} = f(x_i, x_j) \in \mathcal{S}_p$, $a_i = q(x_i) \in \mathcal{O}_p$ and $2a_i \in \mathcal{S}_p$ (as in [1, 3.4]). Let $\psi : M_2 \to L_2$ be a canonical embedding with $\psi(x_i) = x_i$ for $i < m$, and $\varepsilon u_m + \delta v_m$ the component of $\psi(x_m)$ in H_m. The value of e_2 depends on how much control there is of the unit ε. First get $\varepsilon \equiv 1 \bmod 2$ by using $\Phi(\xi^2) \in O'(H_m)$ with $\xi \in \mathcal{U}_2$, so that $e_2 \leqslant 2q = [\mathcal{U}_2 : \mathcal{U}_2^2]$, since interchanging u_m and v_m gives no new inequivalent embedding with $\lambda = \varepsilon\delta$ a unit. When $m = m_2 = 1$ there are exactly $2q$ inequivalent embeddings since $O'(H_m)$ is the group of isometries $\Phi(\xi^2)$. Assume, therefore, $m \geqslant 2$.

Proof of Upper Bounds. In Case 1, except for $r_1 = 3$ and J_1 anisotropic, there exists $y \in M_2$ with $f(x, y) = 2$ and $q(y) \in 4\mathcal{O}_2$. Since $\mathcal{O}_2 x + \mathcal{O}_2 y$ orthogonally splits M_2, a basis exists with $x_{m-1} = y = u_{m-1} + q(y)\, v_{m-1}$, $x_m = x = 2v_{m-1} + u_m + \lambda v_m$ and $\psi(x) = 2v_{m-1} + \varepsilon u_m + \delta v_m$ with $\varepsilon\delta = \lambda$. The Eichler transformations $E(v_m, -\delta\, cr)\, E(u_m, cr)$ with $r = u_{m-1} - q(y)\, v_{m-1}$ and $\varepsilon + 2c + c^2 q(y)\, \delta = 1$, which fixes x_{m-1}, changes ε to 1. Hence $e_2 = 1$. In Case 2 a similar argument with $q(y) \in 2\mathcal{U}_2$ gives the bound $e_2 \leqslant 2$, since ε can be additively changed by $2\wp$. In the remaining part of Case 1 where $r_1 = 3$, take $x_{m-2} = u_{m-2} + 2v_{m-2}$, $x_{m-1} = 2v_{m-2} + u_{m-1} + \mu v_{m-1}$ with μ a unit, and $\psi(x_m) = \psi(u_m + \lambda u_m) = \varepsilon u_m + \delta v_m$. Now reduce ε to 1 by using $E(v_m, dt)\, E(u_m, ct)$ twice, first with $t = u_{m-2} - 2v_{m-2} - 2v_{m-1}$ to get $\varepsilon \equiv 1 \bmod 4$, and then with $t = u_{m-1} - \mu v_{m-1}$.

For Cases 3 and 4, use a modified argument with $x_{m-1} = y \in J_2$, $q(y) \in 2\mathcal{U}_2$ and $\psi(x) = \psi(u_m + \lambda v_m) = \varepsilon u_m + \delta v_m$, to get $\varepsilon \equiv 1 \bmod 4$ so that $e_2 \leqslant 2$. When J_3 is odd in Case 3, take $x_{m-2} = u_{m-2} = +\zeta v_{m-2} \in J_3$ with $\zeta \in 4\mathcal{U}_2$, and use $E(u_m, c(u_{m-2} - \zeta v_{m-2}))$ to get $\varepsilon \equiv 1 \bmod 8$; thus $e_2 = 1$. This can be modified when $r_2 \geqslant 2$ with suitable $x_{m-2} \in J_2$, (similar to $r_1 = 3$ above, possibly changing the choice of x). For Case 5, $e_2 \leqslant q$ since ε can be additively changed by $4\mathcal{O}_2$. ∎

The corresponding lower bounds will be obtained by modifying Lemma 3.8 in [1].

