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Quasitilted One-Point Extensions of Wild Hereditary Algebras[1]

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1. INTRODUCTION AND THE MAIN RESULTS

Throughout the paper K will denote a fixed algebraically closed field. By an algebra we mean a basic connected finite dimension K-algebra (associative with an identity), by mod A we mean the category of all finite dimensional left A-modules, and by ind A we mean the full subcategory of mod A consisting of indecomposable modules. Moreover, we denote by $\Gamma(\text{mod } A)$ the Auslander–Reiten quiver of A and by τ_A and τ_A^- the Auslander–Reiten translations $D\,\text{Tr}$ and $\text{Tr}\,D$ in mod A, respectively. We shall identify an object of ind A with the corresponding vertex of the quiver $\Gamma(\text{mod } A)$. By an abelian category we mean a connected abelian K-category $\mathscr{A}$ with finite dimensional K-vector spaces $\text{Hom}(Y,Y)$ and $\text{Ext}^1(X,Y)$ for all objects X and Y in $\mathscr{A}$. We denote by $D^b(\mathscr{A})$ the derived

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category of bounded complexes over $\mathscr{A}$. Finally, an abelian category $\mathscr{A}$ is called hereditary if the functor $\mathrm{Ext}^2(-,-)$ vanishes on $\mathscr{A}$.

An algebra A is said to be piecewise hereditary (of type $\mathscr{H}$) if there exists a hereditary abelian category $\mathscr{H}$ such that $D^b(\mathrm{mod}\, A)$ and $D^b(\mathscr{H})$ are equivalent as triangulated categories. It is known [47] that an algebra A is piecewise hereditary of type $\mathscr{H}$ if and only if $A \cong \mathrm{End}_{D^b(\mathscr{H})}(T^{\cdot})$ for a tilting complex $T^{\cdot}$ in $D^b(\mathscr{H})$, that is, a complex $T^{\cdot}$ with $\mathrm{Hom}_{D^b(\mathscr{H})}(T^{\cdot}, T^{\cdot}[i]) = 0$ for all $i \neq 0$ such that the additive category add $(T^{\cdot})$ of $T^{\cdot}$ generates $D^b(\mathscr{H})$ as a triangulated category. A special but important class of piecewise hereditary algebras is formed by the quasitilted algebras, being the algebras of the form $\mathrm{End}_{\mathscr{H}}(T)$, where T is a tilting object in a hereditary abelian category $\mathscr{H}$, that is, satisfying the properties $\mathrm{Ext}^1_{\mathscr{H}}(T,T) = 0$ and $\mathrm{Hom}_{\mathscr{H}}(T, X) = 0 = \mathrm{Ext}^1_{\mathscr{H}}(T, X)$ implies $X = 0$. It has been proved in [17] that an algebra A is quasitilted if and only if A is of global dimension at most two and every indecomposable finite dimensional A-module is of projective dimension or injective dimension at most one.

In [34] Lenzing has shown that the only noetherian hereditary abelian categories with tilting objects are those which are derived equivalent to $\mathrm{mod}\, H$ for a hereditary algebra H or to a category $\mathrm{coh}\, \mathbb{X}$ of coherent sheaves over a weighted projective line $\mathbb{X}$. Recently [15] Happel proved that up to derived equivalence these are the only possible types of hereditary abelian categories with tilting objects. In particular, one obtains that every quasitilted algebra is tilted (a tilt of a hereditary algebra) or of canonical type (a tilt of a canonical algebra). Moreover, every piecewise hereditary algebra can be obtained from a hereditary algebra or a canonical algebra by a finite sequence of tilts [38]. The quasitilted algebras of tame representation type (and their module categories) have been completely described already in [54]. On the other hand, our knowledge of quasitilted algebras of wild representation type is still relatively poor. For example, it has been shown in [37] that a quasitilted algebra A is of canonical type if and only if A is a semiregular branch enlargement of a concealed canonical algebra C. But besides the canonical algebras we know only a few concealed canonical algebras of wild type (see [39]). Recall that an algebra C is called concealed canonical if C is the endomorphism algebra of a tilting vector bundle T in the category $\mathrm{coh}\, \mathbb{X}$ of coherent sheaves over a weighted projective line $\mathbb{X}$. Concealed canonical algebras are characterized by the existence of a separating family of stable tubes [36, 53]. Similarly, the tilted algebras given by regular tilting modules are characterized by the existence of a stable nonperiodic (connecting) component with finitely many orbits with respect to the action of the Auslander–Reiten translation [50]. In the both cases, only general information on the structure of such stable components is available.

In the paper we are interested in the structure of quasitilted algebras which are one-point extensions

$$H[M] = \begin{bmatrix} H & M \\ 0 & K \end{bmatrix}$$

of wild hereditary algebras H by non-zero regular H-modules M. It is known that if such a one-point extension is piecewise hereditary then necessarily M is quasi-simple [32] and $\operatorname{End}_H(M) \cong K$ (see [17]). Moreover, we know (see [17, 29]) that the one-point extension $H[M]$ is piecewise hereditary of type $\mathscr{H}$ if and only if all one-point extensions $H[\tau_H^i M]$, $i \in \mathbb{Z}$, are piecewise hereditary of type $\mathscr{H}$. Moreover, Lache has shown in [32] that if $H[M]$ is piecewise hereditary of type $\mathscr{H}$ then there exists a positive integer r such that the algebras $H[\tau_H^i M]$, $i \geq r$, are quasitilted of type $\mathscr{H}$. We know also that there exists a positive integer s such that all algebras $H[\tau_H^{-i} M]$, $i \geq s$, are not quasitilted [17]. Finally, we mention that for any wild hereditary algebra H there exists a quasi-simple regular H-module M such that $H[M]$ is quasitilted (see Section 3).

The aim of this paper is two-fold. On the one hand, we explain the transition from a piecewise hereditary algebra $H[M]$ to the quasitilted algebras $H[\tau_H^i M]$ for $i \gg 0$. On the other hand, we exhibit interesting new properties of stable components with finitely many nonperiodic orbits over wild tilted algebras and of stable tubes over wild concealed canonical algebras.

In order to state our first main result, recall that an indecomposable module X over an algebra A is called sincere (respectively, almost sincere) if all (respectively, all but one) simple A-modules occur as composition factors of X. It is known that if $\mathscr{C}$ is the connecting component of a tilted algebra given by a regular tilting module (respectively, the family of all stable tubes of a wild concealed canonical algebra) then all but finitely many indecomposable modules in $\mathscr{C}$ are sincere.

THEOREM 1. *Let H be a connected wild hereditary algebra, let M be a quasi-simple regular H-module, and assume that $H[M]$ is piecewise hereditary. Take a positive integer d. Then there exists a positive integer r such that, for all $i \geq r$, the algebras $H[\tau_H^i M]$ are either tilted algebras with regular connecting components or concealed canonical algebras. Moreover, for $i \geq r$, we have*

(i) *If $H[\tau_H^i M]$ is tilted then all indecomposable modules in the connecting component of $H[\tau_H^i M]$ are of dimension at least d. Moreover, all but finitely many of them are sincere and the remaining ones are almost sincere.*

(ii) *If* $H[\tau_H^i M]$ *is concealed canonical then all indecomposable modules in the family of stable tubes over* $H[\tau_H^i M]$ *are of dimension at least* d. *Moreover, all but finitely many of them are sincere and the remaining ones are almost sincere.*

Let H be a connected wild hereditary algebra and let M be a quasi-simple regular H-module. It is an interesting open problem to find handy necessary and sufficient conditions for M making $H[M]$ piecewise hereditary. It follows from [17, 32] that in this case M is necessarily elementary (in the sense of [29]). It was also shown in [29] that there are only finitely many orbits of the dimension vectors of elementary modules with respect to action of the Coxeter transformation of H. In particular, there are infinitely many τ_H-orbits of quasi-simple regular H-modules which do not create piecewise hereditary one-point extensions. Recall from [33] that the orbit algebra of a regular H-module M is the $\mathbb{Z}$-graded algebra $\mathscr{O}(M) = \oplus_{i \in \mathbb{Z}} \mathscr{O}_i(M)$, where $\mathscr{O}_i(M) = \mathrm{Hom}_H(M, \tau_H^i M)$ and multiplication is given by $\mathscr{O}_i(M) \times \mathscr{O}_j(M) \to \mathscr{O}_{i+j}(M)$, $(f, g) \to (\tau_H^i g) \cdot f$, where "$\cdot$" denotes the composition of maps. It has been shown in [27] that for a quasi-simple regular H-module M, the algebra $\mathscr{O}(M)$ is a free (noncommutative) algebra if and only if M is orbital elementary. Moreover, it is known that if $H[M]$ is a tilted algebra or a concealed canonical algebra then M is orbital elementary [27, 44]. Therefore, we obtain the following consequence of [32, Corollary 2.2] and Theorem 1.

COROLLARY 2. *Let H be a connected wild hereditary algebra and let M be a quasi-simple regular H-module such that $H[M]$ is piecewise hereditary. Then the orbit algebra $\mathscr{O}(M)$ is a free algebra in infinitely many variables.*

It would be interesting to know whether the converse implication is true. The second main result of the paper is the following.

THEOREM 3. *Let m be a positive integer. Then*

(i) *For each connected wild hereditary algebra H with at least three simple modules there exist infinitely many pairwise nonisomorphic connected wild hereditary algebras C and quasi-simple regular C-modules M such that $C[M]$ are tilted algebras of type H with a regular connecting component $\mathscr{C}$ with this property: for any indecomposable module X in $\mathscr{C}$, each simple $C[M]$-module occurs with multiplicity at least m as a composition factor of X.*

(ii) *For each wild canonical algebra Λ there exist infinitely many pairwise nonisomorphic connected wild hereditary algebras C and quasi-simple regular C-modules M such that $C[M]$ are concealed canonical algebras of type Λ whose family $\mathscr{C}$ of all stable tubes has this property: for any indecomposable module X in $\mathscr{C}$, each simple $C[M]$-module occurs with multiplicity at least m as a composition factor of X.*

As a direct consequence we obtain the following fact.

COROLLARY 4. (i) *For each connected wild hereditary algebra H with at least three simple modules there exist infinitely many pairwise nonisomorphic tilted algebras of type H with regular connecting components without simple modules.*

(ii) *For each wild canonical algebra* Λ *there exist infinitely many pairwise nonisomorphic concealed canonical algebras of type* Λ *without simple modules in the tubes.*

It follows from the theory of quasitilted algebras that if a one-point extension $H[M]$ of a connected wild hereditary algebra H by a quasi-simple regular H-module M is tilted (respectively, quasitilted of canonical type) then the connecting component (respectively, the family of tubes) of the Auslander–Reiten quiver of $H[M]$ has no projective modules but may contain injective modules.

THEOREM 5. *Let H be a connected wild hereditary algebra and M is a quasi-simple regular H-module. Assume that* $H[M]$ *is a tilted algebra (respectively, quasitilted algebra of canonical type) whose connecting component contains at least one injective module (respectively, r tubes contain at least one injective module). Then there exists a preprojective tilting H-module Q such that* $H' = \mathrm{End}_H(Q)$ *is a connected wild hereditary algebra,* $M' = \mathrm{Hom}_A(Q, M)$ *is a quasi-simple regular* H'*-module, and* $H'[M']$ *is a tilted algebra with a regular connecting component containing at least one simple module (respectively, concealed canonical algebra with at least r tubes containing simple modules).*

We also note that such a hereditary algebra H' can be obtained from H by a finite sequence of reflections in the sense of [2, 6].

For basic background from the representation theory we refer to the books [3, 13, 49], for special results on wild hereditary algebras to the survey [25], and on quasitilted algebras to [17].

The outline of the paper is as follows. Section 2 is devoted to basic properties and characterizations of tilted algebras and quasitilted algebras of canonical type needed for our considerations. In Sections 3 and 4 we study properties of special (saturated) regular tilting modules over wild hereditary algebras and the dimension-vectors of the indecomposable modules in the regular connecting components of the associated tilted algebras. Sections 5 and 6 are devoted to the proofs of Theorems 1, 3, and 5 in the tilted case. Finally, Sections 7 and 8 are devoted to the proofs of Theorems 1, 3, and 5 in the canonical case. In the paper, we present also some examples and complementary results illustrating our main results.

2. QUASITILTED ALGEBRAS

Let $\mathscr{H}$ be a connected hereditary abelian K-category with finite dimensional homomorphism and extension spaces. By a tilting object in $\mathscr{H}$ we mean a multiplicity-free object T (direct sum of pairwise nonisomorphic indecomposable objects) such that $\mathrm{Ext}^1_{\mathscr{H}}(T, T) = 0$ and $\mathrm{Hom}_{\mathscr{H}}(T, X) = 0 = \mathrm{Ext}^1_{\mathscr{H}}(T, X)$ implies $X = 0$. Then the algebra $\mathrm{End}_{\mathscr{H}}(T)$ is said to be a quasitilted algebra of type $\mathscr{H}$ [17]. It has been recently shown [15] that there are two classes of quasitilted algebras: tilted algebras and quasitilted algebras of canonical type. We shall present here basic properties of these algebras which are relevant to our considerations in the paper.

Let H be a basic connected hereditary K-algebra, let $\mathscr{H} = \mathrm{mod}\, H$ be category of finite dimensional left H-modules, and let T be a tilting object (module) in $\mathrm{mod}\, H$. Then H is the path algebra $K\Delta$ of a finite connected quiver Δ without oriented cycles, and the number $|\Delta_0|$ of vertices of Δ is the rank of the Grothendieck group $K_0(H)$ of H. Moreover, T is a tilting H-module if and only if $\mathrm{Ext}^1_H(T, T) = 0$ and T is a direct sum of $|\Delta_0|$ pairwise nonisomorphic indecomposable H-modules (see [7, 18]). Further, $A = \mathrm{End}_H(T)$ is called a tilted algebra of type Δ. The tilting H-module T determines a torsion pair $(\mathscr{F}(T), \mathscr{T}(T))$ in $\mathrm{mod}\, H$, with the torsion-free part $\mathscr{F}(T) = \{X \in \mathrm{mod}\, H;\ \mathrm{Hom}_A(T, X) = 0\}$ and the torsion pair $\mathscr{F}(T) = \{X \in \mathrm{mod}\, H;\ \mathrm{Ext}^1_H(T, X) = 0\}$, and a splitting torsion pair $(\mathscr{Y}(T), \mathscr{X}(T))$ in $\mathrm{mod}\, A$, with the torsion-free part $\mathscr{Y}(T) = \{Y \in \mathrm{mod}\, A;\ \mathrm{Tor}^A_1(T, Y) = 0\}$ and the torsion part $\mathscr{X}(T) = \{Y \in \mathrm{mod}\, A;\ T \otimes_A Y = 0\}$. By the Brenner–Butler theorem, the functor $\mathrm{Hom}_H(T, -)$ induces an equivalence of $\mathscr{T}(T)$ with $\mathscr{Y}(T)$, and the functor $\mathrm{Ext}^1_A(T, -)$ induces an equivalence of $\mathscr{F}(T)$ and $\mathscr{X}(T)$ (see [8, 18]). Further, the images $\mathrm{Hom}_H(T, I)$ of indecomposable injective H-modules I via $\mathrm{Hom}_H(T, -)$ belong to one connected component $\mathscr{C}_T$ of the Auslander–Reiten quiver $\Gamma(\mathrm{mod}\, A)$ of A, called the connecting component of $\Gamma(\mathrm{mod}\, A)$ determined by T, and form a faithful section Δ_T of $\mathscr{C}_T$. Recall that a full connected subquiver Σ of a (connected) component $\mathscr{C}$ of $\Gamma(\mathrm{mod}\, A)$ is called a section if Σ has no oriented cycles, is convex in $\mathscr{C}$, and intersects each τ_A-orbit of $\mathscr{C}$ exactly once. Moreover, the section Σ is faithful provided the direct sum of all modules lying on Σ is a faithful A-module. The section Δ_T of the connecting component $\mathscr{C}_T$, defined above, has a distinguished property: it connects the torsion-free part $\mathscr{Y}(T)$ with the torsion part $\mathscr{X}(T)$, because every predecessor in $\mathrm{ind}\, A$ of a module $\mathrm{Hom}_H(T, I)$ from Δ_T lies in $\mathscr{Y}(T)$ and every successor of $\tau_A^- \mathrm{Hom}_H(T, I)$ in $\mathrm{ind}\, A$ lies in $\mathscr{X}(T)$. We have also the following fact established in [50].

PROPOSITION 2. *Let $A = \mathrm{End}_H(T)$ be a tilted algebra and let $\mathscr{C}_T$ be the connecting component of $\Gamma(\mathrm{mod}\, A)$ determined by T. Then we have the* *following*:

(i) $\mathscr{C}_T$ *contains a projective module if and only if T admits a preinjective indecomposable direct summand.*

(ii) $\mathscr{C}_T$ *contains an injective module if and only if T admits a preprojective indecomposable direct summand.*

(iii) $\mathscr{C}_T$ *is regular if and only if T is regular.*

Recall also that a connected hereditary algebra $H = K\Delta$ admits a regular tilting module if and only if Δ has at least three vertices and is a wild quiver, that is, neither a Dynkin quiver nor an Euclidean quiver (see [5, 51]).

