An asymmetric Arzelà–Ascoli theorem

Julia Collins [a], Johannes Zimmer [b,*]

[a] *University of Edinburgh, School of Mathematics, Edinburgh EH9 3JZ, UK*
[b] *Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK*

Abstract

An Arzelà–Ascoli theorem for asymmetric metric spaces (sometimes called quasi-metric spaces) is proved. One genuinely asymmetric condition is introduced, and it is shown that several classic statements fail in the asymmetric context if this assumption is dropped.
© 2007 Elsevier B.V. All rights reserved.

1. Introduction

In this paper we investigate the question of whether the classical theorem of Arzelà–Ascoli can be generalised to the world of asymmetric metric spaces. Asymmetric metric spaces are defined as metric spaces, but without the requirement that the (asymmetric) metric d has to satisfy $d(x, y) = d(y, x)$. The investigations show that both the Arzelà–Ascoli theorem as well as underlying fundamental statements of real analysis rely heavily on the symmetry. Without symmetry, an 'embarrassing richness of material' [13] is revealed, which we try to shed light on by discussing various examples. In particular, it is shown that classic statements of real (symmetric) analysis fail if one crucial asymmetric assumption fails.

In the realms of applied mathematics and materials science we find many recent applications of asymmetric metric spaces; for example, in rate-independent models for plasticity [6], shape-memory alloys [8], and models for material failure [12]. In general, gradient flow models have been proposed for situations where local energy minimisation is relevant. For the investigation of the existence of gradient flows in asymmetric metric spaces [1], a compactness argument such as the Arzelà–Ascoli theorem is a vital ingredient, which we could not find in the literature. Since this result and the framework might be of independent interest, it is described in this note.

There are many versions of the classic Arzelà–Ascoli theorem. For example, one formulation considers sets of equicontinuous functions from k-spaces to Tychonoff spaces [2, 8.2.10]. Another version is cast in a metric setting (see, e.g., [9]), and it is this formulation which we generalise to the asymmetric case. This choice is motivated by the

applications in mathematical materials science; a metric framework is a suitable setting for gradient flow models and rate-independent evolution equations.

There are other applications of asymmetric metrics both in pure and applied mathematics; for example, asymmetric metric spaces have recently been studied with questions of existence and uniqueness of Hamilton–Jacobi equations [7] in mind. In this light, it is quite natural to wonder whether the symmetry requirement of a metric is indeed essential, or a mere limitation of applicability, as remarked by Gromov [3, Introduction].

The study of asymmetric metrics apparently goes back to Wilson [13]. Following his terminology, asymmetric metrics are often called quasi-metrics. Though the analysis of asymmetric metric spaces seems quite natural, the lack of a consistent theory has been pointed out long ago [4], and still there seem to be many open questions.

This paper is organised as follows. The topology of asymmetric metric spaces is described in Section 2; forward and backward convergence is analysed in Section 3, followed by a discussion of completeness and compactness in Section 4. Section 5 is devoted to the Arzelà–Ascoli theorem and its proof. Notation: $\mathbb{R}_0^+$ denotes the nonnegative real numbers, and $\mathbb{Z}_+ := \{1, 2, \ldots\}$.

2. Topology of asymmetric metric spaces

Definition 2.1 *(Asymmetric metric space).* A function $d : X \times X \to \mathbb{R}$ is an *asymmetric metric* (sometimes called a *quasi-metric* [13]) and (X, d) is an *asymmetric metric space* if:

(1) For every $x, y \in X$, $d(x, y) \geqslant 0$ and $d(x, y) = 0$ holds if and only if $x = y$,
(2) For every $x, y, z \in X$, we have $d(x, z) \leqslant d(x, y) + d(y, z)$.

Henceforth, (X, d) shall be an asymmetric metric space. We start with some elementary definitions.

Definition 2.2 *(Forward and backward topologies).* The *forward topology* τ_+ induced by d is the topology generated by the *forward open balls*

$$B^+(x, \varepsilon) := \left\{ y \in X \,\middle|\, d(x, y) < \varepsilon \right\} \quad \text{for } x \in X, \ \varepsilon > 0.$$

Likewise, the *backward topology* τ_- induced by d is the topology generated by the *backward open balls*

$$B^-(x, \varepsilon) := \left\{ y \in X \,\middle|\, d(y, x) < \varepsilon \right\} \quad \text{for } x \in X, \ \varepsilon > 0.$$

Definition 2.3 *(Boundedness).* A set $S \subset X$ is *forward bounded*, respectively *backward bounded*, if there exists $x \in X$ and $\varepsilon > 0$ such that $S \subset B^+(x, \varepsilon)$, respectively $S \subset B^-(x, \varepsilon)$.

Definition 2.4 *(Convergence).* A sequence $\{x_k\}_{k \in \mathbb{N}}$ *forward converges* to $x_0 \in X$, respectively *backward converges* to $x_0 \in X$ if and only if

$$\lim_{k \to \infty} d(x_0, x_k) = 0, \quad \text{respectively} \quad \lim_{k \to \infty} d(x_k, x_0) = 0.$$

Notation: $x_k \xrightarrow{f} x_0$, respectively $x_k \xrightarrow{b} x_0$.

Definition 2.5 *(Continuity).* Suppose (X, d_X) and (Y, d_Y) are asymmetric metric spaces. There are four obvious definitions of continuity for a function $f : X \to Y$ at $x \in X$. We say a function is *ff-continuous*, respectively *fb-continuous*, if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $y \in B^+(x, \delta)$ implies $f(y) \in B^+(f(x), \varepsilon)$, respectively $f(y) \in B^-(f(x), \varepsilon)$.

However, note that uniform ff-continuity and uniform bb-continuity are the same (and analogously uniform fb-continuity and uniform bf-continuity).

Lemma 2.6 *(Sequential definition of continuity).* *A function $f : X \to Y$ is ff-continuous at $x \in X$ if and only if whenever $x_k \xrightarrow{f} x$ in (X, d_X) we have $f(x_k) \xrightarrow{f} f(x)$ in (Y, d_Y). An analogous statement holds for the other three notions of continuity.*

Proof. As in the symmetric case.   □

In the literature, one can find at least seven different notions of Cauchyness [11,7]. We single out the two which are appropriate for the investigations to come.

