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Two subclasses of the class MOBI

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Abstract

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Assuming that all spaces are regular we prove that open and compact images of σ-locally compact metric spaces are σ-locally compact metacompact Moore spaces while open and compact images of σ-locally compact metacompact Moore spaces form the class of spaces with a point-countable base of countable order and a closure-preserving closed cover by σ-compact sets. Moreover, this class is the minimal class of regular spaces which contains all σ-locally compact metric spaces and is invariant under open and compact mappings. The complete version of these results gives a characterization of images of C-scattered metric spaces.

Keywords: σ-locally compact metric spaces, open and compact mappings, neighbornets.

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For a class $\mathscr{P}$ of topological spaces, let MOBI($\mathscr{P}$) be the minimal class of regular spaces containing all metric spaces from $\mathscr{P}$ and invariant under open and compact mappings (see [3]). If all metric spaces are contained in $\mathscr{P}$, then we write MOBI instead of MOBI($\mathscr{P}$) [1, 5.7].

It is easy to observe that a regular space is in MOBI($\mathscr{P}$) if and only if it can be obtained as an image of a metric space from $\mathscr{P}$ under a mapping which is a composition of a finite number of open and compact mappings with regular domains (see [2]).

The purpose of this paper is to prove a characterization of the class MOBI(σ-locally compact). This gives a partial solution to the problem of characterizing MOBI in the class of regular spaces (see [6, Problems 5.3 and 7]) and extends the characterization of MOBI(σ-discrete) from [5]. As a corollary to our result we obtain a description of MOBI(C-scattered) which extends the characterization of MOBI(scattered) from [4].

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Unless stated otherwise, all spaces are assumed to be regular. All mappings are continuous and onto. Open and compact mappings are open mappings with compact fibers. A sequence $\langle G_n : n \geq 1\rangle$ of subsets of a space Z is called decreasing (strictly decreasing) if $G_{n+1} \subset G_n$ ($\bar{G}_{n+1} \subset G_n$) for $n \geq 1$.

Recall that a space Z is said to have a base of countable order if there exists a sequence $\langle \mathcal{G}_n : n \geq 1\rangle$ of bases of Z such that each decreasing sequence $\langle G_n : n \geq 1\rangle$, where $G_n \in \mathcal{G}_n$ for $n \geq 1$, satisfies

$$\text{if } z \in \bigcap\{G_n : n \geq 1\}, \text{ then } \{G_n : n \geq 1\} \text{ is a base for } z \text{ in } Z. \tag{δ}$$

If each decreasing sequence $\langle G_n : n \geq 1\rangle$ satisfies

$$\text{if } z_n \in G_n \text{ for } n \geq 1, \text{ then } \langle z_n : n \geq 1\rangle \text{ has an accumulation point,} \tag{λ_c}$$

then Z is said to be a λ_c-space. If both conditions are satisfied, then Z is said to have a λ-base (see [10, 17, 18]).

It is well known (see [1, 18]) that all spaces in MOBI have a point-countable base of countable order and that a space from MOBI is in MOBI(complete) if and only if it has a λ-base.

A space Z is said to be σ-locally compact if Z is the union of countably many locally compact subspaces. Clearly all σ-discrete spaces are σ-locally compact.

A space Z is called C-scattered [16] if every closed subset F of Z contains a compact set with nonempty interior (in F).

We shall often use the following, well-known,

Lemma 0.1. *If Z is a semistratifiable space* [12, 5], *then every compact subset of Z is metrizable. If a space Z has either a point-countable base or a base of countable order, then every compact subset of Z satisfies, in Z, the second axiom of countability.*

1. The first mapping

We shall start with some results describing properties related to σ-local compactness. These results show the analogies between σ-locally compact and σ-discrete spaces.

Proposition 1.1. *If Z is a perfectly subparacompact space, then Z is σ-locally compact if and only if Z has a σ-discrete cover by compact sets.*

Proof. The "if" part is obvious. To prove the "only if" part, observe that since the space Z is perfect, it suffices to show that it can be covered by a countable collection of relatively discrete families of compact sets. Thus we can assume that Z is a locally compact space and use subparacompactness (which is hereditary in the class of perfect spaces) to construct the required cover. □

Proposition 1.2. *If Z is a perfectly metacompact space, then Z has a σ-discrete cover by compact sets if and only if Z has a closure-preserving cover by compact sets.*

Proof. *Let* $\mathcal{D} = \bigcup\{\mathcal{D}(n): n \geq 1\}$ be a cover of Z such that each $\mathcal{D}(n)$ is a discrete collection of compact sets. Use metacompactness of Z to construct, for each $n \geq 1$, a point-finite open expansion $\mathcal{V}(n) = \{V(D): D \in \mathcal{D}(n)\}$ of $\mathcal{D}(n)$. Clearly $\mathcal{V} = \bigcup_{n\geq 1}\{V\setminus E(n): V \in \mathcal{V}(n)\}$, where $E(n) = \bigcup\{\bigcup \mathcal{D}(m): m < n\}$ is a point-finite open cover of the space Z. For each $z \in Z$ define $C(z) = Z\setminus\bigcup\{V \in \mathcal{V}: z \notin V\}$. Observe that the set $C(z)$ is contained in a finite union of elements of $\mathcal{D}$. Since the collection $\{C(z): z \in Z\}$ is a closure-preserving closed cover of Z (see the proof of [13, 3.18]), this finishes the proof of the "only if" part.

Suppose now that Z is a perfect space with a closure-preserving cover $\mathcal{C}$ by compact subsets. Construct, by induction on $n \geq 0$, for each sequence $t = \langle j_0, \ldots, j_{n-1}\rangle$ of natural numbers, a discrete collection $\mathcal{D}(t)$ of compact sets and an open set $U(t)$ such that $U(\emptyset) = \emptyset$, $\mathcal{D}(t)$ is a maximal pairwise-disjoint subcollection of $\{C\setminus U(t): C \in \mathcal{C}\}$ and $U(t^\frown j) = U(t) \cup U'(t^\frown j)$, where $t^\frown j$ is the extension of t by j and $\bigcup \mathcal{D}(t) = \bigcap\{U'(t^\frown j): j \geq 0\}$.

In order to finish the proof, it suffices to show that each $z \in Z$ is in the union of a certain $\mathcal{D}(t)$. Assume that z is a point of Z not in the union of any $\mathcal{D}(t)$ and fix a $C \in \mathcal{C}$ containing z. Inductively choose a sequence $\langle j_n : n \geq 0\rangle$ such that $z \notin U(t_n)$, where $t_n = \langle j_0, \ldots, j_{n-1}\rangle$ for $n \geq 1$. From our construction, it follows that the collection $\{\bigcup \mathcal{D}(t_n): n \geq 0\}$ is discrete in Z and that its elements intersect the compact set C (see [15]). □

Standard arguments may be used to show that in a σ-locally compact space with a point-countable base every open cover has a refinement which is the union of countably many relatively discrete collections (this covering property is called weak θ-refinability). Since spaces with a σ-discrete cover by compact sets are subparacompact and spaces with a closure-preserving cover by compact sets are metacompact [15], it follows that none of the properties considered in the conclusions of Propositions 1.1 and 1.2 is preserved by open and compact mappings (see [3]). Next, we shall introduce a more stable property related to σ-local compactness. This property is a generalization of the concept of co-countable neighbornet [14, 3].

