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A Quasi-Radial Basis Functions Method for American Options Pricing

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Abstract—Based on the idea of quasi-interpolation and radial basis functions approximation, a fast and accurate numerical method is developed for solving the Black-Scholes equation for valuation of American options prices. Since the method does not require solving a resultant full matrix, the ill-conditioning problem resulting from using the radial basis functions as a global interpolant can be avoided. The method has been shown to be effective in solving problems with free boundary condition. As indicated in the numerical computation for the American option pricing, an excellent approximation of the solution as well as the free optimal exercise boundary can be obtained. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords—Radial basis functions, Quasi-interpolation, American options.

1. INTRODUCTION

In 1973, Black and Scholes [1] proposed an explicit formula for evaluating European call options without dividends. By assuming that the asset price is risk-neutral, Black and Scholes showed that the European call options value satisfies a log-normal diffusion type partial differential equation which is now known as the Black-Scholes equation. It is, however, well known that the American options pricing can be treated as a free boundary problem in which no analytical formula is available. Until recently, there were a number of different numerical methods for valuation of the American options, for instance, the finite difference method by Brennan and Schwartz [2]; the binomial method by Cox *et al.* [3]; the projected successful over-relaxation method by Wilmott *et al.* [4]; the front-fixing finite difference method by Wu and Kwok [5]; the Monte Carlo simulation by Grant *et al.* [6]; the integral equation method by Huang *et al.* [7]; the penalty method by Zvan *et al.* [8]; and, more recently, the linear programming technique by Dempster and Hutton [9]. A comparison of some of these numerical methods can be found in Geske and Shastri's [10] and Broadie and Detemple's [11] review papers.

Some kinds of analytical formulas for valuation of the American options have been proposed by Johnson [12], MacMillan [13], and Barone-Adesi and Whaley [14]. These basically transform the free boundary value problem to an integral equation whose analytical formula is obtained

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by either assuming a numerical approximation of the unknown optimal exercise boundary or a polynomial expansion of the unknown integrand.

Recently, Hon and Mao [15] developed a new numerical scheme by applying the global radial basis functions (RBFs), particularly Hardy's multiquadric (MQ), as a spatial approximation for the numerical solution of the options value and its derivatives in the Black-Scholes equation. This transformed the Black-Scholes equation into a system of first-order equations in time, and the numerical solution can then be approximated by using a high-order backward time integration scheme like fourth-order Runge Kutta method. Numerical results indicated that this RBFs method offers a highly accurate spatial approximation to the solution.

The RBFs scheme is a truly meshless computational method which does not require the generation of a regular grid as in the finite difference or a mesh as in the finite element methods. This makes the RBFs particularly efficient in solving this kind of free boundary problems. Furthermore, since the MQ-RBFs are infinitely differentiable, the higher-order partial derivatives of the options value can be directly computed by using the derivatives of the basis functions.

The RBFs method, however, faces a serious ill-conditioning problem due to the use of the RBFs as a global interpolant. While Dubal *et al.* [16] noted many benefits of using the RBFs to solve a three-dimensional nonlinear Poisson equation without the use of domain decomposition or block decomposition schemes, the matrix resulting from using the RBFs of nearly 2000 knots was extremely ill-conditioned. There are currently several ways to solve the ill-conditioning problem of using RBFs for solving partial differential equations (PDEs). For instance, domain decomposition is a common technique used in traditional numerical schemes and had been shown to reduce the condition number and increase efficiency in using the RBFs for solving PDEs by Kansa and Hon [17] and Wong *et al.* [18]. However, no theoretical discussion of convergence and condition number of the preconditioner is given by these methods.

Wu [19] recently developed classes of compactly supported radial basis functions (CSRBFs) in which the unique existence of the solution was assured. It had also been shown by Wu that, given any space dimension and under any continuity requirement, a series of CSRBFs in the form of polynomials could be constructed. Wendland [20] extended these CSRBFs to classes with minimal polynomial degree. The resultant matrices related to the scattered data interpolation or solving PDEs by using the CSRBFs are then sparse. Recently, Wong *et al.* [21] successfully applied the CSRBFs to improve the efficiency in the solving of the shallow water equation for tide and currents simulation. It was, however, found that the accuracy of these CSRBFs also depended on a suitable choice of the value of radius of support. If the value is too small, the magnitude order of accuracy is not satisfied. If the value is too large, the accuracy will be improved in the expense of a decrease in sparsity of the resultant matrix.

