The Structure Matrix of the Class of r-Multigraphs With a Prescribed Degree Sequence*

T. S. Michael
Mathematics Department
United States Naval Academy
Annapolis, Maryland, 21402

Submitted by Richard A. Brualdi

ABSTRACT

We introduce a structure matrix $S_r(D)$ into the study of the class $\mathfrak{G}_r(D)$ of r-multigraphs with the prescribed degree sequence D. Our structure matrix plays the role of the one used by Ryser, Fulkerson, and others to investigate classes of matrices of 0's and 1's with prescribed row and column sums. We develop a theory that is wholly analogous to the classical one. We show that under a type of monotonicity assumption on $D = (d_1, \dots, d_n)$ the class $\mathfrak{G}_r(D)$ is nonempty if and only if the sum $d_1 + \cdots + d_n$ is even and the structure matrix $S_r(D)$ is nonnegative. We also prove a generalization of the analogue of Ryser's maximum term rank formula. This result includes both a formula for the maximum number of edges in a matching and a formula for the maximum number of edges in a spanning, nearly regular subgraph among all graphs with a prescribed degree sequence.

1. AN OVERVIEW

Throughout this paper r denotes a positive integer, and

$$D = (d_1, d_2, \dots, d_n)$$

*This research was partially supported by a grant from the Louisiana Education Quality Support Fund through the Louisiana State University Board of Regents, by a summer grant from the LSU Council on Research, by NSF grant DMS 851-1521, and by a Wisconsin Alumni Research Fellowship at the University of Wisconsin—Madison.

655 Avenue of the Americas, New York, NY 10010 0024-3795/93/$6.00

denotes a sequence of nonnegative integers. We let $\mathfrak{G}_r(D)$ denote the class of all r-multigraphs with degree sequence D. For each r-multigraph $G = (V, E)$ we agree to select $V = \{1, 2, \dots, n\}$ as our vertex set. No edge occurs with multiplicity greater than r in the edge multiset E. Loops are forbidden. The number of edges, counting multiplicities, that contain vertex i is the *degree* d_i of vertex i, and D is the *degree sequence* of G. The *structure matrix*

$$S = S_r(D) = [s_{ij}] \qquad (i, j = 0, 1, \dots, n)$$

(of *multiplicity* r) of D has as its entries the $(n+1)^2$ *structure constants*

$$s_{ij} = s_{ij}(D, r) = rij - r \cdot \min\{i, j\} + \sum_{k>i} d_k - \sum_{k \leqslant j} d_k,$$

where $i, j = 0, 1, \dots, n$. (Empty summations are assigned the value 0.) The sequence D is *monotone* provided

$$d_i \geqslant d_j \qquad \text{for} \quad 1 \leqslant i < j \leqslant n.$$

The sequence D is *nearly monotone* provided

$$d_i \geqslant d_j - 1 \qquad \text{for} \quad 1 \leqslant i < j \leqslant n.$$

Thus each monotone sequence is nearly monotone. The vertices of a multigraph may always be relabeled so that the degree sequence becomes (nearly) monotone. The class $\mathfrak{G}_r(D)$ is *(nearly) monotone* provided D is (nearly) monotone.

Our first theorem gives necessary and sufficient conditions for the nonemptiness of the class $\mathfrak{G}_r(D)$ in terms of the entries of the structure matrix $S_r(D)$.

THEOREM 1.1. *The nearly monotone class $\mathfrak{G}_r(D)$ is nonempty if and only if $d_1 + d_2 + \cdots + d_n$ is even and the structure matrix $S_r(D)$ is nonnegative.*

This result can be deduced as a special case of a powerful theorem of Fulkerson, Hoffman, and McAndrew [12]. (In [12] Theorem 5.1 is the statement of our Theorem 1.1 for monotone sequences D and $r = 1$.) We shall give a short, self-contained proof of Theorem 1.1 by induction. The advantage of dealing with a nearly monotone sequence instead of a monotone sequence in Theorem 1.1 will become apparent in the proofs of our subsequent results.

Erdős and Gallai [9] (for $r = 1$) and Chungphaisan [8] (for $r \geqslant 1$) have given other necessary and sufficient conditions for the nonemptiness of the monotone class $\mathfrak{G}_r(D)$. We shall indicate how their conditions follow from nonnegativity of the structure matrix $S_r(D)$.

Let $G = (V, E)$ and $G' = (V', E')$ be two multigraphs. Then G' is a *submultigraph* of G provided $V' \subseteq V$ and E' is a submultiset of E. The submultigraph G' is a *spanning*, *nearly regular* submultigraph provided $V' = V$ and the degrees of the vertices in G' differ by at most 1 from one another. The *degree sum* of G, denoted by $\tau(G)$, is the sum of the degrees of all the vertices of G. Thus the degree sum of a multigraph equals twice the number of its edges counting multiplicities.

Our second theorem gives necessary and sufficient conditions for the existence of a multigraph in $\mathfrak{G}_r(D)$ with a spanning, nearly regular submultigraph of prescribed degree sum.

THEOREM 1.2. *Let $S = S_r(D) = [s_{ij}]$ denote the structure matrix of the nonempty, monotone class $\mathfrak{G}_r(D)$. Let τ be an even, nonnegative integer, and suppose that*

$$\tau = nk + a, \tag{1.1}$$

where k and a are integers with $0 \leqslant a \leqslant n$. Then there is a multigraph G in $\mathfrak{G}_r(D)$ with a spanning, nearly regular submultigraph of degree sum τ if and only if the inequality

$$\tau \leqslant s_{ij} + \min\{i, a\} + \min\{j, a\} + k(i + j) \tag{1.2}$$

holds for $i, j = 0, 1, \ldots, n$.

A *matching* in an ordinary graph G is a pairwise vertex-disjoint subset of the edge set E. In particular, a matching may be viewed as a spanning, nearly regular subgraph of G. The *matching number* $\mu(G)$ of the graph G is the maximum number of edges in a matching in G. Suppose that the class $\mathfrak{G}_1(D)$ of graphs with degree sequence D is nonempty. We let

$$\bar{\mu} = \bar{\mu}(D) = \max\{\mu(G) : G \in \mathfrak{G}_1(D)\},$$

$$\tilde{\mu} = \tilde{\mu}(D) = \min\{\mu(G) : G \in \mathfrak{G}_1(D)\}$$

denote the largest and smallest matching numbers, respectively, among all graphs with degree sequence D. One consequence of Theorem 1.2 is the following formula for the maximum number of edges in a matching among all graphs with a prescribed degree sequence.

THEOREM 1.3. *Let $S_1(D) = [s_{ij}]$ denote the structure matrix of the nonempty, monotone class $\mathfrak{G}_1(D)$. Then*

$$\bar{\mu} = \min_{i,j} \left\{ \left\lfloor \frac{s_{ij} + i + j}{2} \right\rfloor \right\}, \tag{1.3}$$

where the minimum extends over $i, j = 0, 1, \ldots, n$.

