

Isolating the real roots of the piecewise algebraic variety[☆]

Xiaolei Zhang^{a,*}, Renhong Wang^b

^a College of Statistics and Mathematics, Zhejiang Gongshang University, Hangzhou, 310018, China

^b Institute of Mathematical Sciences, Dalian University of Technology, Dalian 116024, China

ARTICLE INFO

Article history:

Received 30 October 2007

Received in revised form 10 July 2008

Accepted 10 September 2008

Keywords:

Real root isolation

Piecewise algebraic variety

Semi-algebraic set

Univariate interval polynomial

Interval zero

ABSTRACT

The piecewise algebraic variety, as a set of the common zeros of multivariate splines, is a kind of generalization of the classical algebraic variety. In this paper, we present an algorithm for isolating the zeros of the zero-dimensional piecewise algebraic variety which is primarily based on the interval zeros of univariate interval polynomials. Numerical example illustrates that the proposed algorithm is flexible.

Crown Copyright © 2008 Published by Elsevier Ltd. All rights reserved.

1. Introduction

Isolating real solutions for an algebraic variety or semi-algebraic set is an important problem in both practice and theory. There are several classical algorithms, such as Uspensky algorithm based on Descartes' rule of signs, Tarski's method, and the cylindric algebraic decomposition method (cf. [1–3] and the references therein). Recently, Xia et al. proposed an algorithm based on Wu's method for isolating the real roots of semi-algebraic system with integer coefficients. The algorithm was made more available in their later work [4,5], where they gave a complete algorithm by using interval arithmetic.

Some fundamental definitions and properties of the piecewise algebraic variety and the real zeros of the zero-dimensional piecewise algebraic variety are studied. For details, the readers may refer to [6–10].

Let $\Omega \subset \mathbb{R}^n$ be a simply connected region. Using finite number of hyperplanes in \mathbb{R}^n , we divide Ω into a finite number of simply connected regions. Denote by $\Delta = \{\delta_1, \delta_2, \dots, \delta_T\}$ the partition of Ω , where $\delta_i, i = 1, 2, \dots, T$, are called the cells. By s we denote a multivariate spline defined in Ω and by $s|_{\delta_i}$ a polynomial representing s in the cell $\delta_i (i = 1, 2, \dots, T) : s|_{\delta_i} \in \mathbb{R}[x_1, \dots, x_n]$. Put $P(\Delta) = \{s|_{\delta_i} \mid i = 1, 2, \dots, T\}$. Thus, $S^\mu(\Delta) = \{s \mid s \in C^\mu(\Delta) \cap P(\Delta)\}$ is called a C^μ piecewise polynomial ring, where $s \in C^\mu(\Delta)$ means that s possesses μ order continuous partial derivatives over Δ .

For $F \subseteq S^\mu(\Delta)$, the zero set of F is defined to be $z(F) = \{x \in \Omega \mid s(x) = 0, \forall s \in F\}$. Obviously, if an ideal $I \in S^\mu(\Delta)$ is generated by F , then $z(F) = z(I)$. Since $S^\mu(\Delta)$ is a Noether ring, every ideal I has a finite number of generators s_1, s_2, \dots, s_l , then $z(F)$ can be expressed as the common zeros of the splines s_1, s_2, \dots, s_l .

Let Δ be a partition of the region Ω . If there exist $s_1, s_2, \dots, s_m \in S^\mu(\Delta)$ such that $X = z(s_1, s_2, \dots, s_m) = \bigcap_{i=1}^m z(s_i)$, then X is called a C^μ piecewise algebraic variety (PAV for short) with respect to Δ . If $s \in S^\mu(\Delta)$ and $X = z(s)$, then X is called a C^μ piecewise hypersurface.

[☆] Project supported by the National Natural Science Foundation of China (Project Nos.60373093, 60533060) and the Natural Science Foundation of Zhejiang Province (No. Y7080068).

* Corresponding author.

E-mail address: zhangxl0411@yahoo.com.cn (X. Zhang).

In this paper, we present an algorithm for isolating the zeros of a given zero-dimensional piecewise algebraic variety, i.e., determining a sequence of disjoint hyperrectangles such that each of them contains exactly one real root of the piecewise algebraic variety. Our proposed algorithm which is primarily based on the interval zeros of univariate interval polynomial has the advantage of handling polynomials with real coefficients.