This paper combines the quasi-interpolation and the RBFs methods to solve the option pricing models. Since the undetermined coefficients of the quasi-RBFs interpolants for the solution can then be computed by using simple backward and forward substitutions in solving a resultant banded symmetric matrix, the ill-conditioning problem resulting from using the global RBFs is eliminated. Numerical examples show that the total number of knots can be extended to several thousands, which is impossible for the global RBFs approach. A recent application on combining the quasi-interpolation technique with the dual reciprocal method to solve stiff problems can be found from Hon and Wu [22].

The organization of this paper is as follows. In Section 2, we introduce the quasi-RBFs method for solving the Black-Scholes equation. Numerical comparison with the analytical formula in the European options case is given in Section 3. In Section 4, the method is extended for American options valuation. The optimal exercise boundary is also computed using the efficient Newton's iterative method. Numerical comparisons with the binomial method, front-fixed finite difference method, and the global RBFs method are also given. Conclusions are presented in Section 5.

2. QUASI-RBFS FOR SOLVING BLACK-SCHOLES EQUATION

Consider an interpolation problem: Given a function $f(x) \in C^1[a,b]$ and the data $\{(x_j, f_j)\}_{j=0}^N$ where $a = x_0 < x_1 < \cdots < x_N = b$ with density $h = \max(x_j - x_{j-1})$. Beatson and Powell [23] proposed the following quasi-interpolation formula for the function f:

$$\tilde{f}(x) = f_0'\gamma_0 + f_0\beta_0 + \sum_{j=1}^{N-1} f_j\psi_j(x) + f_N'\gamma_N + f_N\beta_N, \tag{1}$$

where

$$\begin{aligned}
\psi_j(x) &= \frac{\phi_{j+1} - \phi_j}{2(x_{j+1} - x_j)} - \frac{\phi_j - \phi_{j-1}}{2(x_j - x_{j-1})}, \\
\phi_j &= \phi(\|x - x_j\|), \\
\beta_0 &= \frac{1}{2} + \frac{\phi_1 - \phi_0}{2(x_1 - x_0)}, \qquad \beta_N = \frac{1}{2} + \frac{\phi_{N-1} - \phi_N}{2(x_N - x_{N-1})}, \\
\gamma_0 &= \frac{1}{2}(x - x_0) - \frac{1}{2}\phi_0, \qquad \gamma_N = \frac{1}{2}\phi_N - \frac{1}{2}(x_N - x).
\end{aligned}$$

Here, $\phi(\|x - x_j\|)$ is called a radial basis function due to the radial distance ($\|x - x_j\|$). Particularly, the Hardy's multiquadric $\phi_j(x) = \sqrt{c^2 + \|x - x_j\|^2}$ is commonly used. Since the undetermined coefficients f_j are the given function values at x_j, the method does not require solving a resultant matrix for their values. The quasi-interpolant (1), however, requires the function's derivatives at the endpoints, which may not be attainable for practical applications. Wu and Schaback [24] later improved the formula (1) to

$$\tilde{f}(x) = f_0\alpha_0 + f_1\alpha_1 + \sum_{j=2}^{N-2} f_j\psi_j(x) + f_{N-1}\alpha_{N-1} + f_N\alpha_N, \tag{2}$$

where

$$\begin{aligned}
\alpha_0 &= \frac{1}{2} + \frac{\phi_1 - (x - x_0)}{2(x_1 - x_0)}, \qquad \alpha_1 = \frac{\phi_2 - \phi_1}{2(x_2 - x_1)} - \frac{\phi_1 - (x - x_0)}{2(x_1 - x_0)}, \\
\alpha_{N-1} &= \frac{(x_N - x) - \phi_{N-1}}{2(x_N - x_{N-1})} - \frac{\phi_{N-1} - \phi_{N-2}}{2(x_{N-1} - x_{N-2})}, \qquad \alpha_N = \frac{1}{2} + \frac{\phi_{N-1} - (x_N - x)}{2(x_N - x_{N-1})}.
\end{aligned}$$

The quasi-interpolation formula (2) does not require the function's derivatives and has been proven to be shape preserving and with convergence order of $O(h^2 \log h)$. Furthermore, if the data form an infinite uniform grid, a higher-order quasi-interpolant based on radial basis functions ϕ can be obtained [24]

$$\tilde{f}(x) = \sum_{j \in \mathbf{Z}} f_j \Psi_j(x), \tag{3}$$

where (x_j, f_j) are given data and $\Psi_j(x) = \Psi(\|x - x_j\|)$ is a linear combination of the functions $\phi_j(x)$ as

$$\begin{aligned}
\Psi_j &= \frac{\psi_{j+1} - \psi_j}{2(x_{j+2} - x_{j-1})} - \frac{\psi_j - \psi_{j-1}}{2(x_{j+1} - x_{j-2})}, \qquad j \in \mathbf{Z}, \\
\psi_j &= \frac{(\phi_{j+1} - \phi_j)/2(x_{j+1} - x_j) - (\phi_j - \phi_{j-1})/2(x_j - x_{j-1})}{x_{j+1} - x_{j-1}}, \qquad j \in \mathbf{Z},
\end{aligned}$$