The edge $[i, j]$ in an *invariant edge* of the nonempty class $\mathfrak{G}_1(D)$ provided $[i, j]$ occurs as an edge in every graph in $\mathfrak{G}_1(D)$. For instance, if the entry s_{ef} is 0 in the structure matrix $S_1(D)$, then $[i, j]$ is an invariant edge for all distinct vertices i and j with $1 \leqslant i \leqslant e$ and $1 \leqslant j \leqslant f$.

Suppose that the class $\mathfrak{G}_1(D)$ is nonempty. Evidently $0 \leqslant \tilde{\mu} \leqslant \bar{\mu} \leqslant \lfloor n/2 \rfloor$. If μ is an integer with $\tilde{\mu} \leqslant \mu \leqslant \bar{\mu}$, then some graph in $\mathfrak{G}_1(D)$ has matching number μ. This follows from the interchange theorem [12, 8] and the observation that a single interchange can alter the matching number of a graph by at most 1. Our final theorem gives sufficient conditions for the strict inequality $\tilde{\mu} < \bar{\mu}$.

THEOREM 1.4. *Let $\mathfrak{G}_1(D)$ be a nonempty, monotone class. Suppose that the following three conditions hold:*

$$\text{the class } \mathfrak{G}_1(D) \text{ has no invariant edges;} \tag{1.4}$$

$$\bar{\mu} < \lfloor n/2 \rfloor; \tag{1.5}$$

$$d_n \geqslant 2. \tag{1.6}$$

Then we have the strict inequality

$$\tilde{\mu} < \bar{\mu}. \tag{1.7}$$

Theorem 1.4 becomes invalid if any one of the three conditions is dropped. (See Example 5.3).

Let $\mathfrak{A}_r(D)$ denote the class of matrices $A = [a_{ij}]$ of order n such that

$$a_{ij} \in \{0, 1, \ldots, r\} \text{ and } a_{ij} = a_{ji} \quad \text{for} \quad i, j = 1, 2, \ldots, n,$$

$$a_{ii} = 0 \text{ and } \sum_{j=1}^{n} a_{ij} = d_i \qquad \text{for} \qquad i = 1, 2, \ldots, n.$$

Thus each matrix in the class $\mathfrak{A}_r(D)$ is a symmetric $(0, 1, \dots, r)$-matrix with trace 0 and row sum vector D. There is a natural one-to-one correspondence $A \leftrightarrow G_A$ between the *adjacency matrices* in the class $\mathfrak{A}_r(D)$ and multigraphs in the class $\mathfrak{G}_r(D)$; the element a_{ij} of A equals the multiplicity of the edge $[i, j]$ in G_A. It follows that for any result about one class there is a corresponding result about the other. We shall use this correspondence freely throughout this paper.

Our definition of the structure matrix S resembles the definition of the structure matrix T for the class $\mathfrak{A}(R, S)$ of $(0, 1)$-matrices with row sum vector R and column sum vector S. Ryser, Fulkerson [11, 13–15, 26–29], and others [5, 6, 16, 17] discovered several remarkable relationships between the properties of the class $\mathfrak{A}(R, S)$ and the entries of the structure matrix T. (Also see the survey article by Brualdi [4].) The matrices in the class $\mathfrak{A}(R, S)$ are the reduced adjacency matrices of the bipartite graphs with the degree sequences R and S, and thus the classical results about $\mathfrak{A}(R, S)$ have interpretations in terms of bipartite graphs with prescribed degrees. Our goal in this paper is to prove several analogous results for graphs and multigraphs with prescribed degrees. For instance, Theorem 1.1 is the analogue of the generalization of the Ford-Fulkerson theorem [10] enunciated in [23], while the maximum matching formula in Theorem 1.3 is the counterpart to Ryser's maximum term rank formula [27; 30, p. 75].

In Section 2 of this paper we list some elementary properties of structure matrices. In Section 3 we prove Theorem 1.1. We prove Theorem 1.2 in Section 4. Matchings are treated in Section 5. In Section 6 we relate the inequalities of Erdős and Gallai and of Chungphaisan to the nonnegativity of the structure matrix. In Section 7 we suggest some directions for research on structure matrices.

2. SOME ELEMENTARY PROPERTIES OF THE STRUCTURE MATRIX

In this section we list several properties of the structure constants

$$s_{ij} = rij - r \cdot \min\{i, j\} + \sum_{k>i} d_k - \sum_{k \leq j} d_k \tag{2.1}$$

and the structure matrix $S_r(D) = [s_{ij}]$. These results are the analogues of the well-known results concerning the structure constants of the class $\mathfrak{A}(R, S)$ of matrices of 0's and 1's. (See Ryser [27–29] and Brualdi [4].)

LEMMA 2.1. *Suppose that $S = S_r(D) = [s_{ij}]$ is the structure matrix of multiplicity r of the nonnegative sequence D. Then*

$$s_{ij} = rij - r \cdot \min\{i, j\} - \sum_{k \leqslant i} d_k + \sum_{k > j} d_k, \tag{2.2}$$

and thus S is symmetric. For $i = 0, 1, \dots, n$ the entries in row i of S satisfy the recurrence relation

$$s_{ij} - s_{i,j-1} = \begin{cases} r(i-1) - d_j & \text{if } \ j \leqslant i, \\ ri - d_j & \text{if } \ j > i \end{cases} \qquad (j = 1, 2, \dots, n). \tag{2.3}$$

For $j = 0, 1, \dots, n$ the entries in column j of S satisfy the recurrence relation

$$s_{ij} - s_{i-1,j} = \begin{cases} r(j-1) - d_i & \text{if } \ i \leqslant j, \\ rj - d_i & \text{if } \ i > j \end{cases} \qquad (i = 1, 2, \dots, n). \tag{2.4}$$

Moreover, for $i, j = 1, 2, \dots, n$

$$(s_{ij} + s_{i-1,j-1}) - (s_{i-1,j} + s_{i,j-1}) = \begin{cases} r & \text{if } \ i \neq j, \\ 0 & \text{if } \ i = j. \end{cases} \tag{2.5}$$

The entries in row 0 and column 0 of S are

$$\begin{aligned} s_{0n} &= s_{n0} = 0, \\ s_{0,n-1} &= s_{n-1,0} = d_n, \\ s_{0,n-2} &= s_{n-2,0} = d_n + d_{n-1}, \\ &\vdots \\ s_{00} &= s_{00} = d_n + d_{n-1} + \cdots + d_1 \end{aligned} \tag{2.6}$$

The diagonal entries of S satisfy the recurrence relation

$$s_{ii} = s_{i-1,i-1} + 2[r(i-1) - d_i] \qquad (i = 1, 2, \dots, n). \tag{2.7}$$

Suppose that D is monotone. Then the entries in row i and the entries in column j of S each form a convex sequence, that is,

$$2s_{ij} \leqslant s_{i,j-1} + s_{i,j+1} \qquad (j = 1,2,\dots,n-1;\ i = 0,1,\dots,n), \quad (2.8)$$

$$2s_{ij} \leqslant s_{i-1,j} + s_{i+1,j} \qquad (i = 1,2,\dots,n-1;\ j = 0,1,\dots,n), \quad (2.9)$$

and the diagonal entries of S form a strictly convex sequence, that is,

$$2s_{ii} < s_{i-1,i-1} + s_{i+1,i+1} \qquad (i = 1,2,\dots,n-1). \qquad (2.10)$$

We omit the proof of Lemma 2.1; the properties (2.2)–(2.10) are direct consequences of the definition (2.1).