2. Preliminaries

In this section, some results on the computation of an algebraic variety on a convex polyhedron are presented. For more details, the readers may refer to [11].

Let $\delta \subset \mathbb{R}^n$ be a convex polyhedron. By $H_i(x) = 0$ ($i = 1, 2, \dots, m$) we denote hyperplanes which divide Ω , where m is the number of facets of δ . Let u_i be the inward pointing normal of $H_i(x) = 0$, then we can express $H_i(x)$ as

$$H_i(x) = u_i \cdot x + a_i, \quad u_i = (u_{i1}, u_{i2}, \dots, u_{in}), \quad x = (x_1, x_2, \dots, x_n), \quad a_i \in \mathbb{R}.$$

Obviously, every point in the interior of δ can be regarded as an intersection point of the m hyperplanes $H_i(x) - y_i = 0, y_i > 0, i = 1, 2, \dots, m$.

We consider the following semi-algebraic set (SAS for short) with real coefficients, i.e., the algebraic variety in the interior of the convex polyhedron δ

$$\text{SAS} : \begin{cases} f_1(x_1, x_2, \dots, x_n) = 0, \\ f_2(x_1, x_2, \dots, x_n) = 0, \\ \dots \\ f_n(x_1, x_2, \dots, x_n) = 0, \\ H_1(x_1, x_2, \dots, x_n) > 0, \\ \dots \\ H_m(x_1, x_2, \dots, x_n) > 0, \end{cases}$$

where $\{f_1, f_2, \dots, f_n\}$ has only a finite number of common zeros.

Theorem 2.1 ([11]). If $\hat{I} = \langle H_1(x) - y_1, H_2(x) - y_2, \dots, H_m(x) - y_m \rangle$, then the reduced Gröbner bases of \hat{I} with respect to lex order with $x \succ y$ is

$$\{x_1 - p_1(y), x_2 - p_2(y), \dots, x_n - p_n(y), g_{n+1}(y), \dots, g_m(y)\}, \quad y = (y_1, y_2, \dots, y_m),$$

where $p_1(y), p_2(y), \dots, p_n(y)$ are polynomials of degree 1 in the variables y_1, y_2, \dots, y_m .

Put $F = \{x_1 - p_1(y), x_2 - p_2(y), \dots, x_n - p_n(y)\}$, and compute $g_i(y) = \overline{f_i(x)}^F, i = 1, 2, \dots, n$ with respect to lex order with $x \succ y$, where $\overline{f_i}^F$ denotes the remainder on division of f_i by the ordered n -tuple F .

Lemma 2.1 ([11]). Let $I = \langle f_1, f_2, \dots, f_n \rangle$ be an ideal in $\mathbb{R}[x]$, and $\tilde{I} = I + \hat{I}$, then

$$\tilde{I} = \langle g_1(y), g_2(y), \dots, g_m(y), x_1 - p_1(y), x_2 - p_2(y), \dots, x_n - p_n(y) \rangle.$$

Theorem 2.2 ([11]). If $M = \langle g_1(y), g_2(y), \dots, g_n(y), g_{n+1}(y), \dots, g_m(y) \rangle$, then the common zeros of the SAS are given by

$$\{x \mid (x_1, \dots, x_n) = (p_1(y), p_2(y), \dots, p_n(y)), y \in z(M), \forall y_i > 0, i = 1, 2, \dots, m\}.$$

Since the ideal generated by $\{f_1, f_2, \dots, f_n\}$ is zero-dimensional, we can transform the system SAS into a system in triangular form by Wu's method, Gröbner basis method or subresultant method [4,12]. Thus, by Theorem 2.2, the system SAS can be reduced to the following simplified triangular semi-algebraic system (STSAS in short)

$$\text{STSAS} : \begin{cases} h_1(y_1) = 0, \\ h_2(y_1, y_2) = 0, \\ \dots \\ h_m(y_1, y_2, \dots, y_m) = 0, \\ y_1 > 0, y_2 > 0, \dots, y_m > 0. \end{cases}$$

Remark 2.1. The computation often becomes unstable if we use floating-point numbers [13,14]. In this paper, we directly adopt the suggested algorithms by Traverso and Zanoni to deal with unstable systems to compute Gröbner bases [14].

