Nonlinear Boundary-Value Problems Arising in Chemical Reactor Theory*

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1. DYNAMICS OF CERTAIN CHEMICAL REACTORS

Chemical reactions are often housed in long tubes; the reactants are admitted at one end of the tube in a continuous stream, and the products of the reaction are withdrawn at the other end. In many important cases the tube wall is adiabatic so that the heat generated (or absorbed) during the reaction tends to increase (or decrease) the temperature of the reaction mixture. A basic problem in tubular reactor design can be stated: given the flow rate of reactants, the nature of the chemical reaction, and the usual chemical and thermodynamical constants of the process, what is the appropriate length of the reactor tube?

In order to attack this problem effectively it is important to develop a mathematical model of the process, following experience and intuition in the selection and analysis of the significant physical parameters. We shall assume that the problem is one-dimensional along the axis of the tube and that the two significant physical mechanisms are the forced convective flow and a diffusive mechanism superimposed on this flow. The convective flow is characterized by the average axial velocity along the reactor tube and the diffusive mechanism, based on Fick's Law, usually arises from Taylor diffusion, turbulent diffusion, or molecular diffusion. The first two of these diffusive mechanisms are important in industrial reactors while the third plays a role in flame theory.

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Both heat and mass are transported along the reactor tube by these mechanisms. It is known that for a single chemical reaction, $\sum_{i=1}^{n} a_{i} A_{i} = 0$, the appropriate dynamical equations of the process are [I]

$$\frac{\lambda}{d^2 T}{ds^2} - \rho v c_p \frac{dT}{ds} + (-\Delta H_i) f_j = 0,$$

$$D_j \frac{d^2 C_j}{ds^2} - \nu \frac{dC_j}{ds} + f_j = 0, \quad j = 1, \ldots, n.$$ 

Here $T$ is the local temperature at the distance $s$ along the tube of length $\ell$, so $0 \leq s \leq \ell$, $C_j$ is the concentration of the chemical species $A_j$, and $f_j$ is the rate of the chemical production of this species, where $a_j$ is assumed to be positive for a product and negative for a reactant. Thus $T(s)$ and $C_j(s)$ are the unknown functions and $f_j(T, C_1, \ldots, C_n)$ are given nonlinear functions. The other quantities $\lambda, \rho, \nu, c_p, \Delta H_j, D_j$ are the known physical constants of heat diffusion, density, velocity, specific heat, heat of reaction, and mass diffusion.

The boundary conditions for the chemical reaction process are

$$\rho v c_p (T_0 - T) = -\lambda \frac{dT}{ds} \quad \text{at} \quad s = 0$$

$$\nu (C_{j0} - C_j) = -D_j \frac{dC_j}{ds} \quad \text{at} \quad s = 0$$

$$\frac{dC_j}{ds} = 0 = \frac{dT}{ds} \quad \text{at} \quad s = \ell,$$

where $T_0$ and $C_{j0}$ are the constant inlet values of the corresponding physical quantities.

For the case of a single chemical reaction we have

$$f' = f_j / a_i = f_i / a_i$$

and

$$\Delta H = \Delta H_i a_i = \Delta H_i a_i,$$

where $a_i$ is the stoichiometric coefficient of the species $A_i$ and $\Delta H$ is negative for an exothermic reaction. If we also assume that the diffusive mechanisms for each chemical species and heat are the same, we then obtain

$$\lambda = \rho c_p D_j = \rho c_p D.$$
Next introduce the dimensionless variables

\[ y_j = \frac{C_j}{C_{j0}}, \quad y = \frac{T}{T_0} \]

\[ F = -\frac{\Delta H c_p L}{c_p \rho D T_0}, \quad \sigma = \frac{y}{\ell} \]

\[ \alpha_j = \frac{\alpha_j c_p T_0}{-\Delta H C_{j0}}, \quad P_e = \frac{\ell \rho c_v}{\lambda} = \frac{\ell v}{D} \]