Definition 2.7 *(Cauchyness).* Suppose (X, d) is an asymmetric metric space. We say that a sequence $\{x_k\}_{k \in \mathbb{N}} \subset X$ is

(1) *forward Cauchy* if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for $m \geqslant n \geqslant N$, $d(x_n, x_m) < \varepsilon$ holds,
(2) *backward Cauchy* if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for $m \geqslant n \geqslant N$, $d(x_m, x_n) < \varepsilon$ holds.

Forward (backward) Cauchy sequences are called left (right) K-Cauchy sequences in [11], apparently referring to Kelly's investigation of these notions [4, Definition 2.10]. Kelly pointed out that with this definition, forward convergence does not imply forward Cauchyness [4, Example 5.8]. We give another, simpler, example of a forward convergent sequence which is not forward Cauchy; see Example 3.6 below.

3. Convergence and limits

For the sake of completeness, we state a simple lemma (compare [13, Theorem I]).

Lemma 3.1. *If $\{x_k\}_{k \in \mathbb{N}} \subset X$ forward converges to $x_0 \in X$ and backward converges to $y_0 \in X$ then $x_0 = y_0$.*

Proof. Fix $\varepsilon > 0$. By assumption, $x_k \xrightarrow{f} x_0$ so there exists $N_1 \in \mathbb{N}$ such that $d(x_0, x_k) < \frac{\varepsilon}{2}$ for all $k \geqslant N_1$. Also, $x_k \xrightarrow{b} y_0$, so there exists $N_2 \in \mathbb{N}$ such that $d(x_k, y_0) < \frac{\varepsilon}{2}$ for all $k \geqslant N_2$. Then for all $k \geqslant N := \max\{N_1, N_2\}$, $d(x_0, y_0) \leqslant d(x_0, x_k) + d(x_k, y_0) < \varepsilon$. As ε was arbitrary, we deduce that $d(x_0, y_0) = 0$, which implies $x_0 = y_0$.   □

Corollary 3.2. *If forward convergence of a sequence implies backward convergence, then the forward limit is unique.*

Proposition 3.3. *Suppose $d(y, x) \leqslant c(x, y)d(x, y)$ for all $x, y \in X$, where $c : X \times X \to \mathbb{R}$ satisfies the following constraint:*

$$\forall x \in X \; \exists \varepsilon > 0 \text{ such that } y \in B^+(x, \varepsilon) \Rightarrow c(x, y) \leqslant C(x), \tag{1}$$

where C is some function which depends only on x.
In this situation, the existence of forward limits implies the existence of backward limits, and so limits are unique.

Proof. Suppose $x_k \xrightarrow{f} x_0$. From Eq. (1), there exists $\delta > 0$ such that $c(x_0, x_k) \leqslant C(x_0)$ for $d(x_0, x_k) < \delta$. Fix $\varepsilon > 0$. Since $x_k \xrightarrow{f} x_0$, there exists $N \in \mathbb{N}$ such that for every $k \geqslant N$, we have $d(x_0, x_k) < \min\{\delta, \varepsilon/C(x_0)\}$ and thus $d(x_k, x_0) \leqslant c(x_0, x_k)d(x_0, x_k) \leqslant C(x_0)d(x_0, x_k) < \varepsilon$. By Corollary 3.2, limits must be unique.   □

A few examples are in order.

Example 3.4. We begin with the simplest example of an asymmetric metric space. Let $\alpha > 0$. Then $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_0^+$ defined by

$$d(x, y) := \begin{cases} y - x & \text{if } y \geqslant x, \\ \alpha(x - y) & \text{if } y < x \end{cases}$$

is obviously an asymmetric metric. This metric satisfies $d(y, x) \leqslant Cd(x, y)$ for all $x, y \in \mathbb{R}$ where $C := \max\{\alpha, \frac{1}{\alpha}\}$. This metric therefore satisfies the conditions for Proposition 3.3. Note that τ_+ and τ_- are the usual topologies on $\mathbb{R}$.

Example 3.5. The function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_0^+$ defined by

$$d(x, y) := \begin{cases} e^y - e^x & \text{if } y \geqslant x, \\ e^{-y} - e^{-x} & \text{if } y < x \end{cases}$$

is an asymmetric metric [7, Example 2.28]. Both τ_+ and τ_- are the usual topologies on $\mathbb{R}$.

It is easy to see that the assumption of Proposition 3.3 is satisfied with $C(x) := \max\{e^x(e^x + \varepsilon), e^{-x}(e^{-x} + \varepsilon)\}$.

Example 3.6 *(Sorgenfrey asymmetric metric).* Now we exhibit a metric which does not satisfy condition (1). The function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_0^+$ defined by

$$d(x, y) := \begin{cases} y - x & \text{if } y \geqslant x, \\ 1 & \text{if } y < x \end{cases}$$

is an asymmetric metric. Here, τ_+ is the lower limit topology on $\mathbb{R}$ and τ_- is the upper limit topology, i.e., $B^+(x, \varepsilon) = [x, x + \varepsilon)$ and $B^-(x, \varepsilon) = (x - \varepsilon, x]$, provided $\varepsilon \leqslant 1$.

It is easy to find a sequence which forward converges in this metric but does not backward converge; for example, fix $x \in \mathbb{R}$ and let $x_k = x(1 + \frac{1}{k})$. Then $x_k \xrightarrow{f} x$ but $x_k \xrightarrow{b}\!\!\!\!\!/ \; x$. Likewise it is easy to show that the existence of a backward limit does not imply the existence of a forward limit.

The next example shows that the converse of Proposition 3.3 is not true. We are not aware of a necessary and sufficient condition under which forward convergence implies backward convergence.

Example 3.7. Let $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$ be defined by

$$d(x, y) := \begin{cases} d(x, 0) + d(0, y) & \text{if } y \neq x, \\ 0 & \text{if } x = y, \end{cases}$$

where $d(x, 0) = \frac{1}{|x|}$ and $d(0, x) = \frac{1}{|x|^2}$ for $x \neq 0$.