3. Interval polynomial

Interval operations have been first introduced by Moore [15]. It is used to tackle the instability and error analysis of numerical computation. In this section, some related concepts and results of interval polynomial are presented. For details, the readers may refer to [16,17].

An interval is a set of real numbers defined by $[a, b] = \{x \mid x \in [a, b]\}$. For an interval $[a, b]$, its width is defined by $w[a, b] = b - a$.

An interval polynomial of degree n is a polynomial whose coefficients are intervals

$$[f](x) = \sum_{i=0}^n [a_i, b_i]x^i = \left\{ \sum_{i=0}^n f_i x^i : f_i \in [a_i, b_i], i = 0, 1, \dots, n \right\}.$$

The upper bound function and lower bound function of $[f](x)$ are defined by:

$$U[f](x) = \begin{cases} \sum_{i=0}^n b_i x^i, & x \geq 0; \\ \sum_{0 \leq 2i \leq n} b_{2i} x^{2i} + \sum_{0 \leq 2i+1 \leq n} a_{2i+1} x^{2i+1}, & x < 0. \end{cases}$$

$$L[f](x) = \begin{cases} \sum_{i=0}^n a_i x^i, & x \geq 0; \\ \sum_{0 \leq 2i \leq n} a_{2i} x^{2i} + \sum_{0 \leq 2i+1 \leq n} b_{2i+1} x^{2i+1}, & x < 0. \end{cases}$$

The set of real zeros of the interval polynomial $[f](x)$ is defined as

$$R(f) = \{x_0 \in \mathbb{R} \mid \exists f(x) \in [f](x), \text{ s.t. } f(x_0) = 0\}.$$

Obviously, we have

$$R(f) = \{x_0 \in \mathbb{R} : L[f](x_0) \leq 0 \leq U[f](x_0)\}.$$

In this case, the zeros set of $[f](x)$ is actually composed of several closed intervals. We call each of these intervals an interval zero of the univariate polynomial $[f](x)$.

Proposition 3.1 ([18]). *If $[a, b]$ is an interval zero of $[f](x)$, then the endpoints b and a are the zeros of the upper bound function $U[f](x)$ and lower bound function $L[f](x)$, respectively.*

Theorem 3.1 ([16]). *An interval polynomial $[f](x)$ of degree n has at most n interval zeros.*

We directly adopt this numerical algorithm to find a set of intervals which bound the interval zeros of a given interval polynomial $[f](x)$. Furthermore, the interval zeros converge to the exact zeros when computing accuracy tends to infinity [16].

Algorithm 3.1 ([16]). Algorithm for computing the zeros of interval polynomial

Input An interval polynomial $[f](x)$, and a small tolerance ε ($0 < \varepsilon \ll 1$).
Output A set S containing all the interval zeros of $[f](x)$.
Step 1 Set the initial interval $I = [-r_0, r_0]$. Here, $r_0 = 1 + \max\{|a_0|, |b_0|, \dots, |a_n|, |b_n|\}$. Let S be an empty set.
Step 2 For the given interval, compute $[f](I)$. If $0 \notin [f](I)$, discard this interval and process the next interval. Otherwise go to Step 3.
Step 3 If $0 \in [f]((a+b)/2)$, $0 \notin U[f](I)$ and $0 \notin L[f](I)$, or the width of I is less than ε , append I to the set S . Otherwise bisection I into two intervals at midpoints and for each subinterval, go to Step 2.
Step 4 Union all the neighboring intervals in S .

Therefore, for any polynomial $f(x) = \sum_{i=0}^n f_i x^i$, it can be written in the form of an interval polynomial $[f](x) = \sum_{i=0}^n [f_i] x^i$. Here, each $[f_i]$, whose width is less than a given tolerance ε , is an interval containing f_i . Obviously, the interval zeros of the interval polynomial $[f](x)$ converge to the exact zeros of the original polynomial $f(x)$ when computing accuracy tends to infinity.