to obtain the dynamical equations on \( 0 < \sigma < 1 \),

\[ \frac{d^2 y_j}{d\sigma^2} - P_e \frac{dy_j}{d\sigma} + F = 0 \]

\[ \frac{d^2 y_j}{d\sigma^2} - P_e \frac{dy_j}{d\sigma} + \alpha_j F = 0 \quad j = 1, \ldots, n \]

with the boundary conditions

\[ \frac{dy}{d\sigma} = P_e (y - 1) \quad \text{at} \quad \sigma = 0 \]

\[ \frac{dy_j}{d\sigma} = P_e (y_j - 1) \quad \text{at} \quad \sigma = 0, j = 1, \ldots, n \]

\[ \frac{dy}{d\sigma} = 0 = \frac{dy_j}{d\sigma} \quad \text{at} \quad \sigma = 1, j = 1, \ldots, n. \]

Here \( P_e \) and \( \alpha_j \) are constants and \( F(y, y_1, y_2, \ldots, y_n) \) is a nonlinear function.

Define the functions \( w_j(\sigma) \) by

\[ w_j = y - \frac{y_j}{\alpha_j} \quad j = 1, \ldots, n \]

and then verify

\[ \frac{d^2 w_j}{d\sigma^2} - P_e \frac{dw_j}{d\sigma} = 0. \]

The boundary condition at \( \sigma = 1 \) asserts \( dw_j/d\sigma = 0 \) so \( dw_j(\sigma)/d\sigma \equiv 0 \).

From the initial data at \( \sigma = 0 \) we conclude that

\[ w_j(0) = \frac{1}{P_e} \frac{dy}{d\sigma} + 1 - \frac{1}{\alpha_j} \left[ \frac{1}{P_e} \frac{dy_j}{d\sigma} + 1 \right] \]

so

\[ w_j(\sigma) = w_j(0) = 1 - 1/\alpha_j. \]

Hence any solutions \( y(\sigma), y_j(\sigma) \) of our differential system are related by

\[ y_j = \alpha_j (y - 1) + 1. \]
We use this relation to define a function $F(y)$ since each of the $y_j$ may be eliminated from $F(y, y_1, y_2, ..., y_n)$. Thus the final differential system which describes the dynamics of a single chemical reaction on $0 < \sigma < 1$, is

$$\frac{d^2y}{d\sigma^2} - P_e \frac{dy}{d\sigma} + F(y) = 0$$

$$\frac{dy}{d\sigma} = P_e(y - 1) \quad \text{at} \quad \sigma = 0$$

$$\frac{dy}{d\sigma} = 0 \quad \text{at} \quad \sigma = 1.$$ 

If we demonstrate the existence of a unique solution $y(\sigma)$ of this nonlinear boundary-value problem, we shall then have proved the existence of a unique solution $[y(0), y(\ell)]$ of our complete problem. This assertion follows from an easy calculation using the relation

$$y_j = \alpha_j(y - 1) + 1.$$ 

2. Qualitative Theory of a Reactor with a Single Reaction

The dynamics of a single chemical reaction within the theory described above, are

$$\frac{d^2y}{d\sigma^2} - P_e \frac{dy}{d\sigma} + F(y) = 0,$$

with

$$\frac{dy}{d\sigma} = P_e(y - 1) \quad \text{at} \quad \sigma = 0$$

$$\frac{dy}{d\sigma} = 0 \quad \text{at} \quad \sigma = 1.$$ 

The solution $y(\sigma)$ on $0 \leq \sigma \leq 1$ represents the dimensionless local temperature at a distance $\sigma$ along the reactor tube in a steady-state process. Also $P_e = \ell \alpha/D$ and

$$F(y) = \frac{-\Delta H\ell^2}{c_pD_T} f(y),$$

in terms of the usual mechanical and chemical constants of the process.

