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Enumeration of graph embeddings

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Abstract

For a finite connected simple graph G, let Γ be a group of graph automorphisms of G. Two 2-cell embeddings $\imath:G\to S$ and $\jmath:G\to S$ of a graph G into a closed surface S (orientable or nonorientable) are *congruent* with respect to Γ if there are a surface homeomorphism $h:S\to S$ and a graph automorphism $\gamma\in\Gamma$ such that $h\circ\imath=\jmath\circ\gamma$. In this paper, we give an algebraic characterization of congruent 2-cell embeddings, from which we enumerate the congruence classes of 2-cell embeddings of a graph G into closed surfaces with respect to a group of automorphisms of G, not just the full automorphism group. Some applications to complete graphs are also discussed. As an orientable case, the oriented congruence of a graph G into orientable surfaces with respect to the full automorphism group of G was enumerated by Mull et al. (1988).

1. Introduction

Throughout this paper, by a graph G we always mean a finite connected simple graph with vertex set $V(G)$ and edge set $E(G)$. We denote the set of vertices adjacent to $v\in V(G)$ by $N(v)$ and call it the *neighborhood* of a vertex v. An *embedding* of a graph G into a closed surface S (orientable or nonorientable) is a homeomorphism $\imath:G\to S$ of G into S, where G is regarded as a one-dimensional simplicial complex in the space $\mathbb{R}^3$. If every component of $S-\imath(G)$, called a *region*, is homeomorphic to an open disk, then the embedding $\imath:G\to S$ is called a 2-*cell* embedding, and the regions are also called *faces* of the embedding. Note that if G is disconnected, no embedding into a connected surface will be a 2-cell embedding.

Two 2-cell embeddings $\imath:G\to S$ and $\jmath:G\to S$ of a graph G into an oriented closed surface S are said to be *equivalent* if there is an orientation-preserving surface homeomorphism $h:S\to S$ such that $h\circ\imath=\jmath$. This means that the surface

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homeomorphism h must preserve the labeling and direction of edges of the graph. Let $\mathrm{Aut}(G)$ denote the group of graph automorphisms of G. As a weaker notion of equivalence of embeddings, we say that two 2-cell embeddings $\imath: G \to S$ and $\jmath: G \to S$ of a graph G into a closed surface S (orientable or nonorientable) are *congruent with respect to a subgroup* Γ of $\mathrm{Aut}(G)$ if there are a surface homeomorphism $h: S \to S$ and a graph automorphism $\gamma \in \Gamma$ such that $h \circ \imath = \jmath \circ \gamma$. Here, the surface homeomorphism h need not be orientation-preserving even if the surface S is oriented. If two embeddings are congruent with respect to the full group $\mathrm{Aut}(G)$, we say just that they are *congruent*. If the surface S is oriented and the surface homeomorphism h preserves orientation, we call it *oriented congruence*. It is a congruence that is found, for example, in [13, 14].

Mull et al. [13] enumerated the oriented congruence classes of 2-cell embeddings of a graph into orientable surfaces. In this paper, we refine their method for enumeration of the congruence classes of 2-cell embeddings of a graph into orientable or nonorientable closed surfaces. It has possibly seemed that concrete enumeration of nonorientable embeddings would largely depend on essentially new methods, such as the overlap matrix, which was introduced by Mohar [12] and applied concretely by Chen et al. [3]. The present paper demonstrates that with sufficiently complete analysis, the existing enumerative machinery can also yield concrete results for nonorientable embeddings. Unless we explicitly say otherwise, from now on, all embeddings mean 2-cell embeddings, and all surfaces mean closed surfaces.

An *embedding scheme* (ρ, λ) for a graph G consists of a *rotation scheme* ρ which assigns a cyclic permutation ρ_v on $N(v)$ to each $v \in V(G)$ and a *voltage map* λ which assigns a value $\lambda(e)$ in $\mathbb{Z}_2 = \{1, -1\}$ to each $e \in E(G)$. The *voltage covering graph* G^λ derived from the voltage map λ on G has $V(G) \times \mathbb{Z}_2$ as its vertex set and $E(G) \times \mathbb{Z}_2$ as its edge set, so that an edge of G^λ joins a vertex (u, α) to $(v, \lambda(e)\alpha)$ for $e = uv \in E(G)$ and $\alpha \in \mathbb{Z}_2$. In the covering graph G^λ, a vertex (u, α) is denoted by u_α, and an edge (e, α) by e_α. Then the natural projection $p_\lambda: G^\lambda \to G$ is a 2-fold covering projection (see [7] for a precise construction of the covering projection $p_\lambda: G^\lambda \to G$). Stahl [16] showed that every embedding scheme for a graph G determines a 2-cell embedding of G into a surface S (orientable or nonorientable), and every 2-cell embedding of G into a surface S is determined by such a scheme. The orientability of S can be detected by looking at the voltage assignment of cycles of G. In fact, S is orientable if and only if each cycle of G is λ-trivial, that is, the number of edges e with $\lambda(e) = -1$ is even in any cycle of G. In particular, every 2-cell embedding of G into an orientable surface can be determined by an embedding scheme (ρ, λ) with $\lambda(e) = 1$ for each $e \in E(G)$.

Let (ρ, λ) be an embedding scheme for a graph G. The derived rotation scheme ρ^λ for the voltage covering graph G^λ is defined by lifting ρ_u (say ρ_u^1) to u_1 and lifting ρ_u^{-1} to u_{-1} for each $u \in V(G)$, i.e., for $e = uv \in E(G)$ and $u_\alpha \in p_\lambda^{-1}(u)$, where $\alpha \in \{1, -1\}$, let $w = (\rho_u)^\alpha(v) \in N(u)$ and $d = uw \in E(G)$. Then, for $v_{\lambda(e)\alpha} \in N(u_\alpha)$

$$(\rho^\lambda)_{u_\alpha}(v_{\lambda(e)\alpha}) = w_{\lambda(d)\alpha}.$$

The rotation scheme ρ^λ determines an embedding $\tilde{\imath}: G^\lambda \to \tilde{S}$ of G^λ into an *orientable* surface $\tilde{S}$ by inserting a region into every closed walk of G^λ determined by ρ^λ. Here, if G^λ is disconnected and so has two components, then each component of G^λ has a 2-cell embedding by ρ^λ into a closed surface, and $\tilde{S}$ has two components each of which is an orientable surface. To adopt some notations from [16], we describe how (ρ, λ) determines the embedding of G into a surface S as shown in the proof of Theorem 2 in [16]. The regions of the embedding of G^λ into $\tilde{S}$ can be partitioned into pairs $\{\bar{R}, \hat{R}\}$ with $\bar{R} \neq \hat{R}$ so that the oriented boundaries of $\bar{R}$ and $\hat{R}$ project down to inverse walks of G. Let $\mathscr{R}$ be the collection of regions which contains only one region, say $\bar{R}$, from each pair of $\{\bar{R}, \hat{R}\}$. Let $P(\bar{R})$ denote a plane polygon whose oriented boundary is $(e_1, e_2, \ldots, e_n)$ if the oriented boundary of $\bar{R}$ is $(\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_n)$, where $\tilde{e}_i$ is a lift of the edge e_i in G. Then each edge e in G occurs twice as the side of some $P(\bar{R})$; i.e., either there are two regions in $\mathscr{R}$ on each of whose boundaries e occurs once, or else there is a single region in $\mathscr{R}$ on whose boundary e occurs twice. Now, an application of the side identification process of the collection $P(\mathscr{R}) = \{P(\bar{R}): \bar{R} \in \mathscr{R}\}$ yields a 2-cell embedding $\imath: G \to S$ of G into a surface S. Moreover, $\tilde{S}$ is the canonical orientable double covering of S and the graph covering projection $p_\lambda: G^\lambda \to G$ can be extended to the surface covering projection $\pi_\lambda: \tilde{S} \to S$ such that the following diagram commutes:

$$\begin{array}{ccc} G^\lambda & \xrightarrow{\tilde{\imath}} & \tilde{S} \\ \downarrow{\scriptstyle p_\lambda} & & \downarrow{\scriptstyle \pi_\lambda} \\ G & \xrightarrow{\imath} & S \end{array}$$

2. Congruence with respect to the trivial subgroup

We use $|X|$ for the cardinality of a set X. For a connected graph G the number $\beta(G) = |E(G)| - |V(G)| + 1$ is equal to the number of independent cycles in G and it is referred to as the *Betti number* of G. Throughout this paper, let Γ denote a subgroup of $\mathrm{Aut}(G)$, and for any $\gamma \in \Gamma$, γv and γe stand for $\gamma(v)$ and $\gamma(e)$ respectively. Let $C^0(G; \mathbb{Z}_2)$ denote the set of maps from $V(G)$ to $\mathbb{Z}_2$.

Theorem 2.1. *Let G be a graph, and (ρ, λ) and (τ, μ) two embedding schemes for G with the corresponding embeddings $\imath: G \to S$ and $\jmath: G \to S$, respectively. Let Γ be a subgroup of* $\mathrm{Aut}(G)$. *Then the following are equivalent.*

(a) *The two embeddings $\imath$ and $\jmath$ are congruent with respect to Γ.*

(b) *There exist a graph isomorphism* $\Phi: G^{\lambda} \to G^{\mu}$ *and* $\gamma \in \Gamma$ *such that* $p_{\mu} \circ \Phi = \gamma \circ p_{\lambda}$ *and* $(\tau^{\mu})_{\Phi(v_{\alpha})} = \Phi \circ (\rho^{\lambda})_{v_{\alpha}} \circ \Phi^{-1}$ *for all* $v_{\alpha} \in V(G^{\lambda})$.

(c) *There exist* $\gamma \in \Gamma$ *and* $f \in C^0(G; \mathbb{Z}_2)$ *such that* $\tau_{\gamma v} = \gamma \circ (\rho_v)^{f(v)} \circ \gamma^{-1}$ *and* $\mu(\gamma e) = f(u)\lambda(e)f(v)$ *for all* $e = uv \in E(G)$.