In Section 4, we present an algorithm for isolating the interval zeros of SAS. In Section 5, an algorithm is presented to isolate the real zeros of a given piecewise algebraic variety on a convex polyhedron partition, which is primarily based on the computation of interval zeros of univariate interval polynomial.

4. Algorithm for isolating the zeros of SAS

We assume, if not specified, that all the algebraic varieties in this paper are zero-dimensional. The zero-dimensional algebraic variety defined on a convex polyhedron can be viewed as a special and simple semi-algebraic set. The algorithm for isolating the interval zeros of SAS is outlined below.

Algorithm 4.1. Algorithm for isolating the zeros of SAS

Input SAS, and a small tolerance ε ($0 < \varepsilon \ll 1$).
Output All the isolating intervals I of SAS.
Step 1 Put $\hat{I} = \langle H_1(x) - y_1, H_2(x) - y_2, \dots, H_m(x) - y_m \rangle$, compute the Gröbner basis of \hat{I} with respect to lex order with $x \succ y$ and let the obtained basis be $\{x_1 - p_1(y), x_2 - p_2(y), \dots, x_n - p_n(y), g_{n+1}(y), \dots, g_m(y)\}$.

Step 2 Put $F = \{x_1 - p_1(y), x_2 - p_2(y), \dots, x_n - p_n(y)\}$, compute $\overline{f_i(x)}^F = g_i(y)$, $i = 1, 2, \dots, n$ and set $M = \langle g_1(y), g_2(y), \dots, g_n(y), g_{n+1}(y), \dots, g_m(y) \rangle$.

Step 3 Compute the Gröbner basis of M with respect to lex order with $y_1 > y_2 > \dots > y_m$ and let the obtained basis be $\langle h_1(y_1), h_2(y_1, y_2), \dots, h_m(y_1, y_2, \dots, y_m) \rangle$.

Step 4 Compute the interval zeros of univariate interval polynomial $[h_1](y_1)$, and let the result be

$$I^{(1)} = \{[a_i^{(1)}, b_i^{(1)}] \mid w[a_i^{(1)}, b_i^{(1)}] < \varepsilon, i = 1, 2, \dots, n_1\}.$$

Here, if $b_i^{(1)} < 0$, then we discard it from $I^{(1)}$.

Step 5 If $I^{(1)} = \emptyset$, then stop and SAS has no common solutions. Otherwise, substituting $y_1^{(i)} = [a_i^{(1)}, b_i^{(1)}]$ into $h_2(y_1, y_2)$, we obtain an interval polynomial $[h_2]^{(i)}(y_2)$. Compute the interval zeros of univariate interval polynomial $[h_2]^{(i)}(y_2)$, and let the result be

$$I_i^{(2)} = \{[a_{ij}^{(2)}, b_{ij}^{(2)}] \mid a_{ij}^{(2)} > 0, j = 1, 2, \dots, j_i\}.$$

Hence, $I^{(2)}$ can be expressed by

$$I^{(2)} = \bigcup_{i=1}^{n_1} I_i^{(2)} = \{[a_i^{(2)}, b_i^{(2)}] \mid w[a_i^{(2)}, b_i^{(2)}] < \varepsilon, i = 1, 2, \dots, n_2\}.$$

Step 6 Inductively, we continue the similar procedure as in Step 5. If there exists i such that $I^{(1)} \neq \emptyset, \dots, I^{(i-1)} \neq \emptyset, I^{(i)} = \emptyset, i = 2, 3, \dots, m$, then stop, and SAS has no common zeros. Otherwise, we obtain the sequence $\{I^{(1)}, I^{(2)}, \dots, I^{(m)}\}$.

Step 7 Therefore, the isolating intervals of SAS can be expressed as

$$I = \{p_1([y]) \times p_2([y]) \times \dots \times p_n([y]) \mid \forall [y] \in I^{(1)} \times \dots \times I^{(m)}\}.$$

5. Algorithm for isolating the zeros of the PAV

Let $\Delta = \{\delta_1, \delta_2, \dots, \delta_T\}$ be the convex polyhedron partition of the region $\Omega \subset \mathbb{R}^n$. Suppose $s_1(x), s_2(x), \dots, s_n(x) \in S^\mu(\Delta)$, and $z(s_1, s_2, \dots, s_n)$ is assumed to be zero-dimensional, i.e., it consists of only a finite number of points. Here, all the convex polyhedrons $\delta_i, i = 1, 2, \dots, T$ are assumed to be in “general position”, which means none of the zeros lie on their boundary.