We remark that (2.6) and the recurrence relation (2.5) facilitate the computation of the $(n+1)^2$ entries of the structure matrix S. The following example illustrates the above properties.

EXAMPLE 2.2. If $r = 2$ and $D = (4,4,3,3,2)$, then

$$S_r(D) = \begin{bmatrix} 16 & 12 & 8 & 5 & 2 & 0 \\ 12 & 8 & 6 & 5 & 4 & 4 \\ 8 & 6 & 4 & 5 & 6 & 8 \\ 5 & 5 & 5 & 6 & 9 & 13 \\ 2 & 4 & 6 & 9 & 12 & 18 \\ 0 & 4 & 8 & 13 & 18 & 24 \end{bmatrix}.$$

The sequence $\bar{D} = (\bar{d}_1, \bar{d}_2, \dots, \bar{d}_n)$ is the *complement* of D (*with respect to* r) provided

$$\bar{d}_i + d_{n+1-i} = r(n-1) \qquad (i = 1,2,\dots,n). \qquad (2.11)$$

Let G be a multigraph in $\mathfrak{G}_r(D)$ with adjacency matrix A. The *complement of G* (*with respect to* r) is the r-multigraph $\bar{G}$ with adjacency matrix

$$\bar{A} = r(J_n - I_n) - A,$$

where J_n is the matrix of order n each of whose entries is 1, and I_n is the identity matrix of order n. Thus if we relabel the vertices of $\bar{G}$ in reverse order, then $\bar{G}$ has the complementary degree sequence $\bar{D}$. Let $P = [p_{ij}]$

$(i, j = 0, 1, \ldots, n)$ denote the permutation matrix of order $n+1$ with 1's in positions $(0, n)$, $(1, n-1), \ldots, (n, 0)$ and 0's elsewhere. Let $U = [u_{ij}]$ $(i, j = 0, 1, \ldots, n)$ denote the upper triangular matrix of order $n+1$ with 1's on and above the main diagonal and 0's below the main diagonal.

Our next result includes algebraic information about structure matrices.

THEOREM 2.3. *Let $S = S_r(D) = [s_{ij}]$ and $\bar{S} = \bar{S}_r(D) = [\bar{s}_{ij}]$ denote the structure matrices of the complementary sequences D and $\bar{D}$. Then*

$$S = P\bar{S}P. \tag{2.12}$$

The matrix factorization

$$S = U^T B U \tag{2.13}$$

also holds, where

$$B = B_r(D) = \left[b_{ij}\right] = \begin{bmatrix} \tau & -D \\ -D^T & r(J_n - I_n) \end{bmatrix} \qquad (i, j = 0, 1, \ldots, n),$$

and $\tau = d_1 + d_2 + \cdots + d_n$.

Proof. For $i, j = 0, 1, \ldots, n$ the (i, j) element of $P\bar{S}P$ is $\bar{s}_{n-i, n-j}$. Now by (2.11) and (2.2)

$$\begin{aligned} \bar{s}_{n-i, n-j} &= r(n-i)(n-j) - r \cdot \min\{n-i, n-j\} \\ &\quad + \sum_{k > n-i} \left[r(n-1) - d_{n+1-k}\right] - \sum_{k \leqslant n-j} \left[r(n-1) - d_{n+1-k}\right] \\ &= rij - r \cdot \min\{i, j\} - \sum_{k \leqslant i} d_k + \sum_{k > j} d_k = s_{ij}. \end{aligned}$$

Thus (2.12) holds.

For $i, j = 0, 1, \ldots, n$ we have

$$\begin{aligned} s_{ij} &= rij - r\cdot\min\{i,j\} + \sum_{k>i} d_k - \sum_{k\leqslant j} d_k \\ &= rij - r\cdot\min\{i,j\} + \tau - \sum_{k\leqslant i} d_k - \sum_{k\leqslant j} d_k \\ &= \sum_{e=0}^{i}\sum_{f=0}^{j} b_{ef} = \sum_{e=0}^{n}\sum_{f=0}^{n} u_{ei}b_{ij}u_{fj}, \end{aligned}$$

which is the (i,j) element of U^TBU. Thus (2.13) holds. ∎

We remark that the factorization (2.13) can be used to show that the rank of the structure matrix S is either n or $n+1$. Moreover, for a nonempty class $\mathfrak{G}_r(D)$ the structure matrix S always has full rank $n+1$ unless D or $\bar{D}$ is $(0,0,\ldots,0)$.

The final result in this section motivates our definition (2.1) of the structure constants. Our proof uses the usual counting arguments for results of this type.

LEMMA 2.4. *Suppose that the (not necessarily monotone) class $\mathfrak{G}_r(D)$ is nonempty. Then $d_1 + d_2 + \cdots + d_n$ is even, and the structure matrix $S_r(D) = [s_{ij}]$ is nonnegative.*

Proof. Suppose that $G \in \mathfrak{G}_r(D)$. Then $\tau(G) = d_1 + d_2 + \cdots + d_n$, which is twice the number of edges of G. Let A be the adjacency matrix of G. For $i, j = 0, 1, \ldots, n$ consider the decompositions

$$A = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} \quad\text{and}\quad \bar{A} = r(J_n - I_n) - A = \begin{bmatrix} W' & X' \\ Y' & Z' \end{bmatrix},$$

where the submatrices W and W' are of size i by j and the submatrices Z and Z' are of size $n-i$ by $n-j$. (If $i=0$ or n, or if $j=0$ or n, then some of the submatrices are vacuous.) Let $\tau(B)$ denote the sum of the elements of the matrix B. The matrices A and $\bar{A}$ are both nonnegative. Hence

$$\begin{aligned} 0 \leqslant \tau(Z) + \tau(W') &= rij - r\cdot\min\{i,j\} + \tau(Z) - \tau(W) \\ &= rij - r\cdot\min\{i,j\} + \tau(Z) + \tau(Y) - \tau(Y) - \tau(W) \\ &= rij - r\cdot\min\{i,j\} + \sum_{k>i} d_k - \sum_{k\leqslant j} d_k = s_{ij}. \end{aligned}$$

∎

Lemma 2.4 may also be established by the techniques of transversal theory (see Mirsky [24, pp. 204–211]) or by network flows (see Ford and Fulkerson [10, pp. 79–90]); each of these theories also motivates the definition (2.1).

3. PROOF OF THEOREM 1.1

One implication follows from Lemma 2.4. Now suppose that $d_1 + d_2 + \cdots + d_n$ is even and that S is nonnegative. We prove that the class $\mathfrak{A}_r(D)$ is nonempty by induction on the parameter $m = (d_1 + d_2 + \cdots + d_n)/2$. Suppose that $m = 0$. Then $D = (0, 0, \dots, 0)$. The structure matrix $S = [rij - r \cdot \min\{i, j\}]$ is nonnegative, and $\mathfrak{A}_r(D)$ consists of a matrix of 0's.