Let $0 \neq x \in \mathbb{R}^n$. If, for a sequence $\{x_k\}_{k \in \mathbb{N}}$, $x_k \xrightarrow{f} x$, then there exists $N \in \mathbb{N}$ such that $x_k = x$ for $k \geqslant N$. In that case, $x_k \xrightarrow{b} x$. Similarly, if $x = 0$, $x_k \xrightarrow{f} x$ if $|x_k| \to \infty$, in which case $x_k \xrightarrow{b} x$ as well. Thus, forward convergence implies backward convergence. Yet, there is no bound of type (1); take $x = 0$ and $\varepsilon > 0$. Then $B^+(0, \varepsilon) = \{0\} \cup \{y \in \mathbb{R}^n \mid |y| > \sqrt{\frac{1}{\varepsilon}}\}$. So if $y \in B^+(0, \varepsilon)$ then $|y|$ is not bounded above, and

$$\frac{d(y, 0)}{d(0, y)} = \frac{\frac{1}{|y|}}{\frac{1}{|y|^2}} = |y|.$$

Thus any function c satisfying $d(y, x) \leqslant c(x, y)d(x, y)$ for every $x, y \in \mathbb{R}^n$ will be unbounded in any forward ball of 0.

4. Compactness and completeness

Definition 4.1 *(Compactness and completeness).* A set $S \subset X$ is *forward compact* if every open cover of S in the forward topology has a finite subcover. We say that S is *forward relatively compact* if $\bar{S}$ is forward compact, where ̄ denotes the closure in the forward topology. We say S is *forward sequentially compact* if every sequence has a forward convergent subsequence with limit in S. Finally, $S \subset X$ is *forward complete* if every forward Cauchy sequence is forward convergent.

Note that there is a corresponding backward definition in each case, which is obtained by replacing 'forward' with 'backward' in each definition.

Lemma 4.2. *Let $d : X \times X \to \mathbb{R}_0^+$ be an asymmetric metric. If (X, d) is forward sequentially compact and $x_n \xrightarrow{b} x$, then $x_n \xrightarrow{f} x$.*

Proof. Consider a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \xrightarrow{b} x$ for some $x \in X$. By sequential compactness, every subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ has a forward convergent subsequence. Say $x_{n_{k_j}} \xrightarrow{f} y \in X$ as $j \to \infty$. Then $x = y$ by Lemma 3.1.

Suppose $x_n \xcancel{\xrightarrow{f}} x$. Then there exists $\varepsilon_0 > 0$ and a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ with $d(x, x_{n_k}) \geqslant \varepsilon_0$ for all $k \in \mathbb{N}$. But this subsequence has a subsequence $\{x_{n_{k_j}}\}_{j\in\mathbb{N}}$ forward converging to x, so there exists $J \in \mathbb{N}$ such that for $j \geqslant J$, one has $d(x, x_{n_{k_j}}) < \varepsilon_0$. This is obviously a contradiction, so $x_n \xrightarrow{f} x$. $\square$

Consequently, whenever we have compactness we have unique limits of sequences, and backward limits imply forward limits.

The next lemma indicates that we have chosen the correct definition of 'forward complete' (compare [11, Theorem 1]).

Lemma 4.3. *An asymmetric metric space (X, d) is forward complete if every forward Cauchy sequence has a forward convergent subsequence.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a forward Cauchy sequence in (X, d) with $\{x_{n_k}\}_{k\in\mathbb{N}}$ a subsequence forward converging to $x \in X$. Fix $\varepsilon > 0$. First choose $N \in \mathbb{N}$ so that $d(x_n, x_m) < \frac{\varepsilon}{2}$ for every $m \geqslant n \geqslant N$. Then choose $k \in \mathbb{N}$ so that $n_k \geqslant N$ and $d(x, x_{n_k}) < \frac{\varepsilon}{2}$. Then for $n \geqslant n_k \geqslant N$ we have

$$d(x, x_n) \leqslant d(x, x_{n_k}) + d(x_{n_k}, x_n) < \varepsilon.$$

So $x_n \xrightarrow{f} x$, so (X, d) is forward complete. $\square$

Lemma 4.4. *A forward compact set X is forward sequentially compact.*

Since the proof is, except for obvious modifications, the same as in the symmetric situation, it is omitted here.

Definition 4.5 *(Total boundedness).* A subset $S \subset X$ is *forward totally bounded* if, for each $\varepsilon > 0$, it can be covered by finitely many forward balls of radius ε.

Propositions 4.6 and 4.7 are similar to the symmetric situation, and the proof of Proposition 4.6 follows Munkres [9, Theorem 28.2]. Here, the statement is split in two propositions, since one part, Proposition 4.7, involves a genuinely asymmetric condition.

Proposition 4.6. *If (X, d) is forward sequentially compact and forward totally bounded, then X is forward compact.*

Proof. Let $\mathcal{A}$ be an open cover of X in the forward topology. For contradiction, we assume that for any $\delta > 0$ there is a ball of radius less than δ that is not contained in any element of $\mathcal{A}$. In particular, for each $n \in \mathbb{N}$ there exists a ball $C_n := B^+(x_n, r)$, where $r < \frac{1}{n}$, that is not contained in any element of $\mathcal{A}$. By hypothesis, some subsequence $\{x_{n_j}\}_{j\in\mathbb{N}}$ forward converges, say to the point a. Now $a \in A$ for some $A \subset \mathcal{A}$, and since A is open we may choose $\varepsilon > 0$ such that $B^+(a, \varepsilon) \subset A$. If j is chosen large enough that $\frac{1}{n_j} < \frac{\varepsilon}{2}$ then $C_{n_j} \subset B^+(x_{n_j}, \frac{\varepsilon}{2})$; if j is also chosen large enough so that $d(a, x_{n_j}) < \frac{\varepsilon}{2}$ then $C_{n_j} \subset B^+(a, \varepsilon)$. So $C_{n_j} \subset A$, contrary to our assumption.