Recall that a neighbornet for a space Z is a relation $V \subset Z \times Z$ such that for each $z \in Z$, $z \in \operatorname{int} V(z)$, where $R(z) = \{y \in Z: \langle z, y\rangle \in R\}$ for a relation $R \subset Z \times Z$ [13].

A neighbornet V is called co-σ-compact (co-separable) if $V^{-1}(z) = \{y \in Z: z \in V(y)\}$ is contained in a σ-compact subset of Z (is separable) for all $z \in Z$. A neighbornet V in Z is called transitive if the relation V is transitive ($V \circ V = V$). Transitive neighbornets in Z correspond to closure-preserving closed covers of Z (see [13, 3.14]).

Lemma 1.3. *If all compact subsets of a space Z are metrizable, then Z has a closure-preserving closed cover by σ-compact sets if and only if Z has a co-σ-compact neighbornet.*

Proof. If $\mathscr{C}$ is a closure-preserving closed cover of Z by σ-compact sets, then the sets $V(z)=Z\setminus\bigcup\{C\in\mathscr{C}: z\notin C\}$ generate a co-σ-compact transitive neighbornet V in Z (see [13, 3.14]).

To prove the "if" part, let V be a co-σ-compact neighbornet in Z. Clearly we can assume that $V(z)$ is open for each $z\in Z$ and, consequently, $V^{-1}(A)=\bigcup\{V^{-1}(y): y\in A\}=\{z: V(z)\cap A\neq\emptyset\}=\{z: V(z)\cap \bar{A}\neq\emptyset\}=V^{-1}(\bar{A})$ for an arbitrary $A\subset Z$. If $z\in Z$, then $V^{-1}(z)$ has a countable dense subset A and we obtain $V^{-2}(z)=V^{-1}(V^{-1}(z))=V^{-1}(A)$. Thus $V^2=V\circ V$ is a co-σ-compact neighbornet and one can show, by induction, that the transitive neighbornet $W=\bigcup\{V^n: n\geqslant 1\}$ is co-σ-compact. The collection $\{W^{-1}(z): z\in Z\}$ is a closure-preserving closed cover of Z by σ-compact sets (see [14, proposition 1]). □

Proposition 1.4. *If Z is a semistratifiable meta-Lindelöf space, then Z has a σ-discrete cover by compact sets if and only if Z has a closure-preserving closed cover by σ-compact sets*[1].

Proof. Let $\mathscr{D}$ be a σ-discrete cover of Z by compact sets. As in the proof of Proposition 1.2, use the meta-Lindelöf property of Z to construct a point-countable open expansion $\mathscr{V}=\{V(D): D\in\mathscr{D}\}$ of $\mathscr{D}$. For each $z\in Z$ fix a $D\in\mathscr{D}$ containing z and put $V(z)=V(D)$. Clearly this defines a co-σ-compact neighbornet and one can finish the proof of the "only if" part by applying Lemmas 0.1 and 1.3.

Suppose now that Z is a semistratifiable space with a closure-preserving closed cover $\mathscr{C}$ by σ-compact subsets. Let V be a co-σ-compact transitive neighbornet generated by $\mathscr{C}$ (see the proof of Lemma 1.3). Since $R=V\cap V^{-1}$ is an equivalence relation, by [13, 4.8] (see Proposition 3.3), the partition of Z generated by R has a σ-discrete closed refinement. This refinement is a σ-discrete cover of Z by σ-compact sets and one can easily modify it to a σ-discrete cover of Z by compact sets. □

Our next result, together with Lemma 1.3, shows that the property of having a closure-preserving closed cover by σ-compact sets is preserved by any open and compact mapping provided that the compact subsets of the domain and range of the mapping are metrizable.

Lemma 1.5. *The property of having a co-σ-compact neighbornet is invariant under open mappings with separable fibers.*

Proof. Let f be an open mapping with separable fibers of a space Y having a co-σ-compact neighbornet V onto Z. For each $z\in Z$ fix a $y\in f^{-1}(z)$ and define $W(z)=f(V(y))$. It is easy to check that this generates a co-σ-compact neighbornet in Z (see [14, Proposition 2]). □

[1] The author has been informed by H.J.K. Junnila that a modification of the proof from [15] can be used to show that spaces having a closure-preserving closed cover by Lindelöf subsets are meta-Lindelöf.

From Lemmas 1.3, 1.5, the fact that all spaces in MOBI have a point-countable base and Lemma 0.1, we get

Corollary 1.6. *All spaces in* MOBI(σ*-locally compact*) *have a closure-preserving closed cover by* σ*-compact sets.*

The first step towards our characterization of the class MOBI(σ-locally compact) is the characterization of open and compact images of σ-locally compact metric spaces.

Theorem 1.7. *For a space* X *the following conditions are equivalent*:
(a) X *is an open and compact image of a* σ*-locally compact metric space,*
(b) X *is a* σ*-locally compact metacompact Moore space,*
(c) X *is an open finite-to-one image of a* σ*-locally compact metric space.*

Proof. The implication (c)$\Rightarrow$(a) is obvious and, since open and compact images of metric spaces are metacompact Moore spaces, (a)$\Rightarrow$(b) follows from Corollary 1.6 and Propositions 1.4 and 1.1.

To prove (b)$\Rightarrow$(c), let X be a σ-locally compact metacompact Moore space. By Lemma 0.1 and Proposition 1.1, X has a countable cover by closed metrizable subspaces. In [8] it is shown that X is an open finite-to-one image of a metric space M. Use the fact that X is σ-locally compact to represent M as a countable union of inverse images of locally compact subspaces of X under open k-to-one mappings. Since open k-to-one mappings are local homeomorphisms, it follows that M is a σ-locally compact space. □

The complete version of Theorem 1.7 is

Corollary 1.8. *For a space* X *the following conditions are equivalent*:
(a) X *is an open and compact image of a* C*-scattered metric space,*
(b) X *is a* C*-scattered metacompact Moore space,*
(c) X *is an open finite-to-one image of a* C*-scattered metric space.*

Proof. Since having a λ-base is preserved, in both directions, by open and compact mappings between spaces with bases of countable order (see [18] or [10]), one can add the λ-base property to the equivalent conditions in Theorem 1.7. The proof will be finished once we show that a metacompact Moore space is C-scattered if and only if it is σ-locally compact and has a λ-base.