We henceforth suppose that $m > 0$. Without loss of generality $d_i > 0$ for $i = 1, 2, \dots, n$. We define the index

$$e = \min\{i : d_i \geqslant d_k \text{ for } k = 1, 2, \dots, n\}.$$

Suppose that $e = n$. Then n is odd, and $D = (d, d, \dots, d, d + 1)$ for some odd integer d with $1 \leqslant d \leqslant n - 2$. In this case we may verify directly that the class $\mathfrak{A}_r(D)$ is nonempty and that the structure matrix S is nonnegative. We henceforth suppose the $e < n$.

Because D is nearly monotone,

$$1 \leqslant i < e \quad \text{implies that} \quad d_i = d_e - 1. \tag{3.1}$$

Let E_k denote a unit vector with a 1 in position k and 0's in all other positions. We define the sequence

$$\hat{D} = D - E_e - E_n. \tag{3.2}$$

Now $\hat{D}$ is nearly monotone by our choice of e, and half the sum of the components of $\hat{D}$ equals $m - 1$.

Assume that the structure matrix $\hat{S} = S_r(\hat{D}) =]\hat{s}_{ij}]$ is nonnegative. By induction there is a matrix $\hat{A} = [\hat{a}_{ij}]$ in the class $\mathfrak{A}_r(\hat{D})$. Let E_{ij} denote the symmetric matrix of order n with 1's in positions (i, j) and (j, i) and 0's in all other positions. If $\hat{a}_{en} < r$, then $\hat{A} + E_{en} \in \mathfrak{A}_r(D)$. Suppose that $\hat{a}_{en} = r$. The inequality $0 \leqslant s_{en} = e[r(n-1) - (d_e - 1)] - 1$ implies that $d_e - 1 < r(n-1)$. Thus $\hat{a}_{ek} < r$ for some index $k \neq e$. Without loss of generality $d_k > d_n$, and thus $\hat{a}_{hk} > \hat{a}_{hn}$ for some $h \neq n$. Now $\hat{A} - E_{hk} + E_{ek} + E_{hn} \in \mathfrak{A}_r(D)$.

It remains to prove that $\hat{S}$ is nonnegative. We have

$$\hat{s}_{ij} = \begin{cases} s_{ij} - 2 & \text{if } 0 \leqslant i, j < e, \\ s_{ij} - 1 & \text{if } 0 \leqslant i < e \leqslant j < n \\ & \text{or if } 0 \leqslant j < e \leqslant i < n, \end{cases}$$

and $\hat{s}_{ij} \geqslant s_{ij}$ otherwise. Because S is symmetric and nonnegative, it suffices to prove that

$$s_{ij} \geqslant 2 \qquad \text{for} \quad 0 \leqslant j \leqslant i < e \tag{3.3}$$

and

$$s_{ij} \geqslant 1 \qquad \text{for} \quad 0 \leqslant j < e \leqslant i < n. \tag{3.4}$$

Suppose that $j \leqslant i$ and $j < e$. If $d_e - 1 \leqslant r(i-1)$, then by (3.1)

$$\begin{aligned} s_{ij} &= rij - rj + \sum_{k>i} d_k - \sum_{k \leqslant j} d_k \\ &= j[r(i-1) - (d_e - 1)] + \sum_{k>i} d_k, \end{aligned}$$

and the inequalities (3.3) and (3.4) both hold because $d_e \geqslant 1$ and $d_n \geqslant 1$. We henceforth suppose that $d_e - 1 > r(i-1)$. If $j < e \leqslant i$, then by (3.1)

$$s_{ij} = s_{ie} + (e - j)[(d_e - 1) - r(i-1)] + 1,$$

and (3.4) holds. Finally, suppose that $j \leqslant i < e$. Then by (3.1) and (2.3)

$$s_{ij} = s_{i,i+1} + (i - j + 1)[(d_e - 1) - r(i-1)].$$

Assume that the inequality $s_{ij} \geqslant 2$ fails. Then we must have $i = j$, and $s_{i,i+1} = 0$, and $d_e - 1 = r(i-1) + 1$. The recurrence relation (2.3) implies that the diagonal element s_{ii} equals 1. But we known that $s_{00} = d_1 + d_2 + \cdots + d_n$ is even, and hence each diagonal entry of S must be even by (2.7). This contradiction establishes (3.3). Therefore $\hat{S}$ is nonnegative. ∎

4. PROOF OF THEOREM 1.2

LEMMA 4.1. *Suppose that the sequence* $D' = (d'_1, d'_2, \dots, d'_n)$ *satisfies*

$$d_i - d'_i \in \{k, k+1\} \qquad \text{for} \quad i = 1, 2, \dots, n$$

for some nonnegative integer k. *Then there exists a multigraph* G *in* $\mathfrak{G}_r(D)$ *with a submultigraph* G' *in* $\mathfrak{G}_r(D')$ *if and only if both of the classes* $\mathfrak{G}_r(D)$ *and* $\mathfrak{G}_r(D')$ *are nonempty.*

Our Lemma 4.1 is the k-factor theorem for r-multigraphs. See Kundu [21], Koren [20], Lovász [22], Kleitman and Wang [19], and Chen [7] for various proofs in the special case $r = 1$. The generalization to $r \geqslant 1$ presents few new difficulties. For instance, either the interchange method of Lovász or the short proof of Chen can be readily adapted to establish Lemma 4.1 in general. We omit the proof.

LEMMA 4.2. *Suppose that some multigraph* G *in the monotone class* $\mathfrak{G}_r(D)$ *has a spanning, nearly regular submultigraph* H *of degree sum* τ. *Then some multigraph* G' *in* $\mathfrak{G}_r(D)$ *has a spanning, nearly regular submultigraph of degree sum* τ *whose degree sequence is monotone.*

Proof. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be the adjacency matrices of the multigraphs G and H that satisfy the hypothesis of the lemma. Suppose that the sum of row $e + 1$ of B exceeds the sum of row e by 1. Then $b_{ef} < b_{e+1,f}$ for some index $f \neq e$. If $a_{ef} > b_{ef}$, then we define $\hat{A} = A$ and $\hat{B} = B - E_{e+1,f} + E_{ef}$. We henceforth suppose that $a_{ef} = b_{ef}$. By the monotonicity of D there is an index $j \neq f$ such that $a_{ej} - b_{ej} > a_{e+1,j} - b_{e+1,j}$. If $b_{e+1,j} > 0$, then we define $\hat{A} = A$ and $\hat{B} = B + E_{ej} - E_{e+1,j}$. Now suppose that $b_{e+1,j} = 0$. In this final case we define $\hat{B} = B - E_{e+1,f} + E_{ef}$ and $\hat{A} = A - E_{ej} - E_{e+1,f} + E_{ef} + E_{e+1,j}$. In all cases the matrices $\hat{A}$ and $\hat{B}$ are the adjacency matrices of multigraphs that satisfy the hypothesis of the lemma. But the sum of row e of $\hat{B}$ now exceeds the sum of row $e + 1$ by 1. Iteration of this process completes the proof. ∎