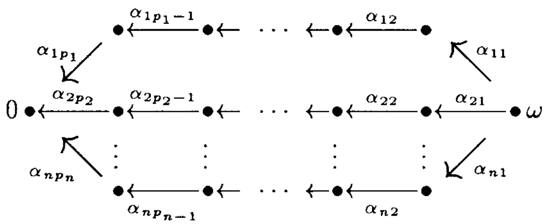
The following criterion for an algebra to be tilted has been established independently in [41, 52].

THEOREM 2.2. *A basic connected algebra A is tilted if and only if* $\Gamma(\operatorname{mod} A)$ *admits a component* $\mathscr{C}$ *with a faithful section* Δ *such that* $\operatorname{Hom}_A(X, \tau_A Y) = 0$ *for all X and Y in* Δ. *Moreover, in this case* $\mathscr{C}$ *is a connecting component* $\mathscr{C}_T$ *of* $\Gamma(\operatorname{mod} A)$ *determined by a tilting module T over a hereditary algebra H with* $A \cong \operatorname{End}_H(T)$.

The structure of the Auslander–Reiten quiver of a tilted algebra $A = \operatorname{End}_H(T)$ is well understood due to results established in [1, 18, 22, 23, 26, 40, 48–51, 55]. First of all, every component of $\Gamma(\operatorname{mod} A)$ different from the connecting component either lies entirely in $\mathscr{Y}(T)$ or lies entirely in $\mathscr{X}(T)$. Further, every component of $\Gamma(\operatorname{mod} A)$ contained in $\mathscr{Y}(T)$ is either preprojective or can be obtained from a stable tube $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$, $r \geq 1$, or a component of the form $\mathbb{Z}\mathbb{A}_\infty$, by a finite number (possibly empty) of ray insertions. Dually, every component of $\Gamma(\operatorname{mod} A)$ contained in $\mathscr{X}(T)$ is either preinjective or can be obtained from a stable tube $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$, $r \geq 1$, or a component of the form $\mathbb{Z}\mathbb{A}_\infty$, by a finite number (possibly empty) of coray insertions.

Let H be a basic connected wild hereditary algebra, let M be a quasi-simple regular H-module, and assume that the one-point extension $A = H[M]$ is a tilted algebra $\operatorname{End}_{H^*}(T^*)$. Then $\Gamma(\operatorname{mod} A)$ admits exactly one preprojective component $\mathscr{P}(A)$, which coincides with the preprojective component $\mathscr{P}(H)$ of $\Gamma(\operatorname{mod} H)$, the connecting component $\mathscr{C}_{T^*}$ of $\Gamma(\operatorname{mod} A)$ determined by T^* has no projective modules (but may contain injective modules), and the family of components of $\Gamma(\operatorname{mod} A)$ contained in $\mathscr{Y}(T^*)$ but different from $\mathscr{P}(A)$ consists of infinitely many components of the form $\mathbb{Z}\mathbb{A}_\infty$ and one nonstable component obtained from a component of type $\mathbb{Z}\mathbb{A}_\infty$ by one ray insertion, containing the indecomposable projective A-module with radical equal M.

Let n be a positive integer ≥ 2, let $p = (p_1, \dots, p_n)$ be an n-tuple of integers $p_i \geq 2$, and let $\lambda = (\lambda_1, \dots, \lambda_n)$ be an n-tuple of pairwise different elements of $\mathbb{P}_1(K) = K \cup \{\infty\}$ normalized such that $\lambda_1 = \infty$, $\lambda_2 = 0$, and $\lambda_3 = 1$. Consider the quiver $\Delta(p_1, \dots, p_n)$



and the admissible ideal $I(\lambda_1, \dots, \lambda_n)$ in the path algebra $K\Delta(p_1, \dots, p_n)$ of the quiver $\Delta(p_1, \dots, p_n)$ generated by $\alpha_{ip_i} \cdots \alpha_{i2}\alpha_{i1} + \alpha_{2p_2} \cdots \alpha_{22}\alpha_{21} + \lambda_i \alpha_{1p_1} \cdots \alpha_{12}\alpha_{11}$, $3 \leq i \leq n$, for $n \geq 3$, and $I(\lambda_1, \lambda_2) = 0$ for $n = 2$. Then

$$\Lambda = \Lambda(p, \lambda) = K\Delta(p_1, \dots, p_n)/I(\lambda_1, \dots, \lambda_n)$$

is called the canonical algebra with the weight sequence p and the parameter sequence λ. Denote by $\Lambda_0 = \Lambda_0(p_1, \dots, p_n)$ the path algebra of the full subquiver of $\Delta(p_1, \dots, p_n)$ consisting of all vertices except the unique source ω. For $n \geq 3$, Λ is a one-point extension $\Lambda = \Lambda_0[R]$ of Λ_0 by a quasi-simple regular Λ_0-module $R = R(\lambda_1, \dots, \lambda_n)$ with the dimension-vector

$$\dim R = 2 \begin{matrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \end{matrix}$$

The representation theory of canonical algebras has been established in [35, 49]. In particular, $\Gamma(\text{mod}\,\Lambda)$ has a decomposition

$$\Gamma(\text{mod}\,\Lambda) = \Gamma_+(\text{mod}\,\Lambda) \vee \Gamma_0(\text{mod}\,\Lambda) \vee \Gamma_-(\text{mod}\,\Lambda)$$

where $\Gamma_+(\text{mod}\,\Lambda)$, $\Gamma_0(\text{mod}\,\Lambda)$, and $\Gamma_-(\text{mod}\,\Lambda)$ consist of components formed by the indecomposable modules of positive, zero, and negative rank, respectively, and $\Gamma_0(\text{mod}\,\Lambda)$ is a $\mathbb{P}_1(K)$-family of pairwise orthogonal standard stable tubes separating $\Gamma_+(\text{mod}\,\Lambda)$ from $\Gamma_-(\text{mod}\,A)$. Moreover,

Λ is of wild representation type if and only if Λ_0 is a wild hereditary algebra. It follows from [11] that the derived category $D^b(\operatorname{mod} \Lambda(p, \lambda))$ is equivalent (as triangulated category) to the derived category $D^b(\operatorname{coh} \mathbb{X}(p, \lambda))$ of coherent sheaves for a weighted projective line $\mathbb{X} = \mathbb{X}(p, \lambda)$. Further, $\mathscr{C} = \operatorname{coh} \mathbb{X}$ is a hereditary abelian K-category having a decomposition

$$\mathscr{C} = \mathscr{C}_+ \sqcup \mathscr{C}_0$$

where $\mathscr{C} = \operatorname{vect} \mathbb{X}$ (respectively, $\mathscr{C}_0 = \operatorname{coh}_0 \mathbb{X}$) is the category of vector bundles (respectively, sheaves of finite length) on $\mathbb{X}$, and there exists a tilting bundle $T \in \operatorname{vect} \mathbb{X}$ such that $\Lambda = \operatorname{End}_{\mathscr{C}}(T)$. Recall also that $\mathscr{C}_0$ decomposes into a coproduct $\coprod_{x \in \mathbb{X}} \mathscr{U}_x$, where $\mathscr{U}_x$ denotes the connected uniserial category of coherent sheaves concentrated at the point $x \in \mathbb{X}$.

Let $\Lambda = \Lambda(p, \lambda)$ be a canonical algebra, let $\mathbb{X} = \mathbb{X}(p, \lambda)$ be the associated weighted projective line, and let $\mathscr{C} = \operatorname{coh} \mathbb{X}$. Following [37], by a quasitilted algebra of canonical type Λ we mean an algebra of the form $\operatorname{End}_{\mathscr{H}}(T)$, where T is a tilting object in a hereditary abelian K-category $\mathscr{H}$ such that $D^b(\mathscr{H})$ is equivalent to $\mathscr{D}^b(\operatorname{coh} \mathbb{X}) \cong \mathscr{D}^b(\operatorname{mod} \Lambda)$. An important class of quasitilted algebras of canonical type Λ is formed by the algebras of the form $C = \operatorname{End}_{\mathscr{C}}(T)$, where T is a tilting bundle from $\mathscr{C}_+ = \operatorname{vect} \mathbb{X}$, called concealed canonical algebras of type Λ. For such an algebra C, we have again a decomposition

$$\Gamma(\operatorname{mod} C) = \Gamma_+(\operatorname{mod} C) \vee \Gamma_0(\operatorname{mod} C) \vee \Gamma_-(\operatorname{mod} C),$$

where $\Gamma_0(\operatorname{mod} C)$ is a $\mathbb{P}_1(K)$-family of pairwise orthogonal stable tubes separating $\Gamma_+(\operatorname{mod} C)$ from $\Gamma_-(\operatorname{mod} C)$. Consider now a decomposition $\mathbb{X} = \mathbb{X}' \vee \mathbb{X}''$ into disjoint subsets, and let $\mathscr{C}_0' = \coprod_{x \in \mathbb{X}'} \mathscr{U}_x$, $\mathscr{C}_0'' = \coprod_{x \in \mathbb{X}''} \mathscr{U}_x$. Then the additive closure $\mathscr{C}(\mathbb{X}, \mathbb{X}'')$ of the category $\mathscr{C}_0'[-1] \vee \mathscr{C}_+ \vee \mathscr{C}_0''$ in $\mathscr{D}^b(\mathscr{C})$ is a hereditary abelian K-category with a tilting object and $D^b(\mathscr{C}(\mathbb{X}', \mathbb{X}'')) \cong \mathscr{D}^b(\mathscr{C})$ as triangulated categories. Note that $\mathscr{C}(\emptyset, \mathbb{X}) = \mathscr{C}$ and $\mathscr{C}(\mathbb{X}, \emptyset) = \mathscr{C}^{\mathrm{op}}$. The following description of quasitilted algebras of canonical type has been established in [37, Theorem 3.4].

THEOREM 2.3. *For a basic connected algebra A, the following conditions are equivalent.*

(i) *A is representation-infinite and quasitilted of canonical type.*

(ii) *A is isomorphic to the endomorphism algebra of a tilting object in a category $\mathscr{C}(\mathbb{X}', \mathbb{X}'')$, for a disjoint decomposition $\mathbb{X} = \mathbb{X}' \vee X''$ of a weighted projective line $\mathbb{X}$.*

(iii) *A is a semiregular branch enlargement of a concealed canonical algebra.*

(iv) *$\Gamma(\operatorname{mod} A)$ admits a sincere separating family of semiregular tubes.*

Recall that a family $\mathscr{T}$ of tubes in $\Gamma(\operatorname{mod} A)$ is called semiregular if no tube in $\mathscr{T}$ contains both a projective module and an injective module. It is known that the family of tubes $\mathscr{T}$ is semiregular if and only if $\mathscr{T}$ decomposes into a disjoint union $\mathscr{T} = \mathscr{T}' \vee \mathscr{T}''$ of tubes such that all tubes in $\mathscr{T}'$ (respectively, in $\mathscr{T}''$) are obtained from stable tubes by a finite number of coray (respectively, ray) insertions. We refer to [37, Section 3] for a definition of a semiregular branch enlargement of a concealed canonical algebra. It is also known (see [36, 45, 53]) that an algebra A is concealed canonical if and only if $\Gamma(\operatorname{mod} A)$ admits a sincere separating family of stable tubes.

The structure of the Auslander–Reiten quiver of a quasitilted algebra A of canonical type is well understood due to results established in [36, 37, 43]. Namely, we have a decomposition

$$\Gamma(\operatorname{mod} A) = \Gamma_+(\operatorname{mod} A) \vee \Gamma_0(\operatorname{mod} A) \vee \Gamma_-(\operatorname{mod} A),$$

where $\Gamma_0(\operatorname{mod} A)$ is a $\mathbb{P}_1(K)$-family of semiregular tubes separating $\Gamma_+(\operatorname{mod} A)$ from $\Gamma_-(\operatorname{mod} A)$. Further, every component in $\Gamma_+(\operatorname{mod} A)$ is either preprojective or can be obtained from a stable tube $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$, $r \geq 1$, or a component of the form $\mathbb{Z}\mathbb{A}_\infty$, by a finite number (possibly empty) of ray insertions. Dually, every component in $\Gamma_-(\operatorname{mod} A)$ is either preinjective or can be obtained from a stable tube $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$, $r \geq 1$, or a component of the form $\mathbb{Z}\mathbb{A}_\infty$, by a finite number (possibly empty) of coray insertions. Moreover, in contrast to the tilted case, $\Gamma(\operatorname{mod} A)$ admits exactly one preprojective component $\mathscr{P}(A)$ and exactly one preinjective component $\mathscr{Q}(A)$. Moreover, we have the following consequence of the proof of Theorem 3.4 in [37].

PROPOSITION 2.4. *Let $\mathbb{X}$ be a weighted projective line, let $\mathbb{X} = \mathbb{X}' \vee \mathbb{X}''$ be a disjoint decomposition of $\mathbb{X}$, let $\mathscr{C} = \operatorname{coh} \mathbb{X}$, let T be a tilting object in the category $\mathscr{C}(\mathbb{X}', \mathbb{X}'')$, and let $A = \operatorname{End}_{\mathscr{C}(\mathbb{X}', \mathbb{X}'')}(T)$. Then we have*

(i) $\Gamma_0(\operatorname{mod} A)$ *contains a nonzero projective module if and only if T admits an indecomposable direct summand from $\mathscr{C}_0''$.*

(ii) $\Gamma_0(\operatorname{mod} A)$ *contains a nonzero injective module if and only if T admits an indecomposable direct summand from $\mathscr{C}_0'[-1]$.*

(iii) $\Gamma_0(\operatorname{mod} A)$ *consists of stable tubes if and only if $T \in \mathscr{C}_+ = \operatorname{vect} \mathbb{X}$.*

Let H be a basic connected wild hereditary algebra, let M be a quasi-simple regular H-module, and assume that the one-point extension $A = H[M]$ is quasitilted of canonical type $\Lambda = \Lambda(p, \lambda)$. Then the preprojective component $\mathscr{P}(H)$ of $\Gamma(\operatorname{mod} H)$ is the preprojective component of $\Gamma(\operatorname{mod} A)$ and the family $\Gamma_0(\operatorname{mod} A)$ of semiregular tubes has no projective modules but may contain injective modules. Therefore, A is isomorphic to an algebra of the form $\operatorname{End}_{\mathscr{C}^{\mathrm{op}}}(T)$ where T is a tilting object in the opposite category $\mathscr{C}^{\mathrm{op}} = \mathscr{C}(\mathbb{X}, 0)$ of the category $\mathscr{C} = \operatorname{coh} \mathbb{X}$, for a weighted projective line $\mathbb{X} = \mathbb{X}(p, \lambda)$. Moreover, $\Gamma_+(\operatorname{mod} A)$ consists of the preprojective component $\mathscr{P}(A) = \mathscr{P}(H)$, an infinite family of stable components of type $\mathbb{Z}\mathbb{A}_\infty$, and one nonstable component obtained from a component of type $\mathbb{Z}\mathbb{A}_\infty$ by one ray insertion, containing the indecomposable projective A-module with radical equal M.

3. SPECIAL REGULAR TILTING MODULES

Let A be a connected wild hereditary algebra with $n \geq 3$ simple modules and let X be a quasi-simple regular A-module with $\operatorname{Ext}_A^1(X, X) = 0$. We denote by $X^\perp$ the right perpendicular category of X [12] consisting of all modules Y in $\operatorname{mod} A$ such that $\operatorname{Hom}_A(X, Y) = 0$ and $\operatorname{Ext}_A^1(X, Y) = 0$. Recall that $X^\perp$ is equivalent to the module category $\operatorname{mod} H$ for a connected wild hereditary algebra H with $n - 1$ simples (see [55]). We shall identify $\operatorname{mod} H$ with the full subcategory $X^\perp$ of $\operatorname{mod} A$. For a H-module M by $\operatorname{dim} M$ we mean the dimension-vector of M considered as an A-module and by $\operatorname{dim}_H M$ we mean the dimension-vector of the H-module M. Moreover, let Z be the middle term of the Auslander–Reiten sequence $0 \to \tau_A X \to Z \to X \to 0$ in $\operatorname{mod} A$. In this section we are interested in regular tilting A-modules of the form $X \oplus \tau_H^{-a} P$, $a \geq 0$, where P is a preprojective tilting H-module.

For a nonprojective indecomposable H-module Y we want to compare the dimension-vectors of the Auslander–Reiten translations $\tau_H Y$ and $\tau_A Y$. Consider the torsion class $\mathscr{G} = \{M \in \operatorname{mod} A \mid \operatorname{Ext}_A^1(X, M) = 0\}$ in $\operatorname{mod} A$ determined by the partial tilting module X. For an indecomposable module M in $\mathscr{G}$ we denote by $\tau_{\mathscr{G}} M$ the largest torsion submodule $t_{\mathscr{G}}(\tau_A M)$ of $\tau_A M$. Observe that $X^\perp \subseteq \mathscr{G}$.

LEMMA 3.1. (i) *For an indecomposable nonprojective H-module Y there exists a short exact sequence*

$$(\eta) \qquad 0 \to \tau_A X^s \to \tau_H Y \to \tau_{\mathscr{G}} Y \to 0,$$

where $s = \dim_K \operatorname{Ext}_A^1(\tau_{\mathscr{G}} Y, \tau_A X)$.