Put $s_i^{(j)} = s_i(x)|_{\delta_j}, i = 1, 2, \dots, n, j = 1, 2, \dots, T$, then, for each $j \in \{1, 2, \dots, T\}$, $z(s_1^{(j)}, s_2^{(j)}, \dots, s_n^{(j)})$ has only a finite number of common zeros in the interior of the convex polyhedron δ_j . With the above preparations, we can easily present the following algorithm for isolating the real roots of a given piecewise algebraic variety on a convex polyhedron partition.

Algorithm 5.1. Algorithm for isolating the zeros of PAV

Input PAV (A piecewise algebraic variety on a convex polyhedron partition).

Output All the isolating intervals I of PAV.

Step 1 Set $j = 1$ and let I be an empty set.

Step 2 For $z(s_1^{(j)}, s_2^{(j)}, \dots, s_n^{(j)})$ on cell δ_j , perform Algorithm 4.1 to obtain the isolating intervals $I^{(j)}$ and set $I := I \cup I^{(j)}$.

Step 3 Set $j = j + 1$. If $j \leq T$ then go to step 2; Else, stop and output I .

6. Numerical example

In this section, an example is provided to illustrate the proposed algorithm for isolating the zeros of a given piecewise algebraic variety.

Example 6.1. Let $\Delta = \{\delta_1, \delta_2, \delta_3, \delta_4\}$ be a convex polyhedron partition of a pentagon $V_A V_B V_C V_D V_E$ in \mathbb{R}^2 , where $\delta_1 = [V_A V_B V_C V_O]$, $\delta_2 = [V_C V_D V_O]$, $\delta_3 = [V_D V_E V_O]$, $\delta_4 = [V_E V_A V_O]$, $V_A = (2, 0)$, $V_B = (\frac{3}{2}, \frac{3}{2})$, $V_C = (0, 1)$, $V_D = (-1, 0)$, $V_E = (1, -1)$ and $V_O = (\frac{1}{2}, 0)$ (see Fig. 1).

Let bivariate splines f and g in $S_3^1(\Delta)$ be defined as follows:

- on cell δ_1 :
$$\begin{cases} f_1(x_1, x_2) = f|_{\delta_1} = x_1^2 + \sqrt{2}x_2^2 - \sqrt{3} \\ g_1(x_1, x_2) = g|_{\delta_1} = x_2^3 - x_1 \end{cases}$$
- on cell δ_2 :
$$\begin{cases} f_2(x_1, x_2) = f|_{\delta_2} = f_1(x_1, x_2) + (2x_1 + x_2 - 1)^2(x_1 + x_2) \\ g_2(x_1, x_2) = g|_{\delta_2} = g_1(x_1, x_2) + (2x_1 + x_2 - 1)^2(2x_2) \end{cases}$$
- on cell δ_3 :
$$\begin{cases} f_3(x_1, x_2) = f|_{\delta_3} = f_2(x_1, x_2) + x_2^2(x_1 - x_2 + 2) \\ g_3(x_1, x_2) = g|_{\delta_3} = g_2(x_1, x_2) + x_2^2(x_2 - 3) \end{cases}$$
- on cell δ_4 :
$$\begin{cases} f_4(x_1, x_2) = f|_{\delta_4} = f_1(x_1, x_2) + x_2^2(x_1 - x_2 + 2) \\ g_4(x_1, x_2) = g|_{\delta_4} = g_1(x_1, x_2) + x_2^2(x_2 - 3). \end{cases}$$