From Lemma 0.1 and the results of [19], it follows that a C-scattered space with a point-countable base has a λ-base and, by [11, Theorem 2], a perfectly subparacompact C-scattered space has a σ-discrete cover by compact sets. On the other hand, any σ-locally compact space satisfying the Baire category theorem hereditarily with respect to closed subsets is C-scattered. □

The main result of this paper is

Theorem 1.9. *For a space Y the following conditions are equivalent*:

(a) *Y is in* MOBI(*σ-locally compact*),

(b) *Y has a closure-preserving closed cover by σ-compact sets and a point-countable base of countable order*[2],

(c) *Y is an open and compact image of a σ-locally compact metacompact Moore space.*

Proof. The implication (c)$\Rightarrow$(a) follows from Theorem 1.7 and (a)$\Rightarrow$(b) follows from Corollary 1.6. In the next section we will prove that (b)$\Rightarrow$(c).

Before passing to the proof of (b)$\Rightarrow$(c), we give the complete version of Theorem 1.9.

Corollary 1.10. *For a space Y the following conditions are equivalent*:

(a) *Y is in* MOBI(*C-scattered*),

(b) *Y is C-scattered and has a point-countable base,*

(c) *Y is an open and compact image of a C-scattered metacompact Moore space.*

Proof. As in the proof of Corollary 1.8, the verification of Corollary 1.10 reduces to showing that a space Y with a point-countable base is C-scattered if and only if it has a closure-preserving closed cover by σ-compact sets and a λ-base.

If Y is a C-scattered space with a point-countable base, then Y has a λ-base (see the proof of Corollary 1.8). Moreover, Y has a well-ordered partition $\mathcal{K}$ such that the union of each initial segment in $\mathcal{K}$ is open and each $K \in \mathcal{K}$ is contained in a compact set (see [16]). Since Y has a point-countable base, $\mathcal{K}$ has a point-countable open expansion in Y and, consequently, Y has a co-σ-compact neighbornet. Thus Lemmas 0.1 and 1.3 show that Y has a closure-preserving closed cover by σ-compact sets.

If Y has a closure-preserving closed cover by σ-compact sets and a λ-base, then one can apply the reasoning used in the proof of [5, Proposition 1.3] to show that Y is C-scattered (see Proposition 3.5). Another way of completing the proof is to observe that, by Theorem 1.9, Y is an open and compact image of a σ-locally compact metacompact Moore space X having a λ-base. Since X is C-scattered and open mappings preserve this property, it follows that Y is C-scattered. □

2. The second mapping

We return to the proof of the implication (b)$\Rightarrow$(c) in Theorem 1.9. The proof is a modification of the construction from [5, 3].

[2] The assumption that Y has a point-countable base follows from the fact that Y is a meta-Lindelöf space with a base of countable order (see the proof of 4.5 in [10]) and, therefore, it can be omitted.

Let Y be a space with a closure-preserving closed cover $\mathscr{C}$ by σ-compact sets and let $\langle \mathscr{G}_n : n \geq 1 \rangle$ be a sequence of bases of Y witnessing the fact that Y has a base of countable order.

For each $C \in \mathscr{C}$ fix a sequence $\langle C(n) : n \in \omega \rangle$ whose terms are compact and cover C. Let $\mathscr{K}$ be the collection of all the $C(n)$ and use an arbitrary well-ordering of $\mathscr{C}$ to generate a lexicographic well-order $<$ on $\mathscr{K}$. Observe that the union of each initial segment of $\mathscr{K}$ is a closed subset of Y.

For each $C \in \mathscr{K}$ put $L(C) = C \setminus \bigcup \{C' \in \mathscr{K} : C' < C\}$. Let $\mathscr{L}$ be the collection of all the nonempty $L(C)$ and consider $\mathscr{L}$ with the natural well-order $<$ inherited from $\mathscr{K}$. Clearly $\mathscr{L}$ is a partition of Y and the union of each initial segment of $\mathscr{L}$ is a closed subset of Y.

Moreover, we have

$$\text{the elements of } \mathscr{L} \text{ have compact closures,} \tag{α}$$

and, since Y has a point-countable base and the elements of $\mathscr{L}$ are separable metric spaces, we can fix for each $L \in \mathscr{L}$ an open set $V(L)$ so that

$$L \subset V(L) \subset Y \setminus \bigcup \{K \in \mathscr{L} : K < L\}, \tag{β}$$

$$V^{-1}(L) = \{K \in \mathscr{L} : V(K) \cap L \neq \emptyset\} \text{ is countable.} \tag{γ}$$

We shall use conditions (α)-(δ) to construct a σ-locally compact metacompact developable regular space X and an open and compact mapping f of X onto Y.

Our plan is to follow [5, 3] by constructing the space X as the union of certain subsets of Y indexed by a tree of finite sequences of elements of $\mathscr{L}$ increasing with respect to $<$ and certain subsets of Y added in order to make the fibers of the natural mapping f of X onto Y compact.

We define by induction, for each $n \geq 0$, a set P_n of increasing n-element sequences in $\mathscr{L}$ and, for each $p \in P_{n+1}$, a finite subcollection $\mathscr{G}(p)$ of $\mathscr{G}_{n+1}$ and an open subset $H(p)$ of Y.

We start with

$$P_0 = \{\emptyset\}, \qquad \mathscr{G}(\emptyset) = \{Y\} \quad \text{and} \quad H(\emptyset) = Y \tag{0}$$

and proceed according to the following four conditions.

$$P_{n+1} = \{p^\frown L : p \in P_n \text{ and } L \cap H(p) \setminus E(p) \neq \emptyset\}, \tag{1}$$

where $p^\frown L$ denotes the extension of p by L and $E(p)$ is the last term of p ($E(\emptyset) = \emptyset$).

Moreover, for $p^\frown L \in P_{n+1}$

$$\mathscr{G}(p^\frown L) \text{ is a finite cover of } \overline{H(p) \cap L} \text{ refining } \mathscr{G}(p) \text{ and consisting of elements of } \mathscr{G}_{n+1} \text{ intersecting } L, \tag{2}$$

$$\overline{H(p^\frown L)} \subset \bigcup \mathscr{G}(p^\frown L) \tag{3}$$

and

$$H(p) \cap L \subset H(p^\frown L) \subset H(p) \cap V(L). \tag{4}$$

To see that the induction works, observe that (0) and (3) imply $\overline{H(p)} \subset \bigcup \mathcal{G}(p)$. Thus ($\alpha$) can be used to define $\mathcal{G}(p^\frown L)$ satisfying (2) while $H(p^\frown L)$ can be constructed by putting $H(p^\frown L) = H \cap H(p) \cap V(L)$, where H is an open set separating the compact set $\overline{H(p) \cap L}$ from the complement of $\bigcup \mathcal{G}(p^\frown L)$.

Note that (4) implies that for $p = r^\frown K$, $H(p) \cap K = H(r) \cap K$. This, together with (0), (1) and (4) gives

$$H(p) = \bigcup \{H(q) \cap L : p \subset q^\frown L\}, \tag{5}$$

where $p \subset q^\frown L$ means that the sequence p is an initial segment of $q^\frown L$.