(ii) *For an indecomposable module M in $\mathscr{G}$ which is not* Ext-*projective, there exists a short exact sequence*

$$(\eta') \qquad 0 \to \tau_{\mathscr{G}} M \to \tau_A M \to \tau_A X^t \to 0,$$

where $t = \dim_K \operatorname{Hom}_A(M, X)$.

Proof. (i) This coincides with Lemma 2 in [9] and (ii) immediately follows from Lemma 2.3 in [23].

LEMMA 3.2. *Let Y be an indecomposable H-module with $\tau_H^i Y \neq 0$ for some $i \geq 1$. Then*

$$\dim \tau_H^i Y = \dim \tau_A^i Y - \sum_{j=0}^{i-1} \langle \tau_H^j Y, Z \rangle \dim \tau_A^{i-j} X$$

where Z is the middle term of the Auslander–Reiten sequence in mod A *ending at X.*

Proof. We proceed by induction on i. Assume first that $i = 1$. Since Y is not projective in mod $H \subseteq \mathscr{G}$, it follows from the above lemma that there exists a short exact sequence $0 \to \tau_{\mathscr{G}} Y \to \tau_A Y \to \tau_A X^t \to 0$ with $t = \dim_K \operatorname{Hom}_A(Y, X)$. Applying the functor $\operatorname{Hom}_A(-, \tau_A X)$ we obtain an exact sequence

$$0 = \operatorname{Ext}_A^1(\tau_A X^t, \tau_A X) \to \operatorname{Ext}_A^1(\tau_A Y, \tau_A X) \to \operatorname{Ext}_A^1(\tau_{\mathscr{G}} Y, \tau_A X) \to 0,$$

and consequently

$$\operatorname{Ext}_A^1(\tau_{\mathscr{G}} Y, \tau_A X) \cong \operatorname{Ext}_A^1(\tau_A Y, \tau_A X) \cong \operatorname{Ext}_A^1(Y, X).$$

Hence, for $s = \dim_K \operatorname{Ext}_A^1(Y, X)$, we obtain

$$t - s = \dim_K \operatorname{Hom}_A(Y, X) - \dim_K \operatorname{Ext}_A^1(Y, X) = \langle Y, X \rangle.$$

From Lemma 3.1(i) we have an exact sequence $0 \to \tau_A X^s \to \tau_H Y \to \tau_{\mathscr{G}} Y \to 0$. Therefore, we get

$$\dim \tau_H Y = \dim \tau_A Y - \langle Y, X \rangle \dim \tau_A X.$$

Applying the functor $\operatorname{Hom}_A(Y, -)$ to the Auslander–Reiten sequence $0 \to \tau_A X \to Z \to X \to 0$ we obtain that $\operatorname{Hom}_A(Y, Z) \cong \operatorname{Hom}_A(Y, X)$ and $\operatorname{Ext}_A^1(Y, Z) \cong \operatorname{Ext}_A^1(Y, X)$ because $Y \in X^{\perp} = {}^{\perp}(\tau_A X)$ and consequently $\langle Y, X \rangle = \langle Y, Z \rangle$. Hence we have

$$\dim \tau_H Y = \dim \tau_A Y - \langle Y, Z \rangle \dim \tau_A X.$$

Assume now that the required formula holds for some $i \geq 1$ and $\tau_H^{i+1} Y \neq 0$. Then, applying the above formula for the module $\tau_H^i Y$, we get

$$\dim \tau_H(\tau_H^i Y) = \dim \tau_A(\tau_H^i Y) - \langle \tau_H^i Y, Z \rangle \dim \tau_A X.$$

Let Φ_A be the Coxeter transformation in $K_0(A)$. Then, applying [49, (2.4)], we obtain the equalities

$$\begin{aligned}\dim \tau_A\big(\tau_H^i Y\big) &= \Phi_A\big(\dim \tau_H^i Y\big)\\ &= \Phi_A\big(\dim \tau_A^i Y\big) - \sum_{j=0}^{i-1} \langle \tau_H^j Y, Z\rangle \Phi_A\big(\dim \tau_A^{i-j} X\big)\\ &= \dim \tau_A^{i+1} Y - \sum_{j=0}^{i-1} \langle \tau_H^j Y, Z\rangle \dim \tau_A^{i+1-j} X.\end{aligned}$$

Therefore, we have

$$\dim \tau_H^{i+1} Y = \dim \tau_A^{i+1} Y - \sum_{j=0}^{i} \langle \tau_H^j Y, Z\rangle \dim \tau_A^{i+1-j} X.$$

This finishes the proof.

COROLLARY 3.3. *Let Y' be an indecomposable H-module which is a regular A-module, and $\tau_H^{-i} Y' \neq 0$. Then*

$$\dim \tau_H^{-i} Y' = \dim \tau_A^{-i} Y' + \sum_{j=0}^{i-1} \langle \tau_H^{j-i} Y', Z\rangle \dim \tau_A^{-j} X.$$

Proof. Applying the above lemma to $Y = \tau_H^{-i} Y'$ we obtain

$$\dim Y' + \sum_{j=0}^{i-1} \langle \tau_H^{j-i} Y', Z\rangle \dim \tau_A^{i-j} X = \dim \tau_A^i \tau_H^{-i} Y'.$$

Then the required equality follows by application of Φ_A^{-i} and [49, (2.4)].

Let $P_1, \dots, P_{n-1}$ be a complete set of indecomposable projective objects in $X^{\perp} = \operatorname{mod} H$. Since at most finitely many indecomposable preprojective H-modules are preprojective A-modules (see [51]), there exists a positive integer r such that the modules $\tau_H^{-t} P_l$, $1 \le l \le n-1$, $t \ge r$, are regular A-modules. Since Z is a regular H-module [43, 55], we conclude that there exists $s \ge r$ such that $\operatorname{Hom}_A(\tau_H^{-t} P_l, Z) = \operatorname{Hom}_H(\tau_H^{-t} P_l, Z) \neq 0$ for all $t \ge s$ and $l = 1, \dots, n-1$ (see, for example, [25]). Further, we know that in the τ_A-orbit of X there exist at most finitely many nonsincere modules.

LEMMA 3.4. *Let P be an indecomposable projective H-module and let m be a positive integer. Then there exists a positive integer $b = b(P, m)$ such that for all $a \ge b$ we have*

(i) *$\tau_H^{-a} P$ is a quasi-simple regular A-module.*

(ii) *The dimension-vectors of all modules $\tau_A^i(\tau_H^{-a} P)$, $i \in \mathbb{Z}$, have all coordinates greater than or equal to m.*

Proof. It follows from our choice of r that all modules $\tau_H^{-t}P$, $t \geq r$, are regular A-modules. Furthermore, each regular component in $\Gamma(\text{mod } A)$ contains at most finitely many modules of the form $\tau_H^{-t}P$, $t \geq r$. Applying now [16, Theorem 2] we conclude that all but finitely many modules $\tau_H^{-t}P$, $t \geq r$, are quasi-simple A-modules. This shows that there exists $b_1 \geq r$ such that all modules $\tau_H^{-t}P$, $t \geq b_1$, are quasi-simple.

We know from the exponential growth of the dimension-vectors of regular A-modules (see [25]) that there exists a positive integer s_1 such that, for all integers i with $|i| \geq s_1$, all coordinates of the dimension-vectors of modules $\tau_A^i X$ are greater than or equal to m. Take now $b_2 = s + 2(s_1 + 1)$, where s is chosen as above. Take $a \geq b_2$ and write $a = s + e$. Hence $e \geq 2(s_1 + 1)$. Let $P' = \tau_H^{-s}P$. Then, applying Corollary 3.3 and [49, (2.4)], we get the equalities

$$\begin{aligned}
\dim \tau_A^i(\tau_H^{-a}P) &= \dim \tau_A^i(\tau_H^{-e}P') = \Phi_A^i(\dim \tau_H^{-e}P') \\
&= \Phi_A^i\left[\dim \tau_A^{-e}P' + \sum_{j=0}^{e-1} \langle \tau_H^{j-e}P', Z\rangle \dim \tau_A^{-j}X\right] \\
&= \dim \tau_A^{i-e}P' + \sum_{j=0}^{e-1} \langle \tau_H^{j-e}P', Z\rangle \dim \tau_A^{j-i}X.
\end{aligned}$$

By the choice of s we have $\langle \tau_H^{j-e}P', Z\rangle = \dim_K \text{Hom}_H\langle \tau_H^{j-e}P', Z\rangle > 0$. Moreover, by the choice of s_1, we know that all coordinates of the dimension-vector of at least one of the modules $\tau_A^{i-j}X$, $j = 0, \ldots, e-1$, are greater than or equal to m. Therefore, the dimension-vectors of all modules $\tau_A^i(\tau_H^{-a}P)$, $i \in \mathbb{Z}$, $a \geq b_1$, are greater than or equal to m. Then, for $b = \max(b_1, b_2)$ the lemma holds.

Let $t(X, m) = \max\{b(P_l, m), 1 \leq l \leq n-1\}$. Then we have

COROLLARY 3.5. *For $a \geq t(X, m)$, the module $X \oplus \tau_H^{-a}H$ is a regular tilting A-module such that each indecomposable direct summand of $\tau_H^{-a}H$ is τ_A-sincere.*

We call a regular tilting A-module $T = X \oplus \tau_H^{-a}P$, with $a \geq t(X, m)$, a saturated completion of the module X. Observe that such a tilting module is a direct sum of pairwise nonisomorphic quasi-simple regular A-modules.

COROLLARY 3.6. *Let m be a positive integer. Then there are infinitely many τ_A-orbits of quasi-simple regular A-modules Y with $\text{Ext}_A^1(Y, Y) = 0$ and all coordinates of the dimension-vectors of all indecomposable modules $\tau_A^i Y$, $i \in \mathbb{Z}$, are greater than or equal to m.*

Proof. Consider the family $\tau_H^{-a}P_l$, $1 \leq l \leq n-1$, $a \geq t(X, m)$. Then all such modules $\tau_H^{-a}P_l$ are quasi-simple regular A-modules and all coordinates of the dimension-vectors of all modules of the form $\tau_A^i(\tau_H^{-a}P_l)$, $i \in \mathbb{Z}$, are greater than or equal to m. Since each regular component of $\Gamma(\operatorname{mod} A)$ contains at most finitely many modules from $X^{\perp}$, we conclude that the assertion of the corollary follows.

4. REGULAR CONNECTING COMPONENTS

The main objective of this section is to study the dimension-vectors of indecomposable modules in the connecting components determined by the saturated regular tilting modules. We start with a short proof of the following lemma established already in [46].

LEMMA 4.1. *Let A be a connected wild hereditary algebra, let T be a regular tilting H-module, let $B = \operatorname{End}_A(T)$, and let $\mathscr{C}$ be the connecting component of $\Gamma(\operatorname{mod} B)$ determined by T. Then all but finitely many indecomposable modules in $\mathscr{C}$ are sincere.*

Proof. Let $T = T_1 \oplus \cdots \oplus T_n$, where $T_1, \ldots, T_n$ are indecomposable and pairwise nonisomorphic. For each $i \in \{1, \ldots, n\}$, there are at most finitely many nonsincere modules in the τ_A-orbit of T_i. Therefore, there are at most finitely many indecomposable preinjective H-modules I with $\operatorname{Hom}_H(T_i, I) = 0$ and at most finitely many indecomposable preprojective H-modules P with $\operatorname{Ext}_H^1(T_i, P) \cong D\operatorname{Hom}_H(P, \tau_H T_i) = 0$. Finally, observe that each indecomposable B-module in $\mathscr{C}$ is either of the form $\operatorname{Hom}_H(T, I)$ with I preinjective or of the form $\operatorname{Ext}_H^1(T, P)$ with P preprojective. Then the claim follows.

Let H be a connected wild hereditary algebra with $n \geq 3$ simple modules and let X be a quasi-simple regular H-module with $\operatorname{Ext}_H^1(X, X) = 0$. Let $X^{\perp} = \operatorname{mod} C$ for a connected wild hereditary algebra C with $n-1$ simple modules. Consider a saturated completion $T = X \oplus \tau_C^{-a}C$ of X, as defined in the previous section. Let $B = \operatorname{End}_H(T)$ be the associated tilted algebra and let $\mathscr{C}$ be the regular connecting component in $\Gamma(\operatorname{mod} B)$ determined by T. Let $P_1, \ldots, P_{n-1}$, P_n be pairwise nonisomorphic indecomposable projective B-modules with $P_n = \operatorname{Hom}_H(T, X)$. Then we have the following

PROPOSITION 4.2. (a) *All indecomposable modules in $\mathscr{C}$ are sincere B-modules if and only if X is a τ_H-sincere module.*

(b) *Assume that X is not τ_H-sincere. Then*

(i) *all nonsincere indecomposable modules in $\mathscr{C}$ are almost sincere,*

(ii) *the number of nonsincere indecomposable modules in* $\mathscr{C}$ *is equal to the sum of numbers of zero coordinates of the dimension-vectors of all modules in the* τ_H*-orbit of* X.

Proof. Let $T = X \oplus \tau_C^{-a} C = X \oplus V_1 \oplus \cdots \oplus V_{n-1}$ with $V_1, \dots, V_{n-1}$ indecomposable. Since $V_1, \dots, V_{n-1}$ are τ_H-sincere quasi-simple regular H-modules, we have $\operatorname{Hom}_H(V_l, I) \neq 0$ for all preinjective indecomposable H-modules I and $\operatorname{Ext}^1_H(V_l, P) \neq 0$ for all preprojective indecomposable H-modules P and all $l = 1, \dots, n-1$. Invoking now the description of the indecomposable modules in the connecting component $\mathscr{C}$ and the formulae for the dimension-vectors of indecomposable B-modules [18] we conclude that $\operatorname{Hom}_B(P_l, M) \neq 0$ for all indecomposable modules M in $\mathscr{C}$ and $l \leq n-1$. Therefore, every indecomposable module in $\mathscr{C}$ is either sincere or almost sincere. Similarly, if X is τ_H-sincere, we conclude that $\operatorname{Hom}_B(P_n, M) \neq 0$ for any indecomposable H-module M in $\mathscr{C}$, and consequently all indecomposable modules in $\mathscr{C}$ are sincere.

Assume that X is not τ_H-sincere. Let $\tau_H^{-i_1} X, \dots, \tau_H^{-i_r} X$, for $0 \leq i_1 < \cdots < i_r$, be all nonsincere modules in the τ_H^{-}-orbit of X. Let $Q_1, \dots, Q_n$ be a complete set of pairwise nonisomorphic indecomposable injective H-modules. Then, for each $i \in \{i_1, \dots, i_r\}$, $\dim \tau_H^{-i} X$ has zero coordinates at the place j if and only if $\operatorname{Hom}_H(X, \tau_H^i Q_j) = 0$, and hence if and only if $\operatorname{Hom}_B(P_n, \operatorname{Hom}_H(T, \tau_H^i Q_j)) = 0$. Therefore, the sum of the numbers of zero coordinates of the dimension-vectors of all modules in the family $\tau_H^{-i} X$, $i \geq 0$, is equal to the number of nonsincere indecomposable modules in $\mathscr{C}$ of the form $\operatorname{Hom}_H(T, I)$ for I, a preinjective H-module. Dually, one shows that the sum of the numbers of zero coordinates of the dimension-vectors of all modules in the family $\tau_H^i X$, $i > 0$, is equal to the number of nonsincere indecomposable modules in $\mathscr{C}$ of the form $\operatorname{Ext}^1_H(T, P)$ for P, a preprojective H-module.

COROLLARY 4.3. *Let m be a positive integer and let $T = X \oplus \tau_C^{-a} C$ be a saturated tilting H-module with $a \geq t(X, m)$. Then* $\dim_K \operatorname{Hom}_B(P_l, M) \geq m$ *for all indecomposable B-modules M in $\mathscr{C}$ and all $l = 1, \dots, n-1$.*

Proof. Let M be an indecomposable module in $\mathscr{C}$. If $M = \operatorname{Hom}_H(T, I)$ for an indecomposable preinjective H-module I, then, for $l \in \{1, \dots, n-1\}$,

$$\operatorname{Hom}_B(P_l, M) = \operatorname{Hom}_B(\operatorname{Hom}_H(T, V_l), \operatorname{Hom}_H(T, I)) \cong \operatorname{Hom}_H(V_l, I)$$

and hence

$$\dim_K \operatorname{Hom}_B(P_l, M) = \dim_K \operatorname{Hom}_H(V_l, I) \geq m.$$

If $M = \operatorname{Ext}^1_H(T, P)$ for an indecomposable preprojective H-module P, then, for $l \in \{1, \ldots, n-1\}$, we have

$$\begin{aligned} \operatorname{Hom}_B(P_l, M) &\cong \operatorname{Hom}_B\big(\operatorname{Hom}_H(T, V_l), \operatorname{Ext}^1_H(T, P)\big) \\ &\cong \operatorname{Ext}^1_H(V_l, P) \cong D\operatorname{Hom}_H(P, \tau_H V_l), \end{aligned}$$

and hence $\dim_K \operatorname{Hom}_B(P_l, M) = \dim_K \operatorname{Hom}_H(P, \tau_H V_l) \geq m$.

We shall illustrate the above considerations by the following example.

EXAMPLE 4.4. Let A be the path algebra of the wild quiver

$$1 \leftleftarrows 2 \leftarrow 3.$$

Since H has three simple modules, it follows from [55] that all indecomposable regular A-modules X with $\operatorname{Ext}^1_A(X, X) = 0$ are quasi-simple. Fix a quasi-simple regular A-module X with $\operatorname{Ext}^1_A(X, X) = 0$. We have three cases to consider.