We now prove Theorem 1.2. By Lemma 4.1 and Lemma 4.2 there exists a multigraph G that satisfies the conditions of Theorem 1.2 if and only if the class $\mathfrak{G}_r(\hat{D})$ is nonempty, where

$$\hat{D} = \left(\hat{d}_1, \hat{d}_2, \dots, \hat{d}_n\right) = D - \Big(\underbrace{k+1, \dots, k+1}_{a}, \underbrace{k, \dots, k}_{n-a}\Big).$$

The vector $\hat{D}$ is nearly monotone. By Theorem 1.1 the class $\mathfrak{G}_r(\hat{D})$ is nonempty if and only if the structure matrix $S_r(\hat{D}) = [\hat{s}_{ij}]$ is nonnegative. Now for $i, j = 0, 1, \ldots, n$ we have

$$\begin{aligned}
\hat{s}_{ij} &= rij - r \cdot \min\{i, j\} + \sum_{p>i} \hat{d}_p - \sum_{p\leqslant j} \hat{d}_p \\
&= rij - r \cdot \min\{i, j\} + \sum_{p>i} (d_p - k) \\
&\qquad - \max\{0, a - i\} - \sum_{p\leqslant j} (d_p - k) + \min\{j, a\} \\
&= rij - r \cdot \min\{i, j\} + \sum_{p>i} d_p \\
&\qquad - \sum_{p\leqslant j} d_p + \min\{i, a\} - a + \min\{j, a\} - k(n - i) + kj \\
&= s_{ij} - \tau + \min\{i, a\} + \min\{j, a\} + k(i + j).
\end{aligned}$$

Therefore $S_r(\hat{D})$ is nonnegative if and only if (1.2) holds and the proof is complete.

The following corollary may be viewed as a structure matrix version of the (uniform) k-factor theorem.

COROLLARY 4.3. *Let $S_r(D) = [s_{ij}]$ denote the structure matrix of the nonempty, monotone class $\mathfrak{G}_r(D)$. Then some multigraph in $\mathfrak{G}_r(D)$ has a submultigraph with every vertex of degree k if and only if nk is even and the inequality*

$$k \leqslant \frac{1}{n - i - j} s_{ij} \tag{4.1}$$

holds for all i and j for which $n > i + j$.

Proof. We select $\tau = nk$ and $a = 0$ in (1.1). By Theorem 1.2 some multigraph in $\mathfrak{G}_r(D)$ has a submultigraph with every vertex of degree k if and only if the inequality

$$nk \leqslant s_{ij} + k(i + j) \tag{4.2}$$

holds for $i, j = 0, 1, \ldots, n$. If $n \leqslant i + j$, then (4.2) clearly holds. Suppose that $n > i + j$. Then (4.2) holds if and only if (4.1) holds. ■

5. MATCHINGS

In this section we study the maximum matching number $\bar{\mu}(D)$ among all graphs with degree sequence D. Our results are the analogues of the results of Ryser [27–29; 30, pp. 72–76] for the maximum term rank $\bar{\rho}(R, S)$ among all (0, 1)-matrices with row sum vector R and column sum vector S. The presence of our parallel theory becomes less surprising when we recall that the term rank of a (0, 1) matrix A equals the maximum cardinality of a matching in the bipartite graph associated with A. Thus the term rank is the bipartite analogue of the matching number. Our Theorem 1.3 is the analogue of Ryser's maximum term rank formula. The decomposition in Theorem 5.2 is analogous to the decomposition obtained by Ryser for (0, 1)-matrices. Also, Theorem 1.4 and the examples we give at the end of this section correspond to Ryser's results in [27; 30, pp. 75–76].

We begin with a proof of Theorem 1.3.

Proof of Theorem 1.3. Let μ be an integer with $0 \leqslant \mu \leqslant \lfloor n/2 \rfloor$. There is a graph in $\mathfrak{G}_1(D)$ with a matching of cardinality μ if and only if some graph G in $\mathfrak{G}_1(D)$ has a spanning, nearly regular subgraph of degree sum 2μ. In Theorem 1.2 we select $r = 1$, $k = 0$, and $\tau = a = 2\mu$. By (1.2) there is a graph in $\mathfrak{G}_1(D)$ with a matching of cardinality μ if and only if the inequality

$$2\mu \leqslant s_{ij} + \min\{i, 2\mu\} + \min\{j, 2\mu\} \tag{5.1}$$

holds for $i, j = 0, 1, \ldots, n$. If $i \geqslant 2\mu$ or if $j \geqslant 2\mu$, then (5.1) clearly holds. Suppose that $i < 2\mu$ and $j < 2\mu$. Then (5.1) holds if and only if $2\mu \leqslant s_{ij} + i + j$. This proves (1.3). ■

A matching M_μ of cardinality μ in a graph G is *canonical* provided

$$M_\mu = \{[1, 2\mu], [2, 2\mu - 1], \ldots, [\mu, \mu + 1]\}.$$

THEOREM 5.1. *If some graph in the monotone class $\mathfrak{G}_1(D)$ has a matching of cardinality μ, then some graph in $\mathfrak{G}_1(D)$ has a canonical matching of cardinality μ.*

Proof. Let $G = (V, E)$ be a graph in $\mathfrak{G}_1(D)$ with a matching M of cardinality μ. By Lemma 5.2 we may suppose that M involves only the vertices $1, 2, \ldots, 2\mu$. Suppose that M is not the canonical matching M_μ. Then there is a minimal positive integer h such that $[h, 2\mu + 1 - h] \notin M$. For $i = 1, 2, \ldots, 2\mu$ define i^* by $[i, i^*] \in M$. Let $k = 2\mu + 1 - h$, and consider the distinct edges $[h, h^*]$ and $[k, k^*]$ of M. By the minimality of h, both h^* and k^* are strictly between h and k. If neither $[h, k]$ nor $[h^*, k^*]$ is in E, or if both $[h, k]$ and $[h^*, k^*]$ are in E, then we readily obtain from G a graph in $\mathfrak{G}_1(D)$ with a matching

$$M' = M \cup [h, k] \cup [h^*, k^*] - [h, h^*] - [k, k^*]$$

of cardinality μ. Suppose that $[h, k] \in E$, but that $[h^*, k^*] \notin E$. Then the inequality $d_{h^*} \geqslant d_k$ implies that $[h^*, j] \in E$, but $[j, k] \notin E$ for some vertex j. We replace E by $E \cup [j, k] \cup [h^*, k^*] - [k, k^*] - [h^*, j]$ and obtain a graph in $\mathfrak{G}_1(D)$ with the matching M'. If $[h^*, k^*] \in E$, but $[h, k] \notin E$, then a similar argument allows us to produce a graph in $\mathfrak{G}_1(D)$ with the matching M'. In all cases we produce a graph in $\mathfrak{G}_1(D)$ with the matching M' of cardinality μ that has more edges of the form $[i, 2\mu + 1 - i]$ than M does. Iteration of this process completes the proof. ∎

Recall that $\tau(B)$ denotes the sum of the elements of the matrix B.