Proof for Cases 5 *and* 6. For Case 6, $J_1 = \mathcal{O}_2 x_m$ with $x_m = u_m + \lambda v_m$, while $x_1, ..., x_{m-1}$ span $\perp_{k \geqslant 3} J_k$. Let $\theta \in O'(L_2)$ satisfy $\theta(x_i) = x_i$ for $i < m$, and $\theta(x_m) = \varepsilon u_m + w$ with ε a unit and $f(w, v_m) = 0$. If we prove $\varepsilon \in \mathcal{U}_2^2$ it then follows that the canonical embeddings ψ_c with ε_c ranging over $\mathcal{U}_2/\mathcal{U}_2^2$ are spinor inequivalent, and $e_2 \geqslant 2q$. The map $\Phi(\varepsilon^{-1})\,\theta$ fixes x_i for $i < m$, while $\Phi(\varepsilon^{-1})\,\theta(x_m) = u_m + \delta v_m + t$ with $t \in H_m^\perp$. Take x_i, v_j as a base for L_2 and let $v_i, s_j = u_j - a_j v_j - \sum_{k > j} a_{jk} v_k$ be the dual base. Then $s_1, ..., s_m$ span the orthogonal complement of M_2 in L_2; this is isometric to $-M_2$. Let $t = \sum_{i<m} (c_i x_i + d_i v_i)$. The map $\phi = E(v_m, \sum_{i<m} c_i s_i)\,\Phi(\varepsilon^{-1})\,\theta$ fixes all x_j. Put $s = s_m = u_m - \lambda v_m \equiv x_m \bmod 2L_2$. Then $\phi(s) = as + 2\sum_{i<m} b_i s_i$ since $f(s, x_j) = 0$. Thus $a^2 \equiv 1 \bmod 32$, using $f(s, s_i) = -a_{im} = 0$ and $q(s_i) = -a_i \in 8\mathcal{O}_2$ for $i < m$. If $a \equiv -1 \bmod 4$, then $\Psi(\phi(s) - s)\,\phi$ lies in $O(L_2)$ and fixes s and all x_i; consequently (using [1, 3.8] on $x_1, ..., x_{m-1}$) this map lies in $O'(L_2)\,O(8L_2) = O'(L_2)$, where $O(8L_2)$ is the congruence subgroup modulo 8, giving the contradiction $\det \phi = -1$. Hence $a \equiv 1 \bmod 16$. Then $\Psi(s)\,\Psi(\phi(s) + s)\,\phi$ fixes s and all x_i, and so is in $O'(L_2)$. It follows from spinor norms that $\varepsilon \in \mathcal{U}_2^2$, and hence $e_2 \geqslant 2q$.

A minor variation of this argument, with weaker congruences, gives $\varepsilon \in (1 + 4\mathcal{O}_2)\,\mathcal{U}_2^2$, and hence $e_2 \geqslant 2q$ for Case 5.

Proof for Case 4. Modify Case 6. Let θ be as above with $\varepsilon \equiv 1 \bmod 4$. If we proof $\varepsilon \in \mathcal{U}_2^2$, it follows that $e_2 \geqslant 2$. Let $x_{m-1} = u_{m-1} + 2\eta v_{m-1} \in J_2$, where $\eta \in \mathcal{U}_2$, and $r = s_{m-1} = u_{m-1} - 2\eta v_{m-1}$. Then, with ϕ, x_m and s as above, $\phi(s) = as + 2br + 2\sum_{i<m-1} b_i s_i$. Since $\lambda \equiv 1 \bmod 4$ and J_3 is even, it follows that $a^2 + 8\eta b^2 \equiv 1 \bmod 32$. Then either $\Psi(\phi(s) - s)\,\phi$ when $a - 1 \in 2\mathcal{U}_2$, or $\tau = \Psi(s)\,\Psi(\phi(s) + s)\,\phi$ when $a + 1 \in 2\mathcal{U}_2$, is integral and fixes s and all x_i, and so by [1, 3.8] lies in $O'(L_2)\,O(4L_2)$. The first possibility violates $\det \phi = 1$. For the second, by spinor norms, $\varepsilon q(s)\,q(\phi(s) + s)$ is in $(1 + 4\mathcal{O}_2)\,F_2^2$. Hence $(1 + a)/2 \equiv \alpha^2 \equiv 1 \bmod 4$ and $a \equiv 1 \bmod 8$. Then $a^2 + 8\eta b^2 \equiv 1 \bmod 32$ shows that b is even and $a \equiv 1 \bmod 16$. Since $r \equiv x_{m-1} \bmod 4L_2$, it follows that $\tau(r) = cr + 4\sum_{i<m-1} d_i s_i$ with $c \equiv 1 \bmod 32$. Now $\Psi(r)\,\Psi(\tau(r) + r)\,\tau$ fixes s, r and all x_i and so is in $O'(L_2)\,O(8L_2) = O'(L_2)$. The product $\Psi(r)\,\Psi(\tau(r) + r)$ is integral, although the individual symmetries are not (check the images of r, v_{m-1} and $t \in H_1 \perp \cdots \perp H_{m-2}$). Finally, by spinor norms, $\varepsilon \in \mathcal{U}_2^2$, and $e_2 \geqslant 2$.

Proof for Case 2. Modify Case 4. Let $x_{m-1} = u_{m-1} + \mu v_{m-1} \in J_1$, where $\mu \equiv 1 \bmod 2$, $-\lambda\mu \not\equiv 1 \bmod 4$ and $r = s_{m-1} = u_{m-1} - \mu v_{m-1}$. For θ, ϕ, x_m and s above, $\phi(s) = as + 2br + 2\sum_{i<m-1} b_i s_i$, and $a^2 + 4\lambda\mu b^2 \equiv 1 \bmod 16$. Either $\sigma = \Psi(\phi(s) - s)\,\phi$ when $a - 1 \in 2\mathcal{U}_2$, or $\tau = \Psi(s)\,\Psi(\phi(s) + s)\,\phi$ when $a + 1 \in 2\mathcal{U}_2$, is integral and fixes s and all x_i. Repeat this for r where now $\sigma(r)$, or $\tau(r)$, equals $cr + 2\sum_{i<m-1} d_i s_i$ so that $c^2 \equiv 1 \bmod 16$. Then either $\Psi(\sigma(r) - r)\,\sigma$ or $\Psi(r)\,\Psi(\tau(r) + r)\,\tau$ fixes all x_i, s and r, and so is in $O'(L_2)\,O(4L_2)$ (the other possibilities violate $\det \phi = 1$). For the first map,

by spinor norms, $\varepsilon q(\sigma(r)-r)\, q(\phi(s)-s)$ lies in $(1+4\mathcal{O}_2)\, F_2^2$. Therefore, $\varepsilon(1-a)/2 \equiv \lambda\mu\alpha^2 \bmod 4$, and $a^2+4\lambda\mu b^2 \equiv 1 \bmod 16$ then gives $\varepsilon \equiv \lambda\mu\alpha^2 + \beta^2 \bmod 4$. Let $-\lambda\mu = 1+2\zeta$ where ζ is a unit since $-\lambda\mu \not\equiv 1 \bmod 4$. If $\varepsilon = 1+2\zeta^{-1}\rho$ with $\rho \notin \wp$, the congruence has no solutions for α, β. Hence $e_2 \geqslant 2$. The second map is similar.

Observation. Let $L_2 = H_1 \perp \cdots \perp H_m$ and $\psi : M_2(1) \to L_2$ be a local canonical embedding with $\psi(x_i) = x_i$ for $i < m$, and the H_m-component of $\psi(x_m)$ equal to $\varepsilon u_m + \delta v_m$ with $\varepsilon \equiv 1 \bmod 2$, and $\varepsilon \equiv 1 \bmod 4$ in Case 4. The above arguments have shown that the spinor orbit of a local canonical embedding is uniquely determined by the group coset $\varepsilon + 2\wp$ in Case 2, by $\varepsilon + 4\wp$ in Cases 4 and 6, and by $\varepsilon + 4\mathcal{O}_2$ in Case 5. We use this to study the action of $\Phi = \Phi(-1)$ and $\Psi = \Psi(u_m - v_m)$ on the spinor orbits of local embeddings. This will later help determine $N(L, M)$.

Corollary 2.2 *Assume the conditions of Theorem* 2.1. *Then* Φ *interchanges the local spinor orbits in pairs in Cases* 5 *and* 6, *and* Φ *interchanges the two local spinor orbits in Case* 2 *except when* $\varepsilon \in \wp$.

Proof. In a local embedding, Φ changes ε to $-\varepsilon$.

Corollary 2.3. *In Case* 4 *let* $J_1 \perp J_2 = \mathcal{O}_2 x \perp \mathcal{O}_2 z$ *with* $\lambda = q(x) \equiv 1 \bmod 4$ *and* $2\eta = q(z) \equiv 2 \bmod 4$. *Then* Φ *leaves the two local spinor orbits invariant if and only if* $\eta + 1 \in 2\wp$.