(i) Assume that at least two modules in the τ_A-orbit of X are nonsincere. Since A has three simples, it follows from [24] that X is an elementary module. In [42] it is shown that there exists a unique τ_A-orbit of elementary A-modules without self-extensions, and this orbit contains a module E with $\dim E = (5, 4, 0)$. Then $\dim \tau_A^2 E = (1, 2, 0)$ and E and $\tau_A^2 E$ are the unique nonsincere modules in the τ_A-orbit of X (see also [48, Section 6]). Therefore, if T is a saturated completion of X and $B = \operatorname{End}_A(T)$, then the regular connecting component $\mathscr{C}$ of $\Gamma(\operatorname{mod} B)$ contains exactly two nonsincere indecomposable modules. Moreover, since $\dim E$ and $\dim \tau_A^2 E$ have zero coordinates at the same place, it follows from the proof of Proposition 4.2 that the nonsincere modules of $\mathscr{C}$ lie in one τ_B-orbit.

(ii) Assume that the τ_A-orbit of X contains exactly one nonsincere module Y. Then direct checking (see also [48, Section 6]) shows that $\dim Y \in \{(m+1, m, 0),\ (r, r+1, 0) \mid m \geq 5,\ r \geq 2\}$. Thus for any saturated completion T of X the connecting component of $\Gamma(\operatorname{mod} \operatorname{End}_H(T))$ contains exactly one nonsincere module.

(iii) Assume that X is τ_A-sincere. Then for any saturated completion T of X the connecting component of $\Gamma(\operatorname{mod} \operatorname{End}_H(T))$ consists entirely of sincere modules.

5. PROOF OF THEOREMS 1 AND 3: TILTED CASE

The aim of this section is to prove Theorems 1 and 3 in the tilted case.

Let H be a connected wild hereditary algebra and let M be a quasi-simple regular H-module. We assume that $H[M]$ is a piecewise hereditary algebra of type $\mathscr{H} = \text{mod}\, A$, for some connected wild hereditary algebra A. The following proposition establishes the proof of Theorem 1 in the tilted case.

PROPOSITION 5.1. *Let m be a positive integer. There exists a positive integer r such that, for each $i \geq r$, $H[\tau_H^i M]$ is a tilted algebra with a regular connecting component $\mathscr{C}_i$ and* $\dim_K \text{Hom}_{H[\tau_H^i M]}(P, N) \geq m$ *for all indecomposable modules N in $\mathscr{C}_i$ and all indecomposable projective H-modules P.*

Proof. By [32, Corollary 2.2] there exists a positive integer s such that the algebras $H[\tau_H^i M]$, $i \geq s$, are tilted. Therefore, we may assume that $H[M]$ is tilted. Hence there exists a tilting A-module $T = X \oplus P$ with $\text{End}_A(T) = H[M]$, where X is a quasi-simple regular A-module, with $\text{Ext}_A^1(X, X) = 0$, and P is a preprojective tilting module in $X^{\perp} = \text{mod}\, C$, for some connected wild hereditary algebra C. Since $\text{End}_C(P) = H$ is hereditary, P is the direct sum of all indecomposable modules lying on a section of the preprojective component of $\Gamma(\text{mod}\, C)$. Then for the positive integer $t(X, m)$, defined in Section 3, and for every $a \geq t(X, m)$, we conclude by Lemma 3.4 that $T_a = X \oplus \tau_C^{-a} P$ is a regular tilting A-module with $\text{End}_A(T_a) \cong H[\text{Hom}_A(\tau_C^{-a} P, X)] \cong H[\tau_H^a, M]$ (see proof of [17, Proposition 5.3]). Moreover, for each indecomposable direct summand V of $\tau_C^{-a} P$, the dimension-vectors of all modules $\tau_A^i V$, $i \in \mathbb{Z}$, have all coordinates greater than or equal to m. Then, as in the proof of Corollary 4.3, we infer that $\dim_K \text{Hom}_{H[\tau_H^a M]}(\text{Hom}_A(T_a, V), N) \geq m$ for all indecomposable modules N in the connecting component of $\Gamma(\text{mod}\, H[\tau_H^a M])$ and all indecomposable direct summands V of $\tau_C^{-a} P$. This finishes the proof.

We shall prove now the tilted part of Theorem 3.

PROPOSITION 5.2. *Let m be a positive integer and let H be a connected wild hereditary algebra with at least three simple modules. Then there exist infinitely many pairwise nonisomorphic wild hereditary algebras C and quasi-simple regular C-modules M such that $C[M]$ are tilted algebras of type H with a regular connecting component $\mathscr{C}$ and all coordinates of the dimension-vectors of all indecomposable modules in $\mathscr{C}$ are greater than or equal to m.*

Proof. By Corollary 3.6 there exist infinitely many quasi-simple regular H-modules Y_j, $j \in \mathbb{N}$, lying in pairwise different τ_H-orbits such that the coordinates of the dimension-vectors of all indecomposable modules of the form $\tau_H^i Y_j$, $i \in \mathbb{Z}$, $j \in \mathbb{N}$, are greater than or equal to m. Let C_j, $j \in \mathbb{N}$, be connected wild hereditary algebras such that $Y_j^{\perp} = \text{mod}\, C_j$. By [16, Proposition 4.2] we may assume that the algebras C_j, $j \in \mathbb{N}$, are pairwise nonisomorphic. For each $j \in \mathbb{N}$, take a saturated completion $T^{(j)} = Y_j \oplus \tau_{C_j}^{-t_j} C_j$ with $t_j = t(Y_j, m)$. Then the tilted algebras $B_j = \text{End}_H(T^{(j)})$, $j \in \mathbb{N}$, satisfy the assertions of the proposition.

6. PASSING FROM NONREGULAR TO REGULAR CONNECTING COMPONENTS

Let H be a connected wild hereditary algebra and let M be a quasi-simple regular H-module such that $H[M]$ is a tilted algebra, say of type A. In general, the connecting component of $\Gamma(\operatorname{mod} H[M])$ may contain injective modules (but clearly does not contain projective modules). We know from Proposition 5.1 that $H[\tau_H^i M]$, for $i \gg 0$, are tilted algebras with regular connecting components without simple modules. We shall show that if the connecting component of $H[M]$ contains at least one injective module then applying suitable reflections to H we obtain a tilted algebra with a regular connecting component containing at least one simple module. This is strongly related to the problem of distribution of simple modules and projective modules in the Auslander–Reiten components of selfinjective algebras of the wild tilted type (see [10, Section 5]).

PROPOSITION 6.1. *Assume that the connecting component of $\Gamma(\operatorname{mod} H[M])$ contains at least one injective module. Then there exists a preprojective tilting H-module Q with $H' = \operatorname{End}_H(Q)$ hereditary such that $H'[\operatorname{Hom}_H(Q, M)]$ is a tilted algebra with a regular connecting component containing a simple module.*

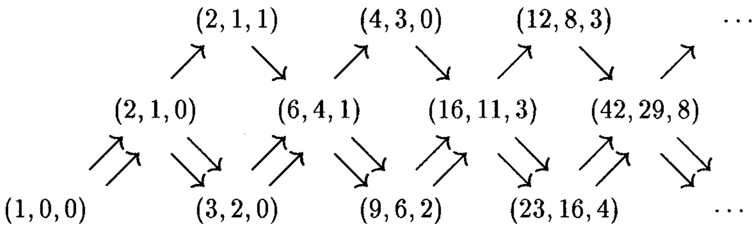
Proof. We know that $H[M] \cong \operatorname{End}_A(X \oplus P)$ where X is a quasi-simple regular A-module with $\operatorname{Ext}_A^1(X, X) = 0$ and P is a preprojective tilting module in $X^{\perp} = \operatorname{mod} C$ with $\operatorname{End}_C(P) = H$. Since the connecting component of $\Gamma(\operatorname{mod} H[M])$ contains an injective module, P contains indecomposable direct summands from the preprojective component $\mathscr{P}(A)$ of $\Gamma(\operatorname{mod} A)$. Since there are only finitely many indecomposable C-modules in $\mathscr{P}(A)$, then there exists an indecomposable preprojective C-module Y such that Y is in $\mathscr{P}(A)$ and no proper successor of Y in $\operatorname{mod} C$ is contained in $\mathscr{P}(A)$. Consider the (unique) section Σ in the preprojective component $\mathscr{P}(C)$ of $\Gamma(\operatorname{mod} C)$ containing Y at its unique source. Let $Q' = \oplus_{Y_i \in \Sigma} Y_i$ with $Y_1 = Y$. Define $Q = \tau_C^- Y_1 \oplus (\oplus_{Y_i \in \Sigma \setminus \{Y_1\}} Y_i)$. Then $T' = X \oplus Q$ is a regular tilting A-module and $\operatorname{End}_A(T') = H'[\operatorname{Hom}_H(Q, M)]$ with $H' = \operatorname{End}_H(Q)$. We claim that the (regular) connecting component of $\Gamma(\operatorname{mod} \operatorname{End}_A(T'))$ contains a simple module. We have $\operatorname{Ext}_A^1(X \oplus (\oplus_{i>1} Y_i), Y) = 0$; hence $\operatorname{Ext}_A^1(T', Y)$ is a simple module in the connecting component of $\operatorname{End}_A(T') = H'[\operatorname{Hom}_H(Q, M)]$.

We illustrate the above procedure by a concrete example.

EXAMPLE 6.2. Let A be the path algebra of the wild quiver

$$1 \leftleftarrows 2 \leftarrow 3.$$

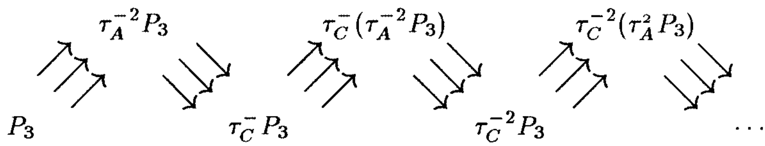
Then $\Gamma(\text{mod } A)$ has the preprojective component $\mathscr{P}(A)$ of the form



where the indecomposable modules are replaced by their dimension-vectors. Let E be the quasi-simple regular A-module with $\dim E = (5, 4, 0)$ (see (4.4)), and consider the Auslander–Reiten sequence $0 \to E \to Z \to X \to 0$. Then $\dim X = \dim \tau_A^- E = \Phi_A^{-1}(\dim E) = (15, 10, 4)$, and hence $\dim Z = (20, 14, 4)$. Furthermore, X is a quasi-simple regular A-module with $\text{Ext}_A^1(X, X) = 0$ (because E has this property) and $X^\perp = \text{mod}\, C$ for a connected wild hereditary algebra C with two simple modules. We know from (4.4) that the τ_A-orbit of X has exactly two nonsincere modules, namely E and $\tau_A^2 E$. As a consequence we obtain that the projective module P_3, of dimension-vector $(2, 1, 1)$, and its shift $\tau_A^{-2} P_3$, of dimension-vector $(12, 8, 3)$, are the unique indecomposable preprojective A-modules lying in the perpendicular category $X^\perp$. Clearly then P_3 and $\tau_A^{-2} P_3$ form a complete family of indecomposable projective objects in $X^\perp$, and consequently $C = \text{End}_A(P_3 \oplus \tau_A^{-2} P_3)$. Since $\text{Hom}_A(\tau_A^{-2} P_3, P_3) = 0$ and $\dim_K \text{Hom}_A(P_3, \tau_A^{-2} P_3) = 3$, we conclude that C is the path algebra of the wild quiver

$$1 \overset{\longleftarrow}{\underset{\longleftarrow}{\longleftarrow}} 2.$$

We know also that Z is a quasi-simple regular C-module. Since $\dim_K \text{Hom}_A(P_3, Z) = 4$ and $\dim_A(\tau_A^{-2} P_3, Z) = \dim_K \text{Hom}_A(P_3, \tau_A^2 Z) = 2$, because $\dim \tau_A^2 Z = (3, 2, 2)$ (see [48]), we conclude that Z as a C-module has dimension-vector $(4, 2)$. Consider the A-module $T = X \oplus P$ with $P = P_3 \oplus \tau_A^{-2} P_3$. Then T is a tilting A-module and $B = \text{End}_A(T) \cong C[Z]$. Moreover, the four indecomposable preprojective A-modules of dimension-vectors $(1, 0, 0)$, $(2, 1, 0)$, $(3, 2, 0)$, and $(4, 3, 0)$ together with $\tau_A X$ are the unique indecomposable objects in the torsion-free class $\mathscr{F}(T) = \{M \in \text{mod}\, A \mid \text{Hom}_A(T, M) = 0\}$ in $\text{mod}\, A$ determined by T. Therefore, the connecting component $\mathscr{C} = \mathscr{C}_T$ of $\Gamma(\text{mod}\, B)$ determined by T is a preinjective component containing all indecomposable injective B-modules. The preprojective component $\mathscr{P}(C)$ of $\Gamma(\bmod C)$ is of the form



and hence $Y = \tau_A^{-2}P_3$ has this property: Y is in $\mathscr{P}(A)$ and no proper successor of Y in $\bmod C$ is contained in $\mathscr{P}(A)$. Hence the modules $Y_1 = Y$ and $Y_2 = \tau_C^{-}P_3$ form the unique slice Σ in $\mathscr{P}(C)$ having Y as its unique source. Take now $Q = \tau_C^{-}Y_1 \oplus Y_2 = \tau_C^{-}(\tau_A^{-2}P_3) \oplus \tau_C^{-}P_3$ and $T' = X \oplus Q$. Then T' is a regular tilting A-module and $B' = \operatorname{End}_A(T') = C'[\operatorname{Hom}_C(Q, Z)]$ with $C' = \operatorname{End}_C(Q) \cong C$. Since $Q = \tau_C^{-}C$ we conclude that $B' = C[\tau_C Z]$. Then it follows from the proof of Proposition 6.1 that B' is a tilted algebra (of type A) with a connecting regular component containing the simple module $\operatorname{Ext}_A^1(T', \tau_A^{-2}P_3)$.

We shall exhibit now a class of tilted algebras $\Lambda = C[\tau_C M]$ such that C is a connected wild hereditary algebra, M is a quasi-simple regular C-module, $\Gamma(\bmod \Lambda)$ admits a regular connecting component without simple modules, and $C[M]$ is not tilted. Hence, we may have immediate jumps from the nontilted algebras $C[M]$ to the tilted algebras $C[\tau_C M]$ having regular connecting components.

Let H be a connected wild hereditary algebra with $n \geq 3$ simple modules. It follows from [28, (2.1)] (or Corollary 3.6) that there exist infinitely many τ_H-orbits of quasi-simple τ_H-sincere modules without self-extensions. Invoking [16, Theorem 2] we conclude that there exist infinitely many τ_H-orbits of quasi-simple τ_H-sincere modules X such that $\operatorname{Ext}_H^1(X, X) = 0$ and $\operatorname{Hom}_H(X, \tau_H^2 X) \neq 0$. Take such a quasi-simple H-module X and consider the Auslander–Reiten sequence $0 \to \tau_H X \to Z \to X \to 0$. We know that $X^{\perp} = \bmod C$ for a connected wild hereditary algebra C with $n - 1$ simple modules and Z is a quasi-simple regular C-module, and let P be the minimal projective cogenerator in $X^{\perp}$. Then we have

PROPOSITION 6.3. (i) *$T = X \oplus P$ is a regular tilting H-module with $\operatorname{End}_H(T) \cong C[Z]$, and the connecting component of $C[Z]$ does not contain any simple module.*

(ii) *$C[\tau_C^{-}Z]$ is not a tilted algebra.*

Proof. (i) Since X is τ_H-sincere, $X^{\perp} = \operatorname{mod} C$ consists entirely of regular H-modules. In particular, T is a regular tilting H-module, and consequently $C[Z] = \operatorname{End}_H(T)$ is a tilted algebra with regular connecting component $\mathscr{C}$ determined by T. Moreover, it follows from [10, Proposition 2.5] that there are no simple modules in $\mathscr{C}$.

(ii) Recall from [17, (III.2.13) and (III.3.3)] that if a one-point extension $C[N]$ of C by an indecomposable regular C-module N is quasitilted then N dominates all indecomposable regular C-modules; that is, for any right minimal morphism $f\colon N^t \to R$ with R an indecomposable regular C-module, the kernel of f is projective. Therefore, in order to prove that $C[\tau_C^- Z]$ is not tilted, it is sufficient to show that $\tau_C^- Z$ does not dominate all indecomposable regular C-modules. Observe first that, for all integers m, $\tau_H^m P$ is a minimal projective generator in $(\tau_H^m X)^{\perp} \cong \operatorname{mod} C$ and $\operatorname{End}_H(\tau_H^m X \oplus \tau_H^m P) = \operatorname{End}_H(\tau_H^m T) \cong C[Z]$. Since the dimensions of the modules $\tau_H^{-i} X$, $i \geq 0$, grow exponentially, replacing (if necessary) X by some module $\tau_H^{-i} X$, we may assume that $\dim_K X > \dim_K \tau_H^2 X$. It follows from our choice of X that $r = \dim_K \operatorname{Hom}_H(X, \tau_H^2 X) > 0$. We know from [23, (1.7)] that $\dim_K \operatorname{Hom}_H(X, \tau_H^2 X) = \dim_K \operatorname{Ext}_H^1(Z, Z)$. Hence we obtain

$$r = \dim_K \operatorname{Ext}_H^1(Z,Z) = \dim_K \operatorname{Ext}_C^1(Z,Z) = \dim_K \operatorname{Hom}_C(Z, \tau_C Z).$$

Further, it follows from the proof of Lemma 4.4 in [30] that there exists a short exact sequence

$$0 \to (\tau_H X)^r \to \tau_C Z \to \tau_H^2 X \to 0.$$

Hence, $\dim_K \tau_C Z = r \dim_K \tau_H X + \dim_K \tau_H^2 X$, and clearly $\dim_K Z = \dim_K \tau_H X + \dim_K X$.