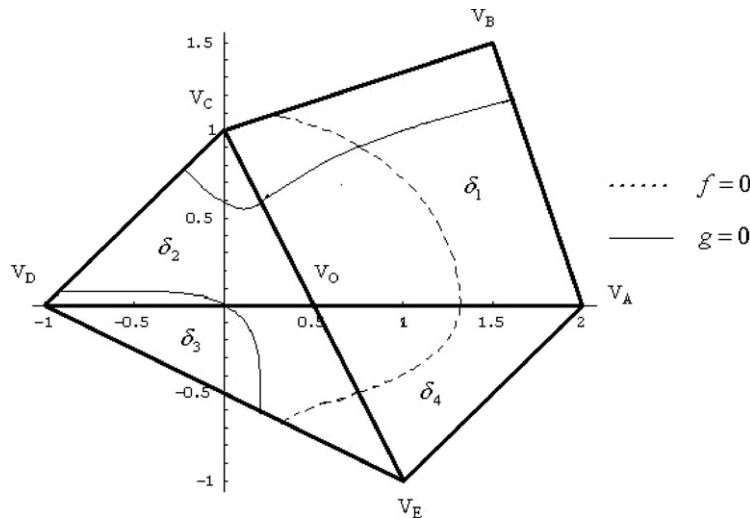


Fig. 1. Two piecewise algebraic curves $f = 0$ and $g = 0$.

In order to illustrate the proposed algorithm, we take the algebraic variety $z(f_1, g_1)$ in the interior of the quadrangle δ_1 for example.

- Step 1 The Gröbner bases of $\hat{I} = \langle x_2 - y_1, 2x_1 + x_2 - 1 - y_2, x_1 - 3x_2 + 3 - y_3, 3x_1 + x_2 - 6 - y_4 \rangle$ with respect to lex order with $x > y_1 > \dots > y_4$ is $\{x_2 - y_1, 1 - 2x_1 - y_1 + y_2, -7 + 7y_1 - y_2 + 6y_3, 9 + y_1 - 3y_2 - 2y_4\}$.
- Step 2 Put $F = \{x_1 - p_1(y_1, y_2), x_2 - p_2(y_1, y_2)\}$, where $p_1(y_1, y_2) = \frac{1}{2}(1 - y_1 + y_2)$, $p_2(y_1, y_2) = y_1$. Compute $\overline{f_1(x)}^F = -\sqrt{3} + \sqrt{2}y_1^2 + \frac{1}{4}(1 - y_1 + y_2)^2$ and $\overline{g_1(x)}^F = y_1^3 - \frac{1}{2}(1 - y_1 + y_2)$ and set $M = \{-\sqrt{3} + \sqrt{2}y_1^2 + \frac{1}{4}(1 - y_1 + y_2)^2, y_1^3 - \frac{1}{2}(1 - y_1 + y_2), -7 + 7y_1 - y_2 + 6y_3, 9 + y_1 - 3y_2 - 2y_4\}$.
- Step 3 The reduced Gröbner bases of M with respect to $y_4 > y_3 > y_2 > y_1$ is $\{h_1(y_1), h_2(y_1, y_2), h_3(y_1, y_2, y_3), h_4(y_1, y_2, y_3, y_4)\} = \{-\sqrt{3} + \sqrt{2}y_1^2 + y_1^6, -1 + y_1 + 2y_1^3 - y_2, -3 + 3y_1 - y_1^3 + 3y_3, -6 + y_1 + 3y_1^3 + y_4\}$.
- Step 4 Set $\varepsilon = 0.01$. Compute the interval zeros of the interval polynomial $[h_1](y_1)$ and the results are $[-0.90918, -0.908203]$ and $[0.908203, 0.90918]$. Obviously, the first interval should be discarded. Thus, $I^{(1)} = \{[0.908203, 0.90918]\}$.
- Step 5 Substituting $y_1 = [0.908203, 0.90918]$ into $h_2(y_1, y_2)$, we obtain an interval polynomial $[h_2](y_2) = [1.40643, 1.41225] - y_2$. Obviously, its interval zero of $[h_2](y_2)$ is $[1.40643, 1.41225]$. That is to say, $I^{(2)} = \{[1.40643, 1.41225]\}$. Inductively, we obtain $I^{(3)} = \{[0.340526, 0.342309]\}$ and $I^{(4)} = \{[2.83622, 2.84445]\}$, respectively.
- Step 6 Therefore, the isolating interval of $z(f_1, g_1)$ in the interior of the quadrangle δ_1 is
- $$p_1([0.908203, 0.90918], [1.40643, 1.41225]) \times p_2([0.908203, 0.90918], [1.40643, 1.41225]) \\ = [0.748628, 0.752023] \times [0.908203, 0.90918].$$

Similarly, we conclude that $z(f_2, g_2)$, $z(f_3, g_3)$ and $z(f_4, g_4)$ have no common zeros in the interior of cells δ_2 , δ_3 and δ_4 , respectively.