with the radial basis function ϕ_j defined by

$$\phi_j = \phi(\|x - x_j\|) = \left(\|x - x_j\|^2 + c^2\right)^{3/2}. \tag{4}$$

The function Ψ_j clearly satisfies

$$\sum_{j\in\mathbf{Z}} \Psi_j(x) = 1, \tag{5}$$

and hence, the value $\Psi_{ij} = \Psi_j(x_i)$ is monotonically approaching zero when the distance between any two points x_i and x_j increases. Given any $\varepsilon > 0$, we can find a positive integer M so that the summation (5) becomes

$$1 - \sum_{j=i-M}^{i+M} \Psi_j(x) < \varepsilon, \tag{6}$$

for any fixed $x = x_i$. This implies that the value of Ψ_{ij} is close to zero if $|i-j| > M$. The resultant coefficient matrix $\{\Psi_{ij}\}$ is then approximately a banded matrix with bandwidth $2M + 1$. Also, its first- and second-order derivatives satisfy

$$\sum_{j=i-M}^{i+M} \frac{\partial \Psi_j(x_i)}{\partial x} < \varepsilon, \qquad i \in \mathbf{Z}, \tag{7}$$

$$\sum_{j=i-M}^{i+M} \frac{\partial^2 \Psi_j(x_i)}{\partial x^2} < \varepsilon, \qquad i \in \mathbf{Z}. \tag{8}$$

For bounded problems with given data points $\{x_j\}_{j=0}^N$, four extra points, namely x_{-2} and x_{-1}, x_{N+1} and x_{N+2} will be added artificially to both endpoints, respectively. Assume that $x_{-2} < x_{-1} < x_0 < x_1 < \cdots < x_{N-1} < x_N < x_{N+1} < x_{N+2}$. For the leftmost four points, from x_{-2} to x_1 and the rightmost four points, from x_{N-1} to x_{N+2}, the cubic splines $\phi_j(x) = (x - x_j)^3$ and $\phi_j(x) = (x_j - x)^3$ are chosen, respectively, as the radial basis functions instead of (4). This ensures that conditions (5)–(8) still hold. The quasi-interpolation formula (3) becomes

$$\tilde{f}(x) = \sum_{j=0}^{N} f_j \Psi_j(x). \tag{9}$$

To illustrate how to apply the quasi-interpolation formula given by (9) for solving the options pricing model, we consider the following Black-Scholes equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{10}$$

where r is the risk-free interest rate, σ is the volatility of the stock price S, and $V(S,t)$ is the option value at time t and stock price S. The terminal condition is given by the maximum payoff valuation

$$V(S,\tau) = \begin{cases} \max\{K - S, 0\}, & \text{for put}, \\ \max\{S - K, 0\}, & \text{for call}, \end{cases} \tag{11}$$

where τ is the time of maturity and K is the strike price of the option. A simple transformation $S = e^x$ changes equation (10) and condition (11) to

$$\frac{\partial U}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 U}{\partial x^2} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial U}{\partial x} - rU = 0, \tag{12}$$

with terminal condition

$$U(x,\tau) = \begin{cases} \max\{K - e^x, 0\}, & \text{for put}, \\ \max\{e^x - K, 0\}, & \text{for call}. \end{cases} \tag{13}$$

The idea of the proposed numerical scheme is to approximate the unknown function U using the quasi-RBFs Ψ by

$$U(x,t) \simeq \sum_{j=0}^{N} U_j(t)\Psi_j(x), \tag{14}$$

where U_j is the unknown option value at $x = x_j$ which depends on time t and $\Psi_j(x) = \Psi(\|x-x_j\|)$ is a linear combination of the radial basis functions $\phi(\|x - x_j\|)$. For the data points $\{x_j\}_{j=0}^{N}$ and the four extra artificial endpoints, it is assumed that $x_{-2} < x_{-1} < x_0 < x_1 < \cdots < x_{N-1} < x_N < x_{N+1} < x_{N+2}$. The terms in the quasi-interpolation formula (14) are then given as

$$\Psi_j = \frac{\psi_{j+1} - \psi_j}{2(x_{j+2} - x_{j-1})} - \frac{\psi_j - \psi_{j-1}}{2(x_{j+1} - x_{j-2})}, \qquad j = 0, \ldots, N, \tag{15}$$

$$\psi_j = \frac{(\phi_{j+1} - \phi_j)/2(x_{j+1} - x_j) - (\phi_j - \phi_{j-1})/2(x_j - x_{j-1})}{x_{j+1} - x_{j-1}}, \qquad j = -1, 0, 1, \ldots, N, N+1, \tag{16}$$

where the radial basis function is chosen to be

$$\phi_j(x) = \begin{cases} (x - x_j)^3, & \text{for } -2 \le j \le 1, \\ \left((x - x_j)^2 + c^2\right)^{3/2}, & \text{for } 2 \le j \le N-2, \\ (x_j - x)^3, & \text{for } N-1 \le j \le N+2. \end{cases} \tag{17}$$