If we introduce the new independent variable $t = \ell - \sigma = \ell - \ell \sigma$ and let $x = y - 1$ we obtain the corresponding nonlinear boundary-value problem:

$$\frac{d^2x}{dt^2} + \frac{v}{D} \frac{dx}{dt} + hf(x) = 0 \quad \text{on} \quad 0 \leq t \leq \ell,$$
with
\[
\frac{dx}{dt} = -\frac{v}{D} x \quad \text{at} \quad t = t^{'},
\]
and
\[
\frac{dx}{dt} = 0 \quad \text{at} \quad t = 0.
\]

Here \( h = -\Delta H/c_p\rho D T_0 \) is a given positive constant and we have written \( f(x) = f(x + 1) \).

For a single exothermic homogeneous chemical reaction the real function \( f(x) \) has the following properties:

- \( f(x) \) is a smooth function increasing from \( f(-1) = 0 \) to a maximum at some point \( x_m \) (which may be positive or negative), and decreasing thereafter to become zero at some \( c > x_m \).

That is, we have the assumptions:

\[
f'(x) > 0 \quad \text{on} \quad -1 < x < x_m,
\]

\[
f'(x) < 0 \quad \text{on} \quad x_m < x < +\infty,
\]

with \( f(-1) = 0, \ f(c) = 0 \).

We are interested in the performance of the reactor when the value of \( x(t) \) at state \( t = 0 \) lies on \( 0 < x(0) < c \), or the final temperature \( y(s) \) at \( s = t^{'}, \) lies in the domain where \( F(y) > 0 \) and \( y(t^{'}) > 1 \). In this case we shall show that \( x(t) \) is monotonic decreasing, or that the temperature \( y(s) \) increases along the tube, corresponding to the exothermic nature of the reaction.

In the next section we shall prove the existence of a unique solution \( x(t) \) of the boundary-value problem in the case where \( x_m \leq 0 \) (or \( x_m > 0 \) is not too large). We shall also permit some nonlinearities in the convective and heat conduction mechanisms. This mathematical analysis increases our confidence in our theoretic model and allows us to make quantitative studies of the dependence of the solution on the parameters of the physical problem.

### 3. Nonlinear Boundary-Value Problems

As we have seen above the analyses of certain chemical processes lead to second-order nonlinear boundary-value problems. In this section we obtain positive solutions or eigenfunctions for a class of nonlinear differential equations of the form

\[
\ddot{x} + g(x) + f(x) = 0,
\]
with nonlinear boundary-values
\[ x(0) = 0, \quad x(T) = \varphi[x(T)], \]
on a finite interval \( 0 \leq t \leq T \). Here the given functions \( f(x), \varphi(x), \) and \( g(y) \)
are in class \( C^1 \) (continuous real functions with continuous first derivatives)
wherever encountered, that is,
\[
\begin{align*}
  f(x) & \in C^1 & \text{on a finite interval } 0 \leq x \leq c \\
  \varphi(x) & \in C^1 & \text{on } 0 \leq x \leq c \\
  g(y) & \in C^1 & \text{on } y \leq 0.
\end{align*}
\]
The boundary-data curve \( y = \varphi(x) \) on \( 0 < x < c \) lies in the (open) fourth
quadrant of the \((x, y = \dot{x})\) phase plane, and we seek a positive eigenfunction
\( x(t) > 0 \) on \( 0 < t < T \) satisfying the two endpoint conditions.

Our principal result is that for each positive \( T > 0 \) there exists a unique
amplitude \( A \) on \( 0 < A < c \) such that the solution \( x(t, A) \) initiating at
\[ x(0, A) = A, \quad \dot{x}(0, A) = 0, \]
is the required positive eigenfunction on \( 0 \leq t \leq T \). In order to apply our
theory to the analyses of chemical processes, we interpret the eigenfunctions
\( x(t, A) \), and their dependence on \( A, T, \) and other parameters, in terms of
the physical models. For instance, \( T = \ell' \) is the length of the reactor tube
and \( A + 1 \) represents the final temperature.