THEOREM 5.2. *Let $S_1(D) = [s_{ij}]$ denote the structure matrix of the nonempty class $\mathfrak{A}_1(D)$, where D is a monotone sequence of positive integers. Then there exist indices e and f with $0 \leqslant e, f \leqslant n$ such that*

$$\bar{\mu} = \left\lfloor \frac{s_{ef} + e + f}{2} \right\rfloor = \left\lfloor \frac{\tau(W') + \tau(Z) + e + f}{2} \right\rfloor. \tag{5.2}$$

The matrices W' and Z in (5.2) are from the decompositions

$$A = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} \quad \text{and}$$

$$\bar{A} = r(J_n - I_n) - A = \begin{bmatrix} W' & X' \\ Y' & Z' \end{bmatrix}, \tag{5.3}$$

where A is an arbitrary matrix in the class $\mathfrak{A}_1(D)$, and the submatrix W is of size e by f. The indices e and f are independent of A, but are not necessarily unique. Moreover, if A is the adjacency matrix of a graph with a canonical *matching of cardinality* $\overline{\mu}$, *then*

$$\tau(W') = 0 \text{ or } 1 \quad \text{and} \quad \overline{\mu} = \left\lfloor \frac{\tau(Z) + e + f}{2} \right\rfloor. \tag{5.4}$$

Proof. The first equality in (5.2) follows from Theorem 1.3, while the second equality follows from our proof of Lemma 2.4. Suppose that A is the adjacency matrix of a graph with a canonical matching of cardinality $\overline{\mu}$. Then A has 1's in positions $(1, 2\overline{\mu}), (2, 2\overline{\mu} - 1), \dots, (2\overline{\mu}, 1)$. We refer to these 1's as *essential*. Consider the decomposition (5.3) of A. If $s_{ef} = 0$, then $\tau(W') = 0$, and (5.4) holds. Now suppose that $s_{ef} > 0$. Then (5.2) implies that $2\overline{\mu} + 1 > e + f$. Thus no essential 1 appears in the e by f submatrix W. Hence X has e essential 1's, Y has f essential 1's, and the remaining $2\overline{\mu} - (e + f)$ essential 1's must appear in Z. Thus $\tau(Z) \geqslant 2\overline{\mu} - (e + f)$. But $s_{ef} = \tau(W') + \tau(Z)$. Hence $s_{ef} \geqslant \tau(W') + 2\overline{\mu} - (e + f)$. Now (5.2) implies that $\overline{\mu} \geqslant \overline{\mu} + \lfloor \tau(W')/2 \rfloor$, and (5.4) follows. ■

We have deduced Theorem 5.2 from Theorem 1.3. Presumably it is also possible to give a proof of Theorem 5.2 along the lines of the proof of Ryser's decomposition theorem [30, p. 75]; the formula (1.3) would then follow from Theorem 5.2.

Suppose that D is a monotone sequence of positive integers and that the strict inequality $\overline{\mu}(D) < \lfloor n/2 \rfloor$ holds. We assert that in this case the indices e and f in Theorem 5.3 are both strictly between 0 and n, that is, no submatrix in the decomposition (5.3) is vacuous. For assume that $e = 0$. Then the submatrices W and X are vacuous. Because D is a positive sequence, each of the $n - f$ columns of Z contains at least one 1. Hence $\tau(Z) \geqslant n - f$. Now (5.2) gives the contradiction

$$\begin{aligned} \left\lfloor \frac{n}{2} \right\rfloor > \overline{\mu} &= \left\lfloor \frac{\tau(W') + \tau(Z) + e + f}{2} \right\rfloor \\ &= \left\lfloor \frac{\tau(Z) + f}{2} \right\rfloor \geqslant \left\lfloor \frac{n}{2} \right\rfloor. \end{aligned}$$

The cases $e = n$, $f = 0$, and $f = n$ lead to contradictions similarly.

We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. Consider the decomposition (5.3) for matrices in $\mathfrak{A}_1(D)$. Because $\overline{\mu} < \lfloor n/2 \rfloor$, we know that W is of size e by f, where $1 \leqslant e, f \leqslant n - 1$. Assume that $e = 1$. The inequalities $n - 1 > d_1$ and

$d_n \geqslant 2$ imply that

$$s_{1f} = f - 1 - \sum_{k \leqslant 1} d_k + \sum_{k > f} d_k > n - f.$$

Now (1.5) and (5.2) give us the contradiction

$$\left\lfloor \frac{n}{2} \right\rfloor > \bar{\mu} = \left\lfloor \frac{s_{1f} + 1 + f}{2} \right\rfloor \geqslant \left\lfloor \frac{n+1}{2} \right\rfloor.$$

Thus $e \neq 1$. Similarly $f \neq 1$.

We now know that $2 \leqslant e, f \leqslant n - 1$ in (5.3). Because the class $\mathfrak{G}_1(D)$ has no invariant edges, there is a graph G in $\mathfrak{G}_1(D)$ whose adjacency matrix $A = [a_{ij}]$ in $\mathfrak{A}_1(D)$ satisfies $a_{12} = a_{21} = 0$. We shall prove that $\mu(G) < \bar{\mu}$. Now $\tau(W') \geqslant 2$ in (5.3). Thus $\tau(Z) \leqslant 2\bar{\mu} - e - f - 1$ by (5.2). Consider the $\mu(G)$ pairs of symmetrically placed 1's in A that correspond to a matching of cardinality $\mu(G)$. No two of these $2\mu(G)$ 1's appear in the same row or column. At most e of these 1's occur in the first e rows of A (that is, in W and X), at most f of these 1's occur in the first f columns of A (that is, in W and Y), and at most $2\bar{\mu} - e - f - 1$ of these 1's occur in Z. Thus

$$2\mu(G) \leqslant e + f + (2\bar{\mu} - e - f - 1) = 2\bar{\mu} - 1,$$

a contradiction. Hence $\mu(G) < \bar{\mu}$. Therefore $\tilde{\mu} < \bar{\mu}$. ■

In the following example we show that the conclusion of Theorem 1.4 does not hold if any one of the hypotheses (1.4), (1.5), and (1.6) is violated.

EXAMPLE 5.3.

(a) Consider the sequence $D = (n - 1, n - 1, 2, 2, \dots, 2)$ with n terms, where $n \geqslant 6$. The class $\mathfrak{G}_1(D)$ consists of a single graph G. Thus $\mathfrak{G}_1(D)$ has invariant edges $\tilde{\mu} = \bar{\mu}$. Hence neither the hypothesis (1.4) nor the conclusion (1.7) holds. However, the hypotheses (1.5) and (1.6) both hold, because $d_n = 2$ and $\tilde{\mu} = \bar{\mu} = \mu(G) = 2$.