In analogy with [5], each sequence $p^\frown L \in P_{n+1}$ should represent a piece of X (the locally compact subset $H(p) \cap L$ of Y). Let $*$ be a point not in Y. The pieces needed to make the fibers of f compact will be indexed by sequences $p^\frown *^\frown L$. Since we want to keep the space X σ-locally compact, we have to make sure that each compactifying piece is a σ-locally compact subset of Y. By splitting each $p^\frown L$ into countably many copies, we will ensure that the compactifying piece indexed by $p^\frown *^\frown L$ is equal to $H(p) \cap L$.

Thus we define, for $n \geq 0$

$$P'_{n+1} = \{\langle k_0, L_0, \ldots, k_n, L_n \rangle : \langle L_0, \ldots, L_n \rangle \in P_{n+1} \text{ and } k_0, \ldots, k_n \in \omega\}$$

and extend the definitions of $E(p)$, $\mathcal{G}(p)$ and $H(p)$ over

$$P = \bigcup \{P'_n : n \geq 1\}$$

by asserting that they do not depend on the k's.

Observe that (γ), (1) and (4) imply that $E : P \rightsquigarrow \mathcal{L}$ is countable-to-one.

The part of X corresponding to $q = p^\frown k^\frown L \in P$ will be the set $X(q) = \{q^\frown y : y \in H(p) \cap L\}$. Put $S = \bigcup \{X(q) : q \in P\}$ and let e be the natural mapping of S onto Y defined by $e(q^\frown y) = y$.

The pieces of X needed to compactify the fibers of e are indexed by

$$P^* = \{p^\frown *^\frown L : p^\frown 0^\frown L \in P\}.$$

The part of X corresponding to $q^* = p^\frown *^\frown L \in P^*$ will be the set $X(q^*) = \{q^{*\frown} y : y \in H(p) \cap L\}$.

Put

$$\Pi = P \cup P^*$$

and extend E over Π by asserting that $E(\pi)$ is the last term of π for $\pi \in \Pi$. Observe that, by the definition of P^*, $E : \Pi \rightsquigarrow \mathcal{L}$ is countable-to-one.

Finally, define

$$X = \bigcup \{X(\pi) : \pi \in \Pi\}$$

and let f be the natural extension of e over X given by $f(\pi^\frown y) = y$.

Since E is countable-to-one, $\mathcal{L}$ is a partition of Y and the restriction of f to each piece $X(\pi)$ of X is a one-to-one function mapping this piece into $E(\pi)$, it follows that

$$f \text{ is countable-to-one.} \tag{6}$$

For $p \in P \cup \{\emptyset\}$ define

$$A(p) = \{\pi \in \Pi : p \subset \pi\} \subset \Pi$$

and

$$B(p) = \{x \in X : p \subset x\} \subset X,$$

where $p \subset \pi$ ($p \subset x$) means that the sequence p is an initial segment of the sequence π (x).

Note that

$$B(p) = \{\pi^\frown y \in X : \pi \in A(p)\} \tag{7}$$

and, by (7) and (5),

$$f(B(p)) = H(p). \tag{8}$$

Before describing the topology of X we shall prove a fact which explains the role of the base of countable order in our construction and will later be used to show that the fibers of f are compact.

Fact 2.1. *If F is an infinite subset of $e^{-1}(y)$, then there exists a $p \in P$ such that the set $F \cap B(p)$ cannot be covered by any finite subcollection of $\{B(p^\frown k^\frown K): p^\frown k^\frown K \in P\}$.*

Proof. Suppose that such a p does not exist. By induction on $n \geq 0$ construct a sequence $\langle p_n : n \geq 0\rangle$ such that $p_0 = \emptyset$, $p_{n+1} = p_n^\frown k_n^\frown K_n$ and $F \cap B(p_{n+1})$ is infinite for $n \geq 0$.

Since $f^{-1}(y) \cap B(p_n) \neq \emptyset$, condition (8) assures that $y \in H(p_n)$. Using (2) and (3) we can find a decreasing sequence $\langle G_n : n \geq 1\rangle$ such that $G_n \in \mathscr{G}(p_n)$ and $y \in G_n$ for $n \geq 1$. From (δ), it follows that $\{G_n : n \geq 1\}$ is a base for y in Y and, in particular, there exists an $n \geq 1$ such that $G_n \subset V(L)$, where L is the element of $\mathscr{L}$ containing y. Since, by condition (2), G_n intersects K_{n-1}, (β) gives $L \preccurlyeq K_{n-1} = E(p_n) < E(p_{n+1}) = K_n$.

On the other hand, from (4), we get $H(p_{n+1}) \subset V(K_n)$. Thus L intersects $V(K_n)$ and this contradicts (β). □

To define the topology of X we need some more notation.

For each $L \in \mathscr{L}$ we can use (α) and Lemma 0.1 to find a strictly decreasing sequence $\langle U(L,j) : j \geq 0\rangle$ of neighborhoods of $\bar{L}$ in Y such that

$$U(L,0) = Y \quad \text{and} \quad \bigcap\{U(L,j) : j \geq 0\} = \bar{L}. \tag{ε}$$

For an open subset U of Y, $p = r^\frown k^\frown L \in P$ and $j \geq 0$ define

$$B(p, U, j) = B(p) \cap f^{-1}(U) \cap f^{-1}(U(L,j)). \tag{9}$$

Note that $B(p, Y, 0) = B(p)$. The sets $B(p, U, j)$ will generate the topology of X in points of $X(p)$. Clearly the traces of these sets on $X(p)$ endow it with the topology making f a homeomorphism of $X(p)$ onto $H(r) \cap L$.

In order to define sets generating the topology of X in points of the compactifying pieces, we need some more notation.

For each $L \in \mathscr{L}$ fix a one-to-one enumeration $m_L: V^{-1}(L) \rightsquigarrow \omega$ of $V^{-1}(L)$ (see (γ)). Whenever we write $m_L(K)$, we assume that $K \in V^{-1}(L)$. If $r = \langle k_0, L_0, \ldots, k_n, L_n \rangle \in E^{-1}(L)$, then (1) and (4) imply $L_i \in V^{-1}(L)$ for $i = 1, \ldots, n$. Thus we can define

$$m_L(r) = \sum_{i=0}^{n} k_i + \sum_{i=0}^{n} m_L(L_i).$$

For $q^* = p^\frown *^\frown L \in P^*$ and $j \geq 0$ define

$$\Pi(q^*, j) = E^{-1}(L) \cap A(p) \setminus \bigcup \{A(p^\frown k^\frown K) : k + m_L(K) < j\}$$

and (10)

$$X(q^*, j) = \{\pi^\frown y \in X : \pi \in \Pi(q^*, j)\}.$$

By (7)

$$X(q^*, j) = f^{-1}(L) \cap B(p) \setminus \bigcup \{B(p^\frown k^\frown K) : k + m_L(K) < j\}. \tag{11}$$

We complete the definition of sets generating the topology on X by putting, for an open subset U of Y, $q^* = p^\frown *^\frown L \in P^*$ and $j \geq 0$,

$$B(q^*, U, j) = (X(q^*, j) \cap f^{-1}(U)) \cup \bigcup \mathscr{B}(q^*, U, j),$$

where (12)

$$\mathscr{B}(q^*, U, j) = \{B(r, U, j + m_L(r)) : r \in \Pi(q^*, j) \cap P\}.$$

Observe that $X(q^*) \subset X(q^*, j)$ and that the traces of the sets $B(q^*, U, j)$ on $X(q^*)$ endow it with the topology making f a homeomorphism of $X(q^*)$ onto $H(p) \cap L$.