The functions $\Psi_j(x)$ satisfy

$$\sum_{j=0}^{N} \Psi_j(x) = \frac{1}{2}, \tag{18}$$

$$\sum_{j=0}^{N} \frac{\partial \Psi_j(x)}{\partial x} = 0, \tag{19}$$

$$\sum_{j=0}^{N} \frac{\partial^2 \Psi_j(x)}{\partial x^2} = 0, \tag{20}$$

and the elements $\Psi_{ij} = \Phi_j(x_i)$, $\frac{\partial \Phi_j(x_i)}{\partial x}$, and $\frac{\partial^2 \Phi_j(x_i)}{\partial x^2}$ decrease monotonically to zero as the distance between the two points x_i and x_j increases. For any $\epsilon > 0$, we can determine a value of M so that

$$\frac{1}{2} - \sum_{j=i-M}^{i+M} \Psi_j(x_i) < \epsilon, \tag{21}$$

$$\sum_{j=i-M}^{i+M} \frac{\partial \Psi_j(x_i)}{\partial x} < \epsilon, \tag{22}$$

$$\sum_{j=i-M}^{i+M} \frac{\partial^2 \Psi_j(x_i)}{\partial x^2} < \epsilon, \tag{23}$$

for all values of $i = 0, 1, \ldots, N-1, N$. The coefficient matrices $\{\Psi_{ij}\}$ and its derivatives $\{\frac{\partial \Psi_{ij}}{\partial x}\}$ and $\{\frac{\partial^2 \Psi_{ij}}{\partial x^2}\}$ can then be treated approximately as banded matrices with bandwidth of $2M + 1$.

The positive constant c^2 contained in the multiquadric (MQ) function given by (17) is called a shape parameter whose magnitude of value affects the accuracy of the approximation. In most applications of using the MQ for scattered data interpolation, a constant shape parameter is assumed for simplicity. Kansa [25], Hon *et al.* [26], and Golberg *et al.* [27] have shown that the use of the MQ for solving partial differential equations is highly effective and accurate. However, the accuracy of the MQ is greatly affected by the choice of the shape parameter c^2 whose optimal value is still unknown. Hickernell and Hon [28] and Golberg *et al.* [27] had successfully used the technique of cross-validation to obtain an optimal value of the shape parameter. Recently, an algorithm for selecting a good value of the parameter based on the total number of data points and the precision of the machine is given by Rippa [29]. In this paper, the quasi-interpolant (14) with the multiquadric function given by (17) is applied as a spatial approximation for the unknown function U in (12). While theoretical studies on the choice of the optimal shape parameter are still ongoing, we choose c^2 to be $0.1d_{\text{ave}}$, where d_{ave} is the average of the minimum distance between any two distinct points x_i and x_j.

Collocating (12) at the $N+1$ points x_i, $i = 0, \dots, N$, and bearing in mind the approximation (14), the following system of linear equations is obtained:

$$\frac{\partial U(x_i,t)}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 U(x_i,t)}{\partial x^2} + \left(r - \frac{1}{2}\sigma^2\right) \frac{\partial U(x_i,t)}{\partial x} - rU(x_i,t) = 0. \tag{24}$$

Since the basis function does not depend on time, the time derivative of U is simply the time derivatives of the coefficients

$$\frac{\partial U(x_i,t)}{\partial t} = \sum_{j=0}^{N} \frac{dU_j(t)}{dt} \Psi_j(x_i). \tag{25}$$

The first and second partial derivatives of U with respect to x are given, respectively, by

$$\frac{\partial U(x_i,t)}{\partial x} = \sum_{j=0}^{N} U_j(t) \frac{\partial \Psi_j(x_i)}{\partial x}, \tag{26}$$

$$\frac{\partial^2 U(x_i,t)}{\partial x^2} = \sum_{j=0}^{N} U_j(t) \frac{\partial^2 \Psi_j(x_i)}{\partial x^2}. \tag{27}$$