Let $f_1, \dots, f_r$ be a K-basis of $\operatorname{Hom}_C(Z, \tau_C Z)$. Since $\operatorname{End}_C(Z) = K$, the morphism $f = (f_1, \dots, f_r)\colon Z^r \to \tau_C Z$ is a minimal right add Z-approximation of $\tau_C Z$ (in the sense of [3, Chapter XI]). Observe that f is not a monomorphism. This follows from the inequalities $\dim_K Z^r = r \dim_K X + r \dim_K \tau_H > \dim_K \tau_H^2 X + r \dim_K \tau_H X = \dim_K \tau_C Z$. Further, since $C[Z]$ is a tilted algebra, Z dominates all indecomposable regular C-modules. Applying now [17, (III.3.1)] we conclude that for the minimal right add Z-approximation $f\colon Z^r \to \tau_C Z$, we have $K = \operatorname{Ker} f$ is projective. Since τ_C^- is an autoequivalence of the category of all regular C-modules, we infer that $\tau_C^- f\colon (\tau_C^- Z)^r \to Z$ is a minimal right add $\tau_C^- Z$-approximation of Z, and obviously $\tau_C^- K$ is contained in $\operatorname{Ker} \tau_C^- f$. Therefore, $\operatorname{Ker} \tau_C^- f$ is not projective, because C is hereditary and $\tau_C^- K$ is not projective. Applying [17, (III.3.1)] again we then conclude that $\tau_C^- Z$ does not dominate the indecomposable regular C-module Z. Hence, the one-point extension $C[\tau_C^- Z]$ is not a tilted algebra.

7. PROOFS OF THEOREMS 1 AND 3: CANONICAL CASE

The aim of this section is to prove Theorems 1 and 3 in the canonical case.

Let H be a connected wild hereditary algebra and let M be a quasi-simple regular H-module. We assume that $H[M]$ is a piecewise hereditary algebra of type $\mathscr{H} = \operatorname{coh}\mathbb{X}$ for some weighted projective line $\mathbb{X} = \mathbb{X}(p, \lambda)$ with a weight sequence $p = (p_1, \dots, p_t)$ and a parameter sequence $\lambda = (\lambda_1, \dots, \lambda_t)$. Denote by $\Lambda = \Lambda(p, \lambda)$ the associated canonical algebra. Recall that $\Lambda = \operatorname{End}_{\mathscr{H}}(T)$, where T is a tilting bundle of the form $T = \mathscr{O}_{\mathbb{X}}(\vec{c}) \oplus P$, with $\mathscr{O}_{\mathbb{X}}(\vec{c})$ as the canonical line bundle and P a minimal projective generator in the perpendicular category $\mathscr{O}_{\mathbb{X}}(\vec{c})^{\perp}$. Moreover, $\mathscr{O}_{\mathbb{X}}(\vec{c})^{\perp} = \operatorname{mod}\Lambda_0$ for the wild hereditary algebra given by the associated star quiver, and $\Lambda = \Lambda_0[R]$ for the corresponding quasi-simple regular Λ_0-module R.

The following proposition establishes the proof of Theorem 1 in the canonical case.

PROPOSITION 7.1. *Let m be a positive integer. There exists a positive integer r such that, for each $i \geq r$, $H[\tau_H^i M]$ is a concealed canonical algebra with a $\mathbb{P}_1(K)$-family $\mathscr{C}_i$ of stable tubes such that* $\dim_K \operatorname{Hom}_H(P, N) \geq m$ *for all indecomposable modules N in $\mathscr{C}_i$ and all indecomposable projective H-modules P.*

Proof. By [32, Corollary 2.2] there exists a positive integer s such that the algebras $H[\tau_H^i M]$, $i \geq s$ are quasitilted. Therefore, we may assume that $H[M]$ is quasitilted. It follows from [37, Section 3] that $H[M] = \operatorname{End}_{\mathscr{H}'}(T)$ for some tilting object T in $\mathscr{H}' = \mathscr{H}^{\mathrm{op}}$. Let $T = T_1 \oplus \cdots \oplus T_n$, where $T_1, \dots, T_n$ are pairwise nonisomorphic indecomposable objects such that $\operatorname{rad}\operatorname{Hom}_{\mathscr{H}'}(T, T_n) = M$. Then $P_i = \operatorname{Hom}_{\mathscr{H}'}(T, T_i)$, $1 \leq i \leq n$, is a complete family of indecomposable projective $H[M]$-modules. Since $\operatorname{Hom}_{H[M]}(P_n, P_i) = 0$ for $i < n$, we deduce that $\operatorname{Hom}_{\mathscr{H}'}(T_n, T_i) = 0$ for $i < n$, and hence $\tilde{T} = \oplus_{i=1}^{n-1} T_i$ is a tilting object in $T_n^{\perp}$. Moreover, $\operatorname{End}_{\mathscr{H}'}(\tilde{T}) \cong H$. Observe also that T_n belongs to a component of the form $\mathbb{Z}\mathbb{A}_{\infty}$ in the Auslander–Reiten quiver of $\Gamma(\mathscr{H}')$ (see [37, 43]). We claim that T_n is quasi-simple. Indeed, if it is not the case, then the category $T_n^{\perp}$ is not connected (see [43]), and so $H \cong \operatorname{End}_{\mathscr{H}'}(\tilde{T})$ is not connected, a contradiction. Since T_n is a quasi-simple object in a component of type $\mathbb{Z}\mathbb{A}_{\infty}$, it follows from [44, Lemma 6.8] that $T_n^{\perp} = \operatorname{mod} C$ for a connected wild hereditary algebra C. Since $\operatorname{End}_{\mathscr{H}'}(\tilde{T})$ is hereditary and $H[M] = \operatorname{End}_{\mathscr{H}'}(\oplus \tilde{T}_n)$, we infer that $\tilde{T}$ is a slice module in the preprojective component $\mathscr{P}(C)$ of $\Gamma(\operatorname{mod} C)$. Since all indecomposable preprojective C-modules have no selfextensions, the projective component $\mathscr{P}(C)$ contains at most finitely many objects from the family $\mathscr{T}'$ of $\Gamma(\mathscr{H}')$. Hence there exists a positive integer t such that $\tau_C^{-t}\tilde{T} \in \operatorname{vect}\mathbb{X}' \cong \operatorname{vect}\mathbb{X}$. Hence, $H[\tau_H^t M] \cong H[\operatorname{Hom}_C(\tau_C^{-t}\tilde{T}, M)] \cong \operatorname{End}_{\mathscr{H}'}(\tau_C^{-t}\tilde{T} \oplus T_n)$ is a concealed canonical algebra. Therefore, we may assume that $H[M]$ is a concealed canonical algebra. Let $\mathscr{T}$ be a stable tube in $\Gamma(\operatorname{mod} H[M])$. Since $\mathscr{T}$ is sincere, there exists a quasi-simple module X in $\mathscr{T}$ with $\operatorname{Hom}_{H[M]}(P_n, X) \neq 0$. Since $\operatorname{pd}_{H[M]} X \leq 1$, X admits a minimal projective resolution of the form

$$0 \to P^{(1)} \to P_n^l \oplus P^{(0)} \to X \to 0,$$

where $l \geq 1$ and $P^{(0)}, P^{(1)}$ are projective H-modules. Moreover, we have a short exact sequence

$$0 \to M \to P_n \to S_n \to 0.$$

Let $p = \text{l.c.m.}(p_1, \dots, p_t)$. Then there exists a positive integer $r > p$ such that $\dim_K \operatorname{Hom}_H(\tau_H^{-a}P, M) \geq m$ for all $a \geq r - p$ and all indecomposable projective H-modules P. Let $a \geq r - p$ and let P be an indecomposable projective H-module. Applying the functor $\operatorname{Hom}_{H[M]}(\tau_H^{-a}P, -)$ to the above exact sequences we obtain monomorphisms

$$\operatorname{Hom}_{H[M]}\left(\tau_H^{-a}P, P_n^l\right) \to \operatorname{Hom}_{H[M]}\left(\tau_H^{-a}P, X\right)$$

and

$$\operatorname{Hom}_{H[M]}\left(\tau_H^{-a}P, M\right) \to \operatorname{Hom}_{H[M]}\left(\tau_H^{-a}P, P_n\right),$$

and consequently

$$\begin{aligned} m &\leq \dim_K \operatorname{Hom}_H\left(\tau_H^{-a}P, M\right) \\ &= \dim_K \operatorname{Hom}_{H[M]}\left(\tau_H^{-a}P, M\right) \leq \dim_K \operatorname{Hom}_{H[M]}\left(\tau_H^{-a}P, X\right). \end{aligned}$$

Take an arbitrary quasi-simple module Y in $\mathscr{T}$. Then $Y = \tau_{H[M]}^{-b}X$ for some $b \leq p$. From the Auslander–Reiten formulae we obtain the inequalities

$$\begin{aligned} \dim_K \operatorname{Hom}_{H[M]}\left(\tau_H^{-a}P, Y\right) &\geq \dim_K \operatorname{Hom}_{H[M]}\left(\tau_{H[M]}\left(\tau_H^{-a}P\right), \tau_{H[M]}Y\right) \\ &\geq \cdots \geq \dim_K \operatorname{Hom}_{H[M]}\left(\tau_{H[M]}^b\left(\tau_H^{-a}P\right), \tau_{H[M]}^b Y\right) \\ &= \dim_K \operatorname{Hom}_{H[M]}\left(\tau_H^{b-a}P, X\right) \geq m, \end{aligned}$$

because $b - a \geq r - p$. Therefore, for each $i \geq r$, the algebra $H[\tau_H^i M] \cong \operatorname{End}_{\mathscr{H}'}(\tau_C^{-i}\tilde{T} \oplus T_n)$ is a concealed canonical algebra with $\mathbb{P}_1(K)$-family $\mathscr{C}_i$

of stable tubes such that $\dim_K \operatorname{Hom}_H(P, N) \geq m$ for all indecomposable modules N in $\mathscr{C}_i$ and all indecomposable projective H-modules P.

It follows from [35, Theorem 6.3(ii)] that if U is an indecomposable regular Λ_0-module, then $\tau_{\Lambda_0}^{-r-i}U = \tau_{\Lambda}^{-i}(\tau_{\Lambda_0}^{-r}U)$ for all $i \geq 0$ and $r \gg 0$. Moreover, if U is quasi-simple regular in $\operatorname{mod}\Lambda_0$ and not in the component of $\Gamma(\operatorname{mod}\Lambda_0)$ containing $\mathscr{O}_{\mathbb{X}}(\vec{c})$, then U is quasi-simple in $\operatorname{vect}\mathbb{X}$. Indeed, suppose there exists an irreducible monomorphism $f\colon U' \to U$ in $\operatorname{vect}\mathbb{X}$. Then one checks directly that U' is in $\mathscr{O}_{\mathbb{X}}(\vec{c})^{\perp} = \operatorname{mod}\Lambda_0$ and that f is an irreducible monomorphism in $\operatorname{mod}\Lambda_0$.

LEMMA 7.2. *Let Y be a quasi-simple regular Λ_0-module with* $\operatorname{Ext}^1_{\Lambda_0}(Y, Y) = 0$ *and such that Y is quasi-simple in* $\operatorname{vect}\mathbb{X}$ *with $\tau_{\Lambda}^{-r}Y = \tau_{\Lambda_0}^{-r}Y$ for all $r \geq -2$. Consider the Auslander–Reiten sequences $0 \to \tau_{\mathscr{H}}Y \to M \to Y \to 0$ in $\mathscr{H}$ and $0 \to \tau_{\Lambda_0}Y \to M' \to Y \to 0$ in* $\operatorname{mod}\Lambda_0$. *Then*

$$\begin{aligned}\dim_K \operatorname{Ext}^1_{\Lambda_0}(M', M') &= \dim_K \operatorname{Hom}_{\Lambda_0}\left(Y, \tau_{\Lambda_0}^2 Y\right)\\ &\leq \dim_K \operatorname{Hom}_{\mathscr{H}}\left(Y, \tau_{\mathscr{H}}^2 Y\right) = \dim_K \operatorname{Ext}^1_{\mathscr{H}}(M, M).\end{aligned}$$

Proof. The equality $\dim_K \operatorname{Ext}^1_{\Lambda_0}(M', M') = \dim_K \operatorname{Hom}_{\Lambda_0}(Y, \tau_{\Lambda_0}^2 Y)$ follows from [23, Lemma 1.7]. Similarly, we have

$$\dim_K \operatorname{Hom}_{\mathscr{H}}\left(Y, \tau_{\mathscr{H}}^2 Y\right) = \dim_K \operatorname{Ext}^1_{\mathscr{H}}(M, M)$$

since Y is a quasi-simple object in $\operatorname{vect}\mathbb{X}$ with $\operatorname{Hom}_{\mathscr{H}}(Y, \tau_{\mathscr{H}}^{-}Y) = 0$. Hence, it remains to show the inequality $\dim_K \operatorname{Hom}_{\Lambda_0}(Y, \tau_{\Lambda_0}^2 Y) \leq \dim_K \operatorname{Hom}_{\mathscr{H}}(Y, \tau_{\mathscr{H}}^2 Y)$. Since we may assume that $\tau_{\Lambda}^{-r}Y = \tau_{\Lambda_0}^{-r}Y$ for all $r \geq -2$, it follows from [35, Proposition 5.3] that there exist exact sequences

$$(\eta) \qquad 0 \to \tau_{\Lambda_0}Y \to \tau_{\mathscr{H}}Y \to \tau_{\mathscr{H}}\mathscr{O}_{\mathbb{X}}(\vec{c})^{q_1} \to 0$$

with $q_1 = \dim_K \operatorname{Hom}_{\mathscr{H}}(Y, \mathscr{O}_{\mathbb{X}}(\vec{c}))$ and

$$(\xi) \qquad 0 \to \tau_{\Lambda_0}^2 Y \to \tau_{\mathscr{H}}\left(\tau_{\Lambda_0}Y\right) \to \tau_{\mathscr{H}}\mathscr{O}_{\mathbb{X}}(\vec{c})^{q_2} \to 0$$

with $q_2 = \dim_K \operatorname{Hom}_{\mathscr{H}}(\tau_{\Lambda_0}Y, \mathscr{O}_{\mathbb{X}}(\vec{c}))$. Since $\tau_{\mathscr{H}}$ is an equivalence on $\mathscr{H}$ we get from (η) a short exact sequence

$$(\eta') \qquad 0 \to \tau_{\mathscr{H}}\tau_{\Lambda_0}Y \to \tau_{\mathscr{H}}^2 Y \to \tau_{\mathscr{H}}^2\mathscr{O}_{\mathbb{X}}(\vec{c})^{q_1} \to 0.$$

Combining (ξ) and (η') we get the following commutative diagram with exact rows and columns

Applying the functor $\operatorname{Hom}_{\mathscr{H}}(Y, -)$ to the second row we obtain an exact sequence

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \tau^2_{\Lambda_0} Y & \longrightarrow & \tau_{\mathcal{H}}(\tau_{\Lambda_0} Y) & \longrightarrow & \tau_{\mathcal{H}} \mathcal{O}_{\mathbb{X}}(\vec{c})^{q_2} & \rightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \tau^2_{\Lambda_0} Y & \longrightarrow & \tau^2_{\mathcal{H}} Y & \longrightarrow & Q & \rightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \tau^2_{\mathcal{H}} \mathcal{O}_{\mathbb{X}}(\vec{x})^{q_1} & = & \tau^2_{\mathcal{H}} \mathcal{O}_{\mathbb{X}}(\vec{c})^{q_1} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

$$0 \to \operatorname{Hom}_{\mathscr{H}}\left(Y, \tau^2_{\Lambda_0} Y\right) \to \operatorname{Hom}_{\mathscr{H}}\left(Y, \tau^2_{\mathscr{H}} Y\right),$$

and consequently we have

$$\dim_K \operatorname{Hom}_{\Lambda_0}\left(Y, \tau^2_{\Lambda_0} Y\right) = \dim_K \operatorname{Hom}_{\mathscr{H}}\left(Y, \tau^2_{\Lambda_0} Y\right) \leq \dim_K \operatorname{Hom}_{\mathscr{H}}\left(Y, \tau^2_{\mathscr{H}} Y\right).$$

We shall prove now the canonical part of Theorem 3.