Besides the hypotheses that \( f, g, \) and \( \varphi \) have continuous derivatives, we
assume certain convexity conditions:
\[
\begin{align*}
  (1) \quad \frac{f(x)}{x} > 0 & \quad \text{and} \quad \frac{d}{dx} \frac{f(x)}{x} < 0 & \quad \text{on } 0 < x < c \\
  (2) \quad \frac{g(y)}{y} > 0 & \quad \text{and} \quad \frac{d}{dy} \frac{g(y)}{y} < 0 & \quad \text{on } y < 0 \\
  (3) \quad \frac{\varphi(x)}{x} < 0 & \quad \text{and} \quad \frac{d}{dx} \frac{\varphi(x)}{x} < 0 & \quad \text{on } 0 < x < c.
\end{align*}
\]
In nonlinear mechanics the first two conditions describe a weak-spring with
dissipative friction, \([2, 3]\). The frictional term \( g(y) \) and the boundary curve
\( \varphi(x) \) may be linear, say
\[ g(y) = ay \quad \text{and} \quad \varphi(x) = -bx \quad \text{for } a, b > 0; \]
but nonlinear functions are also suitable, say
\[ g(y) = ay + \alpha y^2, \quad \varphi(x) = -bx - \beta x^2 \quad \text{for } \alpha, \beta > 0. \]
However, the weak-spring force must be strictly nonlinear, say \( f(x) = 1 - x^3 \) on \( 0 \leq x \leq 1 \), and we shall also demand the asymmetry \( f(0) > 0, f(c) = 0 \).

**Lemma.** Consider the phase plane differential system

\[
\dot{x} = y, \quad \dot{y} = -f(x) - g(y).
\]

Assume

1. \( f(x) \in C^1 \) on \( 0 \leq x \leq c \) with \( f(0) > 0, f(c) = 0 \) and

\[
\frac{f(x)}{x} > 0, \quad \frac{d}{dx} f(x) < 0 \quad \text{on} \quad 0 < x < c.
\]

2. \( g(y) \in C^1 \) on \( y \leq 0 \) with \( g(0) = 0 \) and

\[
\frac{g(y)}{y} > 0, \quad \frac{d}{dy} \frac{g(y)}{y} \leq 0 \quad \text{on} \quad y < 0.
\]

Then the solution \( x(t, A), y(t, A) \) initiating at \( x(0, A) = A, y(0, A) = 0 \) for \( 0 < A < c \) enters the fourth-quadrant for \( t > 0 \), and leaves the fourth-quadrant on the negative y-axis after a finite time. Along this solution \( x(t, A) \) decreases monotonically from \( A \) to 0, and the angular coordinate

\[
\theta(t) = \arctan \frac{y(t, A)}{x(t, A)},
\]

also decreases monotonically from 0 to \(-\pi/2\), with \( \dot{\theta} < 0 \) on \(-\pi/2 \leq \theta \leq 0\).

**Proof.** Since \( f(x) > 0 \) on \( 0 \leq x \leq A \) and \( g(0) = 0 \), the phase-plane velocity vector has a downward component \( \dot{y} = -f(x) - g(0) \) on the x-axis. Hence the solution \( x(t, A), y(t, A) \) must enter the fourth-quadrant for \( t > 0 \), and remain in this quadrant forever unless it leaves on the negative y-axis or tends to infinity in the plane.

In the fourth-quadrant \( \dot{x} = y < 0 \) and so \( x(t, A) \) decreases monotonically from its initial value \( x(0, A) = A \). Moreover, the solution cannot tend toward infinity since the slope is suitably bounded,

\[
\frac{dy}{dx} = \frac{-f(x) - g(y)}{y} \leq \max_{0 < x < A} |f(x)|, \quad \text{when} \quad |y| > 1.
\]

Hence the solution must intersect the negative y-axis after some finite duration.

It remains to be shown that \( \dot{\theta} < 0 \) along the solution \( x(t, A), y(t, A) \). We compute

\[
\dot{\theta}(t) = \frac{x(-f(x) - g(y)) - y^2}{x^2 + y^2} = -\frac{f(x) + \frac{g(y)}{x}}{1 + \left(\frac{y}{x}\right)^2}.
\]
At \( t = 0 \), \( \dot{\theta}(0) = -f(A)/A < 0 \) and we show that \( \dot{\theta}(t) < 0 \) until the solution crosses the negative y-axis as it leaves the fourth-quadrant. At this instant of leaving, \( \dot{x} = y < 0 \) and so \( \dot{\theta} = y/|y| = -1 < 0 \).