In fact, any order of partial derivatives can be computed from the quasi-RBFs Ψ_j which are continuous differentiable over the computational domain. In matrix form, equation (24) can be expressed as

$$\mathbf{\Psi}\dot{\mathbf{U}} + \frac{1}{2}\sigma^2 \mathbf{\Psi}_{xx}\mathbf{U} + \left(r - \frac{1}{2}\sigma^2\right) \mathbf{\Psi}_x \mathbf{U} - r\mathbf{\Psi}\mathbf{U} = 0, \tag{28}$$

where $\mathbf{U}$ denotes the vector containing the unknown option value $U_j = U(x_j,t)$ and $\mathbf{\Psi}$, $\mathbf{\Psi}_x$, and $\mathbf{\Psi}_{xx}$ are the $(N+1) \times (N+1)$ matrices of $\Psi_j(x_i)$, $\frac{\partial \Psi_j(x_i)}{\partial x}$, and $\frac{\partial^2 \Psi_j(x_i)}{\partial x^2}$, respectively. Here, the overdot ($\dot{}$) indicates the time derivative. Equation (28) can then be rewritten as

$$\mathbf{\Psi}\dot{\mathbf{U}} = -\left[\frac{1}{2}\sigma^2 \mathbf{\Psi}_{xx}\mathbf{U} + \left(r - \frac{1}{2}\sigma^2\right) \mathbf{\Psi}_x \mathbf{U} - r\mathbf{\Psi}\mathbf{U}\right] \equiv \mathbf{P}\mathbf{U}, \tag{29}$$

where $\mathbf{P}$ is the $(N+1) \times (N+1)$ matrix

$$\mathbf{P} = r\mathbf{\Psi} - \frac{1}{2}\sigma^2 \mathbf{\Psi}_{xx} - \left(r - \frac{1}{2}\sigma^2\right) \mathbf{\Psi}_x. \tag{30}$$

For fixed points x_j, equation (29) is a linear system of first-order homogeneous ordinary differential equations with constant coefficients. Starting from the terminal condition (13), we can use any backward time integration scheme to obtain the unknown coefficients $\mathbf{U}$ at each time step $\tau - n\Delta t$. For notational convenience, let $\mathbf{U}_n$ denote the vector $[U(x_0,t_n), U(x_1,t_n), \dots, U(x_N,t_n)]'$ at each time step $t_n = \tau - n\Delta t, n = 0, 1, \ldots, T$ where $T = \tau/\Delta t$. The following implicit numerical time integration scheme is used to discretize equation (29) for the valuations of the European and American options:

$$\mathbf{\Psi}\mathbf{U}_n = \mathbf{\Psi}\mathbf{U}_{n-1} - \Delta t\mathbf{P}\left[\theta\mathbf{U}_{n-1} + (1-\theta)\mathbf{U}_n\right]. \tag{31}$$

Let $\mathbf{P}_1 = [\mathbf{\Psi} + (1-\theta)\Delta t\mathbf{P}]$ and $\mathbf{P}_2 = [\mathbf{\Psi} - \theta\Delta t\mathbf{P}]$. Equation (31) can further be rewritten as

$$\mathbf{P}_1\mathbf{U}_n = \mathbf{P}_2\mathbf{U}_{n-1}. \tag{32}$$

Here, the $(N+1) \times (N+1)$ coefficient matrices $\mathbf{P}_1$ and $\mathbf{P}_2$ are approximately symmetrically banded with bandwidth of $2M+1$. An efficient solver for this kind of banded matrix can be found in the book of Golub *et al.* [30], in which its numerical solution can be obtained by simple backward and forward substitutions. In all of the following computations, the value of θ is chosen to be 0.5.

3. NUMERICAL COMPUTATION OF EUROPEAN OPTIONS

For European options pricing, the following boundary conditions are imposed:

$$\begin{aligned} &V(0,t) = Ke^{-r(\tau-t)}, \qquad V(S,t) \to 0, \quad \text{as } S \to \infty \text{ for put},\\ &V(0,t) = 0, \qquad\qquad\quad V(S,t) \to S, \quad \text{as } S \to \infty \text{ for call}. \end{aligned} \tag{33}$$

The exact solution of equation (10) subject to the terminal condition (11) and the boundary conditions (33) is given by

$$\begin{aligned} &V(S,t) = Ke^{-r(\tau-t)}\aleph(-d_2) - S\aleph(-d_1), \qquad \text{for put},\\ &V(S,t) = S\aleph(d_1) - Ke^{-r(\tau-t)}\aleph(d_2), \qquad \text{for call}, \end{aligned} \tag{34}$$

where $\aleph(\cdot)$ is the cumulative standard normal distribution function with

$$d_1 = \frac{\log(S/K) + \left(r + (1/2)\,\sigma^2\right)(\tau - t)}{\sigma\sqrt{\tau - t}} \tag{35}$$

and

$$d_2 = \frac{\log(S/K) + \left(r - (1/2)\,\sigma^2\right)(\tau - t)}{\sigma\sqrt{\tau - t}}. \tag{36}$$

For illustration of the accuracy of the proposed method, we consider a European put option with $K = 10$, $r = 0.05$, $\sigma = 0.20$, and $\tau = 0.5$ (year) as given in the book by Wilmott *et al.* [31]. Let $x \in [x_{\min}, x_{\max}]$, and hence, $S \in [e^{x_{\min}}, e^{x_{\max}}]$. In this computation, we choose $x_{\min} = -3.5$ and $x_{\max} = 4.5$ so that the range for the stock S is sufficiently large to satisfy the boundary condition (33). Let N be a fixed positive integer, $\Delta x = (x_{\max} - x_{\min})/N$ and $x_j = x_{\min} + j\Delta x$ for $j = 0, 1, 2, \ldots, N$. With these fixed values of x_j, the matrices $\mathbf{\Psi}$, $\mathbf{\Psi}_x$, and $\mathbf{\Psi}_{xx}$ in (29) are constant matrices. With $\Delta t = \tau/T$, where T is a positive integer, denote each time step $t_n = \tau - n\Delta t$ for $n = 1, 2, \ldots, T$. From the terminal condition (13) for European put option, the initial elements $U_0(i)$ of the initial vector $\mathbf{U}_0$ are computed by

$$U_0(i+1) = U(x_i, \tau) = \max\left\{K - e^{x_i}, 0\right\}, \qquad i = 0, 1, \ldots, N. \tag{37}$$

From the backward implicit formula (32), the option values $\mathbf{U}_n$, $n = 1, 2, \ldots, T$, can then be obtained iteratively. Since the matrices $\mathbf{P}_1$ and $\mathbf{P}_2$ are approximately banded with bandwidth $2M+1$, the numerical solutions $\mathbf{U}_n$ can be obtained efficiently by using simple backward and forward substitutions. To satisfy the boundary condition (33), at each time step t_n, we update $U_n(1) = U(x_0, t_n) = Ke^{-r(\tau - t_n)}$ and $U_n(N+1) = U(x_N, t_n) = 0$.

Table 1. Comparison between the exact and the quasi-RBFs solutions for the European option. $K = 10$, $r = 0.05$, $\sigma = 0.20$, $\tau = 0.5$, $x_{\min} = -3.5$ and $x_{\max} = 4.5$, $N = 2000$, and $T = 100$.

Stock S	Exact	Quasi-RBFs
2.00	7.7531	7.7531
4.00	5.7531	5.7531
6.00	3.7532	3.7532
7.00	2.7568	2.7568
8.00	1.7987	1.7988
9.00	0.9880	0.9881
10.00	0.4420	0.4420
11.00	0.1606	0.1606
12.00	0.0483	0.0483
13.00	0.0124	0.0124
14.00	0.0028	0.0028
15.00	0.0006	0.0006
16.00	0.0001	0.0001
RMSE		0.00004

The numerical computations are performed on a SUN Sparc workstation by using FORTRAN 77 with double precision. With N equals 2000, T equals 100, and M equals 20 (so that $\epsilon < 10^{-7}$), the CPU time of the computation was about 12 seconds. The result of comparison with the exact solution is given in Table 1.

The numerical comparison shown in Table 1 indicates that the quasi-RBFs method provides a highly accurate approximation to the solution of the European option. With $\Delta x = 0.004$ when $N = 2000$ and $\Delta t = 0.005$ when $T = 100$, the root-mean-square-error (RMSE) defined by

$$\mathrm{RMSE} = \frac{1}{13}\sqrt{\sum_{i=1}^{13}\left(\frac{V(S_i,0) - U(\log(S_i),0)}{V(S_i,0)}\right)^2},$$

where V is the exact solution computed from (34), U is the numerical approximation, and S_i are the stock values in the table, has already been reduced to 0.00004.

4. NUMERICAL COMPUTATION OF AMERICAN OPTIONS

It is well known that the American options valuation can be treated as a free boundary value problem, and until very recently no analytical formula was available. The American options allow early exercise at any time $t \in [0, \tau]$ with optimal exercise stock value $S = B(t)$. The difficulty for most numerical methods to compute an accurate solution for the American options is due to the unknown free boundary $B(t)$. To satisfy this early optimal exercise, the Black-Scholes equation for the American put options valuation is imposed by Wilmott *et al.* [31] as

$$\begin{aligned} \frac{\partial U}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 U}{\partial x^2} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial U}{\partial x} - rU = 0, &\qquad x > x_{\mathrm{opt}}(t), \\ U(x,t) = \max\{U(x,t), U(x,\tau)\}, &\qquad x \le x_{\mathrm{opt}}(t), \end{aligned} \tag{38}$$

where $x_{\mathrm{opt}}(t) = \log(B(t))$ is the corresponding optimal exercise point due to the transformation $S = e^x$ and $U(x,\tau) = K - e^x$ is the maximum payoff value given by the terminal condition (13). The region $x \le x_{\mathrm{opt}}(t)$ corresponds to where the American options should be early exercised to attain the optimal value $U(x,\tau)$.

The difficulty to solve equation (38) is due to the unknown optimal exercise point $x_{\text{opt}}(t)$. To satisfy this early optimal exercise for the valuation of the American put options, we simply update, at each time step t_n in the valuation of the European option, the elements of $\mathbf{U}_n$ by $\mathbf{U}_n(i) = \max\{K - e^{x_i}, \mathbf{U}_n(i)\}$. This makes the valuation of the American options relatively simple. To demonstrate the accuracy of this quasi-interpolation method for the American put options, we consider an American put option with $K = 100$, $r = 0.1$, $\sigma = 0.30$, and $\tau = 1$ (year). Let $x \in [-5, 7]$ so that $S \in [e^{-5}, e^{7}]$. In this computation, we take $N = 2000$, $T = 500$, and $M = 20$ and apply the quasi-RBFs method with the implicit time integration scheme to compute the American put option values. The CPU time of computation is approximately 47 seconds. Table 2 gives the results of comparison among the binomial, front-fixing finite difference (F-F-F), global RBFs, and quasi-RBFs methods for the American put option values.

Table 2. Comparison of accuracy for the American option $K = 100$, $r = 0.1$, $\sigma = 0.30$, $\tau = 1$, $x_{\min} = -5$, $x_{\max} = 7$.

Stock S	Binomial $N = 1000$	F-F-F $N = 134$, $T = 100$	Implicit: Global RBFs $N = 101$, $T = 100$	Implicit: Quasi-RBFs $N = 2000$, $T = 500$
80	20.2689	20.2662	20.2777	20.2655
85	16.3467	16.3396	16.3378	16.3427
90	13.1228	13.1124	13.1142	13.1185
95	10.4847	10.4733	10.4752	10.4813
100	8.3348	8.3277	8.3338	8.3363
105	6.6071	6.5936	6.6010	6.6020
110	5.2091	5.2004	5.2092	5.2079
115	4.0976	4.0872	4.0965	4.0935
120	3.2059	3.2023	3.2108	3.2072
RMSE		0.0090	0.0054	0.0034

It can be observed from Table 2 that the quasi-RBFs method with the implicit time integration schemes provides a better approximation to the American put option compared with the binomial method with $N = 1000$. Although a total of 2000 knots and 500 time steps was employed, the overall CPU time was still less than using the global RBFs with only 101 knots and 100 time steps. This is due to the efficient solver for banded symmetric matrix resulting from using this quasi-RBFs method. It is also noted here that the use of the quasi-RBFs method avoids the ill-conditioning problem caused from using the RBFs globally. Furthermore, since the quasi-RBFs basis functions are infinitely differentiable, the computations of the derivatives of the options values are readily available from the derivatives of the basis functions. For illustration, Table 3 shows a comparison among these methods for the Delta values $\frac{\partial V(S,0)}{\partial S}$ of the same American put option by evaluating equation (26) for $(1/S)\,\frac{\partial U(x,0)}{\partial x}$ at $x = \log(S)$.

It can again be observed from Table 3 that the Delta values obtained from the quasi-RBFs method appeared to be the closest to those obtained by the binomial method.

In most existing numerical methods for the valuation of American options, the determination of the unknown optimal exercise boundary $B(t)$ is difficult. From equations (11) and (38), the boundary condition at the optimal exercise boundary $B(t)$ for an American put option can be stated as

$$V(B(t), t) = K - B(t), \tag{39}$$

or equivalently,

$$U(x_{\text{opt}}(t), t) = K - x_{\text{opt}}(t), \tag{40}$$

Table 3. Comparison of accuracy for the Delta values. $K = 100$, $r = 0.1$, $\sigma = 0.30$, $\tau = 1$, $x_{\min} = -5$, $x_{\max} = 7$.

Stock S	Binomial $N = 1000$	F-F-F $N = 134$, $T = 100$	Implicit: Global RBFs $N = 101$, $T = 100$	Implicit: Quasi-RBFs $N = 2000$, $T = 500$
80	−0.8631	−0.8661	−0.8707	−0.8630
85	−0.7109	−0.7133	−0.7101	−0.7106
90	−0.5829	−0.5848	−0.5836	−0.5827
95	−0.4755	−0.4769	−0.4748	−0.4753
100	−0.3856	−0.3866	−0.3849	−0.3854
105	−0.3108	−0.3116	−0.3104	−0.3107
110	−0.2491	−0.2497	−0.2485	−0.2490
115	−0.1986	−0.1990	−0.1983	−0.1985
120	−0.1575	−0.1578	−0.1573	−0.1575
RMSE		0.0016	0.0026	0.0002

where $B(t) = e^{x_{\rm opt}(t)}$. At each time t_n, the approximated optimal exercise boundary $B(\tau - n\Delta t) = e^{x_n^*}$ can be obtained by applying the fast Newton's iterative method to locate the root x_n^* of the function $F(x) = U(x, \tau - n\Delta t) - K + x$ as given by

$$x_n^{(m)} = x_n^{(m-1)} - \frac{F\left(x_n^{(m-1)}\right)}{F'\left(x_n^{(m-1)}\right)}, \tag{41}$$

where the function F and its derivative F' at any value x can easily be computed by using the derivatives of the basis functions. This is definitely an advantage of this quasi-interpolation method which is not shared by the finite element or finite difference methods. The initial value $x_n^{(0)}$ at each time step $t = t_n$ is taken to be the value of x_{n-1}^* obtained from previous time step $t = t_{n-1}$. In the computation of the American option, it is observed that the computed value of $e^{x_{\rm opt}(\tau)}$ is 76.33, which is very close to the true optimal value of $B(\tau) = 76.25$. The following Figure 1

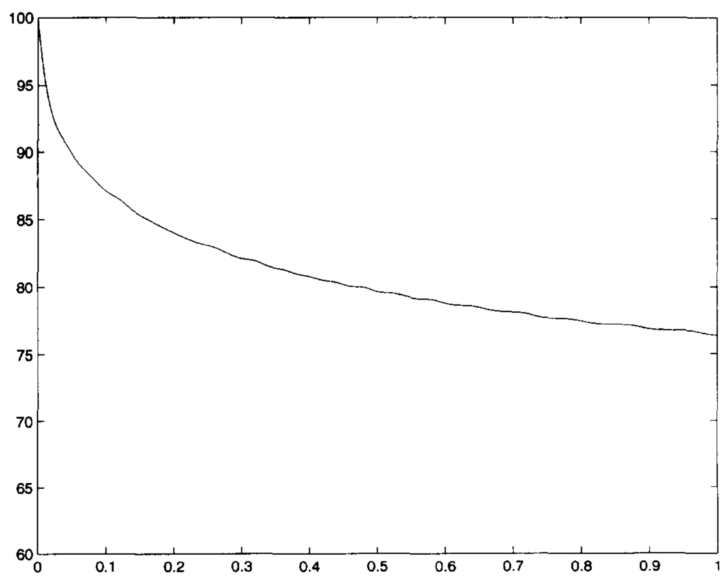

Figure 1. Optimal exercise boundary $K = 100$, $r = 0.1$, $\sigma = 0.30$, $\tau = 1$, $S_{\min} = 1$, $S_{\max} = e^6$, $N = 2000$, and $M = 500$.

gives the graph of the optimal exercise boundary $B(t)$ which is as smooth as the one obtained by Wu and Kwok [5], who used the front-fixing method to transform the unknown boundary $B(t)$ into the continuous Black-Scholes equation so that a continuous curve of $B(t)$ can be obtained by using the finite difference method.

5. CONCLUSIONS

Numerical results show that the quasi-RBFs method, particularly the use of the multiquadric basis functions, offers a very high accuracy in the computations of both European and American options. Unlike the finite element method which interpolates the solution by using low-order piecewise continuous polynomials or the finite difference method where the derivatives of the solution are approximated by finite quotients, the proposed quasi-RBFs method provides a global interpolation formula not only for the solution, but also for its derivatives. This makes the computations of those important indicators like Delta values as a bonus without a need to use extra interpolation technique. The free boundary condition in the valuation of American options usually places a great difficulty to most existing numerical methods for obtaining an accurate approximation. This, however, does not apply to this proposed method. The valuation of American options has shown to be almost the same as European ones except for the update procedure. One disadvantage of the global RBFs method is the full resulting matrix which normally hinders its application to large scale problems. The use of this quasi-RBFs method results in a banded symmetric matrix whose numerical solution can be obtained efficiently from simple backward and forward substitutions. With the encouraging results from our recent numerical computations, we believe that this quasi-RBFs method can provide an improved RBFs scheme for solving large scale problems. The extendability of applying this quasi-RBFs method to the multiple assets Black-Scholes model, however, depends on the modification of the quasi-interpolation formula to handle multivariables, which is the current focus of the author.

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