So we deduce that the open covering $\mathcal{A}$ has a $\delta > 0$ so that for each ball in X with radius less than δ, there exists an element of $\mathcal{A}$ containing it. Let $\varepsilon = \frac{\delta}{2}$. By total boundedness of X we know we can find a finite covering of X by forward ε-balls. Each of these balls has radius $\varepsilon < \delta$, so it lies in an element of $\mathcal{A}$. Choosing one element of $\mathcal{A}$ for each of these ε-balls, we obtain a finite subcollection of $\mathcal{A}$ that covers X. So X is forward compact. $\square$

Proposition 4.7. *If (X, d) is forward sequentially compact, and if forward convergence implies backward convergence, then X is forward totally bounded.*

Proof. We proceed by contradiction. Suppose there is an $\varepsilon > 0$ such that X cannot be covered by finitely many forward ε-balls. Construct a sequence of points as follows: first choose x_1 to be any point of X. Noting that the ball $B^+(x_1, \varepsilon)$ is not all of X, choose $x_2 \in X \backslash B^+(x_1, \varepsilon)$. In general, choose x_{n+1} to be a point not in $B^+(x_1, \varepsilon) \cup \cdots \cup B^+(x_n, \varepsilon)$, using the fact that these balls do not cover X. By construction, $d(x_j, x_{n+1}) \geqslant \varepsilon$ for $j = 1, \ldots, n$. If this sequence had a forward convergent subsequence, then the subsequence would also be backward convergent (by assumption) and thus it would be forward Cauchy, which contradicts the construction. $\quad\square$

The formulation given in the last two propositions suffices for our purposes. More generally, however, one can show that sequentially compact asymmetric metric spaces are compact [5]. We will not use this formulation, but point out that the proof, which is much more sophisticated that the ones presented here, relies on an argument given by Niemytzki [10].

The following proposition has been shown before [11, Theorems 12 and 13], but we prefer to give our own proof.

Proposition 4.8. *An asymmetric metric space* (X, d) *is forward compact if and only if it is forward complete and forward totally bounded.*

Proof. If X is forward compact then it is forward sequentially compact by Lemma 4.4. In particular, every forward Cauchy sequence has a forward convergent subsequence, so by Lemma 4.3 the space X is forward complete. Total boundedness follows because every covering of X by forward ε-balls has a finite subcover.

Conversely, let X be forward complete and totally bounded. We shall prove that X is forward sequentially compact, which, by Proposition 4.6, implies forward compactness.

Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in X. First cover X by finitely many balls of radius 1. One of these balls, say B_1, must contain $\{x_n\}_{n \in \mathbb{N}}$ for infinitely many values of n. Let $J_1 \subset \mathbb{N}$ consist of those indices n for which $x_n \in B_1$. Now B_1 is totally bounded, so we may cover it by finitely many balls of radius $\frac{1}{2}$. At least one of these balls, say B_2, contains $\{x_n\}_{n \in \mathbb{N}}$ for infinitely many values of n in J_1. Proceeding iteratively, we can define a decreasing sequence of balls $B_1 \supset B_2 \supset B_3 \supset \cdots$ and corresponding sets $J_{k+1} := \{n \mid n \in J_k, x_n \in B_{k+1}\}$.

Choose $n_1 \in J_1$. Given n_k, choose $n_{k+1} \in J_{k+1}$ such that $n_{k+1} > n_k$. Then $x_{n_j} \in B_j$ for every $j \in \mathbb{N}$. Let y_j be the centre of B_j, so $B_j := B^+(y_j, \frac{1}{j})$. Now $d(y_j, y_{j+1}) < \frac{1}{j} \to 0$ as $j \to \infty$. Thus, for $j \leqslant k$, $d(y_j, y_k) \to 0$ as $j \to \infty$, so $\{y_k\}_{k \in \mathbb{N}}$ is a forward Cauchy sequence. Since X is forward complete, $y_k \xrightarrow{f} y \in X$ as $k \to \infty$.

Finally, $d(y, x_{n_j}) \leqslant d(y, y_j) + d(y_j, x_{n_j}) \to 0$ as $j \to \infty$. So $x_{n_j} \xrightarrow{f} y \in X$, so we have found a forward convergent subsequence of $\{x_n\}_{n \in \mathbb{N}}$, proving forward sequential compactness. $\quad\square$

5. The Arzelà–Ascoli theorem

In this section we will adapt the Arzelà–Ascoli theorem to the asymmetric case.

5.1. Equicontinuity and compactness

Definition 5.1 *(Forward equicontinuity).* Let $(X, d_X), (Y, d_Y)$ be asymmetric metric spaces. A set $\mathcal{F}$ of functions from X to Y is *forward equicontinuous*, respectively *backward equicontinuous*, if for every $x \in X$ and for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $y \in Y$ and every $f \in \mathcal{F}$ with $d_X(x, y) < \delta$,

$$d_Y\big(f(x), f(y)\big) < \varepsilon, \quad \text{respectively} \quad d_Y\big(f(y), f(x)\big) < \varepsilon$$

holds.

We introduce some further notation. Y^X denotes the space of functions from X to Y; $C(X, Y)$ denotes the set of all ff-continuous functions from X to Y. The *uniform metric* on Y^X is

$$\bar{\rho}(f, g) := \sup\big\{\bar{d}\big(f(x), g(x)\big) \mid x \in X\big\},$$

where $\bar{d}(x, y) := \min\{d(x, y), 1\}$ and d is the asymmetric metric associated with Y. This metric induces the *uniform topology* on Y^X.

Lemma 5.2. *Suppose* (Y, d) *is forward compact, and that forward convergence implies backward convergence in this metric. Then*

(1) *If a set* $\mathcal{F} \in C(X, Y)$ *is forward equicontinuous then it is also backward equicontinuous.*
(2) *If a sequence* $\{f_n\}_{n \in \mathbb{N}} \subset Y^X$ *is uniformly backward convergent then it is uniformly forward convergent* (*and vice versa*).

Before giving the proof of Lemma 5.2, we first demonstrate that the statement (1) in Lemma 5.2 is wrong without the compactness assumption on Y.