Suppose \( \dot{\theta}(t) \) is somewhere zero, and let \( t_0 > 0 \) be the first instant when \( \dot{\theta}(t_0) = 0 \). Then \( \ddot{\theta}(t_0) > 0 \), since the inequality \( \ddot{\theta}(t_0) < 0 \) would imply that \( \dot{\theta}(t) \) decreases to \( \dot{\theta}(t_0) = 0 \). Compute this second derivative to be

\[
\ddot{\theta}(t_0) = -\frac{\frac{d}{dt} \frac{f(x)}{x}}{1 + \left(\frac{y}{x}\right)^2},
\]

where we have used the stationary feature \( \frac{d}{dt} \left(\frac{f(x)}{x}\right) = 0 \). Note that

\[
\frac{d}{dt} \frac{f(x)}{x} = \frac{\frac{d}{dx} f(x)}{x} \cdot \frac{dx}{dt} > 0.
\]

It is also true that

\[
\frac{d}{dt} \frac{g(y)}{y} = \frac{\frac{d}{dy} g(y)}{y} \cdot \frac{dy}{dt} \leq 0.
\]

This second inequality follows from the fact that the solution must be tangent to the ray \( \theta = \theta(t_0) \) at \( t = t_0 \), and here \( \dot{x} < 0, \dot{y} > 0 \). Since \( y/x < 0 \), we have

\[
\frac{d}{dt} \frac{f(x)}{x} + \frac{d}{dt} \frac{g(y)}{y} \cdot \frac{y}{x} > 0,
\]

or \( \ddot{\theta}(t_0) < 0 \).

This contradicts the known non-negativity of \( \ddot{\theta}(t_0) \). Therefore \( \dot{\theta}(t) \) is nowhere zero and must remain negative everywhere along the given solution curve. Hence \( \dot{\theta}(t) \) decreases monotonically from 0 to \(-\pi/2\), as required.

**Theorem 1.** Consider the nonlinear differential equation

\[
\ddot{x} + g(\dot{x}) + f(x) = 0,
\]

with the nonlinear boundary-data curve \( y = \varphi(x) \) in the \((x, y = \dot{x})\) phase plane. Assume

1. \( f(x) \in C^1 \) on \( 0 \leq x \leq c \) with \( f(0) > 0 \), \( f(c) = 0 \) and

\[
\frac{f(x)}{x} > 0, \quad \frac{d}{dx} \frac{f(x)}{x} < 0 \quad \text{on} \quad 0 < x < c.
\]
\( g(y) \in C^1 \) on \( y \leq 0 \) with \( g(0) = 0 \) and
\[
\frac{g(y)}{y} > 0, \quad \frac{d}{dy} \frac{g(y)}{y} \leq 0 \quad \text{on} \quad y < 0.
\]

(3) \( \varphi(x) \in C^1 \) on \( 0 \leq x \leq c \) with \( \varphi(0) = 0 \) and
\[
\frac{\varphi(x)}{x} < 0, \quad \frac{d}{dx} \frac{\varphi(x)}{x} \leq 0 \quad \text{on} \quad 0 < x < c.
\]

Then for each \( T > 0 \) there exists a unique amplitude \( A \) on \( 0 < A < c \) for which the corresponding solution \( x(t, A) \) is a positive eigenfunction satisfying
\[
x(0, A) = A, \quad \dot{x}(0, A) = 0, \quad x(T, A) = \varphi[x(T, A)],
\]
on the interval \( 0 \leq t \leq T \). Furthermore \( A(T) \) is a continuous monotonic increasing function with
\[
\lim_{T \to 0} A(T) = 0 \quad \lim_{T \to \infty} A(T) = c.
\]