PROPOSITION 7.3. *Let m be a positive integer. There exist infinitely many pairwise nonisomorphic connected wild hereditary algebras C and quasi-simple regular C-modules m such that $C[M]$ are concealed canonical algebras of type Λ and all coordinates of the dimension-vectors of all indecomposable modules in the $\mathbb{P}_1(K)$-family $\mathscr{C}$ of stable tubes in $\Gamma(\operatorname{mod} C[M])$ are greater than or equal to m.*

Proof. By [16, Theorem 2] and Lemma 7.2 there exist infinitely many quasi-simple regular Λ_0-modules Y_i, $i \in \mathbb{N}$, with $\operatorname{Ext}^1_{\Lambda_0}(Y_i, Y_i) = 0$ and

$$\dim_K \operatorname{Hom}_{\Lambda_0}\left(Y_i, \tau^2_{\Lambda_0} Y_i\right) \neq \dim_K \operatorname{Hom}\left(Y_j, \tau^2_{\Lambda_0} Y_j\right)$$

for all $i \neq j$. Moreover, we may assume, shifting if necessary in the

τ^{-}-direction that

(1) $\dim_K \operatorname{Hom}_{\Lambda_0}(\tau_{\Lambda_0}^{-l} Y_i, R) \geq m$ for all $l \geq 0$ and $i \in \mathbb{N}$,

(2) $\tau_{\Lambda}^{-t} Y_i = \tau_{\Lambda_0}^{-t} Y_i$ for all $t \geq 0$ and $i \in \mathbb{N}$,

(see [35, Theorem 6.3]). Let $\mathscr{T}$ be a stable tube in $\Gamma(\operatorname{mod} \Lambda)$. Then there exists a unique quasi-simple module X in $\mathscr{T}$ with $\operatorname{Hom}_{\Lambda}(P_\omega, X) \neq 0$. Thus X admits a minimal projective resolution of the form

$$0 \to P_X^{(0)} \to P_\omega \to X \to 0,$$

where $P_X^{(0)}$ is a projective Λ_0-module. Moreover, we have the canonical short exact sequence

$$0 \to R \to P_\omega \to S_\omega \to 0.$$

Let $t \geq 0$ and $i \in \mathbb{N}$. Applying the functor $\operatorname{Hom}_{\Lambda}(\tau_{\Lambda}^{-t} Y_i, -)$ to the above short exact sequences we obtain exact sequences

$$0 = \operatorname{Hom}_{\Lambda}\left(\tau_{\Lambda}^{-t} Y_i, P_X^{(0)}\right) \to \operatorname{Hom}_{\Lambda}\left(\tau_{\Lambda}^{-t} Y_i, P_\omega\right) \to \operatorname{Hom}_{\Lambda}\left(\tau_{\Lambda}^{-t} Y_i, X\right),$$

$$0 \to \operatorname{Hom}_{\Lambda}\left(\tau_{\Lambda}^{-t} Y_i, R\right) \to \operatorname{Hom}_{\Lambda}\left(\tau_{\Lambda}^{-t} Y_i, P_\omega\right),$$

and consequently $\dim_K \operatorname{Hom}_{\Lambda}(\tau_{\Lambda}^{-t} Y_i, X) \geq m$.

Let $p = \text{l.c.m.}(p_1, \ldots, p_t)$ and let $t \geq p$. As in the proof of Proposition 7.1 we infer that $\dim_K \operatorname{Hom}_{\Lambda}(\tau_{\Lambda}^{-t} Y_i, E) \geq m$ for all quasi-simple modules of all stable tubes in $\Gamma(\operatorname{mod} \Lambda)$. Since $\operatorname{mod} \Lambda_0 = \mathscr{O}_{\mathbb{X}}(\vec{c})^{\perp}$, we may consider the modules $Y_i' = \tau_{\Lambda_0}^{-p} Y_i$, $i \in \mathbb{N}$, as quasi-simple objects of the category $\operatorname{vect} \mathbb{X}$. Then $\dim_K \operatorname{Hom}_{\mathscr{H}}(Y_i', E) \geq m$ for all quasi-simple objects E in $\operatorname{coh}_0 \mathbb{X}$ and $i \in \mathbb{N}$. Moreover, $Y_i^{\perp} \cong C_i$ for a connected wild hereditary algebra C_i, $i \in \mathbb{N}$ (see [44, Proposition 6.8]). We shall identify $Y_i^{\perp}$ with $\operatorname{mod} C_i$. Fix $i \in \mathbb{N}$. Consider the Auslander–Reiten sequence $0 \to \tau_{\mathbb{X}} Y_i \to M_i \to Y_i' \to 0$ in $\mathscr{H} = \operatorname{coh} \mathbb{X}$. Then $Y_i' \oplus C_i$ is a tilting vector bundle with $\operatorname{End}_{\mathscr{H}}(Y_i \oplus C_i) \cong C_i[M_i]$. From Proposition 7.1 there exists a positive integer r_i such that the algebra $C_i[\tau_{C_i}^{r_i} M_i] \cong \operatorname{End}_{\mathscr{H}}(Y_i' \oplus \tau_{C_i}^{-r_i} C_i)$ is a concealed canonical algebra such that $\dim_K \operatorname{Hom}_{C_i}(P, N) \geq m$ for all indecomposable modules N in the $\mathbb{P}_1(K)$-family $\mathscr{C}_i$ of stable tubes in $\Gamma(\operatorname{mod} C_i[\tau_{C_i}^{r_i} M_i])$ and all indecomposable projective C_i-modules P. Moreover, it follows from the construction of Y_i' that for the indecomposable projective $C_i[\tau_{C_i}^{r_i} M_i]$-module P' with $\operatorname{rad} P' = \tau_{C_i}^{r_i} M_i$, the same property holds. We have from [14] the following exact sequence of Hochschild cohomologies induced by the one-point extension $C_i[M_i]$

$$0 = \operatorname{End}_{C_i}(M_i)/K \to H^1(C_i[M_i]) \to H^1(C_i)$$

$$\to \operatorname{Ext}^1_{C_i}(M_i, M_i) \to H^2(C_i[M_i]) \to H^2(C_i) = 0.$$

Considering the corresponding exact sequence for the one-point extension $\Lambda = \Lambda_0[R]$ we obtain $H^1(\Lambda) = 0$. Since $H^1(C_i[M_i]) \cong H^1(\Lambda) = 0$ and

$H^2(C_i[M_i]) = H^2(\Lambda)$, we obtain the equality

$$\dim_K H^1(C_i) = \dim_K \mathrm{Ext}^1_{C_i}(M_i, M_i) - \dim_K H^2(\Lambda).$$

Moreover, it follows from Lemma 7.2 that

$$\begin{aligned} \dim_K \mathrm{Ext}^1_{C_i}(M_i, M_i) &= \dim \mathrm{Hom}_{\mathscr{H}}\left(Y_i', \tau_{\mathbb{X}}^2 Y_i'\right) \\ &\geq \dim_K \mathrm{Hom}_{\Lambda_0}\left(Y_i', \tau_{\Lambda_0}^2 Y_i'\right) = \dim_K \mathrm{Hom}_{\Lambda_0}\left(Y_i, \tau_{\Lambda_0}^2 Y_i\right). \end{aligned}$$

Since $\dim_K \mathrm{Hom}_{\Lambda_0}(Y_i, \tau_{\Lambda_0}^2 Y_i) \neq \dim_K \mathrm{Hom}_{\Lambda_0}(Y_j, \tau_{\Lambda_0}^2 Y_j)$ for $i \neq j$, we conclude that there exists an infinite sequence (i_n), $n \in \mathbb{N}$, such that $\dim_K H^1(C_{i_r}) \neq \dim_K H^1(C_{i_s})$ for all $r \neq s$. Therefore the algebras C_{i_n}, $n \in \mathbb{N}$, are pairwise nonisomorphic. This finishes the proof.

8. PASSING FROM NONSTABLE TO STABLE TUBES

Let H be a connected wild hereditary algebra and let M be a quasi-simple regular H-module such that $H[M]$ is a quasi-tilted algebra of canonical type $\Lambda = \Lambda(p, \lambda)$. In general, the $\mathbb{P}_1(K)$-family of tubes of $\Gamma(\mathrm{mod}\, H[M])$ may contain injective modules (but clearly does not contain projective modules). We know from Proposition 7.1 that $H[\tau_H^i M]$, for $i \gg 0$, are concealed canonical algebras of type Λ whose $\mathbb{P}_1(K)$-families of stable tubes do not contain simple modules. If A is a quasitilted algebra of canonical type and the $\mathbb{P}_1(K)$-family of tubes in $\Gamma(\mathrm{mod}\, A)$ contains at least one injective module, applying [37, Theorem 3.4], we conclude that there exists a concealed canonical algebra C such that A is a tubular coextension $[K_r, E_r] \cdots [K_1, E_1]C$ of C, where $E_1, \dots, E_r$, $r \geq 1$, are pairwise nonisomorphic quasi-simple C-modules in the $\mathbb{P}_1(K)$-family of stable tubes of $\Gamma(\mathrm{mod}\, C)$ and $K_1, \dots, K_r$ are branches (see [49, (4.4) and (4.7)]). We shall show that in this case by applying suitable reflections to H we obtain a concealed canonical algebra whose $\mathbb{P}_1(K)$-family of stable tubes contains at least r simple modules. Clearly, from this statement and its proof, the second part (canonical case) of Theorem 5 follows. This is again related with the problem of distribution of simple modules and projective modules in the Auslander–Reiten components of selfinjective algebras of canonical type (see [39]). One could follow the idea of the corresponding proof of Proposition 6.1 and show that there exists a concealed canonical algebra whose $\mathbb{P}_1(K)$-family of stable tubes contains at least one simple module. Our intention is to describe the transformation from nonstable tubes to stable tubes more explicitely and prove the following stronger result.

PROPOSITION 8.1. *Let* $H[M] = [K_r, E_r] \cdots [K_1, E_1]C$ *with* $r \geq 1$ *be a tubular coextension of a concealed canonical algebra* C. *Then there exists a preprojective tilting* H*-module* Q *with* $H' = \operatorname{End}_H(Q)$ *hereditary such that* $H'[\operatorname{Hom}_H(Q, M)]$ *is a concealed canonical algebra and the* $\mathbb{P}_1(K)$*-family of stable tubes of* $\Gamma(H'[\operatorname{Hom}_H(Q, M)])$ *contains at least r simple modules.*

Proof. Let $A = H[M]$ and let H be the path algebra $K\Delta$ of the wild quiver Δ. Then $A \cong K\Omega/J$, where the quiver Ω is obtained from Δ by adding one (extension) vertex ω, being a source of Ω, and J is an ideal in $K\Omega$ generated by linear combinations of paths in Ω having ω as their source. For each vertex a of Ω, we denote by $S(a)$ the simple A-module associated to a and by $P(a)$ and $I(a)$ the projective cover and injective envelope of $S(a)$ in $\operatorname{mod} A$, respectively. Moreover, it follows from the above description of $A = K\Omega/J = [K_r, E_r] \cdots [K_1, E_1]C$ that there are pairwise disjoint convex linear subquivers $\Delta_1, \dots, \Delta_r$ of Δ such that $K_1 = K\Delta_1, \dots, K_r = K\Delta_r$. For each $i \in \{1, \dots, r\}$, denote by ω_i the coextension vertex of $[E_i]C$, being also one of the ends of the linear quiver Δ_i, and denote by m_i the number of vertices of Δ_i. Clearly, then E_i is a direct summand of $I(\omega_i)/S(\omega_i)$, for each $i \in \{1, \dots, r\}$. Since $A = H[M] = [K_r, E_r] \cdots [K_1, E_1]C$ the preprojective component $\mathscr{P}(H) = \mathscr{P}(A)$ of $\Gamma(\operatorname{mod} A)$ contains sectional paths

$$\Sigma_i\colon P(\omega_i) = Z_1^{(i)} \to Z_2^{(i)} \to \cdots \to Z_{m_i}^{(i)}, \qquad 1 \leq i \leq r,$$

such that, for each $i \in \{1, \dots, r\}$, Σ_i consists of the representatives of the τ_H-orbits of the indecomposable projective modules associated to all vertices of Δ_i. Then the quivers $\tau_H^- \Sigma_i$, $1 \leq i \leq r$, together with the projective modules $P(a)$, $a \in \Delta \setminus (\Delta_1 \cup \cdots \cup \Delta_r)$, form a (complete) section Σ of $\mathscr{P}(H)$. Denote by Q the direct sum of all modules lying on Σ. Clearly, then $H' = \operatorname{End}_H(Q)$ is a hereditary algebra of wild type Σ^{op}. Moreover, $T = Q \oplus P(\omega)$ is a tilting A-module such that $B = \operatorname{End}_A(T) \cong H'[\operatorname{Hom}_A(Q, P(\omega))] = H'[\operatorname{Hom}_H(Q, M)]$. We claim that B is a concealed canonical algebra (clearly of type $\Lambda = \Lambda(p, \lambda)$) and the $\mathbb{P}_1(K)$-family of stable tubes in $\Gamma(\operatorname{mod} B)$ contains at least r simple modules. For each $i \in \{1, \dots, r\}$, denote by $\mathscr{C}_i$ the full translation subquiver of $\mathscr{P}(H)$ formed by all predecessors of Σ_i in $\mathscr{P}(H)$, equivalently in $\Gamma(\operatorname{mod} A)$. Observe that every indecomposable module X in $\mathscr{C}_i$ is a module over $K\Delta_i$, and there is a monomorphism $X \to (\oplus_{j=1}^{m_i} Z_j^{(i)})^s$, for some positive integers s, and consequently X is cogenerated by $\oplus_{j=1}^{m_i} Z_j^{(i)}$. Moreover, every morphism from the indecomposable projective module $P(a)$ in $\mathscr{C}_i$ (equivalently, with a being a vertex of Δ_i) to an indecomposable successor Y of Σ factorizes through the module Q^m, for some positive integer m. This shows that the tilting A-module T is separating; that is, every indecomposable A-module belongs either to the torsion-free part

$$\mathscr{F}(T) = \{M \in \operatorname{mod} A \mid \operatorname{Hom}_A(T, M) = 0\} = \operatorname{Cogen} \tau_A T$$
$$= \operatorname{Cogen}\left(\bigoplus_{i=1}^{r} \bigoplus_{j=1}^{m_i} Z_j^{(i)} \right)$$

or to the torsion part

$$\mathscr{T}(T) = \{M \in \operatorname{mod} A \mid \operatorname{Ext}_A^1(T, M) = 0\} = \operatorname{Gen}(T).$$

Note that $\mathscr{F}(T)$ is the additive category of $\mathscr{C}_1 \cup \cdots \cup \mathscr{C}_r$. Further, it follows from the Brenner–Butler theorem that the functor $\operatorname{Hom}_A(T, -)$ induces an equivalence of $\mathscr{T}(T)$ with the torsion-free part

$$\mathscr{Y}(T) = \{N \in \operatorname{mod} B \mid \operatorname{Tor}_1^B(T, N) = 0\}$$

of mod B and the functor $\operatorname{Ext}_A^1(T, -)$ induces an equivalence of $\mathscr{F}(T)$ with the torsion part

$$\mathscr{X}(T) = \{N \in \operatorname{mod} B \mid T \otimes_B N = 0\}$$

of mod B. For each module N in mod B we have also the canonical exact sequence

$$0 \to tN \to N \to N/tN \to 0,$$

where tN is the largest torsion submodule of N with respect to $(\mathscr{X}(T), \mathscr{Y}(T))$ and clearly N/tN is torsion-free. In particular, every simple B-module belongs to $\mathscr{X}(T)$ or $\mathscr{Y}(T)$. Observe also that, for each vertex $a \in \Delta_1 \cup \cdots \cup \Delta_r$, the projective module $P(a)$ is not a direct summand of T and therefore (see [18, Corollary 6.3]) we have in mod B a connecting Auslander–Reiten sequence of the form

$$0 \to \operatorname{Hom}_A(T, I(a)) \to M(a) \to \operatorname{Ext}_A^1(T, P(a)) \to 0,$$

and the canonical sequence $0 \to tM(a) \to M(a) \to M(a)/tM(a) \to 0$ of $M(a)$ is of the form

$$0 \to \operatorname{Ext}_A^1(T, \operatorname{rad} P(a)) \to M(a) \to \operatorname{Hom}_A(T, I(a)/S(a)) \to 0.$$

Fix $i \in \{1, \dots, r\}$. Let $\mathscr{T}$ be the tube of $\Gamma(\operatorname{mod} A)$ containing the injective modules $I(a)$, $a \in \Delta_i$. Then $\mathscr{T}$ admits a maximal infinite sectional path

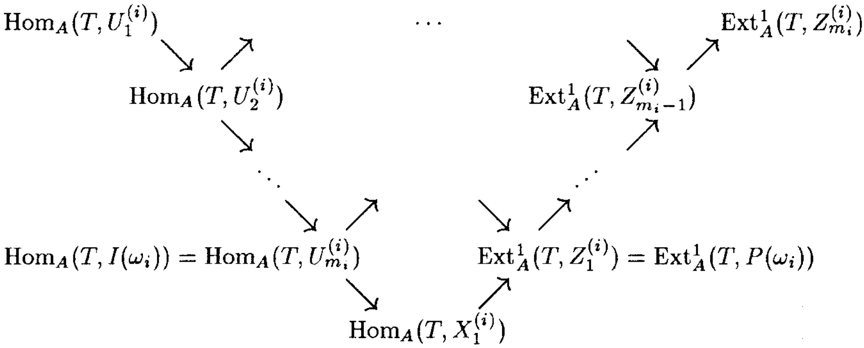
$$U_1^{(i)} \to U_2^{(i)} \to \cdots \to U_{m_i}^{(i)} \to X_1^{(i)} \to X_2^{(i)} \to X_3^{(i)} \cdots,$$

where $U_{m_i}^{(i)} = I(\omega_i)$, $U_1^{(i)}, \dots, U_{m_i}^{(i)}$ are representatives of the τ_A-orbits of the injective modules $I(a)$, $a \in \Delta_i$, $U_1^{(i)} \to U_2^{(i)}$ is the unique arrow in $\mathscr{T}$

starting at $U_1^{(i)}$, and there exists an Auslander–Reiten sequence in mod A of the form

$$0 \to X_1^{(i)} \to X_2^{(i)} \to Y_1^{(i)} \to 0.$$

Denote by $\mathscr{D}_i$ the full translation subquiver of $\mathscr{T}$ given by all indecomposable factor modules of the modules $U_1^{(i)}, \dots, U_{m_i}^{(i)} = I(\omega_i)$, $X_1^{(i)}$ lying in $\mathscr{T}$. Clearly, $\mathscr{D}_i$ is a finite translation subquiver of $\mathscr{T}$ consisting of modules from $\mathscr{T}(T)$, because the modules of the $\mathbb{P}_1(K)$-family of tubes in $\Gamma(\text{mod } A)$ are not predecessors of preprojective modules and $P(\omega)$. We claim that $\Gamma(\text{mod } B)$ contains a full translation subquiver $\mathscr{E}_i$ of the form



obtained by glueing the image $\text{Hom}_A(T, \mathscr{D}_i)$ of $\mathscr{D}_i$ by the functor $\text{Hom}_A(T, -)$ and the image $\text{Ext}_A^1(T, \mathscr{C}_i)$ of $\mathscr{C}_i$ by the functor $\text{Ext}_A^1(T, -)$, via the connecting Auslander–Reiten sequences

$$0 \to \text{Hom}_A(T, I(a)) \to M(a) \to \text{Ext}_A^1(T, P(a)) \to 0$$

with $M(a) \cong \text{Hom}_A(T, I(a)/S(a)) \oplus \text{Ext}_A^1(T, \text{rad } \mathscr{P}(a))$, $a \in \Delta_i$. Assume first that Δ_i consists only of the vertex ω_i or equivalently that $m_i = 1$. Then $P(\omega_i) = S(\omega_i)$, rad $P(\omega_i) = 0$, $I(\omega_i)/S(\omega_i) = X_1^{(i)}$, and hence we have the connecting Auslander–Reiten sequence of the form

$$0 \to \text{Hom}_A(T, I(\omega_i)) \to \text{Hom}_A\big(T, X_1^{(i)}\big) \to \text{Ext}_A^1(T, P(\omega_i)) \to 0.$$

Assume now that $m_i > 1$, and let b_i be the end of the linear quiver Δ_i different from the coextension vertex ω_i. If b_i is a sink of Δ_i, then again $P(b_i) = S(b_i)$, $I(b_i)/S(b_i)$ is indecomposable, and we have the connecting

Auslander–Reiten sequence of the form

$$0 \to \mathrm{Hom}_A(T, I(b_i)) \to \mathrm{Hom}_A(T, I(b_i)/S(b_i)) \to \mathrm{Ext}_A^1(T, P(b_i)) \to 0.$$

Finally, assume that b_i is a source of Δ_i. Then rad $P(b_i)$ is indecomposable, $I(b_i) = S(b_i)$, $I(b_i)/S(b_i) = 0$, and hence we have the connecting Auslander–Reiten sequence of the form

$$0 \to \mathrm{Hom}_A(T, I(b_i)) \to \mathrm{Ext}_A^1(T, \mathrm{rad}\, P(b_i)) \to \mathrm{Ext}_A^1(T, P(b_i)) \to 0.$$

Passing from the end b_i to the end ω_i along the quiver Δ_i we now conclude that the middle terms $M(a)$ of the connecting Auslander–Reiten sequences

$$0 \to \mathrm{Hom}_A(T, I(a)) \to M(a) \to \mathrm{Ext}_A^1(T, P(a)) \to 0$$

are isomorphic to $\mathrm{Hom}_A(T, I(a)/S(a)) \oplus \mathrm{Ext}_A^1(T, \mathrm{rad}\, P(a))$, for all vertices a of Δ_i. Invoking now the fact that $\mathscr{X}(T)$ is closed under factor modules, we deduce that $\mathrm{Ext}_A^1(T, \mathscr{C}_i)$ is a full translation subquiver of $\Gamma(\mathrm{mod}\, B)$. Similarly, using the fact that $\mathscr{Y}(T)$ is closed under submodules, we infer that also $\mathrm{Hom}_A(T, \mathscr{D}_i)$ is a full translation subquiver of $\Gamma(\mathrm{mod}\, B)$. Therefore, $\mathscr{E}_i$ is a full translation subquiver of $\Gamma(\mathrm{mod}\, B)$. Moreover, observe that $\mathrm{Ext}_A^1(T, Z_{m_i}^{(i)})$ is a simple B-module. Indeed, since $\mathrm{Ext}_A^1(T, Z_{m_i}^{(i)})$ belongs to $\mathscr{X}(T)$, every factor module of $\mathrm{Ext}_A^1(T, Z^{(i)})$ also belongs to $\mathscr{X}(T)$ and hence is of the form $\mathrm{Ext}_A^1(T, M)$ for some module M from $\mathscr{F}(T)$. On the other hand, $Z_{m_i}^{(i)}$ is a sink of the translation quiver $\mathscr{C}_i$, $\mathrm{Hom}_A(\mathscr{C}_i, \mathscr{C}_j) = 0$ for all $i \neq j$ from $\{1, \ldots, r\}$, and consequently, applying the Brenner–Butler theorem, we obtain

$$\mathrm{Hom}_B\left(\mathrm{Ext}_A^1\left(T, Z_{m_i}^{(i)}\right), \mathrm{Ext}_A^1(T, M)\right) \cong \mathrm{Hom}_A\left(Z_{m_i}^{(i)}, M\right) = 0$$

for any indecomposable A-module M from $\mathscr{F}(T)$ nonisomorphic to $Z_{m_i}^{(i)}$. Clearly, this shows that $\mathrm{Ext}_A^1(T, Z_{m_i}^{(i)})$ is a simple B-module.

Our next aim is to show that the simple B-module $\mathrm{Ext}_A^1(T, Z_{m_i}^{(i)})$ is not injective. It is enough to show that $\mathrm{Ext}_B^1(\mathrm{Hom}_A(T, Y_1^{(i)}), \mathrm{Ext}_A^1(T, Z_{m_i}^{(i)})) \neq 0$. We have

$$\mathrm{Ext}_B^1\left(\mathrm{Hom}_A\left(T, Y_1^{(i)}\right)\right), \mathrm{Ext}_A^1\left(T, Z_{m_i}^{(i)}\right) \cong \mathrm{Ext}_A^2\left(Y_1^{(i)}, Z_{m_i}^{(i)}\right),$$

by Hoshino's formula [20]. Moreover, since $Z_{m_i}^{(i)}$ lies on the end of the sectional path starting at $Z_1^{(i)} = P(\omega_i)$ and ω_i is the end of the linear quiver Δ_i, we deduce that there exists an exact sequence

$$0 \to Z_{m_i}^{(i)} \to I(\omega_i) \to X_1^{(i)} \to 0.$$

Hence, we obtain $\operatorname{Ext}_A^2(Y_1^{(i)}, Z_{m_i}^{(i)}) \cong \operatorname{Ext}_A^1(Y_1^{(i)}, X_1^{(i)}) \neq 0$, and this implies our claim. In particular, we obtain an Auslander–Reiten sequence

$$0 \to \operatorname{Ext}_A^1\left(T, Z_{m_i}^{(i)}\right) \to W_{1, m_i}^{(i)} \to \tau_B^{-} \operatorname{Ext}_A^1\left(T, Z_{m_i}^{(i)}\right) \to 0$$

in mod B, with $W_{1, m_i}^{(i)}$ indecomposable. Since the sectional path

$$\operatorname{Hom}_A\left(T, X_1^{(i)}\right) \to \operatorname{Ext}_A^1(T, Z^{(i)}) \to \cdots \to \operatorname{Ext}_A^1\left(T, Z_{m_i-1}^{(i)}\right) \to \operatorname{Ext}_A^1\left(T, Z_{m_i}^{(i)}\right)$$

is given by irreducible epimorphisms, we then conclude that $\operatorname{Hom}_A(T, X_1^{(i)})$ and $\operatorname{Ext}_A^1(T, Z_j^{(i)})$, $1 \le j \le m_i$, are noninjective. Moreover, since the branches $K_1, \dots, K_r$ are hereditary algebras, we know that all modules $X_q^{(i)}$, $q \ge 1$, are noninjective, and consequently we have also in $\Gamma(\operatorname{mod} A)$ an infinite sectional path

$$Y_1^{(i)} \to Y_2^{(i)} \to \cdots Y_q^{(i)} \to Y_{q+1}^{(i)} \to \cdots$$

with $Y_q^{(i)} = \tau_A^{-} X_q^{(i)}$ for any $q \ge 1$. Clearly, the modules $Y_q^{(i)}$, $q \ge 1$, belong to $\mathcal{T}(T)$. We shall prove now that there are in $\Gamma(\operatorname{mod} B)$ infinite sectional paths

$$\operatorname{Ext}_A^1\left(T, Z_j^{(i)}\right) = W_{0,j}^{(i)} \to W_{1,j}^{(i)} \to \cdots \to W_{s,j}^{(i)} \to W_{s+1,j}^{(i)} \to \cdots,$$

$s \ge 1$, $1 \le j \le m_i$, and Auslander–Reiten sequences

$$0 \to \operatorname{Hom}_A\left(T, X_s^{(i)}\right) \to \operatorname{Hom}_A\left(T, X_{s+1}^{(i)}\right) \oplus W_{s-1,1}^{(i)} \to W_{s,1}^{(i)} \to 0, \qquad s \ge 1,$$

$$0 \to W_{s-1,j}^{(i)} \to W_{s,j}^{(i)} \oplus W_{s-1,j+1}^{(i)} \to W_{s,j+1}^{(i)} \to 0, \qquad s \ge 1, 1 \le j < m_i,$$

$$0 \to W_{s,m_i}^{(i)} \to W_{s+1,m_i}^{(i)} \oplus \operatorname{Hom}_A\left(T, Y_s^{(i)}\right) \to \operatorname{Hom}_A\left(T, Y_{s+1}^{(i)}\right) \to 0, \qquad s \ge 1.$$

Clearly, then we have $\tau_B^{-} \operatorname{Ext}_A^1(T, Z_{m_i}^{(i)}) = \tau_B^{-} W_{0,m_i}^{(i)} = \operatorname{Hom}_A(T, Y_1^{(i)})$. We proceed by induction on s. Assume that we have defined indecomposable noninjective modules $W_{p,j}^{(i)}$, $0 \le p \le s-1$, $1 \le j \le m_i$, for some $s \ge 1$, satisfying the required conditions. Observe that the module $\operatorname{Hom}_A(T, X_s^{(i)})$ is noninjective in mod B, because we have in $\mathcal{Y}(T)$ a nonsplittable short exact sequence

$$0 \to \operatorname{Hom}_A\big(T, X_s^{(i)}\big) \to \operatorname{Hom}_A\big(T, X_{s+1}^{(i)}\big) \oplus \operatorname{Hom}_A\big(T, Y_{s-1}^{(i)}\big) \to \operatorname{Hom}_A\big(T, Y_s^{(i)}\big) \to 0,$$

where $Y_0^{(i)} = 0$. Consider the indecomposable B-modules

$$W_{s,1}^{(i)} = \tau_B^- \operatorname{Hom}_A\big(T, X_{s-1}^{(i)}\big) \quad \text{and} \quad W_{s,j+1}^{(i)} = \tau_B^- W_{s-1,j}^{(i)}, \quad 1 \le j \le m_i.$$

We have then Auslander–Reiten sequences

$$0 \to W_{s-1,j}^{(i)} \to W_{s,j}^{(i)} \oplus W_{s-1,j}^{(i)} \to W_{s,j+1}^{(i)} \to 0, \quad 1 \le j < m_i,$$

$$0 \to W_{s-1,m_i}^{(i)} \to W_{s,m_i}^{(i)} \oplus \operatorname{Hom}_A\big(T, Y_{s-1}^{(i)}\big) \to W_{s,m_i+1}^{(i)} \to 0,$$

$$0 \to \operatorname{Hom}_A\big(T, X_{s-1}^{(i)}\big) \to W_{s-1,1}^{(i)} \oplus E \to W_{s,1}^{(i)} \to 0,$$

for some B-module E. We have a sectional path of irreducible morphisms

$$\operatorname{Ext}_A^1\big(T, Z_1^{(i)}\big) = W_{0,1}^{(i)} \to \cdots \to W_{s-1,1}^{(i)} \to W_{s,1}^{(i)}$$

with nonzero composition, and hence $tW_{s,1}^{(i)} \neq 0$. Consider the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Hom}_A(T, X_{s-1}^{(i)}) & \longrightarrow & W_{s-1,1}^{(i)} \oplus E & \longrightarrow & W_{s,1}^{(i)} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{Hom}_A(T, X_{s-1}^{(i)}) & \longrightarrow & (W_{s-1,1}^{(i)}/tW_{s-1,1}^{(i)}) \oplus (E/tE) & \longrightarrow & W_{s,1}^{(i)}/tW_{s,1}^{(i)} & \longrightarrow & 0 \end{array}$$

Observe that the lower exact sequence is an Auslander–Reiten sequence in $\mathscr{Y}(T)$, and hence we obtain $W_{s,1}^{(i)}/tW_{s,1}^{(i)} \cong \operatorname{Hom}_A(T, Y_s^{(i)})$ and $W_{s-1,1}^{(i)}/tW_{s-1,1}^{(i)} \oplus E/tE \cong \operatorname{Hom}_A(T, Y_{s-1}^{(i)}) \oplus \operatorname{Hom}_A(T, X_s^{(i)})$. Moreover, from our inductive assumption, we have an exact sequence

$$0 \to \operatorname{Ext}_A^1\big(T, Z_1^{(i)}\big) \to W_{s-1,1}^{(i)} \to \operatorname{Hom}_A\big(T, Y_{s-1}^{(i)}\big) \to 0,$$

and hence $W_{s-1,1}^{(i)}/tW_{s-1,1}^{(i)} \cong \operatorname{Hom}_A(T, Y_{s-1}^{(i)})$. Obviously, then we have $E/tE \cong \operatorname{Hom}_A(T, X_s^{(i)})$. We claim that $tE = 0$. Observe that $tW_{s,1}^{(i)} \cong \operatorname{Ext}_A^1(T, Z_1^{(i)})$. Hence $tE = 0$ if and only if $tW_{s,1}^{(i)} \cong tW_{s-1,1}^{(i)} \cong \operatorname{Ext}_A^1(T, Z_1^{(i)})$. Let N be an indecomposable module from $\mathscr{C}_j$, for some $1 \le j \le r$. Since $\operatorname{Ext}_A^1(T, N)$ is a torsion B-module, we have $\operatorname{Hom}_B(\operatorname{Ext}_A^1(T, N), B_B) = 0$.

Then applying the Auslander–Reiten formula, we obtain isomorphisms of vector spaces

$$\begin{aligned}\mathrm{Hom}_B\big(\mathrm{Ext}^1_A(T,N),W^{(i)}_{s-1}\big) &= \mathrm{Hom}_B\big(\mathrm{Ext}^1_A(T,N),\tau_B^-\ \mathrm{Hom}_A\big(T,X^{(i)}_s\big)\big)\\ &= \underline{\mathrm{Hom}}_B\big(\mathrm{Ext}^1_A(T,N),\tau_B^-\ \mathrm{Hom}_A\big(T,X^{(i)}_s\big)\big)\\ &\cong D\,\mathrm{Ext}^1_B\big(\tau_B^-\ \mathrm{Hom}_A\big(T,X^{(i)}_s\big),\tau_B\ \mathrm{Ext}^1_A(T,N)\big)\\ &\cong \overline{\mathrm{Hom}}_B\big(\tau_B\ \mathrm{Ext}^1_A(T,N),\mathrm{Hom}_A\big(T,X^{(i)}_s\big)\big).\end{aligned}$$

Observe that $\tau_B\ \mathrm{Ext}^1_A(T,N)$ either belongs to the torsion part $\mathscr{X}(T)$ or is of the form $\mathrm{Hom}_A(T,I(a))$, for some vertex $a\in\Delta_j$. Moreover, invoking the Brenner–Butler theorem, for each vertex $a\in\Delta_j$, we have $\mathrm{Hom}_B(\mathrm{Hom}_A(T,I(a)),\mathrm{Hom}_A(T,X^{(i)}_s))\cong \mathrm{Hom}_A(I(a),X^{(i)}_s))$. Since the tubes in $\Gamma(\mathrm{mod}\,A)$ are standard and pairwise orthogonal we conclude that, if $\mathrm{Hom}_B(\mathrm{Ext}^1_B(T,N),W^{(i)}_{s,1})\neq 0$, then $j=i$, $N=P(a)$, $a\in\Delta_i$, and the injective modules $I(a)$ lies on the sectional path $U^{(i)}_1\to\cdots\to U^{(i)}_{m_i}=I(\omega_i)$. Moreover, in this case,

$$\dim_K \mathrm{Hom}_B\big(\mathrm{Ext}^1_A(T,P(a)),W^{(i)}_{s,1}\big)=1,$$
$$\dim_K \mathrm{Hom}_B\big(\mathrm{Ext}^1_A(T,P(a)),W^{(i)}_{s-1,1}\big)=1,$$

and consequently $tE=0$. In particular, we obtain $E\cong\mathrm{Hom}_A(T,X^{(i)}_s)$ and $\tau_B^-W^{(i)}_{s,m_i}\cong W^{(i)}_{s,1}/W^{(i)}_{0,1}=W^{(i)}_{s,1}/tW^{(i)}_{s,1}\cong\mathrm{Hom}_A(T,Y^{(i)}_s)$. Therefore, we have the required Auslander–Reiten sequences

$$0\to\mathrm{Hom}_A\big(T,X^{(i)}_{s-1}\big)\to W^{(i)}_{s-1,1}\oplus\mathrm{Hom}_A\big(T,X^{(i)}_s\big)\to W^{(i)}_{s,1}\to 0,$$
$$0\to W^{(i)}_{s-1,m_i}\to\mathrm{Hom}_A\big(T,Y^{(i)}_{s-1}\big)\oplus W^{(i)}_{s,m_i}\to\mathrm{Hom}_A\big(T,Y^{(i)}_s\big)\to 0.$$

Now let M be an indecomposable B-module such that $0\neq tM\neq M$, and let N be an indecomposable direct summand of M/tM. We claim that there exists $i\in\{1,\dots,r\}$ such that $\mathrm{Hom}_B(\mathrm{Hom}_A(T,Y^{(i)}_1),N)\neq 0$. Observe that tM belongs to the additive category add $(\oplus_{i=1}^r\ \oplus_{j=1}^{m_i}W^{(i)}_{0,j})$. Consider the commutative diagram

$$\begin{array}{ccccccccc}0&\longrightarrow&tM&\longrightarrow&M&\longrightarrow&M/tM&\longrightarrow&0\\ &&\|&&\downarrow&&\downarrow p&&\\ 0&\longrightarrow&tM&\longrightarrow&M'&\longrightarrow&N&\longrightarrow&0\end{array}$$

induced by the canonical projection $p\colon M/tM\to N$. Since M is an indecomposable B-module, the lower exact sequence is not splittable, and hence $\mathrm{Ext}^1_B(N,tM)\neq 0$. Thus there are $i\in\{1,\dots,r\}$, $j\in\{1,\dots,m_i\}$, and a nonsplittable short exact sequence

$$0 \to W_{0,j}^{(i)} \to L \to N \to 0$$

in mod B. We then obtain a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W_{0,j}^{(i)} & \longrightarrow & W_{1,j}^{(i)} & \longrightarrow & \mathrm{Hom}_A(T, Y_1^{(i)}) & \longrightarrow & 0 \\ & & \| & & \downarrow f & & \downarrow g & & \\ 0 & \longrightarrow & W_{0,j}^{(i)} & \longrightarrow & L & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

Since $\operatorname{Ker} f = \operatorname{Ker} g$ and $\mathrm{Hom}_B(\mathrm{Hom}_A(T, Y_1^{(i)}), W_{1,j}^{(i)}) = 0$, we conclude that $g \neq 0$ and hence $\mathrm{Hom}_B(\mathrm{Hom}_A(T, Y_1^{(i)}), N) \neq 0$, as required. We also note that $N = \mathrm{Hom}_A(T, V)$ for some indecomposable A-module V from $\mathcal{T}(T)$, and then $\mathrm{Hom}_B(\mathrm{Hom}_A(T, Y_1^{(i)}), N) \cong \mathrm{Hom}_A(Y_1^{(i)}, V)$. This shows that if we have an Auslander–Reiten sequence

$$0 \to U \to V \to W \to 0$$

in $\mathcal{T}(T)$ such that $\mathrm{Hom}_A(Y_1^{(i)}, W) = 0$ for any $i \in \{1, \dots, r\}$, then

$$0 \to \mathrm{Hom}_A(T, U) \to \mathrm{Hom}_A(T, V) \to \mathrm{Hom}_A(T, W) \to 0$$

is an Auslander–Reiten sequence in mod B. Recall now that the Auslander–Reiten quiver $\Gamma(\mathrm{mod}\, A)$ of A is of the form

$$\Gamma(\mathrm{mod}\, A) = \Gamma_+(\mathrm{mod}\, A) \vee \Gamma_0(\mathrm{mod}\, A) \vee \Gamma_-(\mathrm{mod}\, A),$$

where $\Gamma_+(\mathrm{mod}\, A)$ consists of the preprojective component $\mathcal{P}(A) = \mathcal{P}(H)$ and infinitely many components with stable parts of type $\mathbb{Z}\mathbb{A}_\infty$, $\Gamma_-(\mathrm{mod}\, A)$ consists of a preinjective component $Q(A)$ and infinitely many components with stable parts of type $\mathbb{Z}\mathbb{A}_\infty$, and $\Gamma_0(\mathrm{mod}\, A)$ is a $\mathbb{P}_1(K)$-family of coray tubes (coray insertions of stable tubes) separating $\Gamma_+(\mathrm{mod}\, A)$ from $\Gamma_-(\mathrm{mod}\, A)$ (see [37, 43]). In particular, we have $\mathrm{Hom}_A(Y_1^{(i)}, L) = 0$ for any $i \in \{1, \dots, r\}$ and every indecomposable A-module L from $\Gamma_+(\mathrm{mod}\, A)$ or a stable tube of $\Gamma_0(\mathrm{mod}\, A)$. Then it follows from the above considerations that $\mathrm{Hom}_A(T, \mathcal{T}(T) \cap \mathcal{P}(A))$ is a preprojective component $\mathcal{P}(B)$ of $\Gamma(\mathrm{mod}\, B)$, and, for a component $\mathcal{C}$ in $\Gamma_+(\mathrm{mod}\, A)$ different from $\mathcal{P}(A)$ (respectively, stable tube of $\Gamma_0(\mathrm{mod}\, A)$), $\mathrm{Hom}_A(T, \mathcal{C})$ is a full component of $\Gamma(\mathrm{mod}\, A)$. Further, for each $i \in \{1, \dots, r\}$, there are $j \in \{1, \dots, r\}$ (possibly $i = j$) and $n(i, j) \geq 0$ such that $U_1^{(j)} = \tau_A^{-n(i,j)} Y_1^{(i)}$ and $\mathrm{Hom}_A(T, U_1^{(j)}) = \tau_B^{-n(i,j)} \mathrm{Hom}_A(T, Y_1^{(i)})$, and consequently the images $\mathrm{Hom}_A(T, \mathcal{T})$ of nonstable tubes $\mathcal{T}$ of $\Gamma_0(\mathrm{mod}\, A)$ via $\mathrm{Hom}_A(T, -)$ are modified to stable tubes of $\Gamma(\text{mod } B)$ by insertions of translation quivers $\text{Ext}_A^1(T, \mathscr{C}_i)$ and the rectangles $\mathscr{R}^{(i)}$ formed by the modules $W_{s,j}^{(i)}$, $1 \le i \le r$, $1 \le j \le m_i$. Therefore, $\Gamma(\text{mod } B)$ is of the form

$$\Gamma(\text{mod } B) = \Gamma_+(\text{mod } B) \vee \Gamma_0(\text{mod } B) \vee \Gamma_-(\text{mod } B),$$

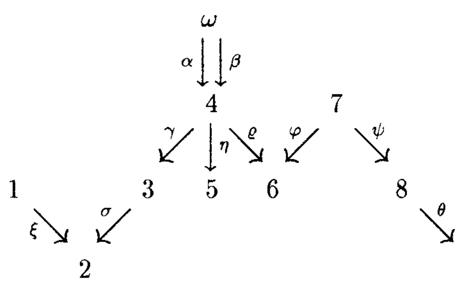
where $\Gamma_+(\text{mod } B)$ is the image of $\Gamma_+(\text{mod } A) \cap \mathscr{T}(T)$ via the functor $\text{Hom}_A(T, -)$ and $\Gamma_0(\text{mod } B)$ is a sincere $\mathbb{P}_1(K)$-family of stable tubes separating $\Gamma_+(\text{mod } B)$ from $\Gamma_-(\text{mod } B)$. In particular,

$$B = H'[\text{Hom}_H(Q, M)]$$

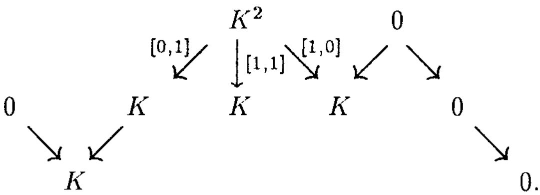
is a concealed canonical algebra of type $\Lambda = \Lambda(p, \lambda)$, and $\text{Hom}_H(Q, M)$ is a quasi-simple regular H'-module. Moreover, the family $\Gamma_0(\text{mod } B)$ of stable tubes contains r simple B-modules $\text{Hom}_A(T, Z_{m_1}^{(1)}), \dots, \text{Hom}_A(T, Z_{m_r}^{(r)})$. This finishes the proof.

We illustrate the above procedure by a concrete example.

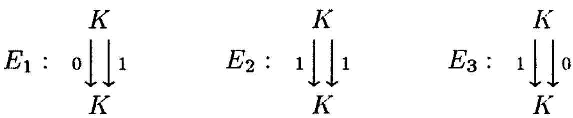
EXAMPLE 8.2. Let A be the bound quiver algebra $K\Omega/J$, where Ω is the quiver



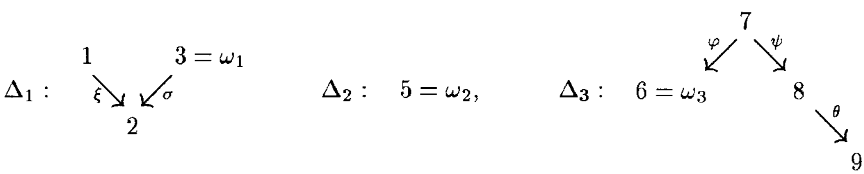
and J is the ideal in $K\Omega$ generated by $\gamma\alpha$, $\varrho\beta$, and $\eta\alpha - \eta\beta$. Denote by H the path algebra $K\Delta$ of the full subquiver Δ of Ω given by the vertices $1, \dots, 9$. Then H is a wild hereditary algebra and $A = H[M]$, where M is the indecomposable H-module of the form



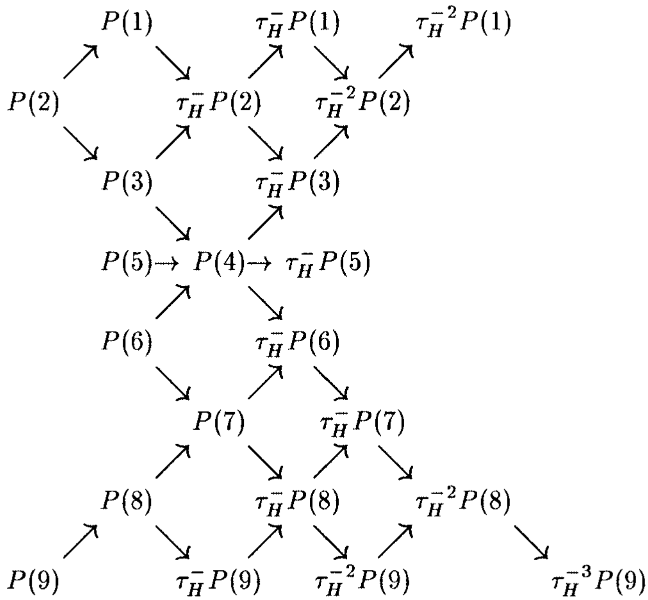
Moreover, A is a tubular coextension $[K_3, E_3][K_2, E_2][K_1, E_1]C$, where C is the path algebra of the Kronecker quiver given by the vertices ω and 4, E_1, E_2, and E_3 are the quasi-simple regular C-modules



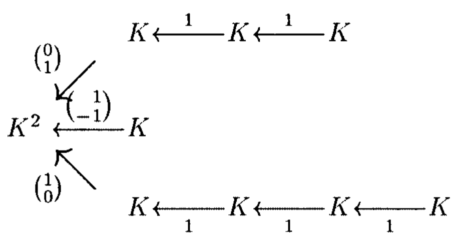
and K_1, K_2, and K_3 are the branches given by the quivers



respectively. Hence A is a quasitilted algebra of wild canonical type $(4, 2, 5)$ and thus is not tilted. In particular, M is a quasi-simple regular H-module. Further, the $\mathbb{P}_1(K)$-family of tubes in $\Gamma(\text{mod}\, A)$ has exactly three nonstable tubes, namely a tube $\mathcal{T}_1$ (with four corays) containing the injective modules $I(1)$, $I(2)$, and $I(3)$, a tube $\mathcal{T}_2$ (with two corays) containing the injective module $I(5)$, and a tube $\mathcal{T}_3$ (with five corays) containing the injective modules $I(6)$, $I(7)$, $I(8)$, and $I(9)$. A simple calculation shows that the preprojective component $\mathscr{P}(H)$ of $\Gamma(\operatorname{mod} H)$ admits a full translation subquiver of the form



and the section Σ of $\mathscr{P}(H)$, described in the proof of the Proposition 8.1, is given by the modules $\tau_H^{-2}P(1)$, $\tau_H^{-2}P(2)$, $\tau_H^{-}P(3)$, $P(4)$, $\tau_H^{-}P(5)$, $\tau_H^{-}P(6)$, $\tau_H^{-}P(7)$, $\tau_H^{-2}P(8)$, and $\tau_H^{-3}P(9)$. Hence, for Q being the direct sum of all modules on Σ, the algebra $H' = \operatorname{End}_H(Q)$ is a hereditary algebra of wild type Σ^{op}, and the H'-module $\operatorname{Hom}_H(Q, M)$ is isomorphic to the representation of Σ^{op} of the form



Therefore we have that the algebra $B = \mathrm{End}_A(T) = H'[\mathrm{Hom}_H(Q, M)]$, for $T = Q \oplus P(\omega)$, is the canonical algebra $\Lambda(p, \lambda)$ with $p = (4, 2, 5)$ and $\lambda = (\lambda_3) = (1)$. Obviously, we have in $\Gamma(\mathrm{mod}\, B)$ exactly three stable tubes with simple modules, obtained from the coray tubes $\mathscr{T}_1$, $\mathscr{T}_2$, and $\mathscr{T}_3$ by the completion procedure described in the proof of Proposition 8.1.

We shall exhibit now some examples of concealed canonical algebras $\Lambda = C[\tau_C M]$ such that C is a connected wild hereditary algebra, M is a quasi-simple regular C-module, and $C[M]$ is not quasitilted. Hence, we may have immediate jumps from the nonquasitilted algebras $C[M]$ to the concealed canonical algebras $C[\tau_C M]$.

EXAMPLE 8.3. Let $\Lambda = \Lambda(p, \lambda)$ be a wild canonical algebra with a weight sequence $p = (p_1, \dots, p_r)$, $r \geq 5$, and let $\Lambda = \Lambda_0[R]$ be the standard presentation of Λ as the one-point extension of the wild hereditary algebra Λ_0 and the quasi-simple regular Λ_0-module R. We shall prove that $\Lambda_0[\tau_{\Lambda_0}^- R]$ is not quasitilted. Hence, for $C = \Lambda_0$ and $M = \tau_{\Lambda_0}^- R$, $C[\tau_C M]$ is concealed canonical but $C[M]$ is not quasitilted. Denote by P_0 the unique simple projective Λ_0-module. Then we have $\dim_K \mathrm{Hom}_{\Lambda_0}(P_0, R) = 2$ and $\dim_K \mathrm{Hom}_{\Lambda_0}(P_0, \tau_{\Lambda_0} R) = r - 2$. Further, we obtain

$$\dim_K \mathrm{End}_{\Lambda_0}(R) - \dim_K \mathrm{Ext}^1_{\Lambda_0}(R, R) = q_{\Lambda_0}(\dim R) = 4 - r,$$

and hence

$$\dim_K \mathrm{Hom}_{\Lambda_0}\big(R, \tau_{\Lambda_0} R\big) = \dim_K \mathrm{Ext}^1_{\Lambda_0}(R, R) = r - 3,$$

because $\mathrm{End}_{\Lambda_0}(R) \cong K$. Hence, the minimal right add R-approximation of $\tau_{\Lambda_0} R$ is of the form $f\colon R^{r-3} \to \tau_{\Lambda_0} R$. Moreover, f is not injective because $r \geq 5$, and then $\dim_K \mathrm{Hom}_{\Lambda_0}(P_0, R^{r-3}) = 2(r-3) > r - 2 = \dim_K \mathrm{Hom}_{\Lambda_0}(P_0, \tau_{\Lambda_0} R)$. Applying now the arguments as in the final part of the proof of Proposition 6.3 we conclude that the kernel of the minimal right add $\tau_{\Lambda_0}^- R$-approximation $\tau_{\Lambda_0}^- f\colon (\tau_{\Lambda_0}^- R)^{r-3} \to R$ is not projective. Therefore, $\Lambda_0[\tau_{\Lambda_0}^- R]$ is not quasitilted, again by [17, (III.2.13) and (III.3.1)].

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