Hence, the isolating interval of $z(f, g)$ is $[0.748628, 0.752023] \times [0.908203, 0.90918]$.

7. Conclusion

From the numerical result, we can easily see that the proposed algorithm for isolating the zeros of a given zero-dimensional piecewise algebraic variety is flexible. It is primarily based on the computation of interval zeros of univariate interval polynomials. Our proposed algorithm dealing with polynomials with real coefficients is easy to understand and implement.

However, how to control the positive number ε under a given tolerance in Algorithm 4.1 and the efficiency of the proposed algorithm remain as our future work.

Acknowledgement

We are very grateful to the referees for their careful reading of the manuscript and many valuable suggestions, which greatly improved this paper.

References

- [1] G.E. Collins, Quantifier elimination for real closed fields by cylindric algebraic decomposition, in: LNCC, Vol. 33, Springer, 1975.
- [2] G.E. Collins, A.G. Akritas, Polynomial real root isolation using Descartes's rule of signs, in: SYMSAC, 1976, pp. 272–275.
- [3] F. Rouillier, P. Zimmermann, Efficient isolation of polynomial's real roots, *Journal of Computational and Applied Mathematics* 162 (2004) 33–50.
- [4] B.C. Xia, L. Yang, An algorithm for isolating the real solutions of semi-algebraic systems, *Journal of Symbolic Computation* 34 (2002) 461–477.
- [5] B.C. Xia, T. Zhang, Real solution isolation using interval arithmetic, *Computers and Mathematics with Applications* 52 (2006) 853–860.
- [6] R.H. Wang, *Multivariate Spline Functions and Their Applications*, Science Press/Kluwer Pub., Beijing, New York, 2001.
- [7] R.H. Wang, Y.S. Lai, Real piecewise algebraic variety, *Journal of Computational Mathematics* 21 (2003) 473–480.
- [8] R.H. Wang, C.G. Zhu, Piecewise algebraic varieties, *Progress in Natural Science* 14 (2004) 568–572.
- [9] Y.S. Lai, R.H. Wang, J.M. Wu, Real zeros of the zero-dimensional parametric piecewise algebraic variety, *Science in China Series A* 52 (2009) (in press).
- [10] Y.S. Lai, Counting positive solutions for polynomial systems with real coefficients, *Computers and Mathematics with Applications* 56 (6) (2008) 1587–1596.
- [11] R.H. Wang, J.M. Wu, Computation of an algebraic variety on a convex polyhedron, *Journal of Information and Computational Science* 3 (2006) 903–911.
- [12] D.A. Cox, J. Little, D. O'shea, *Ideals, Varieties, and Algorithms*, Springer, Berlin, 1992.
- [13] K. Shirayanagi, M. Sweedler, Remarks on automatic algorithm stabilization, *J. Symb. Comput.* 26 (1998) 761–765.
- [14] C. Traverso, A. Zanon, Numerical stability and stabilization of Gröbner basis computation, in: *Proc. ISSAC*, 2002, pp. 262–269.
- [15] R.E. Moore, *Interval Analysis*, Prentice-Hall, Englewood Cliffs, 1966.
- [16] X.C. Fan, J.S. Deng, F.L. Chen, Zeros of univariate interval polynomials, *Journal of Computational and Applied Mathematics* 216 (2008) 563–573.
- [17] F.L. Chen, W. Yang, Applications of interval algorithm in solving algebraic equations with Wu's method, *Science in China. Series A* 35 (2005) 910–921.
- [18] A.C. Bartlett, C.V. Hollot, Huang Lin, Root location of an entire polytope of polynomials: It suffices to check the edges, *Mathematics of Controls, Signals and Systems* 1 (1988) 61–71.