Note that if $q^* = p^\frown *^\frown L \in P^*$, $j \geq 0$ and $B = B(q^*, Y, j)$, then

$$B \subset B(p) \setminus \bigcup \{B(p^\frown k^\frown K) : k + m_L(K) < j \text{ or } K \notin V^{-1}(L)\}. \tag{13}$$

Moreover, for each $\pi \in \Pi$, $j \geq 0$ and $p \in P$, the definitions (9) and (12) of $B(\pi, U, j)$ imply

$$\text{if } \pi^\frown y \in B(p), \quad \text{then } B(\pi, Y, 0) \subset B(p) \tag{14}$$

and

$$f(B(\pi, U, j)) \subset U \cap U(E(\pi), j). \tag{15}$$

Also, if $V \subset U$ and $i \geq j$, then

$$B(\pi, V, i) \subset B(\pi, U, j). \tag{16}$$

Condition (16) enables us to define the topology in X by using the sets $B(\pi, U, j)$ as weak bases in points of $X(\pi)$; that is, a set $B \subset X$ is open in X if and only if for each $x = \pi^\frown y \in B$ there exist a $j \geq 0$ and a neighborhood U of y in Y such that $B(\pi, U, j) \subset B$.

Fact 2.2. *The function $f: X \to Y$ is continuous.*

Proof. The continuity of f follows from (15). □

Fact 2.3. *The sets $B(\pi, U, j)$ are open in X.*

Proof. First observe that (14) implies that, for each $p \in P$, the set $B(p)$ is open in X. Thus, from Fact 2.2 and (9), it follows that, for each $p \in P$, U open in Y and $j \geqslant 0$, the set $B(p, U, j)$ is open in X.

Consider $B(q^*, U, j)$, where $q^* = p^\frown *^\frown L \in P^*$, U open in Y and $j \geqslant 0$. In view of (12) and the first part of the proof, in order to prove that $B(q^*, U, j)$ is open in X, it suffices to consider $x = \pi^\frown y \in X(q^*, j) \cap f^{-1}(U)$. By (10) $\pi \in \Pi(q^*, j)$. If $\pi = r \in P$, then $x \in \bigcup \mathscr{B}(q^*, U, j)$. Suppose that $\pi = r^\frown *^\frown L' \neq q^*$ is in $\Pi(q^*, j) \setminus P$. From (10), it follows that $L = L'$ and $p \subset r$. Moreover, since $\pi \neq q^*$, there exist $k \in \omega$ and $K \in \mathscr{L}$ such that $p^\frown k^\frown K \subset r$ and $k + m_L(K) \geqslant j$. Thus $\Pi(\pi, j) \subset \Pi(q^*, j)$ and (12) gives $B(\pi, U, j) \subset B(q^*, U, j)$. □

In order to show that X is regular, we shall need the next three facts.

Fact 2.4. *If $q \in P$ and $\bar{V} \subset V(E(q))$, then $\overline{B(q) \cap f^{-1}(V)} \subset B(q)$.*

Proof. Suppose $x = \pi^\frown y \in \overline{B(q) \cap f^{-1}(V)}$ and let $L = E(\pi)$ be the element of $\mathscr{L}$ containing y. By fact 2.2, $y \in \bar{V} \subset V(E(q))$ which, together with (β), gives $E(q) \preccurlyeq L$. Assume that $x \notin B(q)$.

If $\pi = p \in P$, then $B(p) \cap B(q) \neq \emptyset$ and $q \not\subset p$ imply that q is a strict extension of p and this contradicts $E(q) \preccurlyeq E(p)$.

If $\pi = p^\frown *^\frown L \in P^*$, then $B(p) \cap B(q) \neq \emptyset$ and $q \not\subset p$ imply that q is a strict extension of p. Let $k \in \omega$ and $K \in \mathscr{L}$ be such that $p^\frown k^\frown K \subset q$. Take a $j > k + m_L(K)$ if $K \in V^{-1}(L)$ or $j = 0$ if $K \notin V^{-1}(L)$. By (13), $B(\pi, Y, j) \cap B(p^\frown k^\frown K) = \emptyset$ and, since $B(q) \subset B(p^\frown k^\frown K)$, this contradicts $\pi^\frown y \in \overline{B(q)}$. □

Fact 2.5. *For each $q^* \in P^*$ we have $\bigcap \{\Pi(q^*, j): j \geqslant 0\} = \{q^*\}$ and $\bigcap \{X(q^*, j): j \geqslant 0\} = X(q^*)$.*

Proof. This follows directly from (10). □

Fact 2.6. *For each $\pi \in \Pi$ we have $\bigcap \{B(\pi, Y, j): j \geqslant 0\} = X(\pi)$.*

Proof. Clearly the intersection contains $X(\pi)$. If $\pi = p \in P$ and $\rho^\frown y \in \bigcap \{B(\pi, Y, j): j \geqslant 0\}$ then, by (15) and (ε), $y \in \overline{E(p)}$ and (β) implies $E(\rho) \preccurlyeq E(p)$. Since, on the other hand, $\rho^\frown y \in B(p)$, we get $p \subset \rho$ and, consequently, $\rho = p$.

Assume $\pi = p^\frown *^\frown L \in P^*$. Since the elements of $\mathscr{B}(\pi, Y, 0)$ are pairwise-disjoint, the first part of Fact 2.5 shows that the intersection is contained in $\bigcap \{X(q^*, j): j \geqslant 0\}$ and the second part completes the proof. □

From Fact 2.6, it follows that each $x \in X$ is the intersection of its neighborhoods, hence X is a T_1-space. We are ready to show that X is a regular space.

Fact 2.7. *If* $\bar{V} \subset U \cap V(E(\pi))$, *then* $\overline{B(\pi, V, j+1)} \subset B(\pi, U, j)$.

Proof. Put $B(\pi, V, j+1) = B$. If $\pi = p \in P$, then (9), Facts 2.2 and 2.4 give

$$\begin{aligned} \bar{B} &= \overline{B(p) \cap f^{-1}(V) \cap f^{-1}(U(E(p), j+1))} \\ &\subset \overline{B(p) \cap f^{-1}(V)} \cap f^{-1}(\bar{V}) \cap f^{-1}(\overline{U(E(p), j+1)}) \\ &\subset B(p) \cap f^{-1}(U) \cap f^{-1}(U(E(p), j)) = B(p, U, j). \end{aligned}$$

If $\pi = p^\frown *^\frown L \in P^*$, then, by (13) and (15)

$$B \subset f^{-1}(V) \cap B(p) \setminus \bigcup \{B(p^\frown k^\frown K): k + m_L(K) < j+1\}.$$

Thus Facts 2.4 and 2.3 give

$$\bar{B} \subset B(p) \setminus \bigcup \{B(p^\frown k^\frown K): k + m_L(K) < j+1\}.$$

Hence Fact 2.2, (11) and (12) imply

$$\begin{aligned} &\bar{B} \cap f^{-1}(L) \\ &\quad \subset f^{-1}(\bar{V}) \cap f^{-1}(L) \cap B(p) \setminus \bigcup \{B(p^\frown k^\frown K): k + m_L(K) < j+1\} \\ &\quad \subset X(\pi, j+1) \cap f^{-1}(U) \subset B(\pi, U, j). \end{aligned}$$

Take an $x = \rho^\frown y \in \bar{B}$ and let K be the element of $\mathscr{L}$ containing y. Since $y \in \bar{V} \subset V(L)$, (β) assures that $L \leqslant K$. If $L = K$, then, as we have just shown, $x \in B(\pi, U, j)$. Assume that $L < K$. Since (β) implies $\bar{L} \cap K = \emptyset$, we can use (ε) to find $i \geqslant 0$ and a neighborhood W of y in Y such that $U(L, i) \cap W = \emptyset$. If x is in the closure of an element of $\mathscr{B}(\pi, V, j+1)$, then, by the first part of the proof, x is in the closure of the corresponding element of $\mathscr{B}(\pi, U, j)$. Thus, in order to finish the proof, it is sufficient to show that the neighborhood $B(\rho, W, 0)$ of x intersects only finitely many elements of $\mathscr{B}(\pi, V, j+1)$.

Suppose that $B(\rho, W, 0)$ intersects $B(r, V, j+1+m_L(r)) \in \mathscr{B}(\pi, V, j+1)$. By (15), $j+1+m_L(r) < i$ and the definition of $m_L(r)$ assures that there is only finitely many such r. □

Fact 2.8. *The collection* $\{B(\pi, Y, 0): \pi \in \Pi\}$ *is point-finite in* X.

Proof. Let $\rho^\frown y \in B(\pi, Y, 0)$. If $\pi = p \in P$, then $p \subset \rho$ and the number of such $p \in P$ is finite. If $\pi = p^\frown *^\frown L \in P^*$, then $p \subset \rho$ and L is a term of ρ. Again, the number of such $p^\frown *^\frown L$ in P^* is finite. □

Fact 2.9. *The space X is a metacompact Moore space.*

Proof. For each $L \in \mathscr{L}$ let $\langle \mathscr{W}(L, i)$: $i \geqslant 0\rangle$ be a sequence of finite covers of $\bar{L}$ by open subsets of Y such that for each $y \in L$ and its neighborhood U in Y, $\mathrm{St}(y, \mathscr{W}(L, i)) \subset U$ for almost all $i \geqslant 0$ (see Lemma 0.1).

Put $\mathscr{D}(\pi, i) = \{B(\pi, W, i)$: $W \in \mathscr{W}(E(\pi), i)\}$. Clearly $\mathscr{D}(\pi, i)$ is a finite cover of $X(\pi)$. From (16) and Fact 2.8 we infer that $\mathscr{D}(i) = \bigcup\{\mathscr{D}(\pi, i)$: $\pi \in \Pi\}$ is a point-finite open cover of X for $i \geqslant 0$.

If $x = \pi^\frown y \in B = B(\pi, U, j)$ and $i \geqslant j$ is such that $\mathrm{St}(y, \mathscr{W}(E(\pi), i)) \subset U$, then (16) implies $\mathrm{St}(x, \mathscr{D}(\pi, i)) \subset B$. Thus Facts 2.6 and 2.8 and (16) assure that $\langle \mathscr{D}(i)$: $i \geqslant 0\rangle$ is a development for X and the proof is finished. □

Fact 2.10. *The space X is σ-locally compact.*

Proof. Fix a $\pi \in \Pi$. The restriction of f to $X(\pi)$ is a homeomorphism of this set onto an open subset of $E(\pi)$ which is a locally compact subset of Y. Thus $X(\pi)$ is a locally compact subset of Y.

For $n \geqslant 1$ put

$$\mathscr{X}_n = \{X(p)\text{: } p \in P \text{ and } \operatorname{dom} p = 2n\}$$

and

$$\mathscr{X}_n^* = \{X(q^*)\text{: } q^* \in P^* \text{ and } \operatorname{dom} q^* = 2n\}.$$

It is easy to see that these collections are relatively discrete and their union covers X. □

We have shown that the space X has the required properties and the function $f: X \to Y$ is continuous. It remains to prove that f is an open and compact mapping.

Fact 2.11. *The mapping $f: X \to Y$ is open.*

Proof. By (8) and (9), $f(B(p, U, j)) = H(p) \cap U \cap U(E(p), j)$ is open in Y. For an open set $B = B(q^*, U, j)$, where $q^* = p^\frown *^\frown L$, by (8), (11) and (12), $f(B) = (H(p) \cap L \cap U) \cup f(\bigcup \mathscr{B}(q^*, U, j))$. Take a $k \geqslant j$ and consider $r = p^\frown k^\frown L \in \Pi(q^*, j) \cap P$. Since $(H(p) \cap L \cap U) \cup f(B(r, U, j + m_L(r))) = H(p) \cap U \cap U(L, j + m_L(r)) \subset f(B)$, it follows that $f(B)$ is open in Y. □

Fact 2.12. *The fibers of f are compact.*

Proof. Take a $y \in Y$ and the $L \in \mathscr{L}$ containing y. If $x = p^\frown *^\frown L^\frown y$ is a point of $f^{-1}(y)\backslash S$ and $B = B(p^\frown *^\frown L, U, j)$ is a neighborhood of x, then B contains the points $p^\frown k^\frown L^\frown y \in S$ for $k \geqslant j$ (see the proof of Fact 2.11). Thus $e^{-1}(y)$ is dense in $f^{-1}(y)$.

If F is an infinite subset of $e^{-1}(y)$, then, by Fact 2.1, there exists a $p \in P$ such that the set $F \cap B(p)$ cannot be covered by any finite subcollection of $\{B(p^\frown k^\frown K)$: $p^\frown k^\frown K \in P\}$. Since $f^{-1}(y) \cap B(p) \neq \emptyset$, (8) implies $y \in H(p)$ and, by (4) and (β), $E(p) \preccurlyeq L$. Since $f^{-1}(y) \cap B(p)$ is, in fact, infinite, we have $E(p) \prec L$ and, by (1) and the definition of P^*, $q^* = p^\frown *^\frown L \in P^*$. Conditions (11) and (12) imply that $q^{*\frown} y$ is an accumulation point of F in $f^{-1}(y)$.

From the last two paragraphs, it follows that every locally finite collection of open subsets of $f^{-1}(y)$ is finite. Since, by (6), the set $f^{-1}(y)$ is countable and the space is regular, this implies compactness of $f^{-1}(y)$. □

3. Remarks

We start with some remarks concerning the construction of X.

Remark 3.1. If the space Y is completely regular (zero-dimensional), then X can be constructed to be completely regular (zero-dimensional).

Proof. Assume that Y is completely regular and fix an $L \in \mathscr{L}$. We shall show that a special choice of the sequence $\langle U(L, j): j \geqslant 0\rangle$ makes it possible to separate points of $\bigcup\{X(\pi): \pi \in E^{-1}(L)\}$ from closed sets by mappings into the unit interval $I = [0, 1]$. Since this can be done for all $L \in \mathscr{L}$, it will follow that X can be made completely regular. If Y is zero-dimensional, the same method produces separating mappings into $\{0, 1\}$ (another, more direct, method of proving that X is zero-dimensional is to use clopen sets $U(L, j)$ and modify Facts 2.4 and 2.7 as in the proof of 4.1 in [5]).

For the fixed L construct, by induction on $j \geqslant 0$, sequences $\langle U_j : j \geqslant 0\rangle$ of neighborhoods of $\bar{L}$ in Y and $\langle \varphi_j : j \geqslant 0\rangle$ of mappings of Y into the unit interval satisfying, for $j \geqslant 0$,

$$U_0 = Y, \tag{17}$$

$$\varphi_j | \bar{L} \equiv 0 \quad \text{and} \quad \varphi_j | X \setminus U_j \equiv 1, \tag{18}$$

$$\bar{U}_{j+1} \subset U_j \cap U(L, j+1), \tag{19}$$

$$\varphi_j(U_{j+1}) \subset [0, 1/j). \tag{20}$$

By (19), the sequence $\langle U_j : j \geqslant 0\rangle$ is strictly decreasing and, since it can be substituted for $\langle U(L, j): j \geqslant 0\rangle$ in (ε), we can use the sets U_j to define the sets $B(\pi, U, j)$ for $\pi \in E^{-1}(L)$ (this affects (9) and, indirectly, (12)).

Consider a $\pi \in E^{-1}(L)$, $x = \pi^\frown y \in X$ and a neighborhood $B = B(\pi, U, j)$ of x in X. We want to separate x from $X \setminus B$ by a mapping $\psi : X \to I$. Clearly we can assume that $\bar{U} \subset V(L)$ and $j \geqslant 1$.

Let $\varphi : Y \to I$ be a mapping satisfying

$$\varphi(y) = 0 \quad \text{and} \quad \varphi \,|\, Y \setminus U \equiv 1. \tag{21}$$

If $\pi = p \in P$, then we define ψ on $B(p)$ to be $(\max\{\varphi, \varphi_j\}) \circ f$. In view of (18) and (21), ψ separates x from $B(p) \setminus B$. Since $B(p)$ is open in X and, by Fact 2.4, $\bar{B} \subset B(p)$, we can extend ψ over X by setting $\psi | X \setminus \bar{B} \equiv 1$.

Assume that $\pi = q^* = p^\frown *^\frown L \in P^*$ and put $O = \bigcup \{B(r): r \in \Pi(q^*, j) \cap P\}$. Since $X(q^*, j) \subset B(q^*, Y, j) \subset X(q^*, j) \cup O$, it follows that the set $X' = X(q^*, j) \cup O$ is open in X. By (16), $B \subset X'$ and Fact 2.7 gives $\bar{B} \subset B(q^*, Y, j-1)$. Moreover, by (13), B is disjoint from the open set $\bigcup \{B(p^\frown k^\frown K): k + m_L(K) = j-1\}$ and, consequently, $\bar{B} \subset X'$.

As in the first part of the proof, it suffices to show that x can be separated from $X' \setminus B$ on X'.

For each $r \in \Pi(q^*, j) \cap P$ put $n(r) = j + m_L(r)$ and define ψ on $B(r)$ to be $(\max\{\varphi, \varphi_{n(r)}\}) \circ f$. This gives ψ on the open set O. Extend ψ by defining it to be $\varphi \circ f$ on $X(q^*, j) \setminus O = X(q^*, j) \setminus S$.

It is easy to check that, by (18) and (21), ψ separates x from $X' \setminus B$. Thus it remains to show that ψ is continuous (in points of $X(q^*, j) \setminus O$).

Consider a $\rho^\frown z \in X(q^*, j) \setminus S$ and a convex neighborhood J of $\varphi(z)$ in I. Let W be a neighborhood of z in Y contained in $\varphi^{-1}(J)$. Take an $l \geq 1$ satisfying $1/l <$ sup J and find an $i > j$ such that $n(r) > l$ for $r \in \Pi(\rho, i) \subset \Pi(q^*, j)$ (see the proof of Fact 2.3). We shall show that $\psi(B') \subset J$, where $B' = B(\rho, W, i) \subset X(\rho, i) \cup O \subset X'$.

Let $x' \in B'$ and put $y' = f(x')$. If $x' \in B' \setminus O$, then, by (15), $\psi(x') = \varphi(y') \in J$. If $x' \in B' \cap O$, then $x' \in B(r)$ for a certain $r \in \Pi(\rho, i) \cap P$ and $\psi(x') = \max\{\varphi(y'), \varphi_{n(r)}(y')\}$. If $\varphi(y') \geq \varphi_{n(r)}(y')$, then $\psi(x') = \varphi(y') \in J$. Assume that $\varphi(y') < \varphi_{n(r)}(y')$. By (15), we have $y' \in U_m$, where $m = i + m_L(r) > n(r) > l$. Thus (20) gives $\varphi(y') < \varphi_{n(r)}(y') = \psi(x') < 1/l$ and the convexity of J assures that $\psi(x') \in J$. □

Our construction is more general than the simple construction described in [7, 1.2]. Thus it may lead to a nonnormal space X even if the space Y is normal.

If the space Y is a Hausdorff space only, then we cannot make the sequences $\langle U(L, j): j \geq 0 \rangle$ strictly decreasing, but everything else works to produce a Hausdorff space X. This, however, is not satisfying, because an open and compact image of a metacompact developable Hausdorff space need not have a base of countable order. Since in the verification of Propositions 1.1, 1.2 and 1.4, Lemmas 1.3 and 1.5 and Corollary 1.6 regularity was not used, it follows that all spaces in $\text{MOBI}_2(\sigma$-locally compact) (the minimal class of Hausdorff spaces which contains all σ-locally compact metric spaces and is invariant under open and compact mappings) have a closure-preserving closed cover by σ-compact sets and a point-countable base. A characterization of $\text{MOBI}_2(\sigma$-locally compact) is given by (see [9])

Theorem 3.2. *For a Hausdorff space Y the following conditions are equivalent*:

(a) *Y is in* MOBI_2(*σ-locally compact*),

(b) *Y has a closure-preserving closed cover by σ-compact sets and a point-countable base,*

(c) *Y is an open and compact image of a σ-locally compact metacompact developable Hausdorff space.*

Proof. The only implication that has to be verified is (b)$\Rightarrow$(c). We shall outline modifications of the construction from the previous section needed to prove this implication.