Example 5.3. Let $X = [0, 1]$ and $Y = [1, \infty)$, with the asymmetric metrics

$$d_X(x, y) := \begin{cases} y - x & \text{if } y \geqslant x, \\ 1 & \text{if } y < x, \end{cases} \qquad d_Y(x, y) := \begin{cases} y - x & \text{if } y \geqslant x, \\ e^x - e^y & \text{if } y < x. \end{cases}$$

Notice that forward convergence does imply backward convergence in Y, but that the space is not forward (or backward) compact. (Y, d_Y) has the usual topology on $\mathbb{R}$.

Let $f_n : X \to Y$ be defined by $x \mapsto x + n$. Fix $x \in X$, $n \in \mathbb{N}$ and $\varepsilon > 0$. If $d_X(x, y) < \delta \leqslant 1$ then $x \leqslant y$ and $d_X(x, y) = y - x$.

Then, $d_Y(f_n(x), f_n(y)) = y + n - (x + n) = y - x < \delta$, and the choice $\delta := \min\{\varepsilon, 1\}$ shows that $\mathcal{F} := \{f_n\}$ is forward equicontinuous. On the other hand, $d_Y(f_n(y), f_n(x)) = e^{y+n} - e^{x+n} = e^n(e^y - e^x)$, so for backward continuity we need $\delta \leqslant \log(\frac{\varepsilon}{e^n} + e^x) - x$. Thus, $\delta \to 0$ as $n \to \infty$ so we cannot choose δ independent of n. Therefore $\mathcal{F}$ is not backward equicontinuous.

Now we prove Lemma 5.2.

Proof. Forward compactness implies forward sequential compactness by Lemma 4.4. Since forward convergence implies backward convergence, we know the space is backward sequentially compact. Now, by Lemma 4.2, we know that backward convergence also implies forward convergence and can therefore conclude, using Propositions 4.6 and 4.7, that the space is backward compact.

We now use the fact that if a space is both forward and backward compact, then the forward and backward topologies are equivalent [7, Corollary 2.17]. Consequently, an easy argument will show that uniform convergence in one topology implies uniform convergence in the other, and similarly with forward and backward equicontinuity. □

There are indeed asymmetric metrics in which the forward topology is compact and the backward one is not:

Example 5.4. This example is similar to Example 3.7 but provides very different topologies.
Let $d : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{R}_0^+$ be defined by

$$d(x, y) := \begin{cases} d(x, 0) + d(0, y) & \text{if } x \neq y, \\ 0 & \text{if } x = y, \end{cases}$$

where $d(x, 0) = x$ and $d(0, x) = \frac{1}{x}$ for $x \neq 0$.

The backward topology for this asymmetric metric is discrete, so $\mathbb{Z}_+$ is not compact because it is infinite. However, $\mathbb{Z}_+$ is compact in the forward topology because any open ball around zero contains all but finitely many points. In this metric, backward convergence implies forward convergence but not vice versa.

Proposition 5.5. *Let* (X, d_X) *and* (Y, d_Y) *be asymmetric metric spaces with the forward topologies both compact. Suppose also that forward convergence implies backward convergence in* (Y, d_Y). *If a subset* $\mathcal{F}$ *of* $C(X, Y)$ *is forward equicontinuous then* $\mathcal{F}$ *is totally forward bounded under the uniform metric* $\bar{\rho}$ *corresponding to* d_Y.

Proof. (Compare [9, Lemma 45.3]; note that again the asymmetric condition of forward convergence implies backward convergence is required.) Assume $\mathcal{F} \in C(X, Y)$ is forward equicontinuous, then $\mathcal{F}$ is also backward equicontinuous by Lemma 5.2. We need to show that for any given $0 < \varepsilon < 1$, there is a cover of $\mathcal{F}$ by finitely many sets that are open ε-balls in the metric $\bar{\rho}$.

Set $\delta := \frac{\varepsilon}{3}$. Given any $a \in X$, there exists $\delta_a > 0$ such that whenever $d_X(a, x) < \delta_a$ we have that $d_Y(f(a), f(x)) < \delta$ and $d_Y(f(x), f(a)) < \delta$ for all $f \in \mathcal{F}$, $x \in X$. By forward compactness of X, we can cover X by finitely many such balls $B^+(a, \delta_a)$, for $a = a_1, \ldots, a_k$. By forward compactness of Y, we can cover Y by finitely many open sets $V_1, \ldots, V_m$ of diameter less than δ. (Here, $\mathrm{diam}(S) := \max_{x, y \in S} d(x, y)$, and every ball of diameter ε is contained in a forward ball of radius 2ε so all the compactness arguments can be used as normal.)

Let J be the collection of functions $\alpha : \{1, \ldots, k\} \to \{1, \ldots, m\}$. Given $\alpha \in J$, if there exists a function $f \in \mathcal{F}$ such that $f(a_j) \in V_{\alpha(j)}$ for all $j = 1, \ldots, k$, choose one such function and label it f_α. The collection $\{f_\alpha\}$ is indexed by a subset J' of the set J and is thus finite. We assert that the open balls $B_{\bar{\rho}}^+(f_\alpha, \varepsilon)$ cover $\mathcal{F}$, for $\alpha \in J'$. To see this, let f be in $\mathcal{F}$, and for each $j = 1, \ldots, k$, choose an integer $\alpha(j)$ such that $f(a_j) \in V_{\alpha(j)}$. Then the function α is in J'. We assert that $f \in B_{\bar{\rho}}^+(f_\alpha, \varepsilon)$.

Pick $x \in X$ and choose j so that $x \in B^+(a_j, \delta_{a_j})$. Then,

$d_Y\big(f_\alpha(x), f_\alpha(a_j)\big) < \delta$ (by equicontinuity),

$d_Y\big(f_\alpha(a_j), f(a_j)\big) < \delta$ (since $f(a_j)$ and $f_\alpha(a_j)$ are in $V_{\alpha(j)}$),

$d_Y\big(f(a_j), f(x)\big) < \delta$ (by equicontinuity).

Then we conclude that $d_Y(f_\alpha(x), f(x)) < \varepsilon < 1$. This inequality holds for every $x \in X$, so

$$\bar{\rho}(f_\alpha, f) = \max\big\{\bar{d}\big(f_\alpha(x), f(x)\big)\big\} < \varepsilon.$$

So $f \in B_{\bar{\rho}}^+(f_\alpha, \varepsilon)$. □

A difference between the symmetric and the asymmetric case will be highlighted in the next example. Namely, the Arzelà–Ascoli theorem relies on the fact that the space of forward continuous functions $C(X, Y)$ is forward complete whenever Y is forward complete. This is true in the symmetric case, but wrong in the asymmetric case unless Y is also forward compact.

Example 5.6. Let $Y = \mathbb{R}_0^+$ with the asymmetric metric

$$d_Y(x, y) := \begin{cases} y - x & \text{if } y \geqslant x, \\ e^x - e^y & \text{if } y < x. \end{cases}$$

This gives Y the usual forward and backward topologies, and forward convergence implies backward convergence.

Claim 1: Y is forward complete. Let $\{y_n\}_{n \in \mathbb{N}} \subset Y$ be a forward Cauchy sequence. It is easy to see that this sequence is forward bounded. Since the topology induced by d_Y is the Euclidean one, this implies forward convergence of a subsequence. By Lemma 4.3 this means (Y, d_Y) is forward complete.

Claim 2: Y^X is not forward complete in the uniform metric $\bar{\rho}$, where $X = \mathbb{R}_0^+$ with the usual metric. We construct a sequence which is forward Cauchy but not forward convergent in the uniform metric. Namely, let $f_n : X \to Y$ be defined by $x \mapsto x + \frac{1}{n}$, which is obviously forward continuous.

If $m \geqslant n \geqslant \frac{1}{\varepsilon}$ then

$$\bar{\rho}(f_n, f_m) = \sup\big\{\bar{d}\big(f_n(x), f_m(x)\big) \mid x \in \mathbb{R}_0^+\big\} = \frac{1}{n} - \frac{1}{m} < \frac{1}{n} \leqslant \varepsilon.$$

So $\{f_n\}_{n \in \mathbb{N}}$ is a forward Cauchy sequence in the uniform metric.

The pointwise limit of this sequence is the identity function f, but the sequence is not uniformly forward convergent. Namely, if $n > N = (\log(\frac{\varepsilon}{e^x} + 1))^{-1}$ then $d(f(x), f_n(x)) = e^x(e^{\frac{1}{n}} - 1) < \varepsilon$, but N cannot be chosen independently of x.

The next statement differs from the symmetric equivalent [9, Theorem 43.5] in the sense that compactness, rather than completeness, is required for the asymmetric case, as shown by Example 5.6.

Lemma 5.7. *If the asymmetric space (Y, d) is forward compact and has the property that forward convergence implies backward convergence, then the space Y^X is complete in the uniform metric $\bar{\rho}$ corresponding to d.*

Proof. If (Y, d) is forward compact then it is forward complete, and so is $(Y, \bar{d})$. Now suppose $f_1, f_2, \ldots$ is a sequence in Y^X that is a forward Cauchy sequence relative to $\bar{\rho}$. Then, for every $x \in X$, the sequence $f_1(x), f_2(x), \ldots$ is a forward Cauchy sequence in $(Y, \bar{d})$ and thus forward and backward converges, say to a point y_x. Let $f : X \to Y$ be the function defined by $f(x) = y_x$.

We now show that $f_n \xrightarrow{f} f$ in the metric $\bar{\rho}$ by first showing that it converges backwards. Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ so that $\bar{\rho}(f_n, f_m) < \frac{\varepsilon}{2}$ for $m \geqslant n \geqslant N$. In particular, then $\bar{d}(f_n(x), f_m(x)) < \frac{\varepsilon}{2}$ for all $x \in X$. Let n and x be fixed and let m become arbitrarily large; then

$$\bar{d}\big(f_n(x), f(x)\big) \leqslant \bar{d}\big(f_n(x), f_m(x)\big) + \bar{d}\big(f_m(x), f(x)\big) < \varepsilon.$$

This inequality holds for all $x \in X$, provided $n \geqslant N$, so $\bar{\rho}(f_n, f) < \varepsilon$ for $n \geqslant N$. So the sequence is uniformly backward convergent, and by Lemma 5.2 that means it is uniformly forward convergent too. $\square$

Lemma 5.8. *Let (X, d_X) and (Y, d_Y) be asymmetric metric spaces, with forward convergence equivalent to backward convergence in Y. If a sequence $\{f_n\}_{n \in \mathbb{N}} \subset C(X, Y)$ forward converges uniformly to f, then $f \in C(X, Y)$.*

Proof. The proof is a modification of the classic $\frac{\varepsilon}{3}$-argument. Fix $\varepsilon > 0$ and $x_0 \in X$. Since $f_n \xrightarrow{f} f$ uniformly, we can choose $N_1 \in \mathbb{N}$ so that $d_Y(f(x), f_n(x)) < \frac{\varepsilon}{3}$ for all $n \geqslant N_1$ and $x \in X$. Now, in particular, $f_n(x_0) \xrightarrow{f} f(x_0)$, and so $f_n(x_0) \xrightarrow{b} f(x_0)$. Thus we can find $N_2 \in \mathbb{N}$ so that $d_Y(f_n(x_0), f(x_0)) < \frac{\varepsilon}{3}$ for $n \geqslant N_2$. Let $N := \max\{N_1, N_2\}$. The function f_N is ff-continuous, and thus fb-continuous by equivalence of forward and backward convergence. This means we can choose $\delta > 0$ so that $d_Y(f_N(x), f_N(x_0)) < \frac{\varepsilon}{3}$ for $d_X(x_0, x) < \delta$. So, if $d_X(x_0, x) < \delta$,

$$d_Y\big(f(x), f(x_0)\big) \leqslant d_Y\big(f(x), f_N(x)\big) + d_Y\big(f_N(x), f_N(x_0)\big) + d_Y\big(f_N(x_0), f(x_0)\big) < \varepsilon.$$

Therefore f is fb-continuous, and by equivalence of convergence it is also ff-continuous, so $f \in C(X, Y)$. $\square$

We point out that this elementary statement is wrong in the asymmetric situation if the hypothesis on forward and backward convergence is dropped. Example 5.9 provides a counterexample which relies on the non-uniqueness of forward limits for metrics where forward convergence does not imply backward convergence. Example 5.10 demonstrates that, with this asymmetric assumption dropped, the uniform limit of a sequence of continuous functions might be discontinuous even if the limit is unique.

Example 5.9. Let $X := [0, 1]$ be equipped with the Euclidean metric, and $Y \subset \mathbb{R}^2$ be the set $Y := \{(y_1, y_2) \mid y_1 = 0$ and $y_2 \in (0, 1]\} \cup \{(-1, 0)\} \cup \{(1, 0)\}$. We choose the metric on Y to be the Sorgenfrey metric of Example 3.6, extended such that, for (y_1, y_2) with $y_2 > 0$, $d((y_1, y_2), (\pm 1, 0)) = d((\pm 1, 0), (\mp 1, 0)) = 1$, $d((\pm 1, 0), (y_1, y_2)) = y_2$. It is easy to see that Y is an asymmetric metric space and that Y is Hausdorff. Let $f : X \to Y$,

$$x \mapsto \begin{cases} (-1, 0) & \text{if } x = 0, \\ (1, 0) & \text{if } x > 0. \end{cases}$$

The sequence $f_n : X \to Y, x \mapsto (0, \frac{1}{n})$ forward converges uniformly to the discontinuous limit f. This is a consequence of the fact that in asymmetric metric spaces, limits are not necessarily unique. Note that there are infinitely many uniform limits in this example, for example $f_n \to \tilde{f}$ forward uniformly, with $\tilde{f} : X \to Y, x \mapsto (1, 0)$ being continuous.

Example 5.10. Let $X := [0, 1]$ be equipped with the Euclidean metric, and $Y := \{(y_1, y_2) \mid 0 \leqslant y_1 \leqslant 1, 0 \leqslant y_2 \leqslant 1\}$. We equip Y with the following variant of the Sorgenfrey metric:

$$d(y, \tilde{y}) := \begin{cases} 0 & \text{if } y = \tilde{y}, \\ |y - \tilde{y}| & \text{if } y_1 \leqslant \tilde{y}_1 \text{ and } \tilde{y}_1 \neq 0, \\ 1 & \text{otherwise}, \end{cases}$$

where $|\cdot|$ denotes the Euclidean metric in $\mathbb{R}^2$. It is easy to check that d defines an asymmetric metric. Then $f_n : X \to Y, x \mapsto (\frac{1}{n}, x)$ forward converges uniformly to a unique limit f, namely $f : X \to Y, x \mapsto (0, x)$. Moreover, f_n is continuous for every $n \in \mathbb{N}$, while f is discontinuous.

Proposition 5.11. *Let (X, d_X) and (Y, d_Y) be asymmetric metric spaces with Y being forward compact and having forward convergence implies backward convergence. Then the set $C(X, Y)$ is complete in the uniform metric $\bar{\rho}$ corresponding to d_Y.*

Proof. If $\{f_n\}_{n \in \mathbb{N}} \subset C(X, Y)$ converges to $f \in Y^X$ relative to $\bar{\rho}$ (i.e., uniform convergence). We have assumed that forward convergence implies backward convergence, and by forward compactness of Y we also know that backward convergence implies forward convergence (Lemma 4.2). Thus, by Lemma 5.8, $f \in C(X, Y)$, which means that $C(X, Y)$ is closed. Since Y^X is complete (Lemma 5.7), $C(X, Y)$ is complete as well. □

5.2. The Arzelà–Ascoli theorem

Now we finally have the tools to prove the Arzelà–Ascoli theorem in the asymmetric case. We introduce one genuinely asymmetric condition; Example 5.13 below shows that the Arzelà–Ascoli theorem does not hold if forward convergence does not imply backward convergence.

Theorem 5.12. *Let (X, d_X) and (Y, d_Y) be asymmetric metric spaces. Suppose that X is forward compact, closed bounded sets are forward compact in Y, and that forward convergence implies backward convergence in Y. Endow $C(X, Y)$ with the uniform metric $\bar{\rho}$ associated with the asymmetric metric d_Y. Then a subset $\mathcal{F}$ of $C(X, Y)$ is forward relatively compact if $\mathcal{F}$ is forward equicontinuous and forward pointwise bounded under d_Y.*

Proof. With the asymmetric framework (and its differences to the symmetric case) being established, we are in a position to follow the argument given in Munkres' book [9].

Step 1: If $\mathcal{F}$ is forward equicontinuous and forward pointwise bounded then so is $\overline{\mathcal{F}}$.

Equicontinuity of $\overline{\mathcal{F}}$: Given $x_0 \in X$ and $\varepsilon > 0$, choose $\delta > 0$ so that $d_X(x_0, x) < \delta$ implies $d_Y(f(x_0), f(x)) < \frac{\varepsilon}{3}$ for all $f \in \mathcal{F}$ and $x \in X$. Given $g \in \overline{\mathcal{F}}$, we can find a sequence of points of $\mathcal{F}$ forward converging to g, and therefore also backward converging to g. So we can choose $f \in \mathcal{F}$ so that $\bar{\rho}(g, f) < \frac{\varepsilon}{3}$ and $\bar{\rho}(f, g) < \frac{\varepsilon}{3}$. Then we have that for $d_X(x_0, x) < \delta$

$$d_Y\big(g(x_0), g(x)\big) \leqslant d_Y\big(g(x_0), f(x_0)\big) + d_Y\big(f(x_0), f(x)\big) + d_Y\big(f(x), g(x)\big) < \varepsilon.$$

Since g was arbitrary, equicontinuity follows.

Pointwise boundedness of $\overline{\mathcal{F}}$: Given $a \in X$, choose $y \in Y$ so that $\{d_Y(y, f(a)) \mid f \in \mathcal{F}\}$ is bounded, i.e., $d_Y(y, f(a)) \leqslant M$ for all $f \in \mathcal{F}$. As before, given $g \in \overline{\mathcal{F}}$, we can find a sequence of points of $\mathcal{F}$ forward and backward converging to g, and so we can choose $f \in \mathcal{F}$ so that $\bar{\rho}(f, g) < 1$. Then $d_Y(y, g(a)) \leqslant d_Y(y, f(a)) + d_Y(f(a), g(a)) < M + 1$. Now, $g \in \overline{\mathcal{F}}$ was arbitrary, so $\{d_Y(y, g(a)) \mid g \in \overline{\mathcal{F}}\}$ is bounded, and so $\overline{\mathcal{F}}$ is pointwise forward bounded.

Step 2: We show that if $\overline{\mathcal{F}}$ is forward equicontinuous and forward pointwise bounded, then there is a forward compact subspace E of Y that contains the union of the sets $g(X)$, for $g \in \overline{\mathcal{F}}$.

Choose, for each $a \in X$, a $\delta_a > 0$ so that $d_X(a, x) < \delta_a$ implies $d_Y(g(a), g(x)) < 1$ for all $g \in \overline{\mathcal{F}}$. Since X is forward compact, we can cover X by finitely many balls $B^+(a, \delta_a)$, say for $a_1, \ldots, a_n$. Because the sets $\{g(a_j) \mid g \in \overline{\mathcal{F}}\}$ are forward bounded, their union is also forward bounded; suppose the union lies in $B^+(y, N)$ for some $y \in Y$, $N \in \mathbb{N}$.

Choose $g \in \overline{\mathcal{F}}$ arbitrarily, and then choose $x \in X$. Now $x \in B^+(a_j, \delta_{a_j})$ for some $j = 1, \ldots, n$, so $d_Y(y, g(x)) \leqslant d_Y(y, g(a_j)) + d_Y(g(a_j), g(x)) < N + 1$. Thus $g(X) \subset B^+(y, N + 1)$. Let E be the closure of this ball. Since closed bounded sets are forward compact in Y, E is forward compact.

Step 3: Assume $\mathcal{F}$ is forward equicontinuous and forward pointwise bounded under d_Y. By Step 2, $\overline{\mathcal{F}}$ is also forward equicontinuous and forward pointwise bounded under d_Y. By Step 3, there is a compact subspace E of Y such that $\overline{\mathcal{F}} \subset C(X, E)$. Proposition 5.5 now reveals that $\overline{\mathcal{F}}$ is forward totally bounded under $\bar{\rho}$.

Since E is forward compact, the set $C(X, E)$ is forward complete in the uniform metric by Proposition 5.11. $\overline{\mathcal{F}}$ is then, as a closed subspace of $C(X, E)$, also forward complete.

Then by Proposition 4.8 we deduce that $\overline{\mathcal{F}}$ is forward compact because it is forward complete and forward totally bounded. So $\mathcal{F}$ is relatively compact. □

We demonstrate that this statement can be false if the asymmetric condition on forward and backward convergence is dropped.

Example 5.13. In the situation of Example 5.10, the set $\mathcal{F} := \{f_n\}_{n \in \mathbb{N}}$ is not forward relatively compact. This is since forward convergence does not imply backward convergence in Y, contrary to the assumption in Theorem 5.12.

We close by pointing out another peculiarity of asymmetric spaces: in the symmetric case, the Arzelà–Ascoli theorem is a necessary and sufficient characterisation of compactness for continuous functions. The reverse direction relies on the fact that a family of continuous functions which is totally bounded under the uniform metric is equicontinuous [9, Lemma 45.2]. It can be shown that this is not true in the asymmetric case, even if forward convergence implies backward convergence. This can, for example, be seen for the metric of Example 3.5, by constructing a sequence of increasingly rapidly oscillating functions.

Acknowledgements

References

[1] I. Chenchiah, M.O. Rieger, J. Zimmer, Gradient flows in asymmetric metric spaces, in preparation.
[2] R. Engelking, General Topology, PWN—Polish Scientific Publishers, Warsaw, 1977. Translated from the Polish by the author, Monografie Matematyczne, Tom 60 [Mathematical Monographs, vol. 60].
[3] M. Gromov, Metric Structures for Riemannian and Non-Riemannian Spaces, Progress in Mathematics, vol. 152, Birkhäuser Boston Inc., Boston, MA, 1999. Based on the 1981 French original [MR0682063 (85e:53051)]. With appendices by M. Katz, P. Pansu and S. Semmes. Translated from French by Sean Michael Bates.
[4] J.C. Kelly, Bitopological spaces, Proc. London Math. Soc. (3) 13 (1963) 71–89.
[5] H.-P.A. Künzi, A note on sequentially compact quasipseudometric spaces, Monatsh. Math. 95 (3) (1983) 219–220.
[6] A. Mainik, A. Mielke, Existence results for energetic models for rate-independent systems, Calc. Var. Partial Differential Equations 22 (1) (2005) 73–99.
[7] A. Mennucci, On asymmetric distances, Technical report, Scuola Normale Superiore, Pisa, 2004.
[8] A. Mielke, T. Roubíček, A rate-independent model for inelastic behavior of shape-memory alloys, Multiscale Model. Simul. 1 (4) (2003) 571–597 (electronic).
[9] J.R. Munkres, Topology: A First Course, Prentice-Hall Inc., Englewood Cliffs, NJ, 1975.
[10] V. Niemytzki, Über die Axiome des metrischen Raumes, Math. Ann. 104 (1) (1931) 666–671.
[11] I.L. Reilly, P.V. Subrahmanyam, M.K. Vamanamurthy, Cauchy sequences in quasipseudometric spaces, Monatsh. Math. 93 (2) (1982) 127–140.
[12] M.O. Rieger, J. Zimmer, Young measure flow as a model for damage, Preprint 11/05, Bath Institute for Complex Systems, Bath, UK, 2005.
[13] W.A. Wilson, On quasi-metric spaces, Amer. J. Math. 53 (3) (1931) 675–684.