(b) Consider the sequence $D = (n - 2, n - 2, \dots, n - 2)$ with n terms, where n is even and $n \geqslant 4$. Each graph in $\mathfrak{G}_1(D)$ arises by deleting $n/2$ vertex-disjoint edges from a complete graph on n vertices. Hence $\mathfrak{G}_1(D)$ has no invariant edges and $\tilde{\mu} = \bar{\mu} = n/2$. Thus neither the hypothesis (1.5) nor the conclusion (1.7) holds, but both of the hypotheses (1.4) and (1.6) hold.

(c) Consider the sequence $D = (n - 3, 1, 1, \dots, 1)$ with n terms, where $n \geqslant 6$. The hypothesis (1.6) is clearly violated. Each graph in the class $\mathfrak{G}_1(D)$ arises by including any two of the vertices $2, 3, \dots, n$ in an edge and then including each of the remaining $n - 3$ of the vertices in an edge with vertex 1. Thus there are no invariant edges in $\mathfrak{G}_1(D)$, and the hypothesis (1.4) holds. Also, $\tilde{\mu} = \overline{\mu} = 2 < \lfloor n/2 \rfloor$. Thus the hypothesis (1.5) also holds, but the conclusion (1.7) fails.

6. THE THEOREMS OF ERDŐS AND GALLAI AND OF CHUNGPHAISAN

In this section we discuss the relationship between the nonnegativity of the structure matrix and the familiar necessary and sufficient conditions given by Erdős and Gallai [9] (for $r = 1$) and by Chungphaisan [8] (for $r \geqslant 1$) for the existence of an r-multigraph with a prescribed degree sequence. The essential idea is that the Erdős-Gallai-Chungphaisan inequalities hold for D and r if and only if the smallest element in each column of the structure matrix $S_r(D)$ is nonnegative.

Throughout this section we assume that D is a monotone sequence with $d_1 + d_2 + \cdots + d_n$ even and $d_1 \leqslant r(n - 1)$. We first recall that Chungphaisan's generalization of the Erdős-Gallai theorem asserts that under these conditions *the class* $\mathfrak{G}_r(D)$ *is nonempty if and only if the inequality*

$$\sum_{k \leqslant j} d_k \leqslant rj(j-1) + \sum_{k > j} \min\{rj, d_k\} \tag{6.1}$$

holds for $j = 1, 2, \dots, n$. These and many other conditions for the nonemptiness of $\mathfrak{G}_r(D)$ are discussed in [32] in the special case $r = 1$.

We now associate with r and D a matrix F and a sequence D^* that are well known to be closely related to the inequalities in (6.1). The *Ferrers matrix* $F = F_r(D)$ of order n is defined as follows. Write $d_i = q_i r + s_i$ $(i = 1, 2, \dots, n)$ where $0 \leqslant s_i < r$. In row i of F the integer r occurs in the q_i leftmost nondiagonal positions, and the integer s_i occurs in the next nondiagonal position $(i = 1, 2, \dots, n)$. The remaining entries of F are 0's. The sequence of column sums of F is denoted by

$$D^* = (d_1^*, d_2^*, \dots, d_n^*).$$

For $j = 1, \dots, n$ let $\omega(j)$ denote the largest nonnegative integer such that the integer r appears in each nondiagonal position of the leading submatrix of size $\omega(j)$ by j of the Ferrers matrix F. Also define $\omega(0) = n$. Thus

$$\sum_{k \leqslant j} d_k^* = r\omega(j)j - r \min\{\omega(j), j\} + \sum_{k > \omega(j)} d_k. \tag{6.2}$$

We immediately note the similarity between the right side of this identity and the definition (2.1) of a structure constant.

EXAMPLE 6.1. If $r = 2$ and $D = (4, 4, 3, 3, 2)$, then

$$F_r(D) = \begin{bmatrix} 0 & 2 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$S_r(D) = \begin{bmatrix} 16 & 12 & 8 & 5 & 2 & 0 \\ 12 & 8 & 6 & 5 & 4 & 4 \\ 8 & 6 & 4 & 5 & 6 & 8 \\ 5 & 5 & 5 & 6 & 9 & 13 \\ 2 & 4 & 6 & 9 & 12 & 18 \\ 0 & 4 & 8 & 13 & 18 & 24 \end{bmatrix},$$

$D^* = (8, 4, 4, 0, 0)$ and $(\omega(0), \dots, \omega(5)) = (5, 5, 2, 2, 0, 0)$. The contour in $F_r(D)$ defined by the positions $(j, \omega(j))$ is shown. We have also superimposed this contour on the structure matrix $S_r(D)$. We observe that the contour in $S_r(D)$ indicates the last occurrence of the smallest entry in each column.

In general one may use the convexity of and recurrence relations for the columns of $S_r(D)$ (Lemma 2.1) to prove that for $j = 0, 1, \dots, n$ the element $s_{\omega(j), j}$ is the last occurrence of the smallest entry in column j of $S_r(D)$. (The similar observation by Anstee [1, p. 105; 4, p. 180] for the class $\mathfrak{A}(R, S)$ was the inspiration for this paper.) Moreover, by (6.2) and the definition of $\omega(j)$ we have

$$s_{\omega(j), j} = r\omega(j)j - r \cdot \min\{\omega(j), j\} + \sum_{k > \omega(j)} d_k - \sum_{k \leqslant j} d_k = \sum_{k \leqslant j} d_k^* - \sum_{k \leqslant j} d_k.$$

Thus we see that the structure matrix S is nonnegative if and only if the smallest entry in each column is nonnegative, and this in turn happens if and only if

$$\sum_{k \leqslant j} d_k \leqslant \sum_{k \leqslant j} d_k^* \qquad (j = 1, 2, \dots, n).$$

However, these inequalities are readily shown to be equivalent to the inequalities in (6.1). (See, e.g., [3, p. 113; 32] for the special case $r = 1$.)

7. SOME RESEARCH PROBLEMS

We conclude with some research problems on structure matrices.

PROBLEM 7.1. Find a formula involving the elements of the structure matrix $S_1(D)$ for the *minimum* matching number $\tilde{\mu}(D)$ among all graphs with degree sequence D.

Haber [16, 17] solved the corresponding problem for the minimum term rank $\tilde{\rho} = \tilde{\rho}(R, S)$ among matrices in $\mathfrak{A}(R, S)$. Thus Haber's formula gives the minimum matching number among all bipartite graphs with degree sequences R and S. Haber's formula and proof were simplified by Brualdi [4, 5]. We expect that the answer to Problem 7.1 will resemble the Haber-Brualdi formula.

PROBLEM 7.2. Determine necessary and sufficient conditions on D for the equality $\tilde{\mu} = \bar{\mu}$ to hold. In other words, characterize the degree sequences for which every graph has the same matching number.

Brualdi [4] has considered the analogous problem for bipartite graphs.