For a Hausdorff space Y with a partition $\mathscr{L}$ satisfying (α)-(γ) put $P_0=\{\emptyset\}$ and use (1), with $H(p)=\bigcap\{V(K)\colon K\in \operatorname{rg} p\}$, to define the sets P_n. These sets give P and S as in the previous section. For each $L\in\mathscr{L}$ we add only one compactifying piece L. Thus $X=S\cup Y$ and f is the combination of e and the identity on Y.

For $p\in P$ the open sets generating the topology of X in points of $X(p)$ will be of the form $B(p, U)=\{q^\frown z\in S\colon p\subset q \text{ and } z\in U\}$, where U is open in Y. The open sets generating the topology in points of $L\subset X$ will consist of $L\cap U$ and the union of all but a finite number of elements of the collection $\{B(p, U)\colon p\in E^{-1}(L)\}$.

It is easy to verify that X is a σ-locally compact metacompact developable Hausdorff space and f is an open and compact mapping of X onto Y. □

In our reasonings we used partitions $\mathscr{L}$ for which there existed a neighbornet V such that for points y and z from different elements of $\mathscr{L}$ either $z\notin V(y)$ or $y\notin V(z)$ (equivalently, $V\cap V^{-1}$ is a subset of the equivalence relation generated by $\mathscr{L}$). Such neighbornets will be called $\mathscr{L}$-separating neighbornets[3]. If the partition $\mathscr{L}$ satisfies (α), then an $\mathscr{L}$-separating neighbornet will be called C-separating.

The class of spaces with a C-separating neighbornet contains all spaces with a σ-discrete cover by compact sets (see the proof of Proposition 1.2), all spaces with a closure-preserving closed cover by σ-compact sets (see the reasoning preceding (α)) and all C-scattered spaces (see the proof of Corollary 1.10). We shall show that, under some additional assumptions, these inclusions can be reversed.

Proposition 3.3. *If Z is a semistratifiable space with a C-separating neighbornet, then Z has a σ-discrete cover by compact sets.*

Proof. It is sufficient to show that if $\mathscr{L}$ is a partition of a semistratifiable space Z and V is an $\mathscr{L}$-separating neighbornet in Z, then $\mathscr{L}$ has a σ-discrete closed refinement (see the proof of Proposition 1.4). This is a slight improvement of [13, 4.8] showing that the assumption that $U\cap U^{-1}$ is an equivalence relation is not necessary in condition (iv) of this result.

Let $\langle V_n : n\geq 0\rangle$ be a co-basic sequence of neighbornets contained in V and assume that $V_n(z)$ is open for all $n\geq 0$ and $z\in Z$ (see [13, 4.1]). For $L\in\mathscr{L}$ define $D_n(L)=\{z\in L\colon V_n^{-1}(z)\subset L\}=Z\setminus\bigcup\{V_n(y)\colon y\notin L\}$. Clearly the collection $\mathscr{D}_n=\{D_n(L)\colon L\in\mathscr{L}\}$ is closed and discrete for each $n\geq 0$. To show that these collections cover Z let z be a point of $L\subset Z$ and find an n such that $V_n^{-1}(z)\subset V(z)$. Since $V_n\subset V$, we have $V_n^{-1}\subset V^{-1}$ and, consequently, $V_n^{-1}(z)\subset (V\cap V^{-1})(z)\subset L$. Thus $z\in D_n(L)$ and the proof is finished. □

[3] The notion of $\mathscr{L}$-separating neighbornet is slightly more general than the notion of unsymmetric neighbornet [13].

From Propositions 3.3 and 1.4, it follows that a semistratifiable meta-Lindelöf space with a C-separating neighbornet has a closure-preserving closed cover by σ-compact sets. Our next result gives another condition sufficient to produce a closure-preserving closed cover by σ-compact sets from a C-separating neighbornet. Moreover, it shows that MOBI(σ-locally compact) is the class of all spaces with a C-separating neighbornet and a point-countable base of countable order (see the proof of (c)$\Rightarrow$(c′) in [5, 2.1]).

Proposition 3.4. *If a space Z has a point-countable base and a C-separating neighbornet, then Z has a closure-preserving closed cover by σ-compact sets.*

Proof. Let V be an $\mathscr{L}$-separating neighbornet in Z for a partition $\mathscr{L}$ of Z satisfying (α). For each $z \in Z$ fix an open set $W(z)$ from a given point-countable base of Z such that $z \in W(z) \subset V(z)$. Observe that if $W(z) = W(y)$, then z and y are in the same element of $\mathscr{L}$ and, consequently, $\{W(z)\colon z \in Z\}$ defines a co-σ-compact neighbornet. By Lemmas 0.1 and 1.3, this completes the proof. □

Finally, we have (see [5, 1.3]).

Proposition 3.5. *If a λ_c-space Z has a C-separating neighbornet, then Z is a C-scattered space.*

Proof. Let $\mathscr{L}$ be a partition of Z satisfying (α) and separated by a neighbornet V. Moreover, let $\langle \mathscr{G}_n : n \geqslant 1\rangle$ be a sequence of bases of Z witnessing the fact that Z is a λ_c-space.

If Z is not C-scattered, then there exists a closed subset F of Z such that no open subset of F is contained in the closure of a finite union of elements of $\mathscr{L}$. Thus one can construct, by induction, sequences $\langle G_n : n \geqslant 0\rangle$ and $\langle z_n : n \leqslant 1\rangle$ such that $G_0 = Z$ and, for $n \geqslant 1$,

(i) $z_n \in F \cap G_{n-1} \setminus \bigcup\{L_m : 0 < m < n\}$,

(ii) $z_n \in G_n \in \mathscr{G}_n$ and $\bar{G}_n \subset G_{n-1}$,

(iii) $G_n \subset V(z_n) \setminus \bigcup\{L_m : 0 < m < n\}$,

where L_m is the element of $\mathscr{L}$ containing z_m.

By (ii), the sequence $\langle G_n : n \geqslant 1\rangle$ satisfies (λ_c) and, consequently, the sequence $\langle z_n : n \geqslant 1\rangle$ has an accumulation point z in Z. The second part of (ii) implies that $z \in \bigcap\{G_n : n \geqslant 1\}$.

Let L be the element of $\mathscr{L}$ containing z and choose an $n \geqslant 1$ such that $z_n \in V(z)$. Since $z \in G_n \subset V(z_n)$, we obtain $L = L_n$ and this contradicts (iii). □

Note that Proposition 3.5 can be used to obtain characterizations of first-countable C-scattered spaces (C-scattered spaces of countable type) analogous to 4.4 (4.5) of [5].

In almost all the results of this paper (σ-)compactness can be replaced with separability. In particular, the class of open and compact images of σ-locally separable metric spaces is the class of σ-locally separable metacompact Moore spaces. Let us finish with a question closely related to [6, 7].

Problem 3.6. Suppose Y is a space with a closure-preserving closed cover by separable subspaces and a point-countable base of countable order. Is Y an open and compact image of a σ-locally separable metacompact Moore space?

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