Proof. (a) $\Leftrightarrow$ (b). Let $h: S \to S$ be a surface homeomorphism and $\gamma \in \sim\Gamma$ a graph automorphism such that $h \circ \imath = \jmath \circ \gamma$, i.e., the diagram

$$\begin{tikzcd} S \arrow[r, "h"] & S \\ G \arrow[u, "\imath"] \arrow[r, "\gamma"] & G \arrow[u, "\jmath"'] \end{tikzcd}$$

commutes. Let $\tilde{S}$ be the canonical oriented double covering of S. We define a surface homeomorphism $\tilde{h}: \tilde{S} \to \tilde{S}$ as follows: Let R be an oriented region of the embedding $\imath: G \to S$. Then $h(R)$ is a region of the embedding $\jmath: G \to S$. We assume that $h(R)$ is oriented with the orientation inherited from that of R. Let $\bar{R}$ and $\hat{R}$ be the oriented regions of the embedding $\tilde{\imath}: G^{\lambda} \to \tilde{S}$ such that $\pi_{\lambda}(\bar{R}) = \pi_{\lambda}(\hat{R}) = R$, and let $\bar{R}'$ and $\hat{R}'$ be the oriented regions of the embedding $\tilde{\jmath}: G^{\mu} \to \tilde{S}$ such that $\pi_{\mu}(\bar{R}') = \pi_{\mu}(\hat{R}') = h(R)$, where $\tilde{\imath}$ and $\tilde{\jmath}$ are the embeddings determined by the derived rotation schemes ρ^{λ} and τ^{μ} respectively. From the construction of the embeddings $\imath$ and $\jmath$ of G into S, we know that the orientations of the polygons $h(P(\bar{R}))$ and $P(\bar{R}')$ are the same or opposite. We define $\tilde{h}: \tilde{S} \to \tilde{S}$ as follows: $\tilde{h}(\bar{R}) = \bar{R}'$, $\tilde{h}(\hat{R}) = \hat{R}'$ if $h(P(\bar{R})) = P(\bar{R}')$ with the same orientation, and $\tilde{h}(\bar{R}) = \hat{R}'$, $\tilde{h}(\hat{R}) = \bar{R}'$ if $h(P(\bar{R})) = P(\bar{R}')$ with the opposite orientation. Clearly, $\tilde{h}: \tilde{S} \to \tilde{S}$ is an orientation-preserving homeomorphism with the property $\pi_{\mu} \circ \tilde{h} = h \circ \pi_{\lambda}$, i.e., the diagram

$$\begin{tikzcd} \tilde{S} \arrow[r, "\tilde{h}"] \arrow[d, "\pi_{\lambda}"'] & \tilde{S} \arrow[d, "\pi_{\mu}"] \\ S \arrow[r, "h"] & S \end{tikzcd}$$

commutes, where π_{λ} and π_{μ} are the canonical surface covering projections corresponding to the embedding schemes (ρ, λ) and (τ, μ) respectively. Let $\Phi: G^{\lambda} \to G^{\mu}$ be the map defined by $\Phi = \tilde{\jmath}^{-1} \circ \tilde{h} \circ \tilde{\imath}$. Then Φ is a graph isomorphism and all the rectangles in the following diagram commute:

$$\begin{tikzcd}
G^{\lambda} \arrow[rrr, "\Phi"] \arrow[dr, "\tilde{\imath}"] \arrow[ddd, "p_{\lambda}"'] & & & G^{\mu} \arrow[dl, "\tilde{\jmath}"'] \arrow[ddd, "p_{\mu}"] \\
& \tilde{S} \arrow[r, "\tilde{h}"] \arrow[d, "\pi_{\lambda}"] & \tilde{S} \arrow[d, "\pi_{\mu}"'] & \\
& S \arrow[r, "h"] & S & \\
G \arrow[ur, "\imath"] \arrow[rrr, "\gamma"] & & & G \arrow[ul, "\jmath"']
\end{tikzcd}$$

Now, it is clear that $(\tau^{\mu})_{\Phi(v_{\alpha})} = \Phi \circ (\rho^{\lambda})_{v_{\alpha}} \circ \Phi^{-1}$.

Conversely, if the condition (b) holds, then the condition $(\tau^{\mu})_{\Phi(v_{\alpha})} = \Phi \circ (\rho^{\lambda})_{v_{\alpha}} \circ \Phi^{-1}$ gives the existence of an orientation-preserving homeomorphism $\tilde{h}: \tilde{S} \to \tilde{S}$ such that $\tilde{\jmath} \circ \Phi = \tilde{h} \circ \tilde{\imath}$. Consider the following commutative diagram

$$\begin{tikzcd}
G^{\lambda} \arrow[rrr, "\Phi"] \arrow[dr, "\tilde{\imath}"'] \arrow[ddd, "p_{\lambda}"'] & & & G^{\mu} \arrow[dl, "\tilde{\jmath}"] \arrow[ddd, "p_{\mu}"] \\
& \tilde{S} \arrow[r, "\tilde{h}"] \arrow[d, "\pi_{\lambda}"'] & \tilde{S} \arrow[d, "\pi_{\mu}"] & \\
& S & S & \\
G \arrow[ur, "\imath"] \arrow[rrr, "\gamma"] & & & G \arrow[ul, "\jmath"']
\end{tikzcd}$$

Let R be an oriented region (or an oriented closed walk of G) of the embedding $\imath: G \to S$. Let $\bar{R}$ and $\hat{R}$ be the oriented regions of the embedding $\tilde{\imath}$ that project down to R in the same and in the opposite orientation to that of R, respectively. Then $\tilde{h}(\bar{R})$ and $\tilde{h}(\hat{R})$ are oriented regions of the embedding $\tilde{\jmath}$ which cover a region $\pi_{\mu}(\tilde{h}(\bar{R})) = \pi_{\mu}(\tilde{h}(\hat{R})) = R'$. Without loss of generality, we may assume that the orientation of R' inherits that of $\tilde{h}(\bar{R})$ and the embedding $\jmath: G \to S$ is induced from these orientations of R's. We define a map $h: S \to S$ by $h(R) = R'$ for each region R in S. Then h is a surface homeomorphism which makes two embeddings $\imath$ and $\jmath$ congruent with respect to Γ.

(b) $\Leftrightarrow$ (c). By assuming (b), we define a map f: $V(G) \to \mathbb{Z}_2$ so that $\Phi(v_{\alpha}) = (\gamma v)_{f(v)\alpha}$ for each $\alpha \in \mathbb{Z}_2$, i.e.,

$$f(v) = \begin{cases} 1 & \text{if } \Phi(v_{\alpha}) = (\gamma v)_{\alpha}, \\ -1 & \text{if } \Phi(v_{\alpha}) = (\gamma v)_{-\alpha}. \end{cases}$$

If u_{α} and v_{β} are joined in G^{λ}, then $\beta = \lambda(uv)\alpha$ and $uv \in E(G)$. Since Φ is a graph isomorphism $(\gamma u)_{f(uf)\alpha}$ and $(\gamma v)_{f(v)\beta}$ are joined in G^{μ}. Thus $\mu(\gamma u\,\gamma v)f(u)_{\alpha}=f(v)\beta=f(v)\lambda(uv)\alpha$ and hence $\mu(\gamma u\,\gamma v)=f(u)\lambda(uv)f(v)$ for each $uv\in E(G)$. Moreover, for any vertex $v\in V(G)$, we know $\Phi(v_1)=(\gamma v)_{f(v)}$. If $f(v)=1$, then $\Phi(v_1)=(\gamma v)_1$, and $(\tau^{\mu})_{(\gamma v)_1}=\Phi\circ(\rho^{\lambda})_{v_1}\circ\Phi^{-1}$ which gives $\tau_{\gamma v}=\gamma\circ\rho_v\circ\gamma^{-1}$ by the definitions of τ^{μ}, ρ^{λ} and the condition $p_{\mu}\circ\Phi=\gamma\circ\rho_{\lambda}$. If $f(v)=-1$, then $\Phi(v_1)=(\gamma v)_{-1}$ and $(\tau^{\mu})_{(\gamma v)_{-1}}=\Phi\circ(\rho^{\lambda})_{v_1}\circ\Phi^{-1}$ which also implies $(\tau_{\gamma v})^{-1}=\gamma\circ\rho_v\circ\gamma^{-1}$. In both cases, we get $\tau_{\gamma_v}=\gamma\circ(\rho_v)^{f(v)}\circ\gamma^{-1}$, which proves (c).

Next, by assuming (c), we define a map $\Phi: G^{\lambda}\to G^{\mu}$ by $\Phi(v_{\alpha})=(\gamma v)_{f(v)\alpha}$ for $v_{\alpha}\in V(G^{\lambda})$. Then Φ preserves the adjacency of vertices, because $\mu(\gamma e)=f(u)\lambda(e)f(v)$ for all $e=uv\in E(G)$. Clearly, Φ is a graph isomorphism and $p_{\mu}\circ\Phi=\gamma\circ p_{\lambda}$. Now, we show that $(\tau^{\mu})_{\Phi(v_{\alpha})}=\Phi\circ(\rho^{\lambda})_{v_{\alpha}}\circ\Phi^{-1}$ for all $v_{\alpha}\in V(G^{\lambda})$. Let $w\in N(v)$, $d=vw\in E(G)$ and $\alpha\in\mathbb{Z}_2$. Then, for $w_{\lambda(d)\alpha}\in N(v_{\alpha})$, we get

$$\begin{aligned}
(\tau^{\mu})_{\Phi(v_{\alpha})}\Phi(w_{\lambda(d)\alpha}) &= (\tau^{\mu})_{(\gamma v)_{f(v)\alpha}}((\gamma w)_{\mu(\gamma d)\,f(v)\alpha})\\
&= ((\tau_{\gamma v})^{f(v)\alpha}(\gamma w))_{\mu(\gamma v\;(\tau_{\gamma v})^{f(v)\alpha}(\gamma w))} \quad [\text{by definition of } \tau^{\mu}]\\
&= (\gamma(\rho_v)^{\alpha}(w))_{\mu(\gamma v\;\gamma(\rho_v)^{\alpha}(w))f(v)\alpha} \quad [\text{since } \gamma(\rho_v)^{\alpha}(w)=(\tau_{\gamma v})^{f(v)\alpha}(\gamma w)]\\
&= (\gamma w')_{f(w')\lambda(vw')\alpha}\\
&= \Phi(w'_{\lambda(vw')\alpha})\\
&= \Phi((\rho^{\lambda})_{v_{\alpha}}(w_{\lambda(d)\alpha})) \quad [\text{by definition of } \rho^{\lambda}],
\end{aligned}$$

where $w'=(\rho_v)^{\alpha}(w)\in V(G)$. Therefore, we get

$$(\tau^{\mu})_{\Phi(v_{\alpha})}=\Phi\circ(\rho^{\lambda})_{v_{\alpha}}\circ\Phi^{-1}. \qquad \square$$

Definition 2.2. Two embedding schemes (ρ,λ) and (τ,μ) for G are *congruent with respect to a subgroup Γ* of Aut(G) if there exist a graph automorphism $\gamma\in\Gamma$ and a map $f\in C^0(G;\mathbb{Z}_2)$ such that $\tau_{\gamma v}=\gamma\circ(\rho_v)^{f(v)}\circ\gamma^{-1}$ and $\mu(\gamma e)=f(u)\lambda(e)f(v)$ for all $v\in V(G)$ and $e=uv\in E(G)$.

Theorem 2.1 says that two embedding schemes for a graph G are congruent with respect to Γ if and only if their corresponding 2-cell embeddings of G are congruent with respect to Γ.

Definition 2.3. Two double coverings G^{λ} and G^{μ} of G are *isomorphic with respect to a subgroup Γ* of Aut(G) if there exist a graph isomorphism $\Phi: G^{\lambda}\to G^{\mu}$ and a graph automorphism $\gamma\in\Gamma$ such that the following diagram commutes:

$$\begin{tikzcd} G^{\lambda} \arrow[r, "\Phi"] \arrow[d, "p_{\lambda}"'] & G^{\mu} \arrow[d, "p_{\mu}"] \\ G \arrow[r, "\gamma"] & G \end{tikzcd}$$

It is known (see [8, 9, 10]) that two double coverings G^{λ} and G^{μ} are isomorphic with respect to a subgroup Γ of $\mathrm{Aut}(G)$ if and only if there exist a graph automorphism $\gamma \in \Gamma$ and a map $f \in C^0(G; \mathbb{Z}_2)$ such that $\mu(\gamma e) = f(u)\lambda(e)f(v)$ for all $e = uv \in E(G)$. Now, this fact and Theorem 2.1 give the following corollary.

Corollary 2.4. *If two 2-cell embeddings $\imath: G \to S$ and $\jmath: G \to S$ are congruent with respect to a subgroup Γ of* $\mathrm{Aut}(G)$, *then their corresponding double coverings of G are isomorphic with respect to Γ.*

Negami [15] showed that if the surface S is the projective plane and the graph G is 3-connected and nonplanar, then the converse of Corollary 2.4 is true.

The *local voltage group* $\mathbb{Z}_2(v)$ of a voltage map λ at a vertex v is the subgroup of $\mathbb{Z}_2$ consisting of all net voltages occurring on v-based closed walks. Note that the number of components of G^{λ} is the index of $\mathbb{Z}_2(v)$ in $\mathbb{Z}_2$. The following theorem might be well known but we have not seen it anywhere. The proof is not difficult.

Theorem 2.5. *Let (ρ, λ) be an embedding scheme for a graph G. Then the following are equivalent.*

(a) *The embedding scheme (ρ, λ) determines an orientable embedding.*

(b) *The derived double covering G^{λ} is disconnected.*

(c) *The local voltage group $\mathbb{Z}_2(v)$* of *λ at any vertex v is trivial.*

Recall that every 2-cell embedding of a graph G into a surface S is determined by an embedding scheme for G. Let $\mathscr{E}(G)$ denote the set of all embedding schemes for G. To define a group action on $\mathscr{E}(G)$ so that their orbits stand for the set of all congruence classes of embedding schemes, we first note that $C^0(G; \mathbb{Z}_2)$ becomes a group isomorphic to $\oplus_{|V(G)|}\mathbb{Z}_2$ under the binary operation given by $(fg)(v) = f(v)g(v)$ for all $v \in V(G)$. Let $C^1(G; \mathbb{Z}_2)$ denote the set of voltage maps from $E(G)$ to $\mathbb{Z}_2$ and Γ a subgroup of $\mathrm{Aut}(G)$. We define Γ-actions on $C^1(G; \mathbb{Z}_2)$ and on $C^0(G; \mathbb{Z}_2)$ as follows: $(\gamma\lambda)(e) = \lambda(\gamma^{-1}e)$ and $(\gamma f)(v) = f(\gamma^{-1}v)$ for $\gamma \in \Gamma$, $\lambda \in C^1(G; \mathbb{Z}_2)$, $f \in C^0(G; \mathbb{Z}_2)$ and $e = uv \in E(G)$.

Let $\Gamma \times C^0(G; \mathbb{Z}_2)$ be the semidirect product group of Γ and $C^0(G; \mathbb{Z}_2)$ with an operation defined by $(\gamma_1, f_1)(\gamma_2, f_2) = (\gamma_1\gamma_2, (\gamma_2^{-1}f_1)f_2)$. Define a group $\Gamma \times C^0(G; \mathbb{Z}_2)$ action on $\mathscr{E}(G)$ by $(\gamma, f)(\rho, \lambda) = ((\gamma, f)\rho, (\gamma, f)\lambda)$ for any $(\rho, \lambda) \in \mathscr{E}(G)$ and

$(\gamma, f) \in \Gamma \times C^0(G; \mathbb{Z}_2)$, where for any $v \in V(G)$ and $e = uv \in E(G)$,

$$[(\gamma, f)\rho]_v = \gamma \circ (\rho_{\gamma^{-1}v})^{f(\gamma^{-1}v)} \circ \gamma^{-1},$$

and

$$[(\gamma, f)\lambda](e) = f(\gamma^{-1}u)\lambda(\gamma^{-1}e)f(\gamma^{-1}v).$$

Let $\mathscr{C}_\Gamma(G) = \mathscr{E}(G)/\Gamma \times C^0(G; \mathbb{Z}_2)$ be the orbit set. Then Theorem 2.1 gives that $\mathscr{C}_\Gamma(G)$ stands for the set of all congruence classes of embedding schemes with respect to Γ. Let I denote the identity element of $\mathrm{Aut}(G)$.

Let T be a fixed spanning tree in G with base vertex v_0. We can assume that all voltage maps λ in the embedding schemes (ρ, λ) in $\mathscr{E}(G)$ satisfy $\lambda(e) = 1$ for each $e \in E(T)$ without loss of generality. To show this, we first define a map $\mathfrak{I}^{\#}: C^1(G; \mathbb{Z}_2) \to C^0(G; \mathbb{Z}_2)$ as follows: for any $v \in V(G)$ there exists a unique path $e_1 e_2 \cdots e_m$ in the tree T from v_0 to v and we define

$$\mathfrak{I}^{\#}(\lambda)(v) = \lambda(e_1) \cdots \lambda(e_m).$$

We write

$$C_T^1(G; \mathbb{Z}_2) = \{\lambda \in C^1(G; \mathbb{Z}_2) \colon \lambda(e) = 1 \quad \text{for each } e \in E(T)\},$$

and define a map $\mathfrak{I}^*: C^1(G; \mathbb{Z}_2) \to C_T^1(G; \mathbb{Z}_2)$ by

$$\mathfrak{I}^*(\lambda)(uv) = \mathfrak{I}^{\#}(\lambda)(u)\, \lambda(uv)\, \mathfrak{I}^{\#}(\lambda)(v).$$

Let $\mathscr{E}_T(G) = \{(\rho, \lambda) \in \mathscr{E}(G) \colon \lambda \in C_T^1(G; \mathbb{Z}_2)\}$. Then the map $\mathfrak{I}^*$ induces a map $\mathscr{E}(G) \to \mathscr{E}_T(G)$ which sends (ρ, λ) to $(\rho', \mathfrak{I}^*(\lambda))$, where $(\rho')_v = (\rho_v)^{\mathfrak{I}^{\#}(\lambda)(v)}$. We also denote this map $\mathfrak{I}^*$. Clearly, $\mathfrak{I}^*$ is the identity map on $\mathscr{E}_T(G)$. Hence, we have the following corollary.

Corollary 2.6. *If T is a spanning tree of G and (ρ, λ) an embedding scheme for G, then there exists an embedding scheme (ρ', λ') for G such that (ρ', λ') is congruent to (ρ, λ) with respect to the trivial subgroup $\{I\}$ and $\lambda'(e) = 1$ for each $e \in E(T)$.*

Corollary 2.6 says that $\mathscr{E}_T(G)$ has all representatives of congruence classes of $\mathscr{E}(G)$. Let T be a spanning tree of G fixed by every automorphism γ in a subgroup Γ of $\mathrm{Aut}(G)$, by what means, $\gamma(T) = T$. Then any two embedding schemes (ρ, λ) and (τ, μ) in $\mathscr{E}_T(G)$ are congruent with respect to Γ if and only if there exist a graph automorphism $\gamma \in \Gamma$ and $\alpha \in \mathbb{Z}_2$ such that $\tau_{\gamma v} = \gamma \circ (\rho_v)^\alpha \circ \gamma^{-1}$ and $\mu(\gamma e) = \lambda(e)$ for all $e = uv \in E(G) - E(T)$, because the map f in Theorem 2.1(c) must be constant. Thus, $\Gamma \times C^0(G; \mathbb{Z}_2)$ action on $\mathscr{E}(G)$ can be reduced to the $\Gamma \times \mathbb{Z}_2$ action on $\mathscr{E}_T(G)$, that is, $|\mathscr{C}_\Gamma(G)| = |\mathscr{E}_T(G)/\Gamma \times \mathbb{Z}_2|$. It is not difficult to show that

$$|\mathscr{E}_T(G)| = 2^{\beta(G)} \prod_{v \in V(G)} (d(v) - 1)!,$$

where $d(v)$ is the degree of vertex v.

Table 1

$\beta(G)$	3	4	5	6	7	8
$\lvert\mathscr{C}_{\{I\}}(G)\rvert$	2^6	2^9	2^{12}	2^{15}	2^{18}	2^{21}

A graph is said to be *irreducible* if either it has only one vertex or else every vertex has degree at least 3.

Theorem 2.7. *Let G be an irreducible graph and $\Gamma=\{I\}$. Then*

$$|\mathscr{C}_{\{I\}}(G)|=2^{\beta(G)-1}\prod_{v\in V(G)}(d(v)-1)!.$$

In particular, if G is regular of degree d,

$$|\mathscr{C}_{\{I\}}(G)|=2^{\beta(G)-1}((d-1)!)^{|V(G)|}.$$

Proof. If $(I,\alpha)(\rho,\lambda)=(\rho,\lambda)$ for $\alpha\in\mathbb{Z}_2$, then $(\rho_v)^\alpha=\rho_v$ for each $v\in V(G)$. Since G is irreducible, $\alpha=1$ for all $v\in V(G)$. Thus, (I,α) fixes an element of $\mathscr{E}_T(G)$ if and only if $\alpha=1$ for all $v\in V(G)$, and $(I,1)$ fixes all elements of $\mathscr{E}_T(G)$. By applying the Burnside lemma, we get

$$\begin{aligned}|\mathscr{C}_{\{I\}}(G)|&=\frac{1}{2}2^{\beta(G)}\prod_{v\in V(G)}(d(v)-1)!\\&=2^{\beta(G)-1}\prod_{v\in V(G)}(d(v)-1)!.\qquad\square\end{aligned}$$

Though the above theorem is stated only for an irreducible graph, it remains true for any graph, because a homeomorphism can eliminate not only the vertices of degree two, but also the vertices of degree one.

Corollary 2.8. *If G is regular of degree 3, then $|\mathscr{C}_{\{I\}}(G)|=2^{|E(G)|}$.*

If a graph G is regular of degree 3, then $3|V(G)|=2|E(G)|$ and $|V(G)|$ is even with greater than 3. Thus, we have

$$\beta(G)=|E(G)|-|V(G)|+1=\frac{|V(G)|+2}{2}\geqslant 3,$$

and $|E(G)|=3\beta(G)-3$. Hence, for a 3-regular graph G, we get Table 1 for $|\mathscr{C}_{\{I\}}(G)|$.

3. The isotropy subgroup Isot $(\Gamma;\rho,\lambda)$

We first recall the $\Gamma\times C^0(G;\mathbb{Z}_2)$ action on the set $\mathscr{E}(G)$ of all embedding schemes for G defined by $(\gamma,f)(\rho,\lambda)=((\gamma,f)\rho,(\gamma,f)\lambda)$ with

$$[(\gamma,f)\rho]_v=\gamma\circ(\rho_{\gamma^{-1}v})^{f(\gamma^{-1}v)}\circ\gamma^{-1},$$

$$[(\gamma,f)\lambda](e)=f(\gamma^{-1}u)\lambda(\gamma^{-1}e)f(\gamma^{-1}v).$$

Let $\mathrm{Isot}(\Gamma;\rho,\lambda)$ denote the isotropy subgroup of $\Gamma\times C^0(G;\mathbb{Z}_2)$ at (ρ,λ), that is, the subgroup of all $(\gamma,f)\in\Gamma\times C^0(G;\mathbb{Z}_2)$ such that $(\gamma,f)(\rho,\lambda)=(\rho,\lambda)$. We study some properties of the isotropy subgroup $\mathrm{Isot}(\Gamma;\rho,\lambda)$ in this section, for use in later sections. Let $\mathbb{N}$ denote the set of natural numbers.

It is clear that if $(\gamma,f)\in\mathrm{Isot}(\Gamma;\rho,\lambda)$, then $(\gamma,f)^n(\rho,\lambda)=(\rho,\lambda)$ for all $n\in\mathbb{N}$. Thus, we have the following lemma.

Lemma 3.1. *Let $(\gamma,f)\in\mathrm{Isot}(\Gamma;\rho,\lambda)$. Then, for all $n\in\mathbb{N}$, we have*
(a) $\rho_v=\gamma^n\circ(\rho_{\gamma^{-n}v})^{\prod_{i=1}^n f(\gamma^{-i}v)}\circ\gamma^{-n}$ *for any $v\in V(G)$.*
(b) $\lambda(e)=(\prod_{i=1}^n f(\gamma^{-i}u))\lambda(\gamma^{-n}e)(\prod_{i=1}^n f(\gamma^{-i}v))$ *for any $e=uv\in E(G)$.*

Theorem 3.2. *Let $(\gamma,f)\in\mathrm{Isot}(\Gamma;\rho,\lambda)$, and let γ^m fix an edge u_0v_0 for some $m\in\mathbb{N}$, which means $\gamma^m u_0=u_0$ and $\gamma^m v_0=v_0$.*
(a) *If $\prod_{i=1}^m f(\gamma^{-i}u_0)=1$, then $\gamma^m=I$ and $\prod_{i=1}^m f(\gamma^{-i}w)=1$ for any $w\in V(G)$.*
(b) *If $\prod_{i=1}^m f(\gamma^{-i}u_0)=-1$, then $\gamma^{2m}=I$ and $\prod_{i=1}^{2m} f(\gamma^{-i}w)=1$ for any $w\in V(G)$.*

Proof. (a) Suppose $\prod_{i=1}^m f(\gamma^{-i}u_0)=1$. Then Lemma 3.1(a) and the hypothesis $\gamma^m u_0=u_0$ give $\rho_{u_0}=\gamma^m\circ\rho_{u_0}\circ\gamma^{-m}$, that is, $\rho_{u_0}\circ\gamma^m=\gamma^m\circ\rho_{u_0}$. Then for each $n\in\mathbb{N}$,

$$\gamma^m((\rho_{u_0})^n(v_0))=(\rho_{u_0})^n(\gamma^m v_0)=(\rho_{u_0})^n(v_0).$$

Thus γ^m must be the identity on the neighborhood $N(u_0)$ of u_0, because $(\rho_{u_0})^n(v_0)$, $n=1,2,\ldots,|N(u_0)|$, runs over all vertices adjacent to u_0. Now Lemma 3.1(b) gives that $\prod_{i=1}^m f(\gamma^{-i}v)=1$ for all $v\in N(u_0)$. By repeating the same process, we can get that γ^m is the identity on the neighbourhood $N(v)$ of all the vertex $v\in N(v_0)$ and $\prod_{i=1}^m f(\gamma^{-i}w)=1$ for any $w\in N(v)$. Now the proof of (a) comes from the connectivity of the graph G.

(b) Suppose $\prod_{i=1}^m f(\gamma^{-i}u_0)=-1$. Clearly, we have $\prod_{i=1}^{2m} f(\gamma^{-i}u_0)=(\prod_{i=1}^m f(\gamma^{-i}u_0))^2=1$. Since $\gamma^{2m}u_0=u_0$ and $\gamma^{2m}v_0=v_0$, it follows from (a) that $\gamma^{2m}=I$ and $\prod_{i=1}^{2m} f(\gamma^{-i}w)=1$ for any $w\in V(G)$. $\square$

Corollary 3.3. *Let $e=uv$ be an edge of G and let $(\gamma,f)\in\mathrm{Isot}(\Gamma;\rho,\lambda)$.*
(a) *If $\gamma u=u$, $\gamma v=v$ and $f(u)=1$, then $\gamma=I$ and $f(w)=1$ for any $w\in V(G)$.*
(b) *If $\gamma u=u$, $\gamma v=v$ and $f(u)=-1$, then $\gamma^2=I$ and $f(w)f(\gamma w)=1$ for any $w\in V(G)$.*
(c) *If $\gamma^m u=v$ and $\gamma^m v=u$, then $\gamma^{2m}=I$ and $\prod_{i=1}^{2m} f(\gamma^{-i}w)=1$ for any $w\in V(G)$.*

Proof. (a) and (b) are immediate consequences of Theorem 3.2. We only prove (c). Clearly, we have $\gamma^{2m}u=u$, $\gamma^{2m}v=v$. Since $\lambda(uv)=\lambda(vu)$ in $\mathbb{Z}_2$, Lemma 3.1(b) gives $\prod_{i=1}^m f(\gamma^{-i}u)=\prod_{i=1}^m f(\gamma^{-i}v)=\pm1$. Hence,

$$\prod_{i=1}^{2m} f(\gamma^{-i}u)=\left(\prod_{i=1}^{m} f(\gamma^{-i}u)\right)\left(\prod_{i=m+1}^{2m} f(\gamma^{-i}u)\right)=\left(\prod_{i=1}^{m} f(\gamma^{-i}u)\right)\left(\prod_{i=1}^{m} f(\gamma^{-i}v)\right)=1.$$

Now, by applying Theorem 3.2(a), we get (c). $\square$

For each $\gamma\in\mathrm{Aut}(G)$, let $l(v;\gamma)$ and $l(e;\gamma)$ denote the length of the vertex cycle $\{\gamma^n v\colon n\in\mathbb{N}\}$ and the length of the edge cycle $\{\gamma^n e\colon n\in\mathbb{N}\}$ induced by γ for $e=uv\in E(G)$ and $v\in V(G)$, respectively. That is,

$$l(v;\gamma)=|\{\gamma^n v\mid n\in\mathbb{N}\}| \quad\text{and}\quad l(e;\gamma)=|\{\gamma^n e\mid n\in\mathbb{N}\}|.$$

Theorem 3.4. *Let $(\gamma,f)\in\Gamma\times C^0(G;\mathbb{Z}_2)$. Then $(\gamma,f)\in\mathrm{Isot}(\Gamma;\rho,\lambda)$ for some $(\rho,\lambda)\in\mathscr{E}(G)$, i.e., (γ,f) has a fixed point (ρ,λ), if and only if the following conditions are satisfied:*

(a) $\prod_{i=1}^{l(e;\gamma)} f(\gamma^{-i}u)=\prod_{i=1}^{l(e;\gamma)} f(\gamma^{-i}v)$ *for any $e=uv\in E(G)$, and*

(b) *for each $v\in V(G)$, there exists a cycle σ_v on $N(v)$ of length $|N(v)|$ such that $\sigma_v=\gamma\circ(\sigma_{\gamma^{-1}v})^{f(\gamma^{-1}v)}\circ\gamma^{-1}$ on $N(v)$.*

Proof. Suppose that $(\gamma,f)\in\mathrm{Isot}(\Gamma;\rho,\lambda)$ for some $(\rho,\lambda)\in\mathscr{E}(G)$. Then Lemma 3.1(b) gives

$$\lambda(e)=\lambda(\gamma^{l(e;\gamma)}e)=\left(\prod_{i=1}^{l(e;\gamma)} f(\gamma^{-i}u)\right)\lambda(e)\left(\prod_{i=1}^{l(e;\gamma)} f(\gamma^{-i}v)\right),$$

for any $e=uv$. This implies (a). Let $\sigma_v=\rho_v$ for all $v\in V(G)$. Then, from Lemma 3.1(a), it follows that

$$\sigma_v=\rho_v=\gamma\circ(\rho_{\gamma^{-1}v})^{f(\gamma^{-1}v)}\circ\gamma^{-1}=\gamma\circ(\sigma_{\gamma^{-1}v})^{f(\gamma^{-1}v)}\circ\gamma^{-1}$$

on $N(v)$ for any $v\in V(G)$. Conversely, suppose that (γ,f) satisfies conditions (a) and (b). By using this, we aim to find an embedding scheme (ρ,λ) which is fixed by (γ,f). Let ρ be the rotation scheme defined by $\rho_v=\sigma_v$ for each $v\in V(G)$. Then $(\gamma,f)\rho=\rho$, because $\sigma_v=\gamma\circ(\sigma_{\gamma^{-1}v})^{f(\gamma^{-1}v)}\circ\gamma^{-1}$ for any $v\in V(G)$. To define a voltage map λ, let $\{\gamma^n e\colon n\in\mathbb{N}\}$ be any edge cycle and define $\lambda(e)$ to be any element in $\mathbb{Z}_2$. To satisfy $(\gamma,f)\lambda=\lambda$, we must have

$$\lambda(e)=\left(\prod_{i=1}^{n} f(\gamma^{-i}u)\right)\lambda(\gamma^{-n}e)\left(\prod_{i=1}^{n} f(\gamma^{-i}v)\right)$$

for any $e=uv\in E(G)$ and for any $n\in\mathbb{N}$, by Lemma 3.1(b). Hence, we define $\lambda(\gamma^n e)$ inductively on $n\geqslant 1$ by

$$\lambda(\gamma^n e)=f(\gamma^{n-1}u)\lambda(\gamma^{n-1}e)f(\gamma^{n-1}v)$$

for $e=uv$. Then, for the length $l(e;\gamma)$ of the edge cycle $\{\gamma^n e\colon n\in\mathbb{N}\}$,

$$\begin{aligned}
\lambda(\gamma^{l(e;\gamma)}e)&=f(\gamma^{l(e;\gamma)-1}u)\lambda(\gamma^{l(e;\gamma)-1}e)f(\gamma^{l(e;\gamma)-1}v)\\
&=f(\gamma^{l(e;\gamma)-1}u)[f(\gamma^{l(e;\gamma)-2}u)\lambda(\gamma^{l(e;\gamma)-2}e)f(\gamma^{l(e;\gamma)-2}v)]f(\gamma^{l(e;\gamma)-1}v)\\
&\;\;\vdots\\
&=\left(\prod_{k=1}^{l(e;\gamma)} f(\gamma^{l(e;\gamma)-k}u)\right)\lambda(e)\left(\prod_{k=1}^{l(e;\gamma)} f(\gamma^{l(e;\gamma)-k}v)\right)\\
&=\lambda(e),
\end{aligned}$$

by condition (a), which shows that the map λ is well-defined on any edge cycle and so on $E(G)$. Now it is clear that $(\gamma, f) \in \mathrm{Isot}(\Gamma; \rho, \lambda)$. $\square$

For each $\gamma \in \mathrm{Aut}(G)$, let $\langle\gamma\rangle$ denote the subgroup of Γ generated by γ. We define a new graph G_γ, whose vertex set is $V(G)/\langle\gamma\rangle = \{[v]\colon v \in V(G)\}$ and edge set $E(G)/\langle\gamma\rangle = \{[e]\colon e \in E(G)\}$, where $[x]$ denotes the orbit of x under the $\langle\gamma\rangle$ action on $V(G)$ or $E(G)$. Then, it is clear that $l(v;\gamma)$ is the cardinality of $[v]$ for any $v \in [v]$. We say that $[v] \in V(G_\gamma)$ *has property* $\boldsymbol{P}$ if either $\gamma^{l(v;\gamma)}$ is not of order 2 on $N(v)$ for all $v \in [v]$, or $l(v;\gamma)$ is even and $v\gamma^{l(v;\gamma)/2}(v) \in E(G)$ for all $v \in [v]$. For $f \in C^0(G; \mathbb{Z}_2)$, we define $f([v]) = \prod_{i=1}^{l(v;\gamma)} f(\gamma^{-i}v)$ for $[v] \in V(G_\gamma)$.

Lemma 3.5. *Let G be an irreducible graph and let $(\gamma, f) \in \mathrm{Isot}(G; \rho, \lambda)$. Suppose that $[v] \in V(G_\gamma)$ has property $\boldsymbol{P}$. Then $f([v]) = 1$.*

Proof. First, let $\gamma^{l(v;\gamma)}$ be not of order 2 on $N(v)$ for all $v \in [v]$. We assume that $f([v]) = -1$. Then

$$\rho_v = [(\gamma, f)^{l(v;\gamma)}\rho]_v = \gamma^{l(v;\gamma)} \circ (\rho_v)^{f([v])} \circ \gamma^{-l(v;\gamma)} = \gamma^{l(v;\gamma)} \circ \rho_v^{-1} \circ \gamma^{-l(v;\gamma)}.$$

But, for a given n-cycle, say σ, in the symmetric group S_n, there are exactly n elements ω in S_n which satisfy $\omega\sigma\omega^{-1} = \sigma^{-1}$, and such n elements are of order 2 (see [11]). Hence $\gamma^{l(v;\gamma)}$ must be of order two on $N(v)$, which is contradictory. Thus $f([v])$ must be 1.

Next, let $l(v;\gamma)$ be even and $v\gamma^{l(v;\gamma)/2}(v) \in E(G)$ for all $v \in [v]$. It is clear that $l(v\gamma^{l(v;\gamma)/2}(v); \gamma) = l(v;\gamma)/2$. If we apply Corollary 3.3(c) to this situation, we have $f([v]) = \prod_{i=1}^{l(v;\gamma)} f(\gamma^{-i}v) = 1$. $\square$

4. Congruence with respect to nontrivial subgroups

In this section, we enumerate congruence classes of embeddings of G into surfaces with respect to any arbitrarily given subgroup Γ of $\mathrm{Aut}(G)$. First, we introduce some notations for $(\gamma, f) \in \Gamma \times C^0(G; \mathbb{Z}_2)$ and a vertex $v \in V(G)$ as follows:

$$\mathrm{Fix}_{(\gamma,f)} = \{(\rho, \lambda) \in \mathscr{E}(G)\colon (\gamma, f)(\rho, \lambda) = (\rho, \lambda)\},$$

$$P(\gamma, f) = \{[v] \in V(G_\gamma)\colon f([v]) = 1\},$$

$$I(\gamma, f) = \{[v] \in V(G_\gamma)\colon f([v]) = -1\},$$

$$P_v(\gamma^n) = \{\sigma\colon \sigma \text{ is a cycle permutation on } N(v) \text{ and } \gamma^n \circ \sigma \circ \gamma^{-n} = \sigma\},$$

$$I_v(\gamma^n) = \{\sigma\colon \sigma \text{ is a cycle permutation on } N(v) \text{ and } \gamma^n \circ \sigma \circ \gamma^{-n} = \sigma^{-1}\},$$

for $n \in \mathbb{N}$. If every automorphism in Γ fixes a given spanning tree T of G, the $\Gamma \times C^0(G; \mathbb{Z}_2)$ action on $\mathscr{E}(G)$ can be reduced to the $\Gamma \times \mathbb{Z}_2$ action on $\mathscr{E}_T(G)$ and

$|\mathscr{C}_\Gamma(G)|=|\mathscr{E}_T(G)/\Gamma\times\mathbb{Z}_2|$. In this case, we adopt the following additional notations:

$$\mathrm{Fix}^T_{(\gamma,\alpha)}=\{(\rho,\lambda)\in\mathscr{E}_T(G)\colon (\gamma,\alpha)(\rho,\lambda)=(\rho,\lambda)\}\quad\text{for } (\gamma,\alpha)\in\Gamma\times\mathbb{Z}_2,$$

$$E^T(G_\gamma)=\{[uv]\in E(G_\gamma)\colon uv\notin E(T)\}.$$

It is easy to show that $\sum_{f\in C^0(G;\mathbb{Z}_2)}|\mathrm{Fix}_{(\gamma_1,f)}|=\sum_{f\in C^0(G;\mathbb{Z}_2)}|\mathrm{Fix}_{(\gamma_2,f)}|$ if γ_1 and γ_2 are conjugate in Aut(G). By using the Burnside lemma with this fact, we have the following theorem.

Theorem 4.1.

$$|\mathscr{C}_\Gamma(G)|=\frac{1}{|\Gamma|2^{|V(G)|}}\sum_{\gamma\in S}|C(\gamma)|\left(\sum_{f\in C^0(G;\mathbb{Z}_2)}|\mathrm{Fix}_{(\gamma,f)}|\right),$$

where S is the set consisting of all representatives of conjugacy classes of Γ, and $C(\gamma)$ denotes the conjugacy class of γ in Γ.

Corollary 4.2. *If every automorphism in Γ fixes a spanning tree T of G, then*

$$|\mathscr{C}_\Gamma(G)|=\frac{1}{2|\Gamma|}\sum_{\gamma\in S}|C(\gamma)|\,(|\mathrm{Fix}^T_{(\gamma,1)}|+|\mathrm{Fix}^T_{(\gamma,-1)}|),$$

where S is the set consisting of all representatives of conjugacy classes of Γ, and $C(\gamma)$ denotes the conjugacy class of γ in Γ.

Now, we aim to calculate $|\mathrm{Fix}_{(\gamma,f)}|$. It is clear that $|\mathrm{Fix}_{(\gamma,f)}|\neq 0$ if and only if (γ,f) satisfies conditions (a) and (b) of Theorem 3.4. We define a map $\psi\colon\Gamma\times C^0(G;\mathbb{Z}_2)\to\{0,1\}$ by $\psi(\gamma,f)=1$ if (γ,f) satisfies the condition (a) of Theorem 3.4, and $\psi(\gamma,f)=0$ otherwise.

Theorem 4.3. *For $(\gamma,f)\in\Gamma\times C^0(G;\mathbb{Z}_2)$, we have*

$$|\mathrm{Fix}_{(\gamma,f)}|=\psi(\gamma,f)\,2^{|E(G_\gamma)|}\prod_{[v]\in P(\gamma,f)}|P_v(\gamma^{l(v;\gamma)})|\prod_{[v]\in I(\gamma,f)}|I_v(\gamma^{l(v;\gamma)})|,$$

where the product over the empty index set is defined to be 1.

Proof. Let $(\rho,\lambda)\in\mathrm{Fix}_{(\gamma,f)}$, or equivalently $(\gamma,f)\in\mathrm{Isot}(\Gamma;\rho,\lambda)$. Then, the voltage map λ satisfies

$$\lambda(e)=\left(\prod_{i=1}^{n}f(\gamma^{-i}u)\right)\lambda(\gamma^{-n}e)\left(\prod_{i=1}^{n}f(\gamma^{-i}v)\right)$$

for any $e=uv\in E(G)$ and for any $n\in\mathbb{N}$, by Lemma 3.1(b). Hence, for any edge $e\in E(G)$, if $\lambda(e)$ is defined to be any element in $\mathbb{Z}_2$, then λ is completely determined on the edge cycle $[e]=\{\gamma^n e\colon n\in\mathbb{N}\}$ containing e by the value $\lambda(e)$, so that there are exactly $2^{|E(G_\gamma)|}$ ways to define such λ's. On the other hand if $(\gamma,f)\rho=\rho$, then

$$\rho_v=\gamma^n\circ(\rho_{\gamma^{-n}v})^{\prod_{i=1}^{n}f(\gamma^{-i}v)}\circ\gamma^{-n}$$

for any $v\in V(G)$ and for any $n\in\mathbb{N}$, by Lemma 3.1(a). Hence, for any vertex $v\in V(G)$, if ρ_v is defined as a cycle permutation on $N(v)$, then ρ is completely determined on the vertex cycle $[v]=\{\gamma^n v\colon n\in\mathbb{N}\}$ containing v by the permutation ρ_v. Moreover, if $\prod_{i=1}^{l(v;\gamma)} f(\gamma^{-i}v)=f([v])=1$, then

$$\begin{aligned}\rho_v&=\gamma^{l(v;\gamma)}\circ(\rho_{\gamma^{-l(v;\gamma)}v})^{\prod_{i=1}^{l(v;\gamma)}f(\gamma^{-i}v)}\circ\gamma^{-l(v;\gamma)}\\&=\gamma^{l(v;\gamma)}\circ\rho_v\circ\gamma^{-l(v;\gamma)},\end{aligned}$$

by Lemma 3.1(a), so that $\rho_v\in P_v(\gamma^{l(v;\gamma)})$. If $\prod_{i=1}^{l(v;\gamma)} f(\gamma^{-i}v)=f([v])=-1$, we can get

$$\rho_v=\gamma^{l(v;\gamma)}\circ(\rho_v)^{-1}\circ\gamma^{-l(v;\gamma)},$$

so that $\rho_v\in I_v(\gamma^{l(v;\gamma)})$. Therefore, the number of all possible ways to define ρ so that $(\gamma,f)\rho=\rho$ is

$$\prod_{[v]\in P(\gamma,f)}|P_v(\gamma^{l(v;\gamma)})|\prod_{[v]\in I(\gamma,f)}|I_v(\gamma^{l(v;\gamma)})|.$$

This completes the proof. $\square$

If $(\gamma,\alpha)\in\Gamma\times\mathbb{Z}_2$, then (γ,α) satisfies the condition (a) of Theorem 3.4. This gives the following corollary.

Corollary 4.4. *If an automorphism $\gamma\in\Gamma$ fixes a spanning tree T of G, i.e., $\gamma(T)=T$, then*

$$|\mathrm{Fix}^T_{(\gamma,1)}|=2^{|E^T(G_\gamma)|}\prod_{[v]\in V(G_\gamma)}|P_v(\gamma^{l(v;\gamma)})|,$$

and

$$|\mathrm{Fix}^T_{(\gamma,-1)}|=2^{|E^T(G_\gamma)|}\prod_{\substack{[v]\in V(G_\gamma)\\|[v]|=even}}|P_v(\gamma^{l(v;\gamma)})|\prod_{\substack{[v]\in V(G_\gamma)\\|[v]|=odd}}|I_v(\gamma^{l(v;\gamma)})|,$$

where the product over the empty index set is defined to be 1.

To complete the calculation of $|\mathrm{Fix}_{(\gamma,f)}|$ or $|\mathrm{Fix}^T_{(\gamma,\alpha)}|$, we need to calculate $|P_v(\gamma^{l(v;\gamma)})|$ and $|I_v(\gamma^{l(v;\gamma)})|$. For a permutation $\sigma\in S_n$, let j_k be the number of cycles of length k in the factorization of σ into disjoint cycles. Then the n-tuple $(j_1,j_2,\ldots,j_n)$ is called the *cycle type* of σ and we denote it by $j(\sigma)$. Let ϕ be the Euler phi-function. In [13], the number $|P_v(\gamma^{l(v;\gamma)})|$ was given as follows.

Theorem 4.5. *Let $[v]\in V(G_\gamma)$ and $|N(v)|=n$. Then*

$$|P_v(\gamma^{l(v;\gamma)})|=\begin{cases}\phi(d)\,((n/d)-1)!\,d^{(n/d)-1} & \text{if } j(\gamma^{l(v;\gamma)}|_{N(v)})=(0,\ldots,0,j_d=n/d,0,\ldots,0),\\ 0 & \text{otherwise.}\end{cases}$$

To calculate $|I_v(\gamma^{I(v;\gamma)})|$, we adopt the following additional notations:

$$J_1=\{\sigma\in S_n\colon j(\sigma)=(1,(n-1)/2,0,\dots,0)\} \quad \text{for odd } n,$$

$$J_2=\{\sigma\in S_n\colon j(\sigma)=(0,n/2,0,\dots,0)\} \quad \text{for even } n,$$

$$J_3=\{\sigma\in S_n\colon j(\sigma)=(2,(n/2)-1,0,\dots,0)\} \quad \text{for even } n,$$

$$I_\alpha=\{\sigma\in S_n\colon \sigma \text{ is an } n\text{-cycle and } \alpha\sigma\alpha^{-1}=\sigma^{-1}\} \quad \text{for } \alpha\in S_n.$$

Lemma 4.6. *For an $\alpha\in S_n$, $|I_\alpha|\neq 0$ if and only if α belongs to J_i for some $i=1,2,3$. In particular, $|I_v(\gamma^{I(v;\gamma)})|\neq 0$ if and only if $\gamma^{I(v;\gamma)}|_{N(v)}$ belongs to J_i for some $i=1,2,3$.*

Proof. Let $\sigma=(a_1a_2\cdots a_n)$ be an n-cycle in S_n. Consider the regular n-gon Q in the plane with vertices $a_1,a_2,\dots,a_n$ labeled consecutively. Then the symmetry group of Q is isomorphic to the dihedral group D_n. Moreover, σ is a generator of the cyclic subgroup of order n of the symmetry group of Q. Note that there are exactly n elements of order two in the symmetry group of Q which are all reflections, and by the conjugation action of such n elements of order 2 in the symmetry group of Q, σ is sent to σ^{-1}. It is not difficult to show (see [11]) that there are exactly n elements $\delta\in S_n$ such that $\delta\sigma_1\delta^{-1}=\sigma_2$ for any two n-cycles σ_1 and σ_2 in S_n. Since σ^{-1} is also an n-cycle and any element of order 2 in the symmetry group of Q is contained in one of the sets J_1, J_2 and J_3, we have the lemma. □

Theorem 4.7. *Let $[v]\in V(G_\gamma)$ and $|N(v)|=n$. Then*

$$|I_v(\gamma^{I(v;\gamma)})|=$$
$$\begin{cases} ((n-1)/2)!\,2^{(n-1)/2} & \text{if } n \text{ is odd and } j(\gamma^{I(v;\gamma)}|_{N(v)})=(1,(n-1)/2,0,\dots,0),\\ (n/2)!\,2^{(n/2)-1} & \text{if } n \text{ is even and } j(\gamma^{I(v;\gamma)}|_{N(v)})=(0,n/2,0,\dots,0),\\ ((n/2)-1)!\,2^{(n/2)-1} & \text{if } n \text{ is even and } j(\gamma^{I(v;\gamma)}|_{N(v)})=(2,(n/2)-1,0,\dots,0),\\ 0 & \text{otherwise.}\end{cases}$$

Proof. Since $\gamma^{I(v;\gamma)}$ is a permutation on $N(v)$ and $|N(v)|=n$, we identify $\gamma^{I(v;\gamma)}$ as a permutation on S_n, say ω. Then, we have $|I_v(\gamma^{I(v;\gamma)})|=|I_\omega|$, and if this is not zero, then $\omega\in J_i$ for some $i=1,2,3$, by Lemma 4.6. It is well known that the number of permutation in S_n of cycle type $(j_1,\dots,j_n)$ is

$$\frac{n!}{\prod_{k=1}^n j_k!\,k^{j_k}}.$$

Hence, we get

$$|J_1|=\frac{n!}{((n-1)/2)!\,2^{(n-1)/2}},\qquad |J_2|=\frac{n!}{(n/2)!\,2^{n/2}}\qquad |J_3|=\frac{n!}{((n/2)-1)!\,2^{n/2}}.$$

But, we note that for any $i=1,2,3$,

$$\sum_{\alpha\in J_i} |I_\alpha| = |\{(\alpha,\sigma)\in J_i\times S_n:\ \sigma \text{ is an } n\text{-cycle and } \alpha\sigma\alpha^{-1}=\sigma^{-1}\}|$$
$$= \sum_{\sigma = n\text{-cycle}} |\{\alpha\in J_i:\ \alpha\sigma\alpha^{-1}=\sigma^{-1}\}|,$$

and

$$|\{\alpha\in J_i:\ \alpha\sigma\alpha^{-1}=\sigma^{-1}\}| = \begin{cases} n & \text{if } i=1,\\ n/2 & \text{if } i=2 \text{ or } 3,\end{cases}$$

which can be shown by an argument similar to the proof of Lemma 4.6. Hence, for any $i=1,2,3$,

$$\sum_{\alpha\in J_i} |I_\alpha| = \begin{cases} (n-1)!\,n & \text{if } i=1\\ (n-1)!\,n/2 & \text{if } i=2 \text{ or } 3.\end{cases}$$

On the other hand, for any two permutations α_1 and α_2 having the same cycle type, we get $|I_{\alpha_1}|=|I_{\alpha_2}|$. Hence, we have

$$\sum_{\alpha\in J_i} |I_\alpha| = |I_{\alpha_0}|\cdot|J_i|$$

for any $\alpha_0\in J_i$ and for $i=1,2,3$. Hence, we get

$$|I_\omega| = \frac{1}{|J_i|}\sum_{\alpha\in J_i}|I_\alpha| = \begin{cases} ((n-1)/2)!\,2^{(n-1)/2} & \text{if } \omega\in J_1,\\ (n/2)!2^{(n/2)-1} & \text{if } \omega\in J_2,\\ ((n/2)-1)!\,2^{(n/2)-1} & \text{if } \omega\in J_3,\end{cases}$$

which completes the proof. □

5. Application to complete graphs

To illustrate some applications of our results, we enumerate the congruence classes of 2-cell embeddings of the complete graph K_n on n vertices $v_1, v_2, \ldots, v_n$ with respect to a subgroup Γ of $\mathrm{Aut}(K_n)$. The automorphism group $\mathrm{Aut}(K_n)$ of K_n is the full symmetric group S_n. We identify $\mathrm{Aut}(K_n)$ with the symmetric group S_n of n elements $1,2,\ldots,n$. For $n=1,2,3$, it is not difficult to enumerate the congruence classes of 2-cell embeddings of K_n:

$$|\mathscr{C}_\Gamma(K_n)| = \begin{cases} 1 & \text{if } n=1,\\ 1 & \text{if } n=2,\\ 2 & \text{if } n=3,\end{cases}$$

for any subgroup Γ of $\mathrm{Aut}(K_n)$. Note that $|\mathscr{C}_\Gamma(K_3)|=2$; one is 2-cell embedding of K_3 into the sphere S^2 and the other is one into the projective plane. In what follows, we assume $n\geqslant 4$. It is not hard to prove the following lemma, which can be found in [8].

Lemma 5.1. *For a $\gamma\in S_n$ with the cycle type $(j_1,\dots,j_n)$, the number $|E(K_{n_\gamma})|$ is given by*

$$\sum_{k=1}^{n}\left(j_k\left\lfloor\frac{k}{2}\right\rfloor+k\binom{j_k}{2}\right)+\sum_{t=2}^{n}\sum_{r=1}^{t-1}j_rj_t\,\gcd(r,t),$$

where $\lfloor x\rfloor$ is the greatest integer less than or equal to x, and $\gcd(r,t)$ is the greatest common divisor of r and t.

First, let Γ be the trivial subgroup $\{I\}$ of $\mathrm{Aut}(K_n)$. Then the following comes from Theorem 2.7.

Theorem 5.2.

$$|\mathscr{C}_{\{I\}}(K_n)|=2^{n(n-3)/2}((n-2)!)^n.$$

To enumerate $|\mathscr{C}_{\{I\}}(K_n)|$ for some nontrivial subgroups Γ of $\mathrm{Aut}(K_n)$, we first start with the following lemma.

Lemma 5.3. *Let $\gamma\in\mathrm{Aut}(K_n)=S_n$ and $f\in C^0(K_n;\mathbb{Z}_2)$.*

(a) *If γ has more than 3 fixed vertices, then $|\mathrm{Fix}_{(\gamma,f)}|\neq 0$ if and only if γ is the identity I and $f(v_i)=1$ for $i=1,2,\dots,n$. For these γ and f,*

$$|\mathrm{Fix}_{(\gamma,f)}|=2^{\binom{n}{2}}((n-2)!)^n,$$

and

$$\sum_{f\in C^0(K_n;\mathbb{Z}_2)}|\mathrm{Fix}_{(\gamma,f)}|=2^{\binom{n}{2}}((n-2)!)^n.$$

(b) *If γ has exactly three fixed vertices, say v_1,v_2,v_3, then $|\mathrm{Fix}_{(\gamma,f)}|\neq 0$ if and only if $f(v_i)=-1$ for $i=1,2,3$, $f([v])=1$ for any $[v]\in V(K_{n_\gamma})-\{v_1,v_2,v_3\}$, n is odd and $j(\gamma)=(3,(n-3)/2,0,\dots,0)$. In this case*

$$|\mathrm{Fix}_{(\gamma,f)}|=2^{(n^2+6n-15)/4}\left(\left(\frac{n-3}{2}\right)!\right)^3((n-2)!)^{(n-3)/2},$$

and

$$\sum_{f\in C^0(K_n;\mathbb{Z}_2)}|\mathrm{Fix}_{(\gamma,f)}|=2^{(n^2+8n-21)/4}\left(\left(\frac{n-3}{2}\right)!\right)^3((n-2)!)^{(n-3)/2}.$$

(c) *If γ has exactly two fixed vertices, say v_1,v_2, then $|\mathrm{Fix}_{(\gamma,f)}|\neq 0$ if and only if $f(v_i)=-1$ for $i=1,2$, $f([v])=1$ for any $[v]\in V(K_{n_\gamma})-\{v_1,v_2\}$, n is even and $j(\gamma)=(2,(n-2)/2,0,\dots,0)$. In this case*

$$|\mathrm{Fix}_{(\gamma,f)}|=2^{(n^2+4n-8)/4}\left(\left(\frac{n-2}{2}\right)!\right)^2((n-2)!)^{(n-2)/2},$$

and

$$\sum_{f\in C^0(K_n;\mathbb{Z}_2)}|\mathrm{Fix}_{(\gamma,f)}|=2^{(n^2+6n-12)/4}\left(\left(\frac{n-2}{2}\right)!\right)^2((n-2)!)^{(n-2)/2}.$$

(d) *If γ has exactly one fixed vertex, say v_1, then $|\mathrm{Fix}_{(\gamma,f)}|\neq 0$ if and only if (γ,f) satisfies one of the following two conditions.*

(i) $f(v_1)=1, f([v])=1$ *for any* $[v]\in V(K_{n_\gamma})-\{v_1\}$ *and* $j(\gamma)=(1,0,\ldots,0,j_d=(n-1)/d,0,\ldots,0)$ *for some* $d|(n-1)$, $d\neq 1$. *In this case*

$$|\mathrm{Fix}_{(\gamma,f)}|=2^{\frac{n-1}{d}\lfloor\frac{d}{2}\rfloor+d\binom{(n-1)/d}{2}+\frac{n-1}{d}}\phi(d)\left(\frac{n-1}{d}-1\right)!d^{((n-1)/d)-1}((n-2)!)^{(n-1)/d},$$

and

$$\sum_{f\in C^0(K_n;\mathbb{Z}_2)}|\mathrm{Fix}_{(\gamma,f)}|=2^{\frac{n-1}{d}\lfloor\frac{d}{2}\rfloor+\binom{(n-1)/d}{2}+n-1}\phi(d)\left(\frac{n-1}{d}-1\right)!\,d^{(n-1)/d-1}((n-2)!)^{(n-1)/d}$$

(ii) $f(v_1)=-1, f([v])=1$ *for any* $[v]\in V(K_{n_\gamma})-\{v_1\}$, *$n$ is odd and* $j(\gamma)=(1,(n-1)/2,0,\ldots,0)$. *In this case*

$$|\mathrm{Fix}_{(\gamma,f)}|=2^{(n^2+2n-7)/4}\left(\frac{n-1}{2}\right)!((n-2)!)^{(n-1)/2},$$

and

$$\sum_{f\in C^0(K_n;\mathbb{Z}_2)}|\mathrm{Fix}_{(\gamma,f)}|=2^{(n^2+4n-9)/4}\left(\frac{n-1}{2}\right)!((n-2)!)^{(n-1)/2}.$$

(e) *If γ has no fixed vertex, then $|\mathrm{Fix}_{(\gamma,f)}|\neq 0$ if and only if (γ,f) satisfies one of the following two conditions.*

(i) *There is a divisor $d\neq 1$ of n such that* $j(\gamma)=(0,\ldots,0,j_d=(n/d),0,\ldots,0)$ *and* $f([v])=1$ *for any* $[v]\in V(K_{n_\gamma})$. *In this case*

$$|\mathrm{Fix}_{(\gamma,f)}|=2^{\frac{n}{d}\lfloor\frac{d}{2}\rfloor+d\binom{n/d}{2}}((n-2)!)^{\frac{n}{d}},$$

and

$$\sum_{f\in C^0(K_n;\mathbb{Z}_2)}|\mathrm{Fix}_{(\gamma,f)}|=2^{\frac{n}{d}(\lfloor\frac{d}{2}\rfloor-1)+\binom{n/d}{2}}((n-2)!)^{n/d}.$$

(ii) $n=6m+3$, $j(\gamma)=(0,0,1,0,0,(n-3)/6,0,\ldots,0)$, $f([v])=-1$ *for v such that* $l(v;\gamma)=3$ *and* $f([v'])=1$ *for* $[v']\in V(K_{n_\gamma})-\{[v]: l(v;\gamma)=3\}$. *In this case*

$$|\mathrm{Fix}_{(\gamma,f)}|=2^{(n^2+6n-15)/12}\left(\frac{n-3}{2}\right)!\,((n-2)!)^{(n-3)/6},$$

and

$$\sum_{f\in C^0(K_n;\mathbb{Z}_2)}|\mathrm{Fix}_{(\gamma,f)}|=2^{(n^2+16n-21)/12}\left(\frac{n-3}{2}\right)!\,((n-2)!)^{(n-3)/6}.$$

Proof. The sufficiency part of each case is trivial by Theorem 4.3, so we prove only the necessity part.

(a) Let γ have more than 3 fixed vertices and let

$$|\mathrm{Fix}_{(\gamma,f)}|=\psi(\gamma,f)\,2^{|E(K_{n_\gamma})|}\prod_{[v]\in P(\gamma,f)}|P_v(\gamma^{l(v;\gamma)})|\prod_{[v]\in I(\gamma,f)}|I_v(\gamma^{l(v;\gamma)})|\neq 0,$$

so that each factor must be nonzero. Choose a fixed vertex, say v, of γ, then j_1 in the cycle type $j(\gamma|_{N(v)})$ is greater than or equal to 3, which implies $|I_v(\gamma^{l(v;\gamma)})|=0$. Hence, v must be contained in $P(\gamma,f)$, i.e., $f(v)=1$, and $|P_v(\gamma^{l(v;\gamma)})|$ cannot be zero; otherwise $|\mathrm{Fix}_{(\gamma,f)}|=0$. Now it follows from Theorem 4.5, that $j(\gamma|_{N(v)})=(n-1,0,\ldots,0)$, i.e., γ is the identity and $|P_v(\gamma^{l(v;\gamma)})|=(n-2)!$. Since $\psi(\gamma,f)=1$ and $f(v)=1$, $f(v_i)=1$ for any vertex v_i. For such $\gamma=I$ and $f=1$, it is easy to show that

$$|\mathrm{Fix}_{(I,1)}|=\sum_{f\in C^0(K_n;\mathbb{Z}_2)}|\mathrm{Fix}_{(\gamma,f)}|=2^{\binom{n}{2}}((n-2)!)^n.$$

(b) Let γ have exactly three fixed vertices v_1,v_2,v_3 and let $|\mathrm{Fix}_{(\gamma,f)}|\neq 0$. Since $n\geqslant 4$, γ is not the identity, and $|P_{v_i}(\gamma^{l(v_i;\gamma)})|=0$ for $i=1,2,3$, by Theorem 4.5. To be $|\mathrm{Fix}_{(\gamma,f)}|\neq 0$, v_i must be contained in $I(\gamma,f)$ and $f(v_i)=-1$ for $i=1,2,3$. Furthermore, $j(\gamma|_{N(v_i)})$ must be $(2,(n-3)/2,0,\ldots,0)$ to be $|I_{v_i}(\gamma^{l(v_i;\gamma)})|\neq 0$ by Theorem 4.7 and $n-1$ must be even. On the other hand, Theorem 3.2(b) gives that γ^2 is the identity and

$$l(v;\gamma)=\begin{cases}1 & \text{if } v\in\{v_1,v_2,v_3\},\\ 2 & \text{otherwise.}\end{cases}$$

Hence, for any $[v]\in V(K_{n_\gamma})-\{v_1,v_2,v_3\}$, $j(\gamma^2|_{N(v)})=(n-1,0,\ldots,0)$, and $f([v])=1$ because $[v]$ has property $\boldsymbol{P}$. For such γ and f,

$$\begin{aligned}|\mathrm{Fix}_{(\gamma,f)}|&=2^{|E(K_{n_\gamma})|}\prod_{[v]\in V(K_{n_\gamma})-\{v_1,v_2,v_3\}}|P_v(\gamma^2)|\prod_{i=1}^{3}|I_{v_i}(\gamma)|\\&=2^{(n^2+6n-15)/4}\left(\left(\frac{n-3}{2}\right)!\right)^3((n-2)!)^{(n-3)/2},\end{aligned}$$

by Lemma 5.1. Clearly, the number of such f's is $2^{(n-3)/2}$ and hence, we have

$$\sum_{f\in C^0(K_n;\mathbb{Z}_2)}|\mathrm{Fix}_{(\gamma,f)}|=2^{(n^2+8n-21)/4}\left(\left(\frac{n-3}{2}\right)!\right)^3((n-2)!)^{(n-3)/2}.$$

(c) follows by a method similar to the proof of (b).

(d) Let γ fix only one vertex, say v_1, and let $|\mathrm{Fix}_{(\gamma,f)}|\neq 0$. Then we have the following two cases.

(i) Let $f(v_1)=1$. Then, by Theorem 4.5, $j(\gamma|_{N(v_1)})=(0,\ldots,0,j_d=(n-1)/d,0,\ldots,0)$ for some $d|(n-1)$ to be $|\mathrm{Fix}_{(\gamma,f)}|\neq 0$, which gives $j(\gamma)=(1,0,\ldots,0,j_d=(n-1)/d,0,\ldots,0)$.

Hence, for any $v \neq v_1$, $l(v;\gamma)=d$ and $j(\gamma^d|_{N(v)})=(n-1,0,\dots,0)$, i.e., γ^d is the identity, which implies $|I_v(\gamma^{l(v;\gamma)})|=0$. To be $|\mathrm{Fix}_{(\gamma,f)}|\neq 0$, $[v]$ must be in $P(\gamma,f)$ and $f([v])=1$ for any $[v]\in V(K_{n_\gamma})-\{v_1\}$. Now, it is easy to show that

$$|\mathrm{Fix}_{(\gamma,f)}|=2^{\frac{n-1}{d}\lfloor \frac{d}{2}\rfloor+\binom{(n-1)/d}{2}+\frac{n-1}{d}}\phi(d)\left(\frac{n-1}{d}-1\right)!\,d^{((n-1)/d)-1}\,((n-2)!)^{(n-1)/d}.$$

Clearly, the number of such f's is $2^{(d-1)(n-1)/d}$ and hence, we have

$$\sum_{f\in C^0(K_n;\mathbb{Z}_2)}|\mathrm{Fix}_{(\gamma,f)}|=2^{\frac{n-1}{d}\lfloor \frac{d}{2}\rfloor+\binom{(n-1)/d}{2}+n-1}\times\phi(d)\left(\frac{n-1}{d}-1\right)!\,d^{((n-1)/d)-1}\,((n-2)!)^{(n-1)/d}.$$

(ii) Let $f(v_1)=-1$. Then, by Theorem 4.7, $j(\gamma|_{N(v_1)})=(0,(n-1)/2,0,\dots,0)$, that is, $j(\gamma)=(1,(n-1)/2,0,\dots,0)$ and n must be odd to be $|\mathrm{Fix}_{(\gamma,f)}|\neq 0$. Since $j(\gamma^2|_{N(v)})=(n-1,0,\dots,0)$ for any $v\neq v_1$, it must be hold that $f([v])=1$ for any $[v]\in V(K_{n_\gamma})-\{v_1\}$. It is also easy to show that

$$|\mathrm{Fix}_{(\gamma,f)}|=2^{(n^2+2n-7)/4}\left(\frac{n-1}{2}\right)!\,((n-2)!)^{(n-1)/2},$$

and

$$\sum_{f\in C^0(K_n;\mathbb{Z}_2)}|\mathrm{Fix}_{(\gamma,f)}|=2^{(n^2+4n-9)/4}\left(\frac{n-1}{2}\right)!\,((n-2)!)^{(n-1)/2}.$$

(e) Let γ have no fixed vertex and let $|\mathrm{Fix}_{(\gamma,f)}|\neq 0$. Let $d=\min\{l(v;\gamma)\colon v\in V(K_{n_\gamma})\}$. Note that $d>1$ and consider the following two cases.

(i) Suppose that d is even and let v be a vertex such that $l(v;\gamma)=d$. Then $v\gamma^{d/2}(v)$ is an edge fixed by γ^d and $f([v])=1$ by Lemma 3.5. Now, Theorem 3.2 gives that γ^d is the identity and $l(v;\gamma)=d$ for all vertex v, which implies $j(\gamma)=(0,\dots,0,j_d=n/d,0,\dots,0)$. Hence, for any $[v]\in V(K_{n_\gamma})$, $[v]\in P(\gamma,f)$ and $|P_v(\gamma^d)|=(n-2)!$. Now, we have

$$|\mathrm{Fix}_{(\gamma,f)}|=2^{\frac{n}{d}\lfloor \frac{d}{2}\rfloor+d\binom{n/d}{2}}((n-2)!)^{n/d}.$$

Note that the number of such f's is $2^{(d-1)n/d}$. This gives that

$$\sum_{f\in C^0(K_n;\mathbb{Z}_2)}|\mathrm{Fix}_{(\gamma,f)}|=2^{\frac{n}{d}(\lfloor \frac{d}{2}\rfloor-1)+d\binom{n/d}{2}+n}((n-2)!)^{n/d}.$$

(ii) Suppose that d is odd, and consider the following two subcases.

(α) $d>3$. Let v be a vertex such that $l(v;\gamma)=d$. Since d is odd and greater than 3, the number j_1 of $j(\gamma^d|_{N(v)})$ is greater than $d-2$ and $d\geqslant 5$. So $|I_v(\gamma^{l(v;\gamma)})|=0$, by Theorem 4.7

and $f([v])$ must be 1 to be $|\mathrm{Fix}_{(\gamma,f)}|\neq 0$. By Theorem 3.2(a), γ^d is the identity and $l(v;\gamma)=d$ for all vertices v. Hence $j(\gamma)=(0,\dots,0,j_d=n/d,0,\dots,0)$ and we have the same type of computations of $|\mathrm{Fix}_{(\gamma,f)}|$ and $\sum_{f\in C^0(K_n;\mathbb{Z}_2)}|\mathrm{Fix}_{(\gamma,f)}|$ as in case (i).

(β) $d=3$. Let v be a vertex such that $l(v;\gamma)=3$. If $f([v])=1$, then, by Theorem 3.2 (a), we have $j(\gamma)=(0,0,n/3,0,\dots,0)$ and $f([v'])=1$ for all $[v']\in V(K_{n_\gamma})$. If $f([v])=-1$, then, by Theorem 4.7, the cycle type $j(\gamma^3|_{N(v)})$ of $\gamma^3|_{N(v)}$ must be $(2,(n-3)/2,0,\dots,0)$ for $v\in[v]$ to be $|\mathrm{Fix}_{(\gamma,f)}|\neq 0$. This implies that $j(\gamma)=(0,0,1,0,0,(n-3)/6,0,\dots,0)$. Thus $n=6m+3$ and γ^6 is the identity. Now, it comes from Theorem 3.2 that $f([v'])=1$ for $[v']\in V(K_{n_\gamma})-\{[v]\}$. For such (γ,f), we have

$$|\mathrm{Fix}_{(\gamma,f)}|=2^{(n^2+6n-15)/12}\left(\frac{n-3}{2}\right)!((n-2)!)^{(n-3)/6}.$$

Note that the number of such f's is $2^{(5n-3)/6}$. Thus, we have

$$\sum_{f\in C^0(K_n;\mathbb{Z}_2)}|\mathrm{Fix}_{(\gamma,f)}|=2^{(n^2+16n-21)/12}\left(\frac{n-3}{2}\right)!((n-2)!)^{(n-3)/6}. \qquad \square$$

Now, we enumerate the congruence classes of 2-cell embeddings of K_n with respect to the subgroup $\mathbb{Z}_n$ of $\mathrm{Aut}(K_n)$ generated by the n-cycle permutation $(1\,2\,3\cdots n)$. Since the subgroup $\mathbb{Z}_n$ acts freely on $V(K_n)$, the cycle type of any element of $\mathbb{Z}_n$ is $(0,\dots,0,j_d=n/d,0,\dots,0)$ for some $d|n$. Moreover, the number of γ's in $\mathbb{Z}_n$ with such cycle type $(0,\dots,0,j_d=n/d,0,\dots,0)$ is $\phi(d)$. Now, Theorem 4.1 and Lemma 5.3 give the following theorem.

Theorem 5.4. *Let $n\geqslant 4$. Then the number of congruence classes of 2-cell embeddings of K_n with respect to $\mathbb{Z}_n$ is*

$$|\mathscr{C}_{\mathbb{Z}_n}(K_n)|=\frac{1}{n}\sum_{d|n}\phi(d)\,2^{n/d(\lfloor d/2\rfloor-1)+d\binom{n/d}{2}}((n-2)!)^{n/d}.$$

In particular, if n is a prime p, then

$$|\mathscr{C}_{\mathbb{Z}_p}(K_p)|=\frac{1}{p}(p-2)!\,2^{(p-3)/2}(2^{(p-1)(p-3)/2}((p-2)!)^{p-1}+(p-1)).$$

It is well known that the number of permutation in S_n of cycle type $(j_1,\dots,j_n)$ is

$$\frac{n!}{\prod_{k=1}^{n}k^{j_k}j_k!}.$$

Now, Theorem 4.1 and Lemma 5.3 with some elementary but laborious calculations give the following theorem.

Theorem 5.5. *Let $n\geqslant 4$. Then the number of congruence classes of 2-cell embeddings of K_n with respect to* $\mathrm{Aut}(K_n)$ *is given as follows:*

(a) *For odd n*

$$
\begin{aligned}
|\mathscr{C}_{\mathrm{Aut}(K_n)}(K_n)| = &\sum_{d|n} 2^{n/d(\lfloor d/2\rfloor-1)+d\binom{n/d}{2}}((n-2)!)^{n/d}/d^{n/d}(n/d)!\\
&+\frac{1}{n-1}\sum_{d|(n-1),d\neq 1} 2^{(n-1)/d\lfloor d/2\rfloor+d\binom{(n-1)/d}{2}-1}\phi(d)((n-2)!)^{(n-1)/d}\\
&+\frac{1}{3!}2^{(n-3)(n+5)/4}\left(\left(\frac{n-3}{2}\right)!\right)^2((n-2)!)^{(n-3)/2}\\
&+2^{(n^2-2n-7)/4}((n-2)!)^{(n-1)/2}+f(n),
\end{aligned}
$$

where

$$
f(n)=\begin{cases} 2^{(n-3)(n+5)/12}\left(\frac{n-3}{2}\right)!\,((n-2)!)^{(n-3)/6}/\left(3^{(n+3)/6}\left(\frac{n-3}{6}\right)!\right) & \text{if } n=3(\mathrm{mod}\ 6),\\ 0 & \text{otherwise.}\end{cases}
$$

(b) *For even n*

$$
\begin{aligned}
|\mathscr{C}_{\mathrm{Aut}(K_n)}(K_n)| = &\sum_{d|n} 2^{n/d\left(\lfloor d/2\rfloor-1\right)+d\binom{n/d}{2}}((n-2)!)^{n/d}/d^{n/d}(n/d)!\\
&+\frac{1}{n-1}\sum_{d|(n-1),d\neq 1} 2^{(n-1)/d\lfloor d/2\rfloor+d\binom{(n-1)/d}{2}-1}\phi(d)((n-2)!)^{(n-1)/d}\\
&+2^{(n^2-12)/4}\left(\frac{n-2}{2}\right)!((n-2)!)^{(n-2)/2}.
\end{aligned}
$$

In particular, if n is a prime p, then

$$
\begin{aligned}
|\mathscr{C}_{\mathrm{Aut}(K_p)}(K_p)| = &\frac{1}{p!}2^{p(p-3)/2}((p-2)!)^p+\frac{1}{p}2^{(p-3)/2}(p-2)!\\
&+\frac{1}{p-1}\sum_{d|(p-1),d\neq 1} 2^{(p-1/d)\lfloor d/2\rfloor+d\binom{p-1/d}{2}-1}\phi(d)((p-2)!)^{(p-1)/d}\\
&+\frac{1}{3!}2^{(p-3)(p+5)/4}\left(\left(\frac{p-3}{2}\right)!\right)^2((p-2)!)^{(p-3)/2}\\
&+2^{(p^2-2p-7)/4}((p-2)!)^{(p-1)/2}.
\end{aligned}
$$

For example, if $n=4$, then $|\mathscr{C}_{\mathrm{Aut}(K_4)}(K_4)|=11$. Hence, there are 11 congruence classes of embeddings of K_4 with respect to $\mathrm{Aut}(K_4)$. But, in the orientable case, K_4 has only 3 oriented congruence classes of embeddings with respect to $\mathrm{Aut}(K_4)$ (see [13]).

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