Proof. The boundary-data curve \( y = \varphi(x) \) lies in the fourth-quadrant and we compute its slope \( \varphi'(x) \). Since
\[
\frac{d}{dx} \frac{\varphi(x)}{x} = \frac{x\varphi'(x) - \varphi(x)}{x^2} \leq 0 \quad \text{on} \quad 0 < x < c,
\]
we find \( \varphi'(x) \leq \varphi(x)/x < 0 \). Thus \( \varphi'(x) \leq 0 \) on \( 0 \leq x \leq c \) and \( \varphi(x) < 0 \) on \( 0 < x < c \), so \( \varphi(x) \) is monotonic nonincreasing from the origin \( \varphi(0) = 0 \).

Next consider the ray from the origin to the point \([x, \varphi(x)]\) on the boundary-data curve. The inclination angle of this ray is
\[
\Theta(x) = \arctan \frac{\varphi(x)}{x} \quad \text{on} \quad 0 < x < c,
\]
and
\[
\Theta(0) = \arctan \lim_{x \to 0} \frac{\varphi(x)}{x} = \arctan \varphi'(0) \leq 0.
\]
We compute
\[
\frac{d\Theta}{dx} = \frac{x\varphi'(x) - \varphi(x)}{x^2 + \varphi(x)^2} \leq 0 \quad \text{on} \quad 0 < x < c.
\]
Hence \( \Theta(x) \) is monotonic nonincreasing on \( 0 \leq x \leq c \).

Now choose an initial amplitude \( A_1 \), on \( 0 < A_1 < c \) and consider the corresponding solution curve \( x(t, A_1), y(t, A_1) = \dot{x}(t, A_1) \). Along this solution \( x(t, A_1) \) decreases monotonically from \( A_1 \) to 0, and so the solution curve
intersects the boundary-data curve \( y = \varphi(x) \). We show that there is just one time \( t = T_1 \) for which the solution meets the curve \( y = \varphi(x) \). Suppose there were two such intersection times \( 0 < T_1 < T_2 \) where

\[
y(T_1, A_1) = \varphi[x(T_1, A_1)] \quad \text{and} \quad y(T_2, A_1) = \varphi[x(T_2, A_1)].
\]

Let \( \theta(T_1) \) and \( \theta(T_2) \) be the corresponding angular coordinates of the intersection points. Then, since \( \dot{\theta}(t) < 0 \), \( \theta(T_2) < \theta(T_1) \). But along the curve \( y = \varphi(x) \), when \( x \) increases from \( \hat{x}_1 = x(T_1, A_1) \) to \( \hat{x}_1 = x(T_2, A_1) \), the inclination \( \Theta(x) \) does not increase. This means that

\[
\arctan \frac{\dot{x}(T_1)}{\dot{y}(T_1)} \geq \arctan \frac{\dot{x}(T_2)}{\dot{y}(T_2)}
\]

or \( \theta(T_2) \geq \theta(T_1) \), which is a contradiction. Therefore each such solution curve \( x(t, A), y(t, A) \) meets the boundary-data curve \( y = \varphi(x) \) in exactly one point, and at exactly one time \( T(A) \).

We next show that \( x(t, A), y(t, A) \) crosses \( y = \varphi(x) \) nontangentially at their unique intersection. Suppose the intersection at time \( T(A) \) were tangential, that is, suppose

\[
y(t, A_1) = \varphi[x(t, A_1)]
\]

and

\[
\varphi'[x(t, A_1)] = \dot{y}(t, A_1)/\dot{x}(t, A_1) \quad \text{at} \quad t = T(A_1).
\]

Then

\[
\Theta'(x) = \frac{x\varphi'(x) - \varphi(x)}{x^2 + \varphi(x)^2} = \frac{x\dot{y} - y\dot{x}}{\dot{x}^2 + \varphi(x)^2} \leq 0
\]

and

\[
\dot{\theta}(t) = \frac{x\dot{y} - y\dot{x}}{x^2 + y^2} < 0,
\]

at the intersection point. Since \( \dot{x} < 0 \) and \( \Theta'(x) \leq 0 \), we find \( x\dot{y} - y\dot{x} > 0 \), which contradicts the sign of \( \dot{\theta} \). Hence each solution curve crosses \( y = \varphi(x) \) nontangentially.