Let the *path number* $\lambda(G)$ [*cycle number* $\gamma(G)$] of the graph G be the maximum number of edges among all paths [cycles] in G. Suppose that the class $\mathfrak{G}_1(D)$ is nonempty. Let

$$\bar{\lambda} = \bar{\lambda}(D) = \max\{\lambda(G) : G \in \mathfrak{G}_1(D)\},$$

$$\bar{\gamma} = \bar{\gamma}(D) = \max\{\gamma(G) : G \in \mathfrak{G}_1(D)\}.$$

PROBLEM 7.3. Find formulas involving the elements of the structure matrix $S_1(D)$ for $\bar{\lambda}$ and $\bar{\gamma}$.

Formulas for the minimal values $\tilde{\lambda}$ and $\tilde{\gamma}$ will presumably be more elusive. We remark that a theorem of Kundu [21] on the existence of Hamiltonian graphs has the following structure matrix reformulation. (Also see [25].)

THEOREM 7.4. *Suppose that the monotone classes $\mathfrak{G}_1(D)$ and $\mathfrak{G}_1(D')$ are nonempty, where $D' = D - (2, 2, \dots, 2)$. Let $S_1(D) = [s_{ij}]$ denote the structure matrix of D. Then*

$$\bar{\gamma} = n \quad \text{if and only if} \quad s_{i, n-i} \geqslant 1 \quad \textit{for } 1 \leqslant i < n/2.$$

Likewise, a structure matrix characterization of the extremal case $\bar{\lambda} = n - 1$ can be extracted from the work of Hakimi and Schmeichel [18, 31].

PROBLEM 7.5. Find relationships between the algebraic properties and parameters of the structure matrix $S_r(D)$ and the combinatorial properties of the multigraphs in the nonempty class $\mathfrak{G}_r(D)$. For instance, what relationships hold between the eigenvalues of $S_r(D)$ and the multigraphs in the class $\mathfrak{G}_r(D)$?

Because $S_r(D)$ is a real, symmetric matrix, its $n + 1$ eigenvalues are real numbers. The remark after Theorem 2.3 implies that 0 is an eigenvalue of $S_r(D)$ if and only if the nonempty class $\mathfrak{G}_r(D)$ consists of the complete r-multigraph on n vertices or the complement of this multigraph. Of course, with structure matrices we are unable to distinguish between multigraphs with the same degree sequence. The permutation matrix P in Theorem 2.3 is equal to its own inverse. Hence by (2.12) the structure matrices S and $\bar{S}$ corresponding to complementary degree sequences are similar. Thus, in Problem 7.4 it may also be difficult to distinguish between two multigraphs with complementary degree sequences by applying algebraic methods in the study of structure matrices.

The author is grateful to Richard Brualdi for providing extensive mathematical and financial assistance during the early stages of the research that led to this paper.

REFERENCES

1 R. P. Anstee, Ph.D. dissertation, California Inst. of Technology, 1980.
2 R. P. Anstee, Properties of a class of (0, 1)-matrices covering a given matrix, *Canad. J. Math.* 34:438–453 (1982).

3 C. Berge, *Graphs*, Elsevier Science, Amsterdam, 1985.

4 R. A. Brualdi, Matrices of zeros and ones with fixed row and column sum vectors, *Linear Algebra Appl.* 33:159–231 (1980).

5 R. A. Brualdi, On Haber's minimum term rank formula, *European J. Combin.* 2:17–20 (1981).

6 R. A. Brualdi and J. A. Ross, On Ryser's maximum term rank formula, *Linear Algebra Appl.* 29:33–38 (1980).

7 Y.-C. Chen, A short proof of Kundu's k-factor theorem, *Discrete Math.* 71:177–179 (1988).

8 V. Chungphaisan, Conditions for sequences to be r-graphic, *Discrete Math.* 7:31–39 (1974).

9 P. Erdős and T. Gallai, Graphs with prescribed valencies (in Hungarian), *Mat. Lapok*, 11:264–274 (1960).

10 L. R. Ford and D. R. Fulkerson, *Flows in Networks.*, Princeton U.P., Princeton, N.J., 1962.

11 D. R. Fulkerson, Zero-one matrices with zero trace, *Pacific J. Math.* 10:831–836 (1960).

12 D. R. Fulkerson, A. J. Hoffman, and M. H. McAndrew, Some properties of graphs with multiple edges, *Canad. J. Math.* 17:166–177 (1965).

13 D. R. Fulkerson and H. J. Ryser, Widths and heights of (0, 1)-matrices, *Canad. J. Math.* 13:239–255 (1961).

14 D. R. Fulkerson and H. J. Ryser, Multiplicities and minimal widths for (0, 1)-matrices, *Canad. J. Math.* 13:498–508 (1961).

15 D. R. Fulkerson and H. J. Ryser, Width sequences for special classes of (0, 1)-matrices, *Canad. J. Math.* 15:371–396 (1963).

16 R. M. Haber, Term rank of (0, 1)-matrices, *Rend. Sem. Math. Padova* 30:24–51 (1960).

17 R. M. Haber, Minimal term rank of a class of (0, 1)-matrices, *Canad. J. Math.* 15:188–192 (1963).

18 S. L. Hakimi and E. F. Schmeichel, Graphs and their degree sequences: A survey, in *Theory and Applications of Graphs (1976)*, Lecture Notes in Math. 642, Springer-Verlag, Berlin, 1978, pp. 225–235.

19 D. J. Kleitman and D. L. Wang, Algorithms for constructing graphs and digraphs with given valences and factors, *Discrete Math.* 6:78–88 (1973).

20 M. Koren, Realization of a sum of sequences by a sum graph, *Israel J. Math.* 15:396–403 (1973).

21 S. Kundu, The k-factor conjecture is true, *Discrete Math.* 6:367–376 (1973).

22 L. Lovász, Valencies of graphs with 1-factors, *Period. Math. Hungar.* 5(2):149–151 (1974).

23 T. S. Michael, The structure matrix and a generalization of Ryser's maximum term rank formula, *Linear Algebra Appl.* 145:21–31 (1991).

24 L. Mirsky, *Transversal Theory*, Academic, New York, 1971.

25 A. R. Rao and S. B. Rao, On factorable degree sequences, *J. Combin. Theory Ser. B* 13:185–191 (1972).

26 H. J. Ryser, Combinatorial properties of matrices of zeros and ones, *Canad. J. Math.* 9:371–377 (1957).
27 H. J. Ryser, The term rank of a matrix, *Canad. J. Math.* 10:57–65 (1958).
28 H. J. Ryser, Traces of matrices of zeros and ones, *Canad. J. Math.* 12:463–476 (1960).
29 H. J. Ryser, Matrices of zeros and ones, *Bull. Amer. Math. Soc.* 66:442–464 (1960).
30 H. J. Ryser, *Combinatorial Mathematics*, Carus Math. Monographs 14, Math. Assoc. Amer., Washington, 1963.
31 E. F. Schmeichel and S. L. Hakimi, On the existence of a traceable graph with prescribed vertex degrees, *Ars Combin.* 4:69–80 (1977).
32 G. Sierksma and H. Hoogeveen, Seven criteria for integer sequences being graphic, *J. Graph Theory* 15:223–231 (1991).

Received 16 August 1991; final manuscript accepted 16 March 1992