Proof. As in the proof 2.1, take $x_m = x = u_m + \lambda v_m$ and $x_{m-1} = z = u_{m-1} + 2\eta v_{m-1}$. Then the orbit of the canonical embedding ψ is determined by $\varepsilon + 4\wp$, where $\psi(x_m) = \varepsilon u_m + \delta v_m$ (with $\varepsilon \equiv \delta \equiv 1 \bmod 4$). Then Φ changes ε to $-\varepsilon \equiv 3 \bmod 4$ in $\psi(x_m)$. Since $E(u_m, u_{m-1} - 2\eta v_{m-1})$ fixes x_{m-1} and changes $-\varepsilon$ to $-\varepsilon - 2\eta\delta \equiv 1 \bmod 4$ in $\Phi\psi(x_m)$, Φ leaves the orbits invariant if and only if $\varepsilon + 4\wp = -\varepsilon - 2\eta\delta + 4\wp$, or $\eta + 1 \in 2\wp$.

Corollary 2.4. *Assume the conditions of Theorem* 2.1 *and* $q = 2$. *Then* Ψ (*resp.* $\Psi(u_m + v_m)$) *leaves the spinor orbits invariant for Cases* 2 *and* 5 *if and only if* $\lambda \equiv 1 \bmod 4$ (*resp.* $\lambda \equiv -1 \bmod 4$), *and for Cases* 4 *and* 6 (*assuming* $L_2 = H_1 \perp \cdots \perp H_m$) *if and only if* $\lambda \equiv 1 \bmod 8$ (*resp.* $\lambda \equiv -1 \bmod 8$).

Proof. In Cases 2 and 5, $\lambda \bmod 4$ is an invariant of M_2, and the spinor orbit of the embedding ψ is determined by the value of $\varepsilon \bmod 4$ in $\psi(x_m) = \varepsilon u_m + \delta v_m$; thus Ψ leaves the orbit invariant if and only if $\varepsilon \equiv \delta \bmod 4$, that is $\lambda = \varepsilon\delta \equiv 1 \bmod 4$. For Cases 4 and 6, $\lambda \bmod 8$ is an invariant of M_2, and the orbit is determined by $\varepsilon \bmod 8$; note λ changes to $\lambda + 4$ if modifications to M_2 and L_2 are needed for the exceptional situation in Theorem 3.1(iii) of [1].

COROLLARY 2.5. *Assume the conditions of Theorem* 2.1. *Then* Ψ *leaves all* $SO(L_2)$*-orbits invariant.*

Proof. The symmetry $\Psi(s)$ with $s=\varepsilon u_m-\delta v_m$ leaves invariant the canonical embedding ψ_c involving $x_c=\varepsilon u_m+\delta v_m$, with $\varepsilon\delta=\lambda$ a unit. Since $\Psi\Psi(s)\in SO(H_m)$, it follows that Ψ leaves the $SO(L_2)$-orbits invariant.

THEOREM 2.6. *Assume* 2 *is prime, the even lattice* L_2 *primitively represents* M_2 *with* $n-m=m_2\geqslant 2$, *and* J_1 *is even. Then*:

1. $e_2=2$ *when* $r_1>0$.
2. $e_2=2q$ *when* $r_1=0$ *and* $r_2>1$.
3. $e_2=4q$ *when* $r_1=0$ *and* $r_2\leqslant 1$.

Proof. This follows from Theorem 3.7 in [1] when M_2 is strongly even. It remains to consider $\mathcal{S}_2=4\mathcal{O}_2$ and J_2 odd. Then $J_1=0$, and either J_2 is split by $\left(\begin{smallmatrix}\alpha & 4\\ 4 & \delta\end{smallmatrix}\right)$ with $\alpha\in 8\mathcal{O}_2$ and $\delta\in 4\mathcal{U}_2$, or $r_2=1$. The first case gives $e_2=2q$ by an argument similar to that used for strongly even lattices in [1]. In the remaining case with $r_2=1$, Proposition 3.6(i) in [1] gives $e_2\leqslant 4q$, while $e_2\geqslant 4q$ follows by an argument similar to that used in the proof of Theorem 2.1(4).

COROLLARY 2.7. *Assume the conditions of Theorem* 2.6. *Then* Φ *and* Ψ *independently interchange the local spinor orbits in pairs, except for* Φ *when* $r_1>0$.

3. GLOBAL ORBITS UNDER $O(L)$ AND $SO(L)$

Assume L is an even unimodular isotropic $\mathbb{Z}$-lattice. The quotient group $O(L)/O'(L)$ is then the Klein 4-group generated by $\Phi(-1)$ and $\Psi(u-v)$ (viewed acting on a hyperbolic plane H_m common to all the local splittings of L_p used in the canonical embeddings).

THEOREM 3.1. *Let* L *be an even unimodular* $\mathbb{Z}$*-lattice with* $n-m\geqslant 3$ *and the Witt index* $i_\infty(L\perp -M)>m$. *Then*

$$2N^+(L,M)=N'(L,M)=\prod_p e_p$$

if and only if either:

1. *There exists a prime* $p \equiv 3 \bmod 4$ *with* $m + m_p \geqslant n - 1$, *or*

2. $m + m_2 = n$, $e_2 \geqslant 2$, *either* $J_1 = 0$ *or* J_1 *is odd, and if* $r_1 = r_2 = 1$ *with* J_3 *even then* $\eta \equiv 1 \bmod 4$.

Otherwise, $N^+(L, M) = N'(L, M)$.

Proof. The conditions given are exactly those needed so that $\Phi(-1)$ does not act trivially on the local spinor orbits. See Theorem 4.1 in [1] for the first part. The rest uses Corollaries from the previous section. ∎

This result is essentially in [2] although expressed differently, and with the notation $e_{++} = N'(L, M)$, $e_{+-} = N^+(L, M)$ and $e = N(L, M)$. Finally, we give necessary and sufficient conditions for $N(L, M) = 1$. Let $p_1, ..., p_s$ be the odd primes where $m + m_p \geqslant n - 1$, and let P be the two rowed matrix over $\mathbb{F}_2$ where the $(1, j)$-entry is 1 if and only if $p_j \equiv 3 \bmod 4$, and the $(2, j)$-entry is 1 if and only if $m + m_{p_j} = n - 1$ and $2dM_{p_j}(0) \neq (-1)^{n-m}\, dL_{p_j}$. This last condition corresponds to $(2\Delta/p_j) = -1$ in [2], where $\Delta \equiv -\lambda \bmod 8$.

THEOREM 3.2. *Let* L *be an even unimodular* $\mathbb{Z}$*-lattice primitively representing* M *with* $n - m \geqslant 3$ *and* $i_\infty(L \perp -M) > m$. *Then* $N(L, M) = 1$ *if and only if* $e_2 \leqslant 4$ *and one of the three following situations occur*:

1. *If* $e_2 = 1$ *and an odd prime* p *exists with* $m + m_p = n$, *then* $p \equiv 3 \bmod 4$, *and* $m + m_p \leqslant n - 2$ *for all other odd primes. At most two primes with* $m + m_p = n - 1$ *exist, with* rank $P = 1$ *when only one exists, and* rank $P = 2$ *when two exist.*

2. *If* $e_2 = 2$, *then* $m + m_p \leqslant n - 1$ *for all odd primes. At most one* p *with* $m + m_p = n - 1$ *exists, with* $p \equiv 3 \bmod 4$ *when* J_1 *is even, and* $2dM_p(0) \neq (-1)^{n-m}\, dL_p$, *with* $\eta \equiv 1 \bmod 4$ *in Case* 4, *when* J_1 *is odd.*

3. *If* $e_2 = 4$, *then* $m + m_p \leqslant n - 2$ *for all odd primes*, $r_1 = 0$ *and* $r_2 > 1$.

Proof. Clearly $\prod_p e_p = N'(L, M) \leqslant 4$ is necessary; Theorem 3.7 in [1] then forces the given restrictions on $m + m_p$ at odd primes. When $e_2 = 1$, the result follows from 3.1 above and Theorem 4.2 (and the remarks following it) in [1]. When $e_2 = 4$, Case 6 cannot occur by 2.5. Thus only 2.6(2) applies, and $\Phi(-1)$ and $\Psi(u - v)$ then act independently on the dyadic spinor orbits. Finally, let $e_2 = 2$. If $r_1 > 0$ with J_1 even, $\Phi(-1)$ acts invariantly on dyadic spinor orbits, so that $p \equiv 3 \bmod 4$ is necessary when p exists with $m + m_p = n - 1$. If J_1 is odd, then $\Psi(u - v)$ acts invariantly on $SO(L_2)$-orbits by 2.5, and also invariantly on $O'(L_p)$-orbits unless $2dM_p(0) \neq (-1)^{n-m}\, dL_p$ (see [1]), so this condition is now necessary.

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