Since the solutions \( x(t, A), y(t, A) \) depend differentiably on the initial amplitude \( A \), we can conclude that the real function \( T(A) \) on \( 0 < A < c \) is continuous (in fact, in class \( C^1 \)). We next show that \( T(A) \) increases monotonically from \( T(0) = 0 \) to \( T(c) = \infty \). The inverse function \( A(T) \) on \( 0 < T < \infty \) will then satisfy all the requirements of the theorem.

Let us consider two solutions \( S_1(t) \) and \( S_2(t) \), with initial amplitudes \( 0 < A_1 < A_2 < c \). Say that \( S_1 \) meets \( y = \varphi(x) \) at \( (x_1, y_1) \) at time \( T(A_1) \) and angular coordinate \( \theta_1 \). Now the solution \( S_2 \) lies "outward from the origin" from \( S_1 \), and so \( S_2 \) meets the ray \( \theta = \theta_1 \) at some point \( (x_2, y_2) \).
with $x_1 < x_2$ (since $\theta < 0$ along each solution, $S_2$ meets this ray in just one point).

We shall note that on the arc of $S_2$ where $x_2 < x \leq A_2$, there is no intersection with $y = \varphi(x)$. For suppose that $S_2$ met $y = \varphi(x)$ at some $x_2 > x_2$. Then the angular coordinate of this intersection point $[\hat{x}_2, \varphi(\hat{x}_2)]$ is
\[
\Theta(\hat{x}_2) = \arctan \varphi(\hat{x}_2)/\hat{x}_2 \leq \Theta(x_2).
\]
But $(x_2, y_2)$ occurs at a later time than does $[\hat{x}_2, \varphi(\hat{x}_2)]$, and since $\theta < 0$ along each solution, we must have
\[
\Theta(x_2) < \Theta(\hat{x}_2).
\]
This contradiction proves that $S_2$ meets $y = \varphi(x)$ at an angular coordinate $\theta_2 \leq \theta_1$.

Finally we show that $S_2$ takes a longer time to reach the ray $\theta_1$ than does $S_1$. The angular speed along a solution is
\[
\dot{\theta} = \frac{x(-f(x) - g(y)) - y^2}{x^2 + y^2} = \frac{f(x) + g(y) y^2}{1 + (y/x)^n},
\]
or
\[
\dot{\theta} = -\frac{f(x)}{x} \frac{g(y) \tan \theta + \tan^2 \theta}{1 + \tan^2 \theta}.
\]
For each fixed value of $\theta$ we see that $|\dot{\theta}|$ decreases as $x$ and $|y|$ increase outward along a ray from the origin. This assertion follows from the hypotheses
\[
\frac{d}{dx} \frac{f(x)}{x} < 0 \quad \text{and} \quad \frac{d}{dy} \frac{g(y)}{y} \leq 0.
\]

Therefore the outside solution $S_2$ has a smaller magnitude of angular speed than has $S_1$, for each angular position on $\theta_1 \leq \theta \leq 0$. Hence $S_2$ takes a greater time duration to reach the ray $\theta_1$ than does $S_1$. Since $S_2$ must pass the ray $\theta_1$ before meeting the boundary-data curve, we conclude that
\[
T(A_2) > T(A_1) \quad \text{when} \quad A_2 > A_1.
\]

Thus $T(A)$ is a continuous monotonic increasing function and has an inverse function $A(T)$ that is also continuous and monotonic increasing. Since $f(0) > 0$ and $\varphi'(0)$ exists, we see that
\[
\lim_{A \to 0} T(A) = 0 \quad \text{and} \quad \lim_{T \to 0} A(T) = 0.
\]
Since $x = c, y = 0$ is an equilibrium or critical point,

$$\lim_{A \to c} T(A) = \infty \quad \text{and} \quad \lim_{T \to \infty} A(T) = c,$$

as required. Q.E.D.

References: