

Available online at www.sciencedirect.com

ScienceDirect

International Journal of Approximate Reasoning
48 (2008) 598–627INTERNATIONAL JOURNAL OF
APPROXIMATE
REASONINGwww.elsevier.com/locate/ijar

A general model of parameterized OWA aggregation with given orness level[☆]

Xinwang Liu

School of Economics and Management, Southeast University, Nanjing, 210096 Jiangsu, PR China

Received 14 May 2007; received in revised form 20 November 2007; accepted 20 November 2007

Available online 5 December 2007

Abstract

The paper proposes a general optimization model with separable strictly convex objective function to obtain the consistent OWA (ordered weighted averaging) operator family. The consistency means that the aggregation value of the operator monotonically changes with the given orness level. Some properties of the problem are discussed with its analytical solution. The model includes the two most commonly used maximum entropy OWA operator and minimum variance OWA operator determination methods as its special cases. The solution equivalence to the general minimax problem is proved. Then, with the conclusion that the RIM (regular increasing monotone quantifier) can be seen as the continuous case of OWA operator with infinite dimension, the paper further proposes a general RIM quantifier determination model, and analytically solves it with the optimal control technique. Some properties of the optimal solution and the solution equivalence to the minimax problem for RIM quantifier are also proved. Comparing with that of the OWA operator problem, the RIM quantifier solutions are usually more simple, intuitive, dimension free and can be connected to the linguistic terms in natural language. With the solutions of these general problems, we not only can use the OWA operator or RIM quantifier to obtain aggregation value that monotonically changes with the orness level for any aggregated set, but also can obtain the parameterized OWA or RIM quantifier families in some specific function forms, which can incorporate the background knowledge or the required characteristic of the aggregation problems.

© 2007 Elsevier Inc. All rights reserved.

Keywords: OWA operator; RIM quantifier; Maximum entropy; Minimum variance; Minimax problem

1. Introduction

The ordered weighted averaging (OWA) operator, which was introduced by Yager [45], has attracted much interest among researchers. It provides a general class of parameterized aggregation operators that include the *min*, *max*, *average*. Many applications in the areas of decision making, expert systems, data mining, approximate reasoning, fuzzy system and control have been proposed [20,21,29,37,53,57,60].

[☆] The work is supported by the National Natural Science Foundation of China (NSFC) under project 70301010 and 70771025, and Program for New Century Excellent Talents in University of China NCET-06-0467.

E-mail address: xwliu@seu.edu.cn

One of the appealing points of OWA operators is the concept of orness [45]. The *orness* measure reflects the *andlike* or *orlike* aggregation result of an OWA operator, which is very important both in theory and applications [13,15,50–52]. The orness of OWA operator, also called “attitudinal-character”, can be used to represent the preference information in aggregation problems [53,54]. It is clear that the actual type of aggregation performed by an OWA operator depends upon the form of the weight vector [8,12–15,49–52]. The weight vector determination is usually a prerequisite step in many OWA related applications, and it has become an active topic in recent years [1,26,31,39,42]. A number of approaches were suggested for obtaining the required OWA operator, i.e., quantifier guided aggregation [45,47], exponential smoothing [13], sample learning [37,56], the weights function method [1], argument dependent methods [41,43] and the preference relation method [2]. The most commonly used method is to obtain the desired OWA operator under a given orness level [12–15,31,35,55], which is usually formulated as a constrained optimization problem. The objective to be optimized can be the (Shannon) entropy [12,14,31,35], the variance [15,26], the maximum dispersion [4,39], the (generalized) Rényi entropy [33] or even the preemptive goal programming [3,40]. O’Hagan [35] suggested the problem of constraint nonlinear programming with a maximum entropy procedure, the solution is called a MEOWA (Maximum Entropy OWA) operator. Filev and Yager [12] further proposed a method to generate MEOWA weight vector by an immediate parameter. Fullér and Majlender [14] transformed the maximum entropy model into a polynomial equation, which can be solved analytically. Liu and Chen [31] proposed general forms of the MEOWA operator with a parametric geometric approach, and discussed its aggregation properties. Apart from maximum entropy OWA operator, Fullér and Majlender [15] suggested the minimal variability OWA operator problem in quadratic programming, and proposed an analytical method for solving it. Liu [26] gave this OWA operator generating method with the equidifferent OWA operator, which seems being a reformulation of [15], but actually is an extension with a more simple and intuitive process [28,34]. A closely related work is that of Wang and Parkan [39]. They proposed a linear programming model with minimax disparity approach to get the OWA operator under the desired orness level. The solution equivalence of the minimum variance problem and the minimax disparity problem was proved by Liu recently [30]. Majlender [33] proposed a maximum Rényi entropy OWA operator problem with exponential objective function, which can include the maximum entropy and minimum variance problem as special cases, and an analytical solution was proposed.

Another important closely related topic is OWA aggregation with Regular Increasing Monotone (RIM) quantifier, which was also proposed by Yager [48]. The linguistic quantifiers were proposed by Zadeh [59], who also classified them with absolute quantifiers, such as “much more than 10”, and relative quantifiers, such as “a half”. Flexibility can be obtained by introducing fuzzy quantifiers which permit a closer representation in the language of daily life. Yager [46,48] further distinguished the relative quantifiers into three classes. They are called Regular Increasing Monotone (RIM) quantifier, Regular Decreasing Monotone (RDM) quantifier and Regular UniModal (RUM) quantifier, where the RIM quantifier is the basis of all kinds of relative quantifiers [46,48]. Some RIM quantifiers in natural language are *most*, *many*, *at least half*, *some* [6,7,11,16,21,19,38]. This RIM quantifier guided aggregation method with OWA operator in natural language [48] has been applied in many areas such as decision analysis, database querying, and computing with words theory [5,6,9,17,18,20,21,44]. Based on this method, Liu [24,29] further analyzed the relationship between the OWA operator and the RIM quantifier with the generating function technique. With the generating function in RIM quantifier playing the role of weight vector in OWA operator, the RIM quantifier can be seen as a dimension free continuous OWA aggregation. The maximum entropy RIM quantifier and minimum variance RIM quantifier were proposed, and some properties of them were discussed [24,27]. A summarization of the OWA operator and the corresponding RIM quantifier determination methods was given in [32].

In the present paper, a general optimization model with strictly convex objective function to obtain the OWA operator under given orness level is proposed. This approach includes the maximum entropy and the minimum variance problems as special cases. The problem is also more general than the Rényi entropy objective function case. The solution methods and the properties of maximum entropy and minimum variance problems were studied separately, but they can be included into this general model now. The consistent property that the aggregation value for any aggregated set monotonically increases with the given orness value is still kept, which gives more alternatives to represent the preference information in the aggregation of decision making. Furthermore, the equivalence to the minimax problem is proved, which is the generalization of the

equivalence of the minimum variance problem and the minimax disparity problem [30], but the proof is simplified. With the generating function in the RIM quantifier playing the role of the weight vector in the OWA operator, a general model that can include the maximum entropy and minimum variance RIM problems is proposed. Some properties are discussed and the solution equivalence to the minimax problem for RIM quantifier is proved. The RIM quantifier has the advantages of being dimension free, having a simple solution, and having the ability to be connected with natural language terms. When we face the problem that the number of arguments changes in different cases, the RIM quantifier based aggregation method can provide a uniform formula with its membership function. With the analytical solution of these general models, we can make the OWA operator become the interpolation series of a given monotonic function or make the RIM quantifier function obey some specific function shapes, which gives more possible alternatives for the OWA operator and RIM quantifier determination. We can also incorporate some prerequisite information such as the background or the characteristic requirements of the aggregation problem into the aggregation process.

The remainder of this paper is organized as follows. Section 2 gives some preliminaries of OWA operators, the RIM quantifier guided OWA aggregation method, and the generating function representation method of RIM quantifier. Section 3 proposes a general model to obtain OWA operator under given orness level. Some properties of the optimal solution are discussed. The solution equivalence of the general model and the corresponding minimax problem is proved. Section 4 can be seen as the continuous extension of Section 3 with RIM quantifier. As both OWA operators and RIM quantifiers have some common characteristics in both the solution process and in their applications, the conclusions are organized in parallel for easy comparison. This similarity proposes a general model to obtain the RIM quantifier under given orness level. Some properties of the optimal solution are discussed and the solution equivalence to the corresponding minimax problem is proved. As the general models of Sections 3 and 4 are improvements and extensions of the minimum variance problems and the minimax disparity problems for OWA operators and RIM quantifiers, respectively, Section 5 summarizes the solutions and properties of these two kinds of problems in this general framework, so that the similarity between these two kinds of problems can be connected and some existing results are extended. Section 6 considers the problems' solutions from another viewpoint, which can make the OWA operator or the RIM quantifier generating function have a specific function shape. Some special function forms for the OWA operator and RIM quantifier solutions are provided, which gives more alternatives for their determination. Section 7 summarizes the main results and draws conclusions.

2. Preliminaries

An OWA operator of dimension n is a mapping $F: \mathbb{R}^n \rightarrow \mathbb{R}$ that has an associated weight vector $W = (w_1, w_2, \dots, w_n)$ having the properties

$$w_1 + w_2 + \dots + w_n = 1; \quad 0 \leq w_i \leq 1, \quad i = 1, 2, \dots, n$$

and such that

$$F_W(X) = F_W(x_1, x_2, \dots, x_n) = \sum_{i=1}^n w_i y_i \quad (1)$$

with y_i being the i th largest of the x_i .

The degree of “orness” associated with this operator is defined as

$$\text{orness}(W) = \sum_{i=1}^n \frac{n-i}{n-1} w_i \quad (2)$$

The *max*, *min* and *average* correspond to W^* , W_* and W_A , respectively, where $W^* = (1, 0, \dots, 0)$, $W_* = (0, 0, \dots, 1)$ and $W_A = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$, that is $F_{W^*}(X) = \min_{1 \leq i \leq n} \{x_i\}$, $F_{W_*}(X) = \max_{1 \leq i \leq n} \{x_i\}$ and $F_{W_A}(X) = \frac{1}{n} \sum_{i=1}^n x_i$. Obviously, $\text{orness}(W^*) = 1$, $\text{orness}(W_*) = 0$ and $\text{orness}(W_A) = \frac{1}{2}$.

In [48], Yager proposed a method for obtaining the OWA weight vectors via fuzzy linguistic quantifiers, especially the RIM quantifiers, which can provide information aggregation procedures guided by verbally expressed concepts and a dimension independent description of the desired aggregation.

Definition 1 [48]. A fuzzy subset Q of the real line is called a Regular Increasing Monotone (RIM) quantifier if $Q(0) = 0$, $Q(1) = 1$, and $Q(x) \geq Q(y)$ for $x > y$.

Examples of this kind of quantifier are *all*, *most*, *many*, *there exists* [48].

The quantifier *all* and *there exists* is represented by Q_* and Q^* , respectively,

$$Q_*(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x \neq 1 \end{cases} \quad Q^*(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \neq 0 \end{cases}$$

With a RIM quantifier Q , the quantifier guided aggregation with OWA operator is [48]

$$F_Q(X) = F_W(X) = \sum_{i=1}^n \left(Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right) \right) y_i \tag{3}$$

where the OWA weight vector $W = (w_1, w_2, \dots, w_n)$ is

$$w_i = Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right) \tag{4}$$

Yager also extended the orness measure of OWA operator, and defined the *orness* of a RIM quantifier [48]. Given a RIM quantifier Q , we can generate the OWA operator with (4). Letting $n \rightarrow \infty$, the orness measure of a RIM quantifier can be obtained

$$orness(Q) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{n-i}{n-1} \left(Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right) \right) = \lim_{n \rightarrow \infty} \sum_{i=1}^{n-1} \frac{Q\left(\frac{i}{n}\right)}{n-1} = \int_0^1 Q(x) dx \tag{5}$$

Thus, the orness degree of a RIM quantifier is equal to the area under it.

To analyze the relationship between OWA operators and RIM quantifiers, a generating function representation of RIM quantifiers was proposed.

Definition 2 [24]. For $f(t)$ on $[0, 1]$ and a RIM quantifier $Q(x)$, $f(t)$ is called generating function of $Q(x)$, if it satisfies

$$Q(x) = \int_0^x f(t) dt \tag{6}$$

where $f(t) \geq 0$ and $\int_0^1 f(t) dt = 1$.

Obviously, for any differentiable RIM quantifier $Q(x)$, its generating function $f(t)$ is equal to its first-order differential function $Q'(x)$.

Using the generating function, the orness of $Q(x)$ can be represented as

$$orness(Q) = \int_0^1 Q(x) dx = \int_0^1 \int_0^x f(t) dt dx = \int_0^1 \int_t^1 f(t) dx dt = \int_0^1 (1-t)f(t) dt \tag{7}$$

Comparing (2) and (7), these two orness measures are similar in their expressions. The generating function $f(x)$ in the RIM fuzzy quantifier plays the role of weights vector W in OWA operator, that the RIM quantifier can be seen as the continuous form of OWA operator with generating function [24,29]. Furthermore, it can be easily seen that Q_* leads to the weight vector W_* , Q^* leads to the weight vector W^* , and the ordinary average RIM quantifier $Q_A(x) = x$ leads to the weight vector W_A . Furthermore, we also have $orness(Q_*) = 0$, $orness(Q^*) = 1$, and $orness(Q_A) = \frac{1}{2}$. Similarly, as the class of RIM quantifiers is bounded by the quantifiers Q_* (quantifier “all”) and Q^* (quantifier “there exists”), thus for any RIM quantifier $Q(x)$, $Q_*(x) \leq Q(x) \leq Q^*(x)$, and for any $X = (x_1, x_2, \dots, x_n)$, $F_{Q^*}(X) = \max_{1 \leq i \leq n} \{x_i\}$, $F_{Q_*}(X) = \min_{1 \leq i \leq n} \{x_i\}$, $F_{Q_A}(X) = \frac{1}{n} \sum_{i=1}^n x_i$.

3. A general model to obtain OWA operator with given orness level

3.1. Problem formulation and its analytical solution properties

Consider the following OWA operator optimization problem with given orness level:

$$\begin{aligned}
 \min \quad & V_{\text{OWA}} = \sum_{i=1}^n F(w_i) \\
 \text{s.t.} \quad & \sum_{i=1}^n \frac{n-i}{n-1} w_i = \alpha, \quad 0 < \alpha < 1 \\
 & \sum_{i=1}^n w_i = 1 \\
 & w_i \geq 0 \quad i = 1, 2, \dots, n
 \end{aligned} \tag{8}$$

where F is a strictly convex function on $[0, 1]$, and it is at least two order differentiable.

As $\alpha = 0$ and $\alpha = 1$ correspond to the unique OWA weight vector W_* and W^* , respectively, they will not be included into the problem.

Problem (8) can be seen as a general model to obtain OWA weights with optimization method. When $F(x) = x \ln(x)$, (8) becomes the maximum entropy OWA operator problem that was extensively discussed in the literature [12,14,31,35]. And $F(x) = x^2$ in (8) corresponds to another commonly discussed minimum variance OWA operator problem [15,26]. More generally, when $F(x) = x^\alpha$ ($\alpha > 1$), (8) becomes the OWA problem of Rényi entropy [33], which includes the maximum entropy and the minimum variance OWA problem as special cases. Some more details of them are discussed in Section 5.

Remark 1. The feasible domain of $F(x)$ becomes $(0, 1)$ if F is meaningless at 0 as in the case of $F(x) = x \ln(x)$. This requires an implicit constraint $w_i > 0$ ($i = 1, 2, \dots, n$) in the problem.

Next, we will discuss some properties of the optimal solution (10) and (11) for problem (8). These properties can be seen as the extensions of the two special cases of the maximum entropy OWA operator [31] and the minimum variance OWA operator [26], with $F(x) = x \ln(x)$ and $F(x) = x^2$, respectively.

Theorem 1. If $W = (w_1, w_2, \dots, w_n)$ is the optimal solution of (8) with given orness level α , then the reversed elements order of W , $\tilde{W} = (w_n, w_{n-1}, \dots, w_1)$ is the optimal solution of (8) with orness value $1 - \alpha$.

Proof. With given orness level α , suppose the optimal solution of (8) is $W = (w_1, w_2, \dots, w_n)$, then

$$\begin{cases} \sum_{i=1}^n \frac{n-i}{n-1} w_i = \alpha \\ \sum_{i=1}^n w_i = 1 \end{cases} \tag{9}$$

We will show that the reversed elements order of W , $\tilde{W} = (w_n, w_{n-1}, \dots, w_1)$ is the optimal solution of (8) with orness value $1 - \alpha$. From the conclusions in [47, p. 127] or (2), it can be verified that $\text{orness}(\tilde{W}) = 1 - \alpha$.

If \tilde{W} is not the optimal solution of (8) with $1 - \alpha$, then there must exist an OWA operator $W^* = (w_1^*, w_2^*, \dots, w_n^*)$ with $\text{orness}(W^*) = 1 - \alpha$, which makes $\sum_{i=1}^n F(w_i^*) < \sum_{i=1}^n F(w_i)$. It is obvious that $\tilde{W}^* = (w_n^*, w_{n-1}^*, \dots, w_1^*)$ with $\text{orness}(\tilde{W}^*) = \alpha$, the objective value is the same as W^* with $\sum_{i=1}^n F(w_i^*)$, which is smaller than that of W with $\sum_{i=1}^n F(w_i)$. This contradicts the assumption that W is the optimal solution of (8) with orness level α . So \tilde{W} is the optimal solution of (8) with $1 - \alpha$. \square

Next, we will give an analytical solution of (8), and some properties will be discussed.

Theorem 2. The optimal solution of (8) is unique, and it can be expressed as $W = (w_1, w_2, \dots, w_n)$ that

$$w_i = \begin{cases} g\left(\frac{n-i}{n-1} \lambda_1 + \lambda_2\right) & \text{if } i \in T \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

where λ_1, λ_2 are determined by

$$\begin{cases} \sum_{i \in T} \frac{n-i}{n-1} g\left(\frac{n-i}{n-1} \lambda_1 + \lambda_2\right) = \alpha \\ \sum_{i \in T} g\left(\frac{n-i}{n-1} \lambda_1 + \lambda_2\right) = 1 \end{cases} \tag{11}$$

and $T = \{i | 1 \leq i \leq n, g\left(\frac{n-i}{n-1} \lambda_1 + \lambda_2\right) > 0\}$ with $g(x) = (F')^{-1}(x)$.

Proof. With the Kuhn–Tucker second-order sufficiency conditions for optimality [10, p. 58], the Lagrange function of the constrained optimization problem (8) gives

$$L(W, \lambda, \mu) = \sum_{i=1}^n F(w_i) + \lambda_1 \left(\sum_{i=1}^n \frac{n-i}{n-1} w_i - \alpha \right) + \lambda_2 \left(\sum_{i=1}^n w_i - 1 \right) - \sum_{i=1}^n \mu_i w_i \tag{12}$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$, and $\mu_i \geq 0$ ($i = 1, 2, \dots, n$).

The optimal solution satisfies that

$$\begin{aligned} \frac{\partial L}{\partial w_i} &= F'(w_i) + \frac{n-i}{n-1} \lambda_1 + \lambda_2 - \mu_i = 0 \quad i = 1, \dots, n \\ \frac{\partial L}{\partial \lambda_1} &= \sum_{i=1}^n \frac{n-i}{n-1} w_i - \alpha = 0 \\ \frac{\partial L}{\partial \lambda_2} &= \sum_{i=1}^n w_i - 1 = 0 \end{aligned} \tag{13}$$

and

$$\mu_i w_i = 0, \quad i = 1, 2, \dots, n \tag{14}$$

where $\mu_i \geq 0$ and $w_i \geq 0$ ($i = 1, 2, \dots, n$).

Because F is strictly convex, that F' is strictly increasing, $(F')^{-1}$ exists and is an increasing function. Observing that if $\mu_i \neq 0$, then $w_i = 0$ and if $w_i \neq 0$, then $\mu_i = 0$, with (13),

$$w_i = (F')^{-1} \left(-\frac{n-i}{n-1} \lambda_1 - \lambda_2 \right) \tag{15}$$

It can be noticed that w_i should be 0 or as (15) if nonzero. An OWA operator weight vector $W = (w_1, w_2, \dots, w_n)$ can be proposed as

$$w_i = \begin{cases} (F')^{-1} \left(-\frac{n-i}{n-1} \lambda_1 - \lambda_2 \right) & \text{if } (F')^{-1} \left(-\frac{n-i}{n-1} \lambda_1 - \lambda_2 \right) > 0 \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

where λ_1, λ_2 are determined by

$$\begin{cases} \sum_{i=1}^n \frac{n-i}{n-1} w_i - \alpha = 0 \\ \sum_{i=1}^n w_i - 1 = 0 \end{cases} \tag{17}$$

Considering that (8) is a problem of separable strictly convex objective function with linear constraints, the Hessian matrix of the Lagrange function is diagonal and positive definite everywhere. There is an unique global optimal minimum solution [10]. This optimal solution is determined by (16) and (17) which is the stationary point of the Lagrangian function (12) that satisfies (13) and (14) with $\mu_i = F'(w_i) + \frac{n-i}{n-1} \lambda_1 + \lambda_2$, $i = 1, 2, \dots, n$. Thus, we have proved that the OWA operator $W = (w_1, w_2, \dots, w_n)$ with (16) and (17) is the unique optimal solution of (8).

Let $(F')^{-1}(x) = g(x)$, and replace $-\lambda_1, -\lambda_2$ with λ_1, λ_2 for a simple expression, the optimal solution (16) and (17) can be expressed in the following form,

$$w_i = \begin{cases} g\left(\frac{n-i}{n-1}\lambda_1 + \lambda_2\right) & \text{if } i \in T \\ 0 & \text{otherwise} \end{cases} \quad (18)$$

where λ_1, λ_2 are determined by

$$\begin{cases} \sum_{i \in T} \frac{n-i}{n-1} g\left(\frac{n-i}{n-1}\lambda_1 + \lambda_2\right) = \alpha \\ \sum_{i \in T} g\left(\frac{n-i}{n-1}\lambda_1 + \lambda_2\right) = 1 \end{cases} \quad (19)$$

where $T = \{i | 1 \leq i \leq n, g\left(\frac{n-i}{n-1}\lambda_1 + \lambda_2\right) > 0\}$. \square

As the unique optimal solution of (8) depends on the given orness level α , the objective function of (8) $V_{\text{OWA}} = \sum_{i=1}^n F(w_i)$ can be seen as the function of the given orness level α , $V_{\text{OWA}}(\alpha)$. Considering that $\tilde{W} = (w_1, w_2, \dots, w_n)$ and $\tilde{W} = (w_n, w_{n-1}, \dots, w_1)$ have the same objective value for (8), from Theorems 1 and 2, we have

Corollary 1. Let $V_{\text{OWA}}(\alpha) = \sum_{i=1}^n F(w_i)$ be the objective function of (8) with orness level α , then $V_{\text{OWA}}(\alpha) = V_{\text{OWA}}(1 - \alpha)$, which means $V_{\text{OWA}}(\alpha)$ is symmetrical for α at $\alpha = \frac{1}{2}$.

Theorem 3. λ_1, λ_2 in (10) and (11) can be seen as the functions of the orness level α with $\lambda_1(\alpha)$ and $\lambda_2(\alpha)$, $\lambda_1(\alpha)$ monotonically increases with α and $\lambda_2(\alpha)$ monotonically decreases with α . And furthermore, the objective value of (8), $V_{\text{OWA}}(\alpha) = \sum_{i=1}^n F(w_i)$ is a convex function of orness level α .

Proof. With Theorem 2, the parameters λ_1, λ_2 in (10) and (11) can be uniquely determined by the orness level α . Let us make a differential operation for α on the both sides of (11),

$$\begin{cases} \sum_{i \in T} \frac{n-i}{n-1} g'\left(\frac{n-i}{n-1}\lambda_1 + \lambda_2\right) \left(\frac{n-i}{n-1}\lambda'_1 + \lambda'_2\right) = 1 \\ \sum_{i \in T} g'\left(\frac{n-i}{n-1}\lambda_1 + \lambda_2\right) \left(\frac{n-i}{n-1}\lambda'_1 + \lambda'_2\right) = 0 \end{cases} \quad (20)$$

that is

$$\begin{cases} \lambda'_1 \sum_{i \in T} \left(\frac{n-i}{n-1}\right)^2 g'\left(\frac{n-i}{n-1}\lambda_1 + \lambda_2\right) + \lambda'_2 \sum_{i \in T} \frac{n-i}{n-1} g'\left(\frac{n-i}{n-1}\lambda_1 + \lambda_2\right) = 1 \\ \lambda'_1 \sum_{i \in T} \frac{n-i}{n-1} g'\left(\frac{n-i}{n-1}\lambda_1 + \lambda_2\right) + \lambda'_2 \sum_{i \in T} g'\left(\frac{n-i}{n-1}\lambda_1 + \lambda_2\right) = 0 \end{cases} \quad (21)$$

Solving these linear equations,

$$\begin{cases} \lambda'_1 = \frac{\sum_{i \in T} g'\left(\frac{n-i}{n-1}\lambda_1 + \lambda_2\right)}{\sum_{i \in T} \left(\frac{n-i}{n-1}\right)^2 g'\left(\frac{n-i}{n-1}\lambda_1 + \lambda_2\right) - \left(\sum_{i \in T} \frac{n-i}{n-1} g'\left(\frac{n-i}{n-1}\lambda_1 + \lambda_2\right)\right)^2} \\ \lambda'_2 = -\frac{\sum_{i \in T} \frac{n-i}{n-1} g'\left(\frac{n-i}{n-1}\lambda_1 + \lambda_2\right)}{\sum_{i \in T} \left(\frac{n-i}{n-1}\right)^2 g'\left(\frac{n-i}{n-1}\lambda_1 + \lambda_2\right) - \left(\sum_{i \in T} \frac{n-i}{n-1} g'\left(\frac{n-i}{n-1}\lambda_1 + \lambda_2\right)\right)^2} \end{cases} \quad (22)$$

Considering that

$$\begin{aligned} & \sum_{i \in T} \left(\frac{n-i}{n-1}\right)^2 g'\left(\frac{n-i}{n-1}\lambda_1 + \lambda_2\right) \sum_{i \in T} g'\left(\frac{n-i}{n-1}\lambda_1 + \lambda_2\right) - \left(\sum_{i \in T} \frac{n-i}{n-1} g'\left(\frac{n-i}{n-1}\lambda_1 + \lambda_2\right)\right)^2 \\ &= \frac{1}{2} \left(\sum_{i \in T} \left(\frac{n-i}{n-1}\right)^2 g'\left(\frac{n-i}{n-1}\lambda_1 + \lambda_2\right) \sum_{j \in T} g'\left(\frac{n-j}{n-1}\lambda_1 + \lambda_2\right) \right. \\ & \quad \left. + \sum_{j \in T} \left(\frac{n-j}{n-1}\right)^2 g'\left(\frac{n-j}{n-1}\lambda_1 + \lambda_2\right) \sum_{i \in T} g'\left(\frac{n-i}{n-1}\lambda_1 + \lambda_2\right) \right) \end{aligned}$$

$$\begin{aligned}
 & -2 \sum_{i \in T} \frac{n-i}{n-1} g' \left(\frac{n-i}{n-1} \lambda_1 + \lambda_2 \right) \sum_{j \in T} \frac{n-j}{n-1} g' \left(\frac{n-j}{n-1} \lambda_1 + \lambda_2 \right) \\
 & = \frac{1}{2} \sum_{i \in T} \sum_{j \in T} \left(\frac{i-j}{n-1} \right)^2 g' \left(\frac{n-i}{n-1} \lambda_1 + \lambda_2 \right) g' \left(\frac{n-j}{n-1} \lambda_1 + \lambda_2 \right)
 \end{aligned}$$

where $T = \{i | 1 \leq i \leq n, g(\frac{n-i}{n-1} \lambda_1 + \lambda_2) > 0\}$ or $T = \{j | 1 \leq j \leq n, g(\frac{n-j}{n-1} \lambda_1 + \lambda_2) > 0\}$ depends on the variable name of the sum computation.

Then, (22) becomes

$$\begin{cases} \lambda'_1 = \frac{2 \sum_{i \in T} g'(\frac{n-i}{n-1} \lambda_1 + \lambda_2)}{\sum_{i \in T} \sum_{j \in T} (\frac{i-j}{n-1})^2 g'(\frac{n-i}{n-1} \lambda_1 + \lambda_2) g'(\frac{n-j}{n-1} \lambda_1 + \lambda_2)} \\ \lambda'_2 = - \frac{2 \sum_{i \in T} \frac{n-i}{n-1} g'(\frac{n-i}{n-1} \lambda_1 + \lambda_2)}{\sum_{i \in T} \sum_{j \in T} (\frac{i-j}{n-1})^2 g'(\frac{n-i}{n-1} \lambda_1 + \lambda_2) g'(\frac{n-j}{n-1} \lambda_1 + \lambda_2)} \end{cases} \tag{23}$$

As $g = (F')^{-1}$ is an strictly increasing function, $g' \geq 0$, it can be obtained that $\lambda'_1 \geq 0$ and $\lambda'_2 \leq 0$, so λ_1 increases with α and λ_2 decreases with α .

With (10) and $g = (F')^{-1}$, it can be obtained that

$$\begin{aligned}
 V'_{OWA}(\alpha) & = \sum_{i \in T} F' \left(g \left(\frac{n-i}{n-1} \lambda_1 + \lambda_2 \right) \right) \frac{\partial g(\frac{n-i}{n-1} \lambda_1 + \lambda_2)}{\partial \alpha} \\
 & = \sum_{i \in T} F' \left(g \left(\frac{n-i}{n-1} \lambda_1 + \lambda_2 \right) \right) g' \left(\frac{n-i}{n-1} \lambda_1 + \lambda_2 \right) \left(\frac{n-i}{n-1} \lambda'_1 + \lambda'_2 \right) \\
 & = \sum_{i \in T} \left(\frac{n-i}{n-1} \lambda_1 + \lambda_2 \right) g' \left(\frac{n-i}{n-1} \lambda_1 + \lambda_2 \right) \left(\frac{n-i}{n-1} \lambda'_1 + \lambda'_2 \right) \\
 & = \lambda_1 \sum_{i \in T} \frac{n-i}{n-1} g' \left(\frac{n-i}{n-1} \lambda_1 + \lambda_2 \right) \left(\frac{n-i}{n-1} \lambda'_1 + \lambda'_2 \right) + \lambda_2 \sum_{i \in T} g' \left(\frac{n-i}{n-1} \lambda_1 + \lambda_2 \right) \left(\frac{n-i}{n-1} \lambda'_1 + \lambda'_2 \right)
 \end{aligned}$$

where $T = \{i | 1 \leq i \leq n, g(\frac{n-i}{n-1} \lambda_1 + \lambda_2) > 0\}$.

Considering (20), then $V'_{OWA}(\alpha) = \lambda_1$, with λ_1 increasing with α . Thus, $V_{OWA}(\alpha)$ is a convex function for α . \square

With Corollary 1 and Theorem 3, it can be obtained that

Corollary 2. The objective function of orness level α for (8), $V_{OWA}(\alpha) = \sum_{i=1}^n F(w_i)$ decreases for $\alpha \in (0, \frac{1}{2}]$, and increases for $\alpha \in [\frac{1}{2}, 1)$. $V_{OWA}(\alpha)$ reaches its minimum value at $\alpha = \frac{1}{2}$.

As $W = (w_1, w_2, \dots, w_n)$ is determined by the orness level α , it can be obtained that

Theorem 4. For the OWA operator F_W with a weight vector $W = (w_1, w_2, \dots, w_n)$ determined by (10) of orness level α , $\sum_{i=1}^k w_i$ monotonically increases with α for any $k(1 \leq k \leq n)$, and furthermore $\forall X = (x_1, x_2, \dots, x_n)$, the aggregation value $F_W(X)$ also monotonically increases with α .

Proof. With (10) and (23),

$$\begin{aligned}
 \frac{\partial \sum_{i=1}^k w_i}{\partial \alpha} & = \sum_{i \in D} g' \left(\frac{n-i}{n-1} \lambda_1 + \lambda_2 \right) \left(\frac{n-i}{n-1} \lambda'_1 + \lambda'_2 \right) \\
 & = \lambda'_1 \sum_{i \in D} \frac{n-i}{n-1} g' \left(\frac{n-i}{n-1} \lambda_1 + \lambda_2 \right) + \lambda'_2 \sum_{i \in D} g' \left(\frac{n-i}{n-1} \lambda_1 + \lambda_2 \right) \\
 & = \frac{2 \sum_{i \in T} g' \left(\frac{n-i}{n-1} \lambda_1 + \lambda_2 \right) \sum_{i \in D} \frac{n-i}{n-1} g' \left(\frac{n-i}{n-1} \lambda_1 + \lambda_2 \right)}{\sum_{i \in T} \sum_{j \in T} (\frac{i-j}{n-1})^2 g' \left(\frac{n-i}{n-1} \lambda_1 + \lambda_2 \right) g' \left(\frac{n-j}{n-1} \lambda_1 + \lambda_2 \right)} - \frac{2 \sum_{i \in T} \frac{n-i}{n-1} g' \left(\frac{n-i}{n-1} \lambda_1 + \lambda_2 \right) \sum_{i \in D} g' \left(\frac{n-i}{n-1} \lambda_1 + \lambda_2 \right)}{\sum_{i \in T} \sum_{j \in T} (\frac{i-j}{n-1})^2 g' \left(\frac{n-i}{n-1} \lambda_1 + \lambda_2 \right) g' \left(\frac{n-j}{n-1} \lambda_1 + \lambda_2 \right)}
 \end{aligned}$$

where $D = \{i | 1 \leq i \leq k, g(\frac{n-i}{n-1} \lambda_1 + \lambda_2) > 0\}$.

As $k \leq n$, D is a subset of $T = \{i | 1 \leq i \leq n, g(\frac{n-i}{n-1} \lambda_1 + \lambda_2) > 0\}$, such that $T - D = \{i | k + 1 \leq i \leq n, g(\frac{n-i}{n-1} \lambda_1 + \lambda_2) > 0\}$, so

$$\begin{aligned} \frac{\partial \sum_{i=1}^k w_i}{\partial \alpha} &= \frac{2 \sum_{i \in T-D} g'(\frac{n-i}{n-1} \lambda_1 + \lambda_2) \sum_{i \in D} \frac{n-i}{n-1} g'(\frac{n-i}{n-1} \lambda_1 + \lambda_2)}{\sum_{i \in T} \sum_{j \in T} (\frac{i-j}{n-1})^2 g'(\frac{n-i}{n-1} \lambda_1 + \lambda_2) g'(\frac{n-j}{n-1} \lambda_1 + \lambda_2)} - \frac{2 \sum_{i \in T-D} \frac{n-i}{n-1} g'(\frac{n-i}{n-1} \lambda_1 + \lambda_2) \sum_{i \in D} g'(\frac{n-i}{n-1} \lambda_1 + \lambda_2)}{\sum_{i \in T} \sum_{j \in T} (\frac{i-j}{n-1})^2 g'(\frac{n-i}{n-1} \lambda_1 + \lambda_2) g'(\frac{n-j}{n-1} \lambda_1 + \lambda_2)} \\ &= \frac{2 \sum_{i \in T-D} \sum_{j \in D} \frac{i-j}{n-1} g'(\frac{n-i}{n-1} \lambda_1 + \lambda_2) g'(\frac{n-j}{n-1} \lambda_1 + \lambda_2)}{\sum_{i \in T} \sum_{j \in T} (\frac{i-j}{n-1})^2 g'(\frac{n-i}{n-1} \lambda_1 + \lambda_2) g'(\frac{n-j}{n-1} \lambda_1 + \lambda_2)} \end{aligned}$$

For $i \in T - D, j \in D$, it holds that $i \geq k + 1 > k \geq j$, and g is an increasing function, $g' \geq 0$, so $\frac{\partial \sum_{i=1}^k w_i}{\partial \alpha} \geq 0$, which means $\sum_{i=1}^k w_i$ monotonically increase with orness level α for any $k (1 \leq k \leq n)$.

Let $s_i = \sum_{k=1}^i w_k, i = 1, 2, \dots, n$, and $s_0 = 0$, then $s_n = 1$. Let us suppose that $x_1 \geq x_2 \geq \dots \geq x_n$, with (1), $F_W(X) = \sum_{i=1}^n w_i x_i = \sum_{i=1}^n (s_i - s_{i-1}) x_i = s_n x_n + \sum_{i=1}^{n-1} s_i (x_i - x_{i+1}) = x_n + \sum_{i=1}^{n-1} s_i (x_i - x_{i+1})$. As s_i monotonically increases with orness level α , so $F_W(X)$ also monotonically increases with α . \square

Furthermore, we can observe the OWA operator weight vector changes with orness level α .

Corollary 3. For the OWA operator weight vector W determined by the optimal solution (8) with orness level α , if $\alpha = \frac{1}{2}$, then $\lambda_1 = 0, W = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}) = W_A$, and $F_W(X) = F_{W_A}(X) = \frac{1}{n} \sum_{i=1}^n x_i$. If $\alpha < \frac{1}{2}$, then $\lambda_1 < 0, w_i$ s have the following form $w_1 = w_2 = \dots = w_{n-r} = 0 < w_{n-r+1} < w_{n-r+2} < \dots < w_n$, and $\forall X, F_W(X) < F_{W_A}(X) = \frac{1}{n} \sum_{i=1}^n x_i$. If $\alpha > \frac{1}{2}$, then $\lambda_1 > 0, W = (w_1, w_2, \dots, w_n)$ has the following form $w_1 > w_2 > \dots > w_r > w_{r+1} = w_{r+2} = \dots = w_n$, and $\forall X, F_W(X) > F_{W_A}(X) = \frac{1}{n} \sum_{i=1}^n x_i$.

Proof. With (10), since $g = (F')^{-1}$ is increasing, the relationships among the OWA operator weight elements of w_i also monotonically change with i . Whether it is increasing or decreasing depends on the sign of λ_1 .

When $\lambda_1 = 0$, from (10), w_i becomes a constant, so $w_1 = w_2 = \dots = w_n = \frac{1}{n}$, then $\alpha = \frac{1}{2}$. From Theorem 3, λ_1 monotonically increases with orness level α , so if $\alpha = \frac{1}{2}$, then $\lambda_1 = 0, W = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}) = W_A$, and $F_W(X) = F_{W_A}(X) = \frac{1}{n} \sum_{i=1}^n x_i$.

With the increasing property of λ_1 for orness level α , when $\alpha > \frac{1}{2}, \lambda_1 > 0$, from (10), $W = (w_1, w_2, \dots, w_n)$ has the following form $w_1 > w_2 > \dots > w_r > w_{r+1} = w_{r+2} = \dots = w_n$, and from Theorem 4, $\forall X, F_W(X) > F_{W_A}(X) = \frac{1}{n} \sum_{i=1}^n x_i$. When $\alpha < \frac{1}{2}, \lambda_1 < 0$, then $W = (w_1, w_2, \dots, w_n)$ has the following form $w_1 = w_2 = \dots = w_{n-r} = 0 < w_{n-r+1} < w_{n-r+2} < \dots < w_n$, and $\forall X, F_W(X) < F_{W_A}(X) = \frac{1}{n} \sum_{i=1}^n x_i$. \square

From these properties, it can be seen that the optimal solutions of (8) with different orness level compose a parameterized OWA operator family, which always includes the ordinary arithmetic mean (average operator) $F_{W_A}(X) = \frac{1}{n} \sum_{i=1}^n x_i$ as a special case with orness being $\frac{1}{2}$. In addition, the aggregation values always monotonically change with the orness level, which make it possible to use the orness level as the control parameter to obtain consistent aggregation results. This is especially useful in real OWA based aggregation problems when the orness level is used as the index of OWA determination or to reflect the preference information [23,25,60]. Note that this consistency property does not hold for ordinary OWA operators, Liu [31, p. 172] once gave a negative example.

3.2. The solution equivalence to the minimax problem

The first minimax problem for OWA operator, called minimax disparity problem, was proposed by Wang and Parkan [39]. The objective is to minimize the maximum disparity, where the disparities between two adjacent weights are made as small as possible:

$$\begin{aligned} \text{minimize} & \left\{ \max_{1 \leq i \leq n-1} |w_i - w_{i+1}| \right\} \\ \text{s.t.} & \sum_{i=1}^n \frac{n-i}{n-1} w_i = \alpha, \quad 0 < \alpha < 1 \end{aligned}$$

$$\sum_{i=1}^n w_i = 1$$

$$w_i \geq 0, \quad i = 1, 2, \dots, n \tag{24}$$

The solution equivalence to the minimum variance problem of Fullér and Majlender [15] was verified theoretically by Liu [30] with the dual theory of linear programming.

The general minimax problem for OWA operators tries to obtain the desired OWA weight vector under given orness level to minimize the maximum difference between the adjacent elements after a monotonic function transformation, which includes the minimax disparity problem as special case. The general minimax problem corresponding to (8) is

$$\min \quad M_{\text{OWA}} = \left\{ \max_{1 \leq i \leq n-1} |F'(w_i) - F'(w_{i+1})| \right\}$$

$$\text{s.t.} \quad \sum_{i=1}^n \frac{n-i}{n-1} w_i = \alpha, \quad 0 < \alpha < 1 \tag{25}$$

$$\sum_{i=1}^n w_i = 1$$

$$w_i \geq 0, \quad i = 1, 2, \dots, n$$

Problem (24) becomes a special case of (25) by setting $F(x) = x^2$ with coefficient 2 being omitted. Comparing the objective functions of the original optimization problem (8) and that of the minimax problem (25), the former minimizes the sum of $F(w_i)$ and the latter tries to minimize the maximum differences between the adjacent $F'(w_i)$ s.

Theorem 5. *If $W = (w_1, w_2, \dots, w_n)$ is the optimal solution of the minimax problem (25) with given orness level α , then the reversed elements order of W , $\tilde{W} = (w_n, w_{n-1}, \dots, w_1)$ is the optimal solution of (25) with orness value $1 - \alpha$.*

Proof. Similar to Theorem 1, omitted. \square

Next, we will prove that problems (8) and (25) have the same optimal solution, which include the results of [30] as a special case and with much more simplified proofs.

Theorem 6. *There is an unique optimal solution for (25), and the optimal solutions of problems (8) and (25) are the same. That is they both have the following form (10), (11) with $W^{\text{opt}} = (w_1^{\text{opt}}, w_2^{\text{opt}}, \dots, w_n^{\text{opt}})$:*

$$w_i^{\text{opt}} = \begin{cases} g\left(\frac{n-i}{n-1}\lambda_1 + \lambda_2\right) & \text{if } i \in T \\ 0 & \text{otherwise} \end{cases} \tag{26}$$

where $g = (F')^{-1}$, λ_1, λ_2 is determined by the constraints of (8):

$$\begin{cases} \sum_{i \in T} \frac{n-i}{n-1} g\left(\frac{n-i}{n-1}\lambda_1 + \lambda_2\right) = \alpha \\ \sum_{i \in T} g\left(\frac{n-i}{n-1}\lambda_1 + \lambda_2\right) = 1 \end{cases} \tag{27}$$

with $T = \{i | 1 \leq i \leq n, g\left(\frac{n-i}{n-1}\lambda_1 + \lambda_2\right) > 0\}$.

Proof. It is obvious that W_{opt} is a feasible solution of (25), as both (25) and (8) have the same constraints. We only need to prove that W_{opt} is the optimal solution of (25). Suppose that there exists another OWA operator $W = (w_1, w_2, \dots, w_n)$ such that $W \neq W^{\text{opt}}$, and

$$\max_{1 \leq i \leq n-1} |F'(w_i) - F'(w_{i+1})| \leq \max_{1 \leq i \leq n-1} |F'(w_i^{\text{opt}}) - F'(w_{i+1}^{\text{opt}})| \tag{28}$$

with $\sum_{i=1}^n w_i = 1$. We will prove that W does not satisfy the constraint $\sum_{i=1}^n \frac{n-i}{n-1} w_i = \alpha$.

First, we will prove that

$$\max_{1 \leq i \leq n-1} |F'(w_i^{\text{opt}}) - F'(w_{i+1}^{\text{opt}})| = \left| \frac{\lambda_1}{n-1} \right| \tag{29}$$

It can be verified in the following three cases.

Case 1: If both $i, i + 1 \in T$. From (26),

$$\begin{aligned} |F'(w_i^{\text{opt}}) - F'(w_{i+1}^{\text{opt}})| &= \left| F' \left(g \left(\frac{n-i}{n-1} \lambda_1 + \lambda_2 \right) \right) - F' \left(g \left(\frac{n-i-1}{n-1} \lambda_1 + \lambda_2 \right) \right) \right| \\ &= \left| F' \left((F')^{-1} \left(\frac{n-i}{n-1} \lambda_1 + \lambda_2 \right) \right) - F' \left((F')^{-1} \left(\frac{n-i-1}{n-1} \lambda_1 + \lambda_2 \right) \right) \right| \\ &= \left| \left(\frac{n-i}{n-1} \lambda_1 + \lambda_2 \right) - \left(\frac{n-i-1}{n-1} \lambda_1 + \lambda_2 \right) \right| = \left| \frac{\lambda_1}{n-1} \right| \end{aligned}$$

Case 2: If only one of the i and $i + 1$ belongs to T .

Let us assume that $i \notin T, i + 1 \in T$, then $g(\frac{n-i}{n-1} \lambda_1 + \lambda_2) \leq 0$ and $g(\frac{n-i-1}{n-1} \lambda_1 + \lambda_2) > 0$ that $w_i^{\text{opt}} = 0$, so $g(\frac{n-i}{n-1} \lambda_1 + \lambda_2)$ is an decreasing function for i . Considering that g is increasing, we must have $\lambda_1 < 0$. Then there exists ξ with $\frac{n-i}{n-1} \lambda_1 + \lambda_2 \leq \xi < \frac{n-i-1}{n-1} \lambda_1 + \lambda_2$, that makes $g(\xi) = 0$. Similar with Case 1, by considering $g = (F')^{-1}$, it can be obtained that

$$|F'(w_i^{\text{opt}}) - F'(w_{i+1}^{\text{opt}})| = \left| \xi - \left(\frac{n-i-1}{n-1} \lambda_1 + \lambda_2 \right) \right| \leq \left| \frac{\lambda_1}{n-1} \right|$$

Case 3: If both $i, i + 1 \notin T$, then $w_i^{\text{opt}} = w_{i+1}^{\text{opt}} = 0, |F'(w_i^{\text{opt}}) - F'(w_{i+1}^{\text{opt}})| = 0$.

Consider these three cases together, it can be obtained that

$$\max_{1 \leq i \leq n-1} |F'(w_i^{\text{opt}}) - F'(w_{i+1}^{\text{opt}})| = \left| \frac{\lambda_1}{n-1} \right| \tag{30}$$

$$|F'(w_i^{\text{opt}}) - F'(w_{i+1}^{\text{opt}})| = \left| \frac{\lambda_1}{n-1} \right| \quad \text{if } i, i + 1 \in T \tag{31}$$

Our next step is proving the optimal solution violation of W for (25). The proof will be presented in the following two cases.

Case 1: If $\alpha = \frac{1}{2}$. From Corollary 3, $\lambda_1 = 0$. In this case, $w_i^{\text{opt}} = \frac{1}{n}$ becomes a constant, the objective value reaches its lower bound 0. With (28), it must have $\max_{1 \leq i \leq n-1} |F'(w_i) - F'(w_{i+1})| = 0$. As F' is strictly monotonic increasing, all the w_i s become a constant, that $w_i = \frac{1}{n}$, so w_i becomes the same as w_i^{opt} , this is a contradiction.

Case 2: If $\alpha \neq \frac{1}{2}$. For simplification, we will only prove the case of $\alpha > \frac{1}{2}$, the condition of $\alpha < \frac{1}{2}$ can be obtained directly with the symmetrical property of Theorems 1 and 5.

From Corollary 3, when $\alpha > \frac{1}{2}, \lambda_1 > 0$. As g is a continuous and strictly monotonic increasing function, $g(\frac{n-i}{n-1} \lambda_1 + \lambda_2)$ monotonically decreases with $i, T = \{i | 1 \leq i \leq n, g(\frac{n-i}{n-1} \lambda_1 + \lambda_2) > 0\}$ has the following form $\{1, 2, \dots, r\}$. w_i^{opt} also has the following form $w_1^{\text{opt}} > w_2^{\text{opt}} > \dots > w_r^{\text{opt}} > 0 = w_{r+1}^{\text{opt}} = w_{r+2}^{\text{opt}} = \dots = w_n^{\text{opt}} = 0$, that $F'(w_1^{\text{opt}}) > F'(w_2^{\text{opt}}) > \dots > F'(w_r^{\text{opt}}) > 0 = F'(w_{r+1}^{\text{opt}}) = F'(w_{r+2}^{\text{opt}}) = \dots = F'(w_n^{\text{opt}}) = F'(0)$. From (28), (30), (31),

$$\max_{1 \leq i \leq n-1} (F'(w_i) - F'(w_{i+1})) \leq \max_{1 \leq i \leq n-1} (F'(w_i^{\text{opt}}) - F'(w_{i+1}^{\text{opt}})) = \frac{\lambda_1}{n-1} \tag{32}$$

$$F'(w_i) - F'(w_{i+1}) \leq F'(w_i^{\text{opt}}) - F'(w_{i+1}^{\text{opt}}) = \frac{\lambda_1}{n-1}, \quad 1 \leq i \leq r-1 \tag{33}$$

We can claim that $F'(w_1) < F'(w_1^{\text{opt}})$, otherwise, $F'(w_1) \geq F'(w_1^{\text{opt}})$. Considering that

$$F'(w_i) = F'(w_1) - \sum_{k=1}^{i-1} (F'(w_k) - F'(w_{k+1})) \tag{34}$$

$$F'(w_i^{\text{opt}}) = F'(w_1^{\text{opt}}) - \sum_{k=1}^{i-1} (F'(w_k^{\text{opt}}) - F'(w_{k+1}^{\text{opt}}))$$

combining (33) and (34), we will have $F'(w_i) \geq F'(w_i^{\text{opt}})$ for $1 \leq i \leq r$, so $w_i \geq w_i^{\text{opt}}$, $\sum_{i=1}^r w_i \geq \sum_{i=1}^r w_i^{\text{opt}}$. Considering that $\sum_{i=1}^r w_i^{\text{opt}} = 1$, and $\sum_{i=1}^n w_i = 1$, $w_i \geq 0$, we must have $w_i \geq w_i^{\text{opt}}$ for $1 \leq i \leq r$ and $w_i = 0$ for $r + 1 \leq i \leq n$, which imply that $w_i = w_i^{\text{opt}}$ for $i = 1, 2, \dots, n$. This is a contradiction. So we must have $F'(w_1) < F'(w_1^{\text{opt}})$,

Next, we will show that there exists a $m, 1 \leq m < n$, that makes

$$\begin{cases} F'(w_i) < F'(w_i^{\text{opt}}) & 1 \leq i \leq m \\ F'(w_i) \geq F'(w_i^{\text{opt}}) & m + 1 \leq i \leq n \end{cases} \tag{35}$$

It will be proved in two cases of $r < n$ and $r = n$, respectively.

If $r < n$, considering that $w_i \geq 0 = w_i^{\text{opt}}$ for $r + 1 \leq i \leq n$, then $F'(w_i) \geq F'(0) = F'(w_i^{\text{opt}})$ for $r + 1 \leq i \leq n$. If $\forall i = 1, 2, \dots, r, F'(w_i) < F'(w_i^{\text{opt}})$, just by setting $m = r$, then (35) stands. Otherwise, there exists a $k, 1 < k \leq r$, that makes $F'(w_k) \geq F'(w_k^{\text{opt}})$. Combining with (33), (34) and $F'(w_1) < F'(w_1^{\text{opt}})$, there has to exist a $m, 1 \leq m < k$, that makes

$$\begin{cases} F'(w_i) < F'(w_i^{\text{opt}}) & 1 \leq i \leq m \\ F'(w_i) \geq F'(w_i^{\text{opt}}) & m + 1 \leq i \leq k \end{cases} \tag{36}$$

and furthermore $F'(w_i) \geq F'(w_i^{\text{opt}})$ for $k \leq i \leq r$, with $F'(w_i) \geq F'(0) = F'(w_i^{\text{opt}})$ for $r + 1 \leq i \leq n$, then

$$\begin{cases} F'(w_i) < F'(w_i^{\text{opt}}) & 1 \leq i \leq m \\ F'(w_i) \geq F'(w_i^{\text{opt}}) & m + 1 \leq i \leq n \end{cases} \tag{37}$$

On the other hand, if $r = n$, we will show that $F'(w_n) \geq F'(w_n^{\text{opt}})$. Otherwise, $F'(w_n) < F'(w_n^{\text{opt}})$. As

$$F'(w_i) = F'(w_n) + \sum_{k=i}^{n-1} (F'(w_k) - F'(w_{k+1})) \tag{38}$$

$$F'(w_i^{\text{opt}}) = F'(w_n^{\text{opt}}) + \sum_{k=i}^{n-1} (F'(w_k^{\text{opt}}) - F'(w_{k+1}^{\text{opt}}))$$

considering (33), we will have that $F'(w_i) < F'(w_i^{\text{opt}})$, $i = 1, 2, \dots, n$, then $w_i < w_i^{\text{opt}}$, thus $\sum_{i=1}^n w_i < \sum_{i=1}^n w_i^{\text{opt}}$, this contradicts the condition that $\sum_{i=1}^n w_i = \sum_{i=1}^n w_i^{\text{opt}} = 1$. With $F'(w_1) < F'(w_1^{\text{opt}})$, $F'(w_n) \geq F'(w_n^{\text{opt}})$ and (33), (38), we can also obtain that there exists a $m, 1 \leq m < n$, that makes

$$\begin{cases} F'(w_i) < F'(w_i^{\text{opt}}) & 1 \leq i \leq m \\ F'(w_i) \geq F'(w_i^{\text{opt}}) & m + 1 \leq i \leq n \end{cases} \tag{39}$$

Combine these two cases of r together, and with F' being strictly increasing, there always exists a $m, 1 \leq m < n$, that makes

$$\begin{cases} w_i < w_i^{\text{opt}} & 1 \leq i \leq m \\ w_i \geq w_i^{\text{opt}} & m + 1 \leq i \leq n \end{cases} \tag{40}$$

With $\sum_{i=1}^n w_i = \sum_{i=1}^n w_i^{\text{opt}} = 1$,

$$\begin{aligned} \sum_{i=1}^n \frac{n-i}{n-1} w_i - \sum_{i=1}^n \frac{n-i}{n-1} w_i^{\text{opt}} &= \sum_{i=1}^m \frac{n-i}{n-1} (w_i - w_i^{\text{opt}}) + \sum_{i=m+1}^n \frac{n-i}{n-1} (w_i - w_i^{\text{opt}}) \\ &< \sum_{i=1}^m \frac{n-m}{n-1} (w_i - w_i^{\text{opt}}) + \sum_{i=m+1}^n \frac{n-m}{n-1} (w_i - w_i^{\text{opt}}) = \frac{n-m}{n-1} \sum_{i=1}^n (w_i - w_i^{\text{opt}}) \\ &= 0 \end{aligned}$$

That is $\sum_{i=1}^n \frac{n-i}{n-1} w_i < \sum_{i=1}^n \frac{n-i}{n-1} w_i^{\text{opt}} = \alpha$. This contradicts the constraint $\sum_{i=1}^n \frac{n-i}{n-1} w_i = \alpha$. Therefore, W_{opt} is the optimal solution of (25), and this optimal solution is unique. \square

Similar to (8), the optimal solution of (25) also depends on the orness level α , from Theorems 5 and 6, we also have

Corollary 4. Let $M_{\text{OWA}}(\alpha) = \max_{1 \leq i \leq n-1} |F'(w_i) - F'(w_{i+1})|$ be the objective function value of (25) with orness level α , then $M_{\text{OWA}}(\alpha) = M_{\text{OWA}}(1 - \alpha)$, which means $M_{\text{OWA}}(\alpha)$ is symmetrical for α at $\alpha = \frac{1}{2}$.

Theorem 7. The objective value of the minimax problem (25), $M_{\text{OWA}}(\alpha) = \max_{1 \leq i \leq n-1} |F'(w_i) - F'(w_{i+1})|$ decreases for $\alpha \in (0, \frac{1}{2}]$, and increases for $\alpha \in [\frac{1}{2}, 1)$. $M_{\text{OWA}}(\alpha)$ reaches its possible minimum value 0 at $\alpha = \frac{1}{2}$.

Proof. From (30), with the optimal solution (26) and (27), the objective function value of the minimax problem (25) is

$$M_{\text{OWA}}(\alpha) = \max_{1 \leq i \leq n-1} |F'(w_i) - F'(w_{i+1})| = \left| \frac{\lambda_1}{n-1} \right| \quad (41)$$

From Corollary 3, when $\alpha = \frac{1}{2}$, $\lambda_1 = 0$, $M_{\text{OWA}}(\alpha) = 0$. From Theorem 3, λ_1 increases with orness level α , so $\lambda_1 < 0$ for $\alpha \in (0, \frac{1}{2})$ and $\lambda_1 > 0$ for $\alpha \in (\frac{1}{2}, 1)$, that $M_{\text{OWA}}(\alpha) = |\lambda_1|$ decreases for $\alpha \in (0, \frac{1}{2})$, and it increases for $\alpha \in (\frac{1}{2}, 1)$, $M(\alpha)$ reaches its possible minimum value 0 at $\alpha = \frac{1}{2}$. \square

4. A general model to obtain RIM quantifier with given orness level

Compared with the various OWA operator determination methods [42,57], the research on quantifier based aggregation and its applications is relatively rare. As the RIM quantifier can be seen as the continuous form of OWA operator with generating function [24,29], all the conclusions in Section 3 can be extended to the RIM quantifier case, which are the extensions of the minimum variance and maximum entropy RIM quantifiers [24,27]. The problem and conclusions are given in parallel to that of the OWA case for comparison.

4.1. Problem formulation and analytical solution properties

The general model for RIM quantifier determination under given orness level can be formulated as

$$\begin{aligned} \min \quad & V_{\text{RIM}} = \int_0^1 F(f(x)) dx \\ \text{s.t.} \quad & \int_0^1 (1-x)f(x) dx = \alpha, \quad 0 < \alpha < 1 \\ & \int_0^1 f(x) dx = 1 \\ & f(x) \geq 0 \end{aligned} \quad (42)$$

where F is a strictly convex function in $[0, +\infty)$,¹ and it is at least two order differentiable.

¹ Similar to the OWA operator case, the feasible domain can be $(0, +\infty)$ if F is meaningless at 0 as in the case of $F(x) = x \ln(x)$. This means an implicit constraint $f(x) > 0$ in the problem.

As $\alpha = 0$ or $\alpha = 1$ correspond to the unique RIM quantifier generating function solution of $Q_*(x)$ and $Q^*(x)$ respectively, we will not include these two special cases into the problem.

Theorem 8. *If $f(x)$ is the optimal solution of (42) with given orness level α , then $f(1 - x)$ is the optimal solution of (42) with $1 - \alpha$.*

Proof. With given orness level α , suppose the optimal solution of (42) is $f(x)$, then

$$\begin{cases} \int_0^1 (1 - x)f(x)dx = \alpha \\ \int_0^1 f(x)dx = 1 \end{cases}$$

We will show that $h(x) = f(1 - x)$ is the optimal solution of (42) with $1 - \alpha$. It can be verified that

$$\begin{cases} \int_0^1 (1 - x)h(x)dx = \int_0^1 (1 - x)f(1 - x)dx = \int_0^1 tf(t)dt = 1 - \alpha \\ \int_0^1 h(x)dx = \int_0^1 f(1 - x)dx = \int_0^1 f(t)dt = 1 \end{cases}$$

If $h(x) = f(1 - x)$ is not the optimal solution of (42) with $1 - \alpha$, then there exists $r(x)$, $r(x) \neq h(x)$ and

$$\begin{cases} \int_0^1 (1 - x)r(x)dx = 1 - \alpha \\ \int_0^1 r(x)dx = 1 \end{cases}$$

which makes $\int_0^1 F(r(x))dx < \int_0^1 F(h(x))dx = \int_0^1 F(f(1 - x))dx$. It can be verified that $r(1 - x)$ satisfies

$$\begin{cases} \int_0^1 (1 - x)r(1 - x)dx = \alpha \\ \int_0^1 r(1 - x)dx = 1 \end{cases}$$

and

$$\int_0^1 F(r(1 - x))dx = \int_0^1 F(r(t))dt < \int_0^1 F(h(x))dx = \int_0^1 F(f(1 - x))dx$$

This contradicts the assumption that $f(x)$ is the optimal solution of (42) with orness level α . So $f(1 - x)$ is the optimal solution of (42) with $1 - \alpha$. \square

Theorem 9. *The optimal solution of (42) is unique, and it can be expressed as*

$$f(x) = \begin{cases} g(\lambda_1 x + \lambda_2) & \text{if } x \in E \\ 0 & \text{otherwise} \end{cases} \tag{43}$$

where λ_1, λ_2 is determined by the constraints of (45):

$$\begin{cases} \int_E xg(\lambda_1 x + \lambda_2)dx = 1 - \alpha \\ \int_E g(\lambda_1 x + \lambda_2)dx = 1 \end{cases} \tag{44}$$

and $E = \{x | 0 \leq x \leq 1, g(\lambda_1 x + \lambda_2) \geq 0\}$ with $g(x) = (F')^{-1}(x)$.

Proof. An alternative form of Problem (42) is

$$\begin{aligned} \min \quad & \int_0^1 F(f(x))dx \\ \text{s.t.} \quad & \int_0^1 xf(x)dx = 1 - \alpha, \quad 0 < \alpha < 1 \\ & \int_0^1 f(x)dx = 1 \\ & f(x) \geq 0 \end{aligned} \tag{45}$$

Similar to the transformation in [24], (45) can be transformed into an equivalent optimal control problem

$$\begin{aligned}
\min \quad & J = \int_0^1 F(f(x)) dx \\
\text{s.t.} \quad & \frac{dw}{dx} = \begin{pmatrix} xf(x) \\ f(x) \end{pmatrix} \quad x \in [0, 1] \\
& w(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad w(1) = \begin{pmatrix} 1 - \alpha \\ 1 \end{pmatrix}
\end{aligned} \tag{46}$$

and the control constraint $f(x) \geq 0$.

As F is strictly convex, with the optimal control theory [36], there exist an unique optimal solution $f^*(x)$ for (46).

The Hamiltonian is

$$H = F(f(x)) + \lambda_1 x f(x) + \lambda_2 f(x) \tag{47}$$

Since F is convex that F' is increasing, $(F')^{-1}$ exists. The optimal solution has the following form:

$$f(x) = \begin{cases} (F')^{-1}(-\lambda_1 x - \lambda_2) & \text{if } F'^{-1}(-\lambda_1 x - \lambda_2) \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{48}$$

Let $(F')^{-1}(x) = g(x)$, and replace $-\lambda_1, -\lambda_2$ with λ_1, λ_2 for simple expression, (48) becomes

$$f(x) = \begin{cases} g(\lambda_1 x + \lambda_2) & \text{if } x \in E \\ 0 & \text{otherwise} \end{cases} \tag{49}$$

where λ_1, λ_2 is determined by the constraints of (45):

$$\begin{cases} \int_E x g(\lambda_1 x + \lambda_2) dx = 1 - \alpha \\ \int_E g(\lambda_1 x + \lambda_2) dx = 1 \end{cases} \tag{50}$$

and $E = \{x | 0 \leq x \leq 1, g(\lambda_1 x + \lambda_2) \geq 0\}$. \square

As $\int_0^1 F(f(1-x)) dx = \int_0^1 F(f(x)) dx$, from Theorems 8 and 9, we can get that

Corollary 5. Let $V_{\text{RIM}}(\alpha) = \int_0^1 F(f(x)) dx$ be the objective function of orness level α for (42), then $V_{\text{RIM}}(\alpha) = V_{\text{RIM}}(1 - \alpha)$, which means $V_{\text{RIM}}(\alpha)$ is symmetrical for α at $\alpha = \frac{1}{2}$.

Theorem 10. λ_1, λ_2 in (43) and (44) can be seen as the functions the orness level α with $\lambda_1(\alpha), \lambda_2(\alpha)$. $\lambda_1(\alpha)$ monotonically decreases with α and $\lambda_2(\alpha)$ monotonically increases with α . The objective value of (42), $V_{\text{RIM}}(\alpha) = \int_0^1 F(f(x, \alpha)) dx$ is a convex function of orness level α .

Proof. With Theorem 9, the parameters λ_1, λ_2 in (43) and (44) can be uniquely determined by the orness level α . Let us make a differential operation for α on the both sides of (44),

$$\begin{cases} \int_E x g'(\lambda_1 x + \lambda_2)(\lambda_1' x + \lambda_2') dx = -1 \\ \int_E g'(\lambda_1 x + \lambda_2)(\lambda_1' x + \lambda_2') dx = 0 \end{cases} \tag{51}$$

that is

$$\begin{cases} \lambda_1' \int_E x^2 g'(\lambda_1 x + \lambda_2) dx + \lambda_2' \int_E x g'(\lambda_1 x + \lambda_2) dx = -1 \\ \lambda_1' \int_E x g'(\lambda_1 x + \lambda_2) dx + \lambda_2' \int_E g'(\lambda_1 x + \lambda_2) dx = 0 \end{cases} \tag{52}$$

Solving these linear equations,

$$\begin{cases} \lambda'_1 = - \frac{\int_E g'(\lambda_1 x + \lambda_2) dx}{\int_E x^2 g'(\lambda_1 x + \lambda_2) dx \int_E g'(\lambda_1 x + \lambda_2) dx - \left(\int_E x g'(\lambda_1 x + \lambda_2) dx\right)^2} \\ \lambda'_2 = \frac{\int_E x g'(\lambda_1 x + \lambda_2) dx}{\int_E x^2 g'(\lambda_1 x + \lambda_2) dx \int_E g'(\lambda_1 x + \lambda_2) dx - \left(\int_E x g'(\lambda_1 x + \lambda_2) dx\right)^2} \end{cases} \tag{53}$$

Considering that

$$\begin{aligned} & \int_E x^2 g'(\lambda_1 x + \lambda_2) dx \int_E g'(\lambda_1 x + \lambda_2) dx - \left(\int_E x g'(\lambda_1 x + \lambda_2) dx\right)^2 \\ &= \frac{1}{2} \left(\int_E x^2 g'(\lambda_1 x + \lambda_2) dx \int_E g'(\lambda_1 y + \lambda_2) dy + \int_E y^2 g'(\lambda_1 y + \lambda_2) dx \int_E g'(\lambda_1 x + \lambda_2) dx \right. \\ & \quad \left. - 2 \int_E x g'(\lambda_1 x + \lambda_2) dx \int_E y g'(\lambda_1 y + \lambda_2) dy \right) \\ &= \frac{1}{2} \mathcal{E} (x^2 - 2xy + y^2) g'(\lambda_1 x + \lambda_2) g'(\lambda_1 y + \lambda_2) dx dy = \frac{1}{2} \mathcal{E} (x - y)^2 g'(\lambda_1 x + \lambda_2) g'(\lambda_1 y + \lambda_2) dx dy \end{aligned}$$

where $E = \{x | 0 \leq x \leq 1, g(\lambda_1 x + \lambda_2) \geq 0\}$ or $E = \{y | 0 \leq y \leq 1, g(\lambda_1 y + \lambda_2) \geq 0\}$ depends on the variable name of the integrand function, and $\mathcal{E} = \{(x, y) | 0 \leq x \leq 1, 0 \leq y \leq 1, g(\lambda_1 x + \lambda_2) \geq 0, g(\lambda_1 y + \lambda_2) \geq 0\}$.

Then (53) becomes

$$\begin{cases} \lambda'_1 = - \frac{2 \int_E g'(\lambda_1 x + \lambda_2) dx}{\mathcal{E} (x-y)^2 g'(\lambda_1 x + \lambda_2) g'(\lambda_1 y + \lambda_2) dx dy} \\ \lambda'_2 = \frac{2 \int_E x g'(\lambda_1 x + \lambda_2) dx}{\mathcal{E} (x-y)^2 g'(\lambda_1 x + \lambda_2) g'(\lambda_1 y + \lambda_2) dx dy} \end{cases} \tag{54}$$

Since g is an increasing function, $g' \geq 0$ and E is not empty, it follows that $\lambda'_1 < 0, \lambda'_2 > 0$, so λ_1 decreases with α and λ_2 increases with α .

With (43) and $g = (F')^{-1}$,

$$\begin{aligned} V'_{\text{RIM}}(\alpha) &= \int_E F'(f(x, \alpha)) \frac{\partial g(\lambda_1 x + \lambda_2)}{\partial \alpha} dx = \int_E F'(g(\lambda_1 x + \lambda_2)) g'(\lambda_1 x + \lambda_2) (\lambda'_1 x + \lambda'_2) dx \\ &= \int_E (\lambda_1 x + \lambda_2) g'(\lambda_1 x + \lambda_2) (\lambda'_1 x + \lambda'_2) dx \\ &= \lambda_1 \int_E x g'(\lambda_1 x + \lambda_2) (\lambda'_1 x + \lambda'_2) dx + \lambda_2 \int_E g'(\lambda_1 x + \lambda_2) (\lambda'_1 x + \lambda'_2) dx \end{aligned}$$

Considering (51), $V'_{\text{RIM}}(\alpha) = -\lambda_1$, with λ_1 decreasing with α , so $V_{\text{RIM}}(\alpha)$ is a convex function for α . □

From Corollary 5 and Theorem 10, it can be obtained that

Corollary 6. The objective function of orness level α for (42), $V_{\text{RIM}}(\alpha) = \int_0^1 F(f(x, \alpha)) dx$ decreases for $\alpha \in (0, \frac{1}{2})$, and increases in $\alpha \in [\frac{1}{2}, 1)$. $V_{\text{RIM}}(\alpha)$ reaches its minimum value at $\alpha = \frac{1}{2}$.

With $Q(x) = \int_0^x f(t) dt$,

$$Q(x) = \int_D g(\lambda_1 t + \lambda_2) dt, \quad D = \{t | 0 \leq t \leq x, g(\lambda_1 t + \lambda_2) \geq 0\} \tag{55}$$

It is obvious that the shape of $f(x)$ and $Q(x)$ is determined by the orness level α . If $Q(x)$ is regarded as a parameterized function family of $Q(x, \alpha)$, it holds that

Theorem 11. For the RIM quantifier function $Q(x, \alpha)$ with orness level α , it holds that $\forall x \in [0, 1], Q(x, \alpha)$ monotonically increases with α , and furthermore, $\forall X = (x_1, x_2, \dots, x_n)$, the aggregation value $F_Q(X)$ also monotonically increases with orness level α .

Proof. From (55),

$$\frac{\partial Q(x, \alpha)}{\partial \alpha} = \int_D g'(\lambda_1 t + \lambda_2)(\lambda_1' t + \lambda_2') dt = \lambda_1' \int_D t g'(\lambda_1 t + \lambda_2) dt + \lambda_2' \int_D g'(\lambda_1 t + \lambda_2) dt$$

With (54), and replacing the integrand variable x, y in λ_1', λ_2' with t, s , $E = \{x | 0 \leq x \leq 1, g(\lambda_1 x + \lambda_2) \geq 0\}$ becomes $E = \{t | 0 \leq t \leq 1, g(\lambda_1 t + \lambda_2) \geq 0\}$, then

$$\frac{\partial Q(x, \alpha)}{\partial \alpha} = \frac{-2 \int_D t g'(\lambda_1 t + \lambda_2) dt \int_E g'(\lambda_1 t + \lambda_2) dt}{\int_D (t-s)^2 g'(\lambda_1 s + \lambda_2) g'(\lambda_1 t + \lambda_2) ds dt} + \frac{2 \int_D g'(\lambda_1 t + \lambda_2) dt \int_E t g'(\lambda_1 t + \lambda_2) dt}{\int_D (t-s)^2 g'(\lambda_1 s + \lambda_2) g'(\lambda_1 t + \lambda_2) ds dt}$$

As $D = \{t | 0 \leq t \leq x, g(\lambda_1 t + \lambda_2) \geq 0\}$ is a subset of E , $E - D = \{t | x \leq t \leq 1, g(\lambda_1 t + \lambda_2) \geq 0\}$,

$$\begin{aligned} \frac{\partial Q(x, \alpha)}{\partial \alpha} &= \frac{-2 \int_D t g'(\lambda_1 t + \lambda_2) dt \int_{E-D} g'(\lambda_1 t + \lambda_2) dt}{\int_D (t-s)^2 g'(\lambda_1 s + \lambda_2) g'(\lambda_1 t + \lambda_2) ds dt} + \frac{2 \int_D g'(\lambda_1 t + \lambda_2) dt \int_{E-D} t g'(\lambda_1 t + \lambda_2) dt}{\int_D (t-s)^2 g'(\lambda_1 s + \lambda_2) g'(\lambda_1 t + \lambda_2) ds dt} \\ &= \frac{2 \int_D (s-t) g'(\lambda_1 s + \lambda_2) g'(\lambda_1 t + \lambda_2) ds dt}{\int_D (s-t)^2 g'(\lambda_1 s + \lambda_2) g'(\lambda_1 t + \lambda_2) ds dt} \end{aligned}$$

where $\mathcal{D} = \{(s, t) | x \leq s \leq 1, 0 \leq t \leq x, g(\lambda_1 s + \lambda_2) \geq 0, g(\lambda_1 t + \lambda_2) \geq 0\}$, $\mathcal{E} = \{(s, t) | 0 \leq s \leq 1, 0 \leq t \leq 1, g(\lambda_1 s + \lambda_2) \geq 0, g(\lambda_1 t + \lambda_2) \geq 0\}$.

Since g is an increasing function, $g' \geq 0$, and $s \geq t$ on \mathcal{D} , $\frac{\partial Q(x, \alpha)}{\partial \alpha} \geq 0$, and $Q(x, \alpha)$ increases with α .

From (3), let us suppose that $x_1 \geq x_2 \geq \dots \geq x_n$, then $F_Q(X) = \sum_{i=1}^n x_i (Q(\frac{i}{n}) - Q(\frac{i-1}{n})) = x_n Q(1) + \sum_{i=1}^{n-1} (x_i - x_{i+1}) Q(\frac{i}{n}) = x_n + \sum_{i=1}^{n-1} (x_i - x_{i+1}) Q(\frac{i}{n})$. As $\forall i, i = 1, 2, \dots, n-1, x_i - x_{i+1} \geq 0$ and $\forall x \in [0, 1]$, $Q(x, \alpha)$ increases with α , so $F_Q(X)$ increases with orness level α . \square

As $g = (F')^{-1}$ is a strictly increasing function, from (43), $f(x)$ is also a monotonic function. Whether it is increasing or decreasing depends on the sign of λ_1 . Furthermore, it can be obtained that

Corollary 7. For the RIM quantifier $Q(x)$ and its generating function $f(x)$ determined by the optimal solution (42) with orness level α , if $\alpha = \frac{1}{2}$, then $\lambda_1 = 0, f(x) = 1, Q(x) = x$, and $F_Q(X) = F_{Q_A}(X) = \frac{1}{n} \sum_{i=1}^n x_i$. If $\alpha < \frac{1}{2}$, then $\lambda_1 > 0, f(x)$ is increasing, $Q(x)$ is convex and $\forall X, F_Q(X) < F_{Q_A}(X) = \frac{1}{n} \sum_{i=1}^n x_i$. If $\alpha > \frac{1}{2}$, then $\lambda_1 < 0, f(x)$ is decreasing, $Q(x)$ is concave and $\forall X, F_Q(X) > F_{Q_A}(X) = \frac{1}{n} \sum_{i=1}^n x_i$.

Proof. When $\lambda_1 = 0$, from (43), $f(x)$ becomes a constant, with Definition 2 and (7), $f(x) = 1$, and $\alpha = \frac{1}{2}$. From Theorem 10, λ_1 decreases with orness level α . So if $\alpha = \frac{1}{2}$, then $\lambda_1 = 0, f(x) = 1, Q(x) = x$, and $F_Q(X) = F_{Q_A}(X) = \frac{1}{n} \sum_{i=1}^n x_i$.

Considering the decreasing property of λ_1 with orness level α , when $\alpha > \frac{1}{2}, \lambda_1 < 0$, then $f(x)$ is decreasing, $Q(x)$ is concave, from Theorem 11, $\forall X, F_Q(X) > F_{Q_A}(X) = \frac{1}{n} \sum_{i=1}^n x_i$. When $\alpha < \frac{1}{2}, \lambda_1 > 0$, then $f(x)$ is increasing, $Q(x)$ is convex and $\forall X, F_Q(X) < F_{Q_A}(X) = \frac{1}{n} \sum_{i=1}^n x_i$. \square

4.2. The solution equivalence to the minimax problem

Corresponding to (42), consider the minimax problem for RIM quantifier:

$$\begin{aligned} \min \quad M_{\text{RIM}} &= \left\{ \max_{0 \leq x \leq 1} |F''(f(x))f'(x)| \right\} \\ \text{s.t.} \quad &\int_0^1 (1-x)f(x)dx = \alpha, \quad 0 < \alpha < 1 \\ &\int_0^1 f(x)dx = 1 \\ &f(x) \geq 0 \end{aligned} \tag{56}$$

Problem (42) minimizes the overall integral of $F(f(x))$, while (56) tries to minimize the absolute maximum local differential value of $F'(f(x))$, that is $|F''(f(x))f'(x)|$.

Theorem 12. *If $f(x)$ is the optimal solution of (56) with given orness level α , then $f(1 - x)$ is the optimal solution of (56) with $1 - \alpha$.*

Proof. Similar to Theorem 8, omitted. \square

Theorem 13. *There is an unique optimal solution for (56), and the optimal solutions of the two kinds problems (42) and (56) are the same. That is, they both have the form of (43) as*

$$f_{\text{opt}}(x) = \begin{cases} g(\lambda_1 x + \lambda_2) & \text{if } g(\lambda_1 x + \lambda_2) \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{57}$$

where $g = (F')^{-1}$, λ_1, λ_2 is determined by the constraints:

$$\begin{cases} \int_0^1 x f_{\text{opt}}(x) dx = 1 - \alpha \\ \int_0^1 f_{\text{opt}}(x) dx = 1 \end{cases} \tag{58}$$

Proof. We only need to prove that f_{opt} is the optimal solution of (56). Assume that there exists a function $f(x)$ such that

$$\begin{aligned} f(x) \not\equiv f_{\text{opt}}(x), \quad f(x) \geq 0, \\ \max_{0 \leq x \leq 1} |F''(f(x))f'(x)| \leq \max_{0 \leq x \leq 1} |F''(f_{\text{opt}}(x))f'_{\text{opt}}(x)| \end{aligned} \tag{59}$$

and the constraint $\int_0^1 f(x) = 1$. We will prove that $f(x)$ does not satisfy the constraint $\int_0^1 (1 - x)f(x) = \alpha$. From (57),

$$F''(f_{\text{opt}}(x))f'_{\text{opt}}(x) = (F'(f_{\text{opt}}(x)))' = \begin{cases} \frac{\partial F'(g(\lambda_1 x + \lambda_2))}{\partial x} & x \in E \\ 0 & \text{otherwise} \end{cases} \tag{60}$$

where $E = \{x | 0 \leq x \leq 1, g(\lambda_1 x + \lambda_2) \geq 0\}$. With $g = (F')^{-1}$, $\frac{\partial F'(g(\lambda_1 x + \lambda_2))}{\partial x} = \frac{\partial(\lambda_1 x + \lambda_2)}{\partial x} = \lambda_1$, thus

$$F''(f_{\text{opt}}(x))f'_{\text{opt}}(x) = \begin{cases} \lambda_1 & x \in E \\ 0 & \text{otherwise} \end{cases} \tag{61}$$

So $\max_{0 \leq x \leq 1} |F''(f_{\text{opt}}(x))f'_{\text{opt}}(x)| = |\lambda_1|$. As $F''(f(x))f'(x) = (F'(f(x)))'$ and $F''(f_{\text{opt}}(x))f'_{\text{opt}}(x) = (F'(f_{\text{opt}}(x)))'$, let $R(x) = F'(f(x))$ and $R_{\text{opt}}(x) = F'(f_{\text{opt}}(x))$, from (59),

$$\max |R'(x)| \leq |R'_{\text{opt}}(x)| = |\lambda_1| \quad x \in E \tag{62}$$

With Theorems 8 and 12, we will discuss in the following two cases.

Case 1: If $\alpha = \frac{1}{2}$, from Corollary 7, $\lambda_1 = 0, f_{\text{opt}}(x)$ becomes a constant, $E = [0, 1], f_{\text{opt}} = 1$. We also have $\max_{x \in [0, 1]} |R'(x)| = 0$, then $R'(x) = (F'(f(x)))' = 0$ for $x \in [0, 1], F'(f(x))$ is a constant. As F is convex, F' is increasing, $f(x)$ is also a constant. With $\int_0^1 f(x) dx = 1, f(x) = 1$, thus $f(x) \equiv f_{\text{opt}}(x)$ on $[0, 1]$, this is a contradiction.

Case 2: If $\alpha \neq \frac{1}{2}$. For simplification, we will only prove the case of $\alpha < \frac{1}{2}$, the condition of $\alpha > \frac{1}{2}$ can be obtained directly with the symmetrical property of Theorems 8 and 12.

From Corollary 7, if $\alpha < \frac{1}{2}$, then $\lambda_1 > 0$. As g is a continuous and monotonic increasing function, $g(\lambda_1 x + \lambda_2)$ is also continuous and monotonic increasing, $E = \{x | 0 \leq x \leq 1, g(\lambda_1 x + \lambda_2) \geq 0\}$ is a continuous and compact subset of $[0, 1]$. Let $\inf\{E\} = a$, and $\sup\{E\} = b$, then $E = [a, b]$ and it has $b = 1$ and $f(a) = g(\lambda_1 a + \lambda_2) = 0$ if $a \neq 0$. From (62),

$$R'(x) \leq R'_{\text{opt}}(x) = \lambda_1 \quad x \in [a, 1] \tag{63}$$

We can claim that $R(1) < R_{\text{opt}}(1)$, otherwise, $R(1) \geq R_{\text{opt}}(1)$. As

$$\begin{aligned} R(x) &= R(1) - \int_x^1 R'(t) dt \\ R_{\text{opt}}(x) &= R_{\text{opt}}(1) - \int_x^1 R'_{\text{opt}}(t) dt \end{aligned} \quad (64)$$

Combining (63) and (64), we will have $R(x) \geq R_{\text{opt}}(x)$ on $[a, 1]$, that is $F'(f(x)) \geq F'(f_{\text{opt}}(x))$, so $f(x) \geq f_{\text{opt}}(x)$, and $\int_a^1 f(x) dx \geq \int_a^1 f_{\text{opt}}(x) dx$. Considering that $\int_0^1 f_{\text{opt}}(x) dx = \int_a^1 f_{\text{opt}}(x) dx = 1$, and $\int_0^1 f(x) dx = 1$, $f(x) \geq 0$, we must have $f(x) = f_{\text{opt}}(x)$ on $[a, 1]$ and $f(x) = 0$ on $[0, a]$, which imply that $f(x) \equiv f_{\text{opt}}(x)$ on $[0, 1]$. This is a contradiction. So we must have $R(1) < R_{\text{opt}}(1)$.

Next, we will show that there exists x_0 that makes

$$\begin{cases} R(x) \geq R_{\text{opt}}(x) & \forall x \in [0, x_0] \\ R(x) < R_{\text{opt}}(x) & \forall x \in (x_0, 1] \end{cases} \quad (65)$$

It will be proved with the two cases $a > 0$ and $a = 0$ respectively.

If $a > 0$, then $f_{\text{opt}}(a) = 0$, with $f(a) \geq 0$, then $f(a) \geq f_{\text{opt}}(a) = 0$, thus $R(a) \geq R_{\text{opt}}(a)$, considering (63), (64) and $R(1) < R_{\text{opt}}(1)$, there exists a $x_0 \in [a, 1]$, that makes

$$\begin{cases} R(x) \geq R_{\text{opt}}(x) & \forall x \in [a, x_0] \\ R(x) < R_{\text{opt}}(x) & \forall x \in (x_0, 1] \end{cases} \quad (66)$$

For $x \in [0, a]$, with $f(x) \geq 0 = f_{\text{opt}}(x)$, then $R(x) \geq R_{\text{opt}}(x)$. Combining with (66),

$$\begin{cases} R(x) \geq R_{\text{opt}}(x) & \forall x \in [0, x_0] \\ R(x) < R_{\text{opt}}(x) & \forall x \in (x_0, 1] \end{cases} \quad (67)$$

If $a = 0$, we will show that $R(0) \geq R_{\text{opt}}(0)$, otherwise $R(0) < R_{\text{opt}}(0)$. Considering that

$$\begin{aligned} R(x) &= R(0) + \int_0^x R'(t) dt \\ R_{\text{opt}}(x) &= R_{\text{opt}}(0) + \int_0^x R'_{\text{opt}}(t) dt \end{aligned} \quad (68)$$

combining (63), we will have $R(x) < R_{\text{opt}}(x)$ on $[0, 1]$, that is $F'(f(x)) < F'(f_{\text{opt}}(x))$, so $f(x) < f_{\text{opt}}(x)$, and $\int_0^1 f(x) dx < \int_0^1 f_{\text{opt}}(x) dx$. This contradicts with the condition that $\int_0^1 f(x) dx = \int_0^1 f_{\text{opt}}(x) dx = 1$. With $R(0) \geq R_{\text{opt}}(0)$ and $R(1) < R_{\text{opt}}(1)$ and (63), it can also be obtained that there exists a $x_0 \in [0, 1]$, that makes

$$\begin{cases} R(x) \geq R_{\text{opt}}(x) & \forall x \in [0, x_0] \\ R(x) < R_{\text{opt}}(x) & \forall x \in (x_0, 1] \end{cases} \quad (69)$$

As $R(x) = F'(f(x))$ and $R_{\text{opt}}(x) = F'(f_{\text{opt}}(x))$, and F' is strictly increasing, thus

$$\begin{cases} f(x) \geq f_{\text{opt}}(x) & \forall x \in [0, x_0] \\ f(x) < f_{\text{opt}}(x) & \forall x \in (x_0, 1] \end{cases} \quad (70)$$

With $\int_0^1 f_{\text{opt}}(x) dx = \int_0^1 f(x) dx = 1$,

$$\begin{aligned} \int_0^1 (1-x)f_{\text{opt}}(x) dx - \int_0^1 (1-x)f(x) dx &= \int_0^{x_0} x[f(x) - f_{\text{opt}}(x)] dx + \int_{x_0}^1 x[f(x) - f_{\text{opt}}(x)] dx \\ &< \int_0^{x_0} x_0[f(x) - f_{\text{opt}}(x)] dx + \int_{x_0}^1 x_0[f(x) - f_{\text{opt}}(x)] dx \\ &= x_0 \int_0^1 [f(x) - f_{\text{opt}}(x)] dx = 0 \end{aligned}$$

That is $\int_0^1 (1-x)f(x)dx > \int_0^1 (1-x)f_{\text{opt}}(x)dx = \alpha$, this contradicts the constraint $\int_0^1 (1-x)f(x)dx = \alpha$. Therefore, $f_{\text{opt}}(x)$ is the optimal solution of (56), and the optimal solution is unique. \square

From Theorems 12 and 13, we can get that

Corollary 8. Let $M_{\text{RIM}}(\alpha) = \max_{0 \leq x \leq 1} |F''(f(x))f'(x)|$ is the objective function of orness level α for (56), then $M_{\text{RIM}}(\alpha) = M_{\text{RIM}}(1 - \alpha)$, which means $V_{\text{RIM}}(\alpha)$ is symmetrical for α at $\alpha = \frac{1}{2}$.

Theorem 14. The objective value of the minimax problem (56), $M_{\text{RIM}}(\alpha) = \max_{0 \leq x \leq 1} |F''(f(x))f'(x)|$ decreases for $\alpha \in (0, \frac{1}{2}]$, and decreases for $\alpha \in [\frac{1}{2}, 1)$. $M_{\text{RIM}}(\alpha)$ reaches its possible minimum value 0 at $\alpha = \frac{1}{2}$.

Proof. From (61), with the unique optimal solution of (57) and (58), the objective function value of the minimax problem (56) is

$$M_{\text{RIM}}(\alpha) = \max_{0 \leq x \leq 1} |F''(f_{\text{opt}}(x))f'_{\text{opt}}(x)| = |\lambda_1| \tag{71}$$

From Corollary 7, when $\alpha = \frac{1}{2}$, $\lambda_1 = 0$, $M_{\text{RIM}}(\alpha) = 0$. From Theorem 10, λ_1 decreases with orness level α , so $\lambda_1 > 0$ for $\alpha \in (0, \frac{1}{2})$, $\lambda_1 < 0$ for $\alpha \in (\frac{1}{2}, 1)$, that $M_{\text{RIM}}(\alpha) = |\lambda_1|$ decreases for $\alpha \in (0, \frac{1}{2})$, and it increases for $\alpha \in (\frac{1}{2}, 1)$, $M_{\text{RIM}}(\alpha)$ reaches its possible minimum value 0 at $\alpha = \frac{1}{2}$. \square

5. The solutions of two special cases

Here we will discuss the solution expression of two special cases of (8) and (42) with $F(x) = x \ln(x)$ and $F(x) = x^2$, which correspond to the maximum entropy problem and the minimum variance problem respectively. The solutions of these two problems in OWA operator case were discussed separately [12,14,15,26,31,35]. The results of this paper can be seen as an extension of them and an effort of trying to connect these two problems together [33]. Most properties of these two kinds problems for OWA operator and RIM quantifier [24,26,27,31] can be deduced directly from the conclusions of this general model. Similar to the conclusions in Sections 3 and 4, the relationship between OWA operator and RIM quantifier can also be observed and compared.

For the optimization problems (8), with $\sum_{i=1}^n w_i = 1$, $\sum_{i=1}^n (aF(w_i) + bw_i) = a\sum_{i=1}^n F(w_i) + b$, similarly, for (42), with $\int_0^1 f(x)dx = 1$, $\int_0^1 (aF(f(x)) + bf(x))dx = a\int_0^1 F(f(x))dx + b$, $F(x)$ and $aF(x) + bx(a > 0)$ have the same optimal solutions for (8) and (42), so the parameters $a(a > 0), b$ in $aF(x) + bx$ of (8) and (42) can be neglected in some way. Please also note that the case of $F(x) = x \ln(x)$ is a maximum problem with an additional negative sign in the objective function.

5.1. Case 1: $F(x) = x \ln(x)$

Problem (8) becomes the maximum entropy OWA (MEOWA) operator problem (72).

$$\begin{aligned} \max \quad & - \sum_{i=1}^n w_i \ln w_i \\ \text{s.t.} \quad & \sum_{i=1}^n \frac{n-i}{n-1} w_i = \alpha, \quad 0 < \alpha < 1 \\ & \sum_{i=1}^n w_i = 1 \end{aligned} \tag{72}$$

As $F'(x) = 1 + \ln(x)$, $g(x) = (F')^{-1}(x) = e^{x-1}$, from (10) and (11), the optimal solution is

$$w_i = e^{\frac{n-i}{n-1}\lambda_1 + \lambda_2 - 1} \quad i = 1, 2, \dots, n.$$

Let $\frac{w_i}{w_{i+1}} = e^{\frac{1}{n-1}\lambda_1} = \frac{1}{q}$, the solution can be expressed in geometric form as [8,31]

$$w_i = \frac{q^{i-1}}{\sum_{j=0}^{n-1} q^j} \quad (73)$$

where q is the unique positive real root of the following equation:

$$(n-1)\alpha q^{n-1} + \sum_{i=2}^n ((n-1)\alpha - i + 1)q^{n-i} = 0$$

With the relationship between λ_1 and q , from the conclusions of Section 3.1, we can also get the same conclusions about the MEOWA operator that were once obtained in [31].

Corresponding to (25), the solution equivalence minimax problem of (72) is

$$\begin{aligned} \min \quad & \left\{ \max_{1 \leq i \leq n-1} |\ln(w_i) - \ln(w_{i+1})| \right\} \\ \text{s.t.} \quad & \sum_{i=1}^n \frac{n-i}{n-1} w_i = \alpha, \quad 0 < \alpha < 1 \\ & \sum_{i=1}^n w_i = 1. \end{aligned} \quad (74)$$

Furthermore, the solution equivalence minimax problem (74) can be replaced with a more simple minimax ratio problem (75) without the absolute value operator. Similar to the minimax disparity problem (24), we can call (75) as minimax ratio problem, which minimizes the maximum of the ratios between two adjacent weight elements.

Theorem 15. *The solution of the maximum entropy problem (72) is also equivalent to the following minimax problem solution:*

$$\begin{aligned} \min \quad & \left\{ \max_{1 \leq i \leq n-1} \frac{w_i}{w_{i+1}} \right\} \\ \text{s.t.} \quad & \sum_{i=1}^n \frac{n-i}{n-1} w_i = \alpha, \quad 0 < \alpha < 1 \\ & \sum_{i=1}^n w_i = 1 \\ & w_i \geq 0, \quad i = 1, 2, \dots, n \end{aligned} \quad (75)$$

Proof. We will show that the optimal solution of maximum entropy OWA operator problem (72) with $W^{\text{opt}} = (w_1^{\text{opt}}, w_2^{\text{opt}}, \dots, w_n^{\text{opt}})$ in (73) is also the unique optimal solution of (75).

Let $W = (w_1, w_2, \dots, w_n)$ be a OWA weight vector such that $W \neq W^{\text{opt}}$, and

$$\max_{1 \leq i \leq n-1} \frac{w_i}{w_{i+1}} \leq \max_{1 \leq i \leq n-1} \frac{w_i^{\text{opt}}}{w_{i+1}^{\text{opt}}} = \frac{1}{q} \quad (76)$$

and the constraint $\sum_{i=1}^n w_i = 1$. We will prove that w_i does not satisfy the constraint $\sum_{i=1}^n \frac{n-i}{n-1} w_i = \alpha$.

We claim that $w_n > w_n^{\text{opt}}$, otherwise, $w_n \leq w_n^{\text{opt}}$. As

$$w_i = w_n \prod_{k=i}^{n-1} \frac{w_k}{w_{k+1}}, \quad w_i^{\text{opt}} = \frac{w_n^{\text{opt}}}{q^{n-i}} \quad \text{for } i = 1, 2, \dots, n \quad (77)$$

considering (76), we have $\prod_{k=i}^{n-1} \frac{w_k}{w_{k+1}} \leq \frac{1}{q^{n-i}}$, so $w_i \leq w_i^{\text{opt}}$, $i = 1, 2, \dots, n$. Since $\sum_{i=1}^n w_i = \sum_{i=1}^n w_i^{\text{opt}} = 1$, we must have $w_i = w_i^{\text{opt}}$ ($i = 1, 2, \dots, n$). This contradicts to $W \neq W^{\text{opt}}$. Thus, we have proved that $w_n > w_n^{\text{opt}}$.

Similarly, with

$$w_i = w_1 \left/ \prod_{k=1}^{i-1} \frac{w_k}{w_{k+1}} \right., \quad w_i^{\text{opt}} = w_1^{\text{opt}} q^{i-1}, \quad i = 1, 2, \dots, n \tag{78}$$

we can also prove that $w_1 < w_1^{\text{opt}}$. Combining these with (77) and (76), we can find k , $1 < k \leq n$, such that

$$\begin{cases} w_i \leq w_i^{\text{opt}} & i = 1, 2, \dots, k-1 \\ w_i > w_i^{\text{opt}} & i = k, k+1, \dots, n \end{cases}$$

With the same proof method as Theorem 6 after (40), we can obtain that $\sum_{i=1}^n \frac{n-i}{n-1} w_i < \sum_{i=1}^n \frac{n-i}{n-1} w_i^{\text{opt}} = \alpha$. Therefore, $W^{\text{opt}} = (w_1^{\text{opt}}, w_w^{\text{opt}}, \dots, w_n^{\text{opt}})$ is the unique optimal solution of (75). \square

Remark 2. Similarly, the objective function in (75) can also be replaced with $\min \left\{ \max_{1 \leq i \leq n-1} \frac{w_{i+1}}{w_i} \right\}$.

For Problem (42), when $F(x) = x \ln(x)$, it becomes the maximum entropy RIM quantifier problem (79).

$$\begin{aligned} \max \quad & - \int_0^1 f(x) \ln f(x) dx \\ \text{s.t.} \quad & \int_0^1 (1-x)f(x) dx = \alpha, \quad 0 < \alpha < 1 \\ & \int_0^1 f(x) dx = 1. \end{aligned} \tag{79}$$

The solution and its properties were discussed in [27].

With $F'(x) = 1 + \ln(x)$, $(F')^{-1}(x) = e^{x-1}$, as $e^{x-1} > 0$, from (43) and (44), the optimal solution is:

$$f(x) = e^{\lambda_1 x + \lambda_2 - 1}$$

With the constraints of (79), the optimal solution can be expressed as

$$f(x) = \frac{\lambda_1 e^{\lambda_1 x}}{e^{\lambda_1} - 1}$$

where λ_1 is the root of the equation $\frac{e^{\lambda_1} - \lambda_1 - 1}{\lambda_1(e^{\lambda_1} - 1)} = \alpha$.

Remark 3. In [27], the solution of the maximum entropy RIM quantifier is expressed as $f(x) = \frac{\lambda e^{\lambda(1-x)}}{e^\lambda - 1}$, where λ is the root of the equation $\frac{\lambda e^\lambda - e^\lambda + 1}{\lambda(e^\lambda - 1)} = \alpha$. It can be easily verified that these two solution forms are equivalent with $\lambda = -\lambda_1$.

As $F''(x) = \frac{1}{x}$, corresponding to (56), the solution equivalence minimax problem of (79) is:

$$\begin{aligned} \min \quad & \left\{ \max_{0 \leq x \leq 1} \left| \frac{f'(x)}{f(x)} \right| \right\} \\ \text{s.t.} \quad & \int_0^1 (1-x)f(x) dx = \alpha, \quad 0 < \alpha < 1 \\ & \int_0^1 f(x) dx = 1 \\ & f(t) > 0 \end{aligned} \tag{80}$$

Furthermore, as in the discrete case of OWA operator, (80) can be replaced with a problem without absolute value operator.

Theorem 16. *The solution of the maximum entropy problem (79) is also equivalent to the following minimax problem solution without the absolute value operator.*

$$\begin{aligned}
\min \quad & \left\{ \max_{0 \leq x \leq 1} \frac{f'(x)}{f(x)} \right\} \\
\text{s.t.} \quad & \int_0^1 (1-x)f(x)dx = \alpha, \quad 0 < \alpha < 1 \\
& \int_0^1 f(x)dx = 1 \\
& f(t) > 0
\end{aligned} \tag{81}$$

Proof. We will show that the optimal solution of maximum entropy OWA operator problem (79), $f_{\text{opt}}(x)$ is also the unique optimal solution of (81).

Let $f(x)$ be a RIM quantifier such that $f(x) \neq f_{\text{opt}}$, and

$$\max_{0 \leq x \leq 1} \frac{f'(x)}{f(x)} \leq \max_{0 \leq x \leq 1} \frac{f'_{\text{opt}}(x)}{f_{\text{opt}}(x)} = \lambda_1 \tag{82}$$

and the constraint $\int_0^1 f(x)dx = 1$. We will prove that $f(x)$ does not satisfy the constraint $\int_0^1 (1-x)f(x)dx = \alpha$.

We claim that $f(0) > f_{\text{opt}}(0)$, otherwise, $f(0) \leq f_{\text{opt}}(0)$.

Let $R(x) = \ln(f(x))$, $R_{\text{opt}}(x) = \ln(f_{\text{opt}}(x))$, then $R(0) \leq R_{\text{opt}}(0)$, and $R'(x) = \frac{f'(x)}{f(x)}$, furthermore, for $x \in [0, 1]$, $R'_{\text{opt}}(x) = \frac{f'_{\text{opt}}(x)}{f_{\text{opt}}(x)} = \lambda_1$. Considering (82),

$$R'(x) \leq R'_{\text{opt}}(x) = \lambda_1 \quad \forall x \in [0, 1] \tag{83}$$

As in Case 2 of the Theorem 13 proof, it can be proved that there exists x_0 , which makes

$$\begin{cases} R(x) \leq R_{\text{opt}}(x) & \forall t \in [0, x_0] \\ R(x) > R_{\text{opt}}(x) & \forall t \in (x_0, 1] \end{cases} \tag{84}$$

thus

$$\begin{cases} f(x) \leq f_{\text{opt}}(x) & \forall t \in [0, x_0] \\ f(x) > f_{\text{opt}}(x) & \forall t \in (x_0, 1] \end{cases} \tag{85}$$

and $\int_0^1 (1-x)f(x)dx < \int_0^1 (1-x)f_{\text{opt}}(x)dx = \alpha$ at last, which contradicts the constraint $\int_0^1 (1-x)f(x)dx = \alpha$. Therefore, $f_{\text{opt}}(x)$ is the unique optimal solution of (81). \square

5.2. Case 2: $F(x) = x^2$

Problem (8) becomes the alternative form of the minimum variance problems for OWA operator [15]:

$$\begin{aligned}
\min \quad & D^2(W) = \frac{1}{n} \sum_{i=1}^n w_i^2 - \frac{1}{n^2} \\
\text{s.t.} \quad & \sum_{i=1}^n \frac{n-i}{n-1} w_i = \alpha, \quad 0 < \alpha < 1 \\
& \sum_{i=1}^n w_i = 1, \\
& w_i \geq 0, \quad i = 1, 2, \dots, n.
\end{aligned} \tag{86}$$

As $F'(x) = 2x$, $(F')^{-1}(x) = \frac{x}{2}$, the optimal solution is:

$$w_i = \begin{cases} \frac{\frac{n-i}{n-1}\lambda_1 + \lambda_2}{2} & \text{if } \frac{\frac{n-i}{n-1}\lambda_1 + \lambda_2}{2} > 0 \\ 0 & \text{otherwise} \end{cases} \tag{87}$$

with

$$\begin{cases} \sum_{i=1}^n \frac{n-i}{n-1} w_i - \alpha = 0 \\ \sum_{i=1}^n w_i - 1 = 0 \end{cases} \tag{88}$$

We will discuss the determination of $W = (w_1, w_2, \dots, w_n)$ in different cases.

Case 1: If $\alpha \geq \frac{1}{2}$. The OWA operator weight vector has the form $W = (w_1, w_2, \dots, w_m, 0, 0, \dots, 0)$, where m is the nonzero elements of W . Observing that $m = 1$ corresponds to the unique case $W^* = (1, 0, \dots, 0)$ of $\alpha = 1$, we will assume $m \geq 2$ in the following. From (88), it can be obtained that

$$\begin{cases} \lambda_1 = \frac{12(n-1)(-2n+2zn+1-2\alpha+m)}{m(m-1)(m+1)} \\ \lambda_2 = \frac{-4(-6n^2+6zn^2-9zn+6n+6mn-3mzn-1+3m\alpha+3\alpha-2m^2-3m)}{m(m-1)(m+1)} \end{cases} \tag{89}$$

With $w_m = \frac{n-m}{n-1} \lambda_1 + \lambda_2 > 0$ and $\frac{n-(m+1)}{n-1} \lambda_1 + \lambda_2 \leq 0$ ($m \geq 2$), we can get that

$$\frac{3n-m-1}{3(n-1)} > \alpha \geq \frac{3n-m-2}{3(n-1)} \tag{90}$$

This is the orness interval α lies in when W has m nonzero elements for $\alpha \geq \frac{1}{2}$. Observing that when $m = 2, 3, \dots, n$, α only changes in $[\frac{2}{3}, 1)$, we can get a division of $[\frac{2}{3}, 1)$ for $m = 1, 2, \dots, n$. It is obvious that when all the w_i s are nonzero, we have $m = n$. From (90), for $\alpha \in [\frac{1}{2}, \frac{2}{3}]$, it certainly has $m = n$. Thus, with given orness level $\alpha \in [\frac{1}{2}, 1)$, m can be determined as

$$m = \begin{cases} [3n - 3(n-1)\alpha - 1] & \alpha \in (\frac{2}{3}, 1) \\ n & \alpha \in [\frac{1}{2}, \frac{2}{3}] \end{cases} \tag{91}$$

Case 2: If $\alpha < \frac{1}{2}$. The OWA operator weight vector has the form $W = (0, 0, \dots, 0, w_{n-m+1}, w_{n-m+2}, \dots, w_n)$. m can be determined in a similar way

$$m = \begin{cases} [3(n-1)\alpha + 2] & \alpha \in (0, \frac{1}{3}) \\ n & \alpha \in [\frac{1}{3}, \frac{1}{2}) \end{cases} \tag{92}$$

Combining (87), (91) and (92), the solution is the maximum spread equidifferent OWA operator exactly [26]:

Algorithm 1

Step 1: Determine m with (93).

$$m = \begin{cases} [3\alpha(n-1) + 2] & \text{if } 0 < \alpha < \frac{1}{3} \\ n & \text{if } \frac{1}{3} \leq \alpha \leq \frac{2}{3} \\ [3n - 3\alpha(n-1) - 1] & \text{if } \frac{2}{3} < \alpha < 1. \end{cases} \tag{93}$$

Step 2: Determine d with (94).

$$d = \begin{cases} \frac{6(2\alpha-2n\alpha+m-1)}{m(m^2-1)} & \text{if } 0 < \alpha < \frac{1}{3} \\ \frac{6(1-2\alpha)}{n(n+1)} & \text{if } \frac{1}{3} \leq \alpha \leq \frac{2}{3} \\ \frac{6(2\alpha-2n\alpha+2n-m-1)}{m(m^2-1)} & \text{if } \frac{2}{3} < \alpha < 1 \end{cases} \tag{94}$$

Step 3: Determine $W = (w_1, w_2, \dots, w_n)$ with (95).

$$\begin{aligned}
 \text{Case 1: } & 0 < \alpha < \frac{1}{3}, \quad w_i = \begin{cases} 0 & \text{if } 1 \leq i \leq n - m \\ \frac{-dm^2 + dm + 2}{2m} + (i - n + m - 1)d & \text{if } n - m + 1 \leq i \leq n \end{cases} \\
 \text{Case 2: } & \frac{1}{3} \leq \alpha \leq \frac{2}{3}, \quad w_i = \frac{-dn^2 + dn + 2}{2n} + (i - 1)d, \quad i = 1, 2, \dots, n \\
 \text{Case 3: } & \frac{2}{3} < \alpha < 1, \quad w_i = \begin{cases} \frac{-dm^2 + dm + 2}{2m} + (i - 1)d & \text{if } 1 \leq i \leq m \\ 0 & \text{if } n - m + 1 \leq i \leq n \end{cases}
 \end{aligned} \tag{95}$$

Similar to the maximum entropy problem, the properties of minimum variance problem that was proposed in [26] can also be obtained from the discussion of Section 3.1. The similarities between the maximum entropy and the minimum variance problems can be understood naturally as they are just two special cases of the general problem (8).

With Theorem 6 and $F(x) = x^2$, the solution equivalence minimax problem of (86) and the minimax disparity problem (24) can be verified with an additional constant 2 in (24)'s objective function, which improves the complicated process of the dual linear programming method [30].

For the RIM quantifier case, when $F(x) = x^2$, problem (42) becomes the minimum variance RIM operator problem [24]:

$$\begin{aligned}
 \min \quad & D^2(f(x)) = \int_0^1 f^2(x) dx - \left(\int_0^1 f(x) dx \right)^2 \\
 \text{s.t.} \quad & \int_0^1 (1-x)f(x) dx = \alpha \quad 0 < \alpha < 1 \\
 & \int_0^1 f(x) dx = 1 \\
 & f(x) \geq 0
 \end{aligned} \tag{96}$$

Problem (96) can be solved with the optimal control technique. The solution is expressed as an equidifferent RIM quantifier. Some properties of it were discussed [24].

As $F'(x) = 2x$, $(F')^{-1}(x) = \frac{x}{2}$, the optimal solution is:

$$f(x) = \begin{cases} \frac{\lambda_1 x + \lambda_2}{2} & \text{if } \frac{\lambda_1 x + \lambda_2}{2} \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

with

$$\begin{cases} \int_0^1 x f(x) dx = 1 - \alpha \\ \int_0^1 f(x) dx = 1 \end{cases}$$

This is just the equidifferent RIM quantifier [24]. The optimal solution is

$$\begin{aligned}
 \text{Case 1: } & \text{If } 0 < \alpha \leq \frac{1}{3}, f(x) = \begin{cases} 0 & 0 \leq x \leq 1 - 3\alpha \\ \frac{2(x-1+3\alpha)}{9\alpha^2} & 1 - 3\alpha < x \leq 1 \end{cases} \\
 \text{Case 2: } & \text{If } \frac{1}{3} < \alpha \leq \frac{2}{3}, f(x) = (6 - 12\alpha)x + (6\alpha - 2), \quad 0 \leq x \leq 1 \\
 \text{Case 3: } & \text{If } \frac{2}{3} < \alpha < 1, f(x) = \begin{cases} \frac{2(3-3\alpha-x)}{9(1-\alpha)^2} & 0 \leq x \leq 3 - 3\alpha \\ 0 & 3 - 3\alpha < x \leq 1 \end{cases}
 \end{aligned} \tag{97}$$

As $F''(x) = 2$, corresponding to (56), the solution equivalence minimax problem of (96) is formulated with the constant 2 being omitted:

$$\begin{aligned}
 \min \quad & \left\{ \max_{0 \leq x \leq 1} |f'(x)| \right\} \\
 \text{s.t.} \quad & \int_0^1 (1-x)f(x)dx = \alpha, \quad 0 < \alpha < 1 \\
 & \int_0^1 f(x) = 1 \\
 & f(x) \geq 0
 \end{aligned} \tag{98}$$

Remark 4. Similar to the conclusion of [Theorem 15](#) and [Remark 2](#) for the maximum entropy OWA operator case, the solution equivalence can be kept without the absolute value operates in the minimax problems (24) and (98) when the orness level $\alpha \in [\frac{1}{3}, \frac{2}{3}]$.

For these two cases, the RIM quantifier membership function can be obtained with $Q(x) = \int_0^x f(t)dt$ directly.

6. Another view of the problems solutions and some discussions

From above, we can see that with given a strictly convex function $F(x)$, a parameterized OWA operator or RIM quantifier family with orness level as its control parameter can always be obtained, which is the unique optimal solution of (8) or (42). The OWA weight vector or the RIM quantifier generating function is determined by the increasing function $g(x) = (F')^{-1}(x)$. On the other hand, with an increasing function $g(x)$, there also exists a strictly convex function $F(x)$, that makes $g(x) = (F')^{-1}(x)$. The OWA operator generated with (10), (11) is the unique optimal solutions of (8), and the RIM quantifier generating function determined by (43) and (44) is the unique optimal solutions of (42). This gives us a very broad way to obtain the parameterized OWA operator or RIM quantifier families with different orness levels. The aggregation values of these OWA operator or RIM quantifiers for any aggregated set are also consistently (monotonically) changes with the orness level. Furthermore, we can control the relationships between the adjacent elements of OWA operator weight vector or the shape of the RIM quantifier function by selecting $g(x)$ appropriately. This means we can not only make the OWA operator or RIM quantifier based aggregation represent the preference information, but also can incorporate the background or problem structure information with $g(x)$.

With $(F')^{-1}(x) = g(x)$, the expression of $F(x)$ can be easily obtained. We can observe how the form of $g(x)$ affects the OWA operator or the RIM quantifier under given orness level. As discussed at the beginning of Section 5, for the OWA operator problems (8) and (42), $F(x)$ and $aF(x) + bx(a > 0)$ have the same optimal solution. Similarly, for the RIM quantifier problems (42) and (56), $F(x)$ and $aF(x) + bx(a > 0)$ also have the same optimal solution. Both of these two cases imply that for the optimal solutions determined by (10), (11), or (43), (44), $a\lambda_1 + b$ or $a\lambda_2 + b(a > 0)$ can be replaced with λ_1 and λ_2 , which means the constants a, b can be neglected in some way.

Table 1 gives some examples of commonly used function forms for $g(x)$ and $F(x)$ respectively. Example I corresponds to the maximum entropy OWA operator (RIM quantifier) problem, and Example II corresponds to the minimum variance OWA operator (RIM quantifier) problem that were discussed previously. Example III and IV can be solved analytically with the method similar to that of Example II. Example II, III, IV are the special cases of Example V. An alternative analytic solution of Example V was proposed by Majlender [33] for OWA operator. For simplification, their analytical solutions forms are omitted. For OWA operator, except the proposed analytical solution method with $g(x)$ in (10) and (11), considering $F(x)$ in the objective function of (8), these problems can also be solved with the optimization software such as Lingo or Maple.² Fig. 1 shows the OWA operator solution under orness levels $\alpha = 0.2, 0.5, 0.8$ for these five cases with $n = 20$ respectively.

Unlike the OWA operator case of (8), the analytical solutions is complicated sometimes. For any strictly convex function $F(x)$ or monotonic increasing function $g(x)$, the analytical solutions of (43) or (42) is usually

² www.lindo.com, www.maplesoft.com.

Table 1
Some examples of $g(x)$ and the possible corresponding $F(x)$

	$g(x)$ in (10) and (43)	$F(x)$ in (8) and (42)
I	$g(x) = e^x$	$F(x) = x \ln(x)$
II	$g(x) = x$	$F(x) = x^2$
III	$g(x) = \sqrt{x}$	$F(x) = x^3$
IV	$g(x) = x^2$	$F(x) = x^{\frac{3}{2}}$
V	$g(x) = x^\alpha (\alpha > 0)$	$F(x) = x^{1+\frac{1}{\alpha}}$
VI	$g(x) = \ln(x)$	$F(x) = e^x$

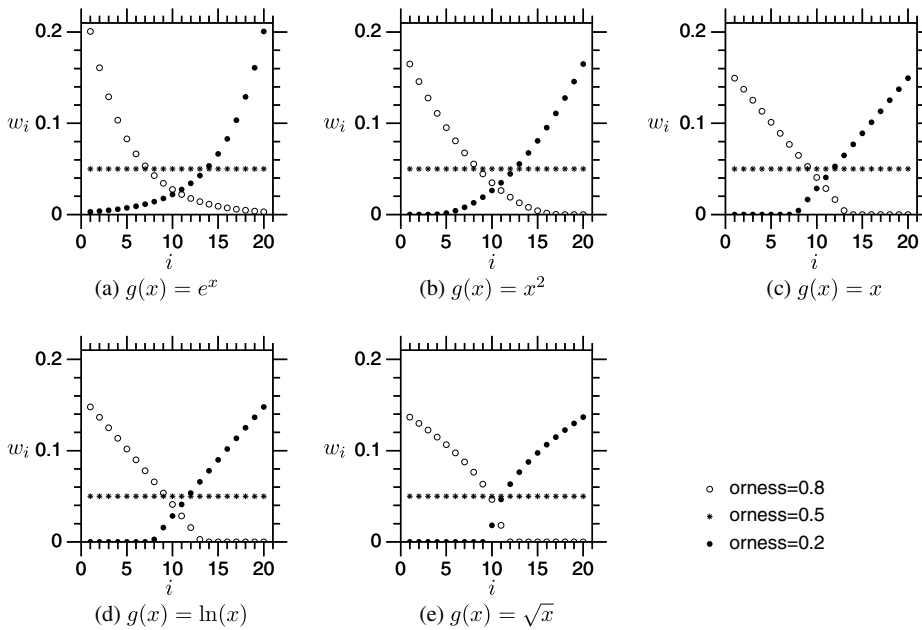


Fig. 1. The OWA operators for different forms of $g(x)$ with $n = 20$.

a continuous function. Table 2 shows the RIM quantifier generating functions under orness levels $\alpha = 0.2, 0.5, 0.8$ for these five cases, with their plots shown in Fig. 2.

From Figs. 1, 2 and Corollary 3, 7, for the optimal solution of (8) or (42), when orness level $\alpha = \frac{1}{2}$, we will always have $W = W_A = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$, or $f(x) = 1$ (which means $Q(x) = Q_A(x) = x$). And for any

Table 2
The RIM quantifier generating functions for different forms of $g(x)$

	$\alpha = 0.8$	$\alpha = 0.5$	$\alpha = 0.2$
$g(x) = e^x$	$f(x) = 4.841e^{-4.801x}$	$f(x) = 1$	$f(x) = 0.398e^{4.801x}$
$g(x) = x^2$	$f(x) = \begin{cases} \frac{15(5x-4)^2}{64} & x \in [0, \frac{4}{5}] \\ 0 & x \in [\frac{4}{5}, 1] \end{cases}$	$f(x) = 1$	$f(x) = \begin{cases} 0 & x \in [0, \frac{1}{5}] \\ \frac{15(5x-1)^2}{64} & x \in [\frac{1}{5}, 1] \end{cases}$
$g(x) = x$	$f(x) = \begin{cases} -\frac{50}{9}x + \frac{10}{3} & x \in [0, \frac{3}{5}] \\ 0 & x \in [\frac{3}{5}, 1] \end{cases}$	$f(x) = 1$	$f(x) = \begin{cases} 0 & x \in [0, \frac{2}{5}] \\ \frac{50}{9}x - \frac{20}{9} & x \in [\frac{2}{5}, 1] \end{cases}$
$g(x) = \ln x^a$	$f(x) = \begin{cases} \ln(-33.71x + 17.54) & x \in [0, 0.49] \\ 0 & x \in [0.49, 1] \end{cases}$	$f(x) = 1$	$f(x) = \begin{cases} 0 & x \in [0, 0.51] \\ \ln(33.71x - 16.16) & x \in [0.51, 1] \end{cases}$
$g(x) = \sqrt{x}$	$f(x) = \begin{cases} 3\sqrt{1-2x} & x \in [0, \frac{1}{2}] \\ 0 & x \in [\frac{1}{2}, 1] \end{cases}$	$f(x) = 1$	$f(x) = \begin{cases} 0 & x \in [0, \frac{1}{2}] \\ 3\sqrt{2x-1} & x \in [\frac{1}{2}, 1] \end{cases}$

^a The coefficients of $f(x)$ for $g(x) = \ln x$ are given numerically with the solution of nonlinear equations.

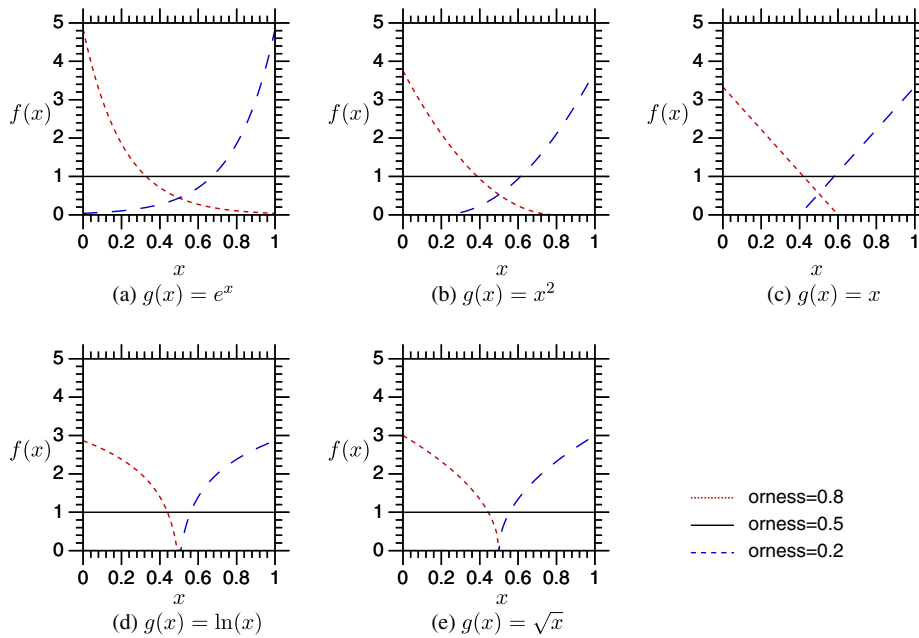


Fig. 2. The RIM quantifiers for different forms of $g(x)$.

$X = (x_1, x_2, \dots, x_n)$, $F_W(X) = F_Q(X) = \frac{1}{n} \sum_{i=1}^n x_i$. The aggregation becomes the ordinary arithmetic mean (average operator). When $\alpha \rightarrow 0$, $W \rightarrow W_*$, $F_W(X) \rightarrow \min_{1 \leq i \leq n} \{x_i\}$, or $Q \rightarrow Q_*$, $F_Q(X) \rightarrow \min_{1 \leq i \leq n} \{x_i\}$. When $\alpha \rightarrow 1$, $W \rightarrow W^*$, $F_W(X) \rightarrow \max_{1 \leq i \leq n} \{x_i\}$ or $Q \rightarrow Q^*$, $F_Q(X) \rightarrow \max_{1 \leq i \leq n} \{x_i\}$. So the solution of (8) or (42) can be seen as a parameterized extension of the ordinary arithmetic mean ranging between maximum and minimum in OWA operator and RIM quantifier forms respectively. The forms of the solutions for (8) or (42) are determined by the strictly convex function $F(x)$. The relationships between the elements of OWA operator or the shape of the RIM quantifier generating function (which determines the membership function) can be observed from the shape of $g(x) = (F')^{-1}(x)$ intuitively.

Comparing the solutions of these problems for OWA operator and RIM quantifier respectively, the optimal solutions for RIM quantifier are usually more simple and intuitive than that of the OWA operator. The RIM quantifier solutions are also dimension independent in the aggregation process. They can be interpreted with natural language terms, and can be connected with the computing with words (CW) paradigm potentially [21,22,44,58]. However, if they are used to generate the OWA weight vector, the weight elements usually are not as accurate as that of the direct OWA generating methods unless the elements number approaches infinity.

7. Conclusions

The paper proposes a general model to obtain the OWA operator with orness as its control parameter. This general model includes the maximum entropy OWA operator and minimum variance OWA operator as special cases. Some properties of its solution are discussed. The solution equivalence to the minimax problem are proved, which is also a generalization of the solution equivalence for the minimum variance and minimax disparity problems. Then, these results are extended to the RIM quantifier case, which corresponds to the OWA operator in continuous form. A general model to obtain the parameterized RIM quantifiers of given orness level is proposed, with the property discussions and the solution equivalence proof to the corresponding minimax problem. With the analytical optimal solution expression of these two kinds problems, the relationship between the OWA operator vector elements or the shape of the RIM quantifier membership function can be observed intuitively. We can not only use the OWA operator or RIM quantifier to get aggregation results consistent to the preference information (orness level), but also can make the obtained optimal OWA operator or

RIM quantifier obey some specific function forms by considering the structure or the background information of the aggregation problem. The parameterized OWA operator and RIM quantifier families of some commonly used function forms are provided for possible applications. Whatever the forms of these optimal solutions, they can always be seen as a parameterized extension of the arithmetic mean between the maximum and minimum. Comparing with the case of the OWA operators, the parameterized RIM quantifier families are dimension free in aggregation and can be connected with natural language interpretation.

Acknowledgements

The author is most grateful to Editor-in-Chief, Thierry Denoeux, the Associate Editor, and the anonymous reviewers for their many valuable comments and suggestions that improve the previous version of this paper.

References

- [1] B.S. Ahn, On the properties of OWA operator weights functions with constant level of orness, *IEEE Transactions on Fuzzy Systems* 14 (4) (2006) 511–515.
- [2] B.S. Ahn, Preference relation approach for obtaining OWA operators weights, *International Journal of Approximate Reasoning* 47 (2) (2008) 166–178.
- [3] G.R. Amin, Note on a preemptive goal programming method for aggregating OWA operator weights in group decision making, *Information Sciences* 177 (17) (2007) 3636–3638.
- [4] G.R. Amin, A. Emrouznejad, An extended minimax disparity to determine the OWA operator weights, *Computers and Industrial Engineering* 50 (3) (2006) 312–316.
- [5] S. Barro, A.J. Bugarín, P. Cariñena, F. Díaz-Hermida, A framework for fuzzy quantification models analysis, *IEEE Transactions on Fuzzy Systems* 11 (1) (2003) 89–99.
- [6] D. Ben-Arieh, Sensitivity of multi-criteria decision making to linguistic quantifiers and aggregation means, *Computers & Industrial Engineering* 48 (2) (2005) 289–309.
- [7] G. Bordogna, M. Fedrizzi, G. Pasi, A linguistic modeling of consensus in group decision making based on OWA operators, *IEEE Transactions on Systems, Man and Cybernetics, Part A* 27 (1) (1997) 126–132.
- [8] M. Carbonell, M. Mas, G. Mayor, On a class of monotonic extended OWA operators, in: *Proceedings of the Sixth IEEE International Conference on Fuzzy Systems, Barcelona, 1997*.
- [9] S. Chang, R.C. Wang, S. Wang, Applying fuzzy linguistic quantifier to select supply chain partners at different phases of product life cycle, *International Journal of Production Economics* 100 (2) (2006) 348–359.
- [10] V. Chankong, Y.Y. Haimes, *Mutiobjective Decision Making: Theory and Methodology*, Elsevier Science B.V., 1983.
- [11] F. Chiclana, F. Herrera, E. Herrera-Viedma, Integrating multiplicative preference relations in a multipurpose decision-making model based on fuzzy preference relations, *Fuzzy Sets and Systems* 122 (2) (2001) 277–291.
- [12] D. Filev, R.R. Yager, Analytic properties of maximum entropy OWA operators, *Information Sciences* 85 (1–3) (1995) 11–27.
- [13] D. Filev, R.R. Yager, On the issue of obtaining OWA operator weights, *Fuzzy Sets and Systems* 94 (2) (1998) 157–169.
- [14] R. Fullér, P. Majlender, An analytic approach for obtaining maximal entropy OWA operator weights, *Fuzzy Sets and Systems* 124 (1) (2001) 53–57.
- [15] R. Fullér, P. Majlender, On obtaining minimal variability OWA operator weights, *Fuzzy Sets and Systems* 136 (2) (2003) 203–215.
- [16] J.M. Galindo, J. Medina, J.C. Cubero, M.T. Garcia, Relaxing the universal quantifier of the division in fuzzy relational databases, *International Journal of Intelligent Systems* 16 (6) (2001) 713–742.
- [17] F. Herrera, E. Herrera-Viedma, Aggregation operators for linguistic weighted information, *IEEE Transactions on Systems, Man and Cybernetics, Part A* 22 (5) (1997) 646–656.
- [18] F. Herrera, E. Herrera-Viedma, J.L. Verdeev, Linguistic measures based on fuzzy coincidence for reaching consensus in group decision making, *International Journal of Approximate Reasoning* 16 (3–4) (1997) 309–334.
- [19] F. Herrera, E. Herrera-Viedma, J.L. Verdegay, A sequential selection process in group decision making with a linguistic assessment approach, *Information Science* 85 (4) (1995) 223–239.
- [20] E. Herrera-Viedma, O. Cordón, M. Luque, A.G. Lopez, A.M. Muñoz, A model of fuzzy linguistic IRS based on multi-granular linguistic information, *International Journal of Approximate Reasoning* 34 (2–3) (2003) 221–239.
- [21] J. Kacprzyk, S. Zadrozny, Computing with words in intelligent database querying: standalone and internet-based applications, *Information Sciences* 134 (1) (2001) 71–109.
- [22] J. Lawry, A methodology for computing with words, *International Journal of Approximate Reasoning* 28 (2–3) (2001) 51–89.
- [23] X. Liu, On the methods of decision making under uncertainty with probability information, *International Journal of Intelligent Systems* 19 (12) (2004) 1217–1238.
- [24] X. Liu, On the properties of equidifferent RIM quantifier with generating function, *International Journal of General Systems* 34 (5) (2005) 579–594.
- [25] X. Liu, Preference solution of probability decision making with RIM quantifiers, *International Journal of Intelligent Systems* 20 (12) (2005) 1253–1271.

- [26] X. Liu, On the properties of equidifferent OWA operator, *International Journal of Approximate Reasoning* 43 (1) (2006) 90–107.
- [27] X. Liu, On the properties of RIM quantifiers with maximum entropy, *International Journal of General Systems*, doi:10.1080/03081070701192675.
- [28] X. Liu, Reply to “Comments on the paper: on the properties of equidifferent OWA operator”, *International Journal of Approximate Reasoning* 43 (1) (2006) 113–115.
- [29] X. Liu, Some properties of the weighted OWA operator, *IEEE Transactions on Systems, Man and Cybernetics, Part B* 36 (1) (2006) 118–127.
- [30] X. Liu, The solution equivalence of minimax disparity and minimum variance problems for OWA operator, *International Journal of Approximate Reasoning* 45 (1) (2007) 68–81.
- [31] X. Liu, L. Chen, On the properties of parametric geometric OWA operator, *International Journal of Approximate Reasoning* 35 (2) (2004) 163–178.
- [32] X. Liu, S. Han, Orness and parameterized RIM quantifier aggregation with OWA operators: a summary, *International Journal of Approximate Reasoning* (2007), doi:10.1016/j.ijar.2007.05.006.
- [33] P. Majlender, OWA operators with maximal Rényi entropy, *Fuzzy Sets and Systems* 155 (3) (2005) 340–360.
- [34] P. Majlender, Comments on the paper: “On the properties of equidifferent OWA operator by Xinwang Liu”, *International Journal of Approximate Reasoning* 43 (1) (2006) 108–112.
- [35] M. O’Hagan, Aggregating template or rule antecedents in real-time expert systems with fuzzy set, in: P. Grove (Ed.), *Proceedings of the 22nd Annual IEEE Asilomar Conference on Signals, Systems, Computers, CA*, 1988.
- [36] S.P. Sethi, G.L. Thompson, *Optimal Control Theory: Applications to Management Science*, Martinus Nijhoff Publishing, Boston, 1981.
- [37] V. Torra, OWA operators in data modeling and reidentification, *IEEE Transactions on Fuzzy Systems* 12 (5) (2004) 652–660.
- [38] J. Wang, Y.I. Lin, A fuzzy multicriteria group decision making approach to select configuration items for software development, *Fuzzy Sets and Systems* 134 (3) (2003) 343–363.
- [39] Y. Wang, C. Parkan, A minimax disparity approach for obtaining OWA operator weights, *Information Sciences* 175 (1) (2005) 20–29.
- [40] Y. Wang, C. Parkan, A preemptive goal programming method for aggregating OWA operator weights in group decision making, *Information Sciences* 177 (2007) 1867–1877.
- [41] J. Wu, C.Y. Liang, Y.Q. Huang, An argument-dependent approach to determining OWA operator weights based on the rule of maximum entropy, *International Journal of Intelligent Systems* 22 (2) (2007) 209–221.
- [42] Z. Xu, An overview of methods for determining OWA weights, *International Journal of Intelligent Systems* 20 (8) (2005) 843–865.
- [43] Z. Xu, *Dependent OWA operators*, in: *Modeling Decisions for Artificial Intelligence*, Springer-Verlag, Berlin, 2006.
- [44] R. Yager, On the retranslation process in Zadeh’s paradigm of computing with words, *IEEE Transactions on Systems, Man and Cybernetics, Part B* 34 (2) (2004) 1184–1195.
- [45] R.R. Yager, On ordered weighted averaging aggregation operators in multicriteria decision making, *IEEE Transactions on Systems, Man and Cybernetics* 18 (1) (1988) 183–190.
- [46] R.R. Yager, Connectives and quantifiers in fuzzy sets, *Fuzzy Sets and Systems* 40 (1) (1991) 39–76.
- [47] R.R. Yager, Families of OWA operators, *Fuzzy Sets and Systems* 59 (2) (1993) 125–143.
- [48] R.R. Yager, Quantifier guided aggregation using OWA operators, *International Journal of Intelligent Systems* 11 (1) (1996) 49–73.
- [49] R.R. Yager, On the analytic representation of the Leximin ordering and its application to flexible constraint propagation, *European Journal of Operational Research* 102 (1) (1997) 176–192.
- [50] R.R. Yager, Decision making with fuzzy probability assessments, *IEEE Transactions on Fuzzy Systems* 7 (4) (1999) 462–467.
- [51] R.R. Yager, Fuzzy modeling for intelligent decision making under uncertainty, *IEEE Transactions on Systems, Man and Cybernetics, Part B* 30 (1) (2000) 60–70.
- [52] R.R. Yager, On the valuation of alternatives for decision-making under uncertainty, *International Journal of Intelligent Systems* 17 (7) (2002) 687–707.
- [53] R.R. Yager, OWA aggregation over a continuous interval argument with applications to decision making, *IEEE Transactions on Systems, Man and Cybernetics, Part B* 34 (5) (2004) 1952–1963.
- [54] R.R. Yager, An extension of the naive bayesian classifier, *Information Sciences* 176 (5) (2006) 577–588.
- [55] R.R. Yager, D.P. Filev, Parameterized and-like and or-like OWA operators, *International Journal of General Systems* 22 (3) (1994) 297–316.
- [56] R.R. Yager, D.P. Filev, Induced ordered weighted averaging operators, *IEEE Transactions on Systems, Man and Cybernetics, Part B* 29 (2) (1999) 141–150.
- [57] R.R. Yager, J. Kacprzyk, *The Ordered Weighted Averaging Operators—Theory and Applications*, Kluwer Academic Publishers, 1997.
- [58] L. Zadeh, Toward a perception-based theory of probabilistic reasoning with imprecise probabilities, *Journal of Statistical Planning and Inference* 105 (1) (2002) 233–264.
- [59] L.A. Zadeh, A computational approach to fuzzy quantifiers in natural languages, *Computing and Mathematics with Applications* 9 (1) (1983) 149–184.
- [60] S. Zadrozny, J. Kacprzyk, On tuning OWA operators in a flexible querying interface, in: *Proceedings of the 7th International Conference on Flexible Query Answering Systems, LNAI*, vol. 4027, Milan, 2006.