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On Modular Forms of Characteristic $p > 0$

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We compare modular forms of characteristic $p > 0$ (i.e., Drinfeld's modular

0. INTRODUCTION

(0.1) Let K be a global field of characteristic $p > 0$ (i.e., a function field of one variable over a finite field of characteristic p) with a marked place, denoted by ∞. For any place v of K, we denote by K_v the completion of K at v and by $\mathcal{O}_v$ the valuation ring of K_v. Let A be the subring of K of regular elements away from ∞ (i.e., of $\lambda \in K$ such that $\lambda \in \mathcal{O}_v$ for all $v \neq \infty$).

(0.2) G denotes the group-scheme GL_2 and Z is its center.

(0.3) The ring of adèles of K, denoted by $\mathbb{A}$, can be written $\mathbb{A} = \mathbb{A}_f \times K_\infty$, where $\mathbb{A}_f$ is the restricted product of $\{K_v\}_v$, v running over the set of places of K not equal to ∞ (the elements of $\mathbb{A}_f$ are called finite adèles). One sets also $\mathcal{O} = \prod_v \mathcal{O}_v$ (v runs over the se of all places of K) and $\mathcal{O}_f = \prod_{v \neq \infty} \mathcal{O}_v$.

(0.4) Following Harder, we will underline elements of adelic nature: for instance, an element $\underline{g} \in G(\mathbb{A}) = G(\mathbb{A}_f) \times G(K_\infty)$ may be decomposed as $\underline{g} = (\underline{g}_f, g_\infty)$ with $\underline{g}_f \in G(\mathbb{A}_f)$ and $g_\infty \in G(K_\infty)$. Elements of $G(K)$, viewed as diagonally embedded in $G(\mathbb{A})$, are not underlined.

(0.5) Definition. An automorphic form with respect to an open compact subgroup $\mathfrak{K}$ of $G(\mathcal{O})$ is a (complex-valued) function f: $G(\mathbb{A}) \to \mathbb{C}$ such that, for all $\gamma \in G(K)$, $\underline{g} \in G(\mathbb{A})$, and $\underline{k} \in \mathfrak{K}Z(K_\infty)$, the equality $f(\gamma \underline{g} \underline{k}) = f(\underline{g})$ holds.

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Moreover, it is called a cusp form if for all $\underline{g} \in G(\mathbb{A})$,

$$\int_{K\backslash\mathbb{A}} f\left(\begin{pmatrix} 1 & \underline{u} \\ 0 & 1 \end{pmatrix} \underline{g}\right) d\underline{u} = 0$$

($d\underline{u}$ is the normalized Haar measure on the compact group $K\backslash\mathbb{A}$).

These notions were first used intensively in positive characteristic by Drinfeld [Dr], although many of its main properties for general reductive groups were given by Harder [Ha]. Recall that cusp forms that transform like the special representation led to a Galois reciprocity law (Drinfeld [Dr]; see also [vdPRe]).

It is clear that the field $\mathbb{C}$ does not play any role in this definition of an automorphic form, so one may replace it by any commutative ring of characteristic zero (with unit). Moreover, the integral in this definition is indeed a finite sum. So in the definition of a cusp form, one can replace $\mathbb{C}$ by any commutative ring containing $\mathbb{Q}$ (as a subring with unit).

(0.6) In what follows, the compact subgroups $\mathfrak{K}$ of $G(\mathcal{O})$ will be of the form $\mathfrak{K} = \mathfrak{K}_f \times \mathfrak{K}_\infty$, with $\mathfrak{K}_f$ and $\mathfrak{K}_\infty$ open compact subgroups of $G(\mathcal{O}_f)$ and $G(\mathcal{O}_\infty)$ respectively.

Modular forms also exist in positive characteristic. They were introduced concretely by Gekeler and Goss ([Gek1, Go], see also [Co]). Their definition, close to the classical one, will be given in Section 1.8. For our purposes, we just recall that they are functions defined on the Drinfeld upper half-plane $\Omega = \mathbb{P}^1_C(C) - \mathbb{P}^1_C(K_\infty)$ with values in C (C is the completion of an algebraic closure of K_∞).

It is possible to make a parallel with the classical case: K with ∞ is the analog of $\mathbb{Q}$ equipped with its ordinary absolute value, K_∞ and C are the analogs of $\mathbb{R}$ and $\mathbb{C}$ respectively, A looks like $\mathbb{Z}$, and Ω is the analog of the Poincaré half-plane. This parallel ends here (in our context). There is no direct link between modular forms and automorphic forms in positive characteristic. On the contrary, it is well known that the two notions of modular forms and automorphic forms coincide in the classical case (see [Gel, Sect. 3]). The main reason for the difference between modular forms and automorphic forms in positive characteristic is probably that there are no tools to go from Ω to $G(\mathbb{A})$ and then to translate functions defined on Ω to functions defined on $G(\mathbb{A})$ (this can be easily accomplished in the classical case).

We do not know how to pass from Ω to $G(\mathbb{A})$, but there are both related to the Bruhat–Tits tree τ of $G(K_\infty)$. On the one hand, τ is isomorphic to the intersection graph of the analytic reduction of Ω, viewed as a rigid analytic space over C ([FevdP, Chap. 5; GekRe, Sect. 1], see also [vdP]). On the other hand, if $\mathfrak{K}$ is an open compact subgroup of $G(\mathbb{A})$ of the form

$\mathfrak{R} = \mathfrak{R}_f \times \mathfrak{R}_\infty$ (see (0.6)), where $\mathfrak{R}_\infty$ is the stabilizer in $G(O_\infty)$ of an edge of τ, we have a one-to-one map between $G(K)\backslash G(\mathbb{A})/\mathfrak{R}Z(K_\infty)$ (compare with (0.5)) and a finite disjoint union of quotients of the set of edges of τ by arithmetic subgroups of $G(K)$ (this will be stated precisely in the next paragraph).

There exist functions on the set of edges of τ that are of particular interest, namely, the harmonic cocycles. They were first introduced in our context by Drinfeld [Dr], who proved that when they take *values in a field of characteristic zero* they are indeed the automorphic forms that transform like the special representation (this result appears in the proof of his reciprocity law in [Dr]; see also [vdPRe] and (1.13) below).

Harmonic cocycles, more precisely a generalization of the above ones, were compared with modular forms by Schneider in the p-adic context [Sc] and by Teitelbaum in positive characteristic. In [Te] (see also (1.9) below), Teitelbaum proves that spaces of harmonic cocycles taking *values in characteristic p* are isomorphic to the spaces of modular forms. It seems to be difficult to lift directly these harmonic cocycles to characteristic zero and then, using Drinfeld's result, to compare them with automorphic forms.

The first result comparing modular and automorphic forms appeared in [GekRe, Section 6.5] (recalled in (1.10)): it relies on modular forms of *weight* 2, doubly cuspidal, with cusp forms (using Teitelbaum's result [Te]).

The purpose of this paper is to study the relationships between automorphic forms and modular forms (*in positive characteristic*). Then, using Teitelbaum's result, we try to interpret harmonic cocycles of equal characteristic (i.e., with values in characteristic p, the same as the base field K) as automorphic forms.

In Section 2 we introduce a notion of automorphic forms of equal characteristic, i.e., taking values ins paces of the same characteristic p as the global field K. In Section 2, we also introduce a notion of special representation (of equal characteristic), which is a variant of the usual one. Then we compare harmonic cocycles and automorphic forms, both of equal characteristic (Theorem 2.4); indeed, we prove that the harmonic cocycles of equal characteristic are also, in some sense, automorphic forms that transform like the special representations (see (2.11)).

The automorphic forms of equal characteristic that we introduce in Section 2 are, as can easily be seen, the reduction modulo p of "automorphic forms" taking values in spaces of characteristic zero. These latter forms are not exactly automorphic forms in the sense of Drinfeld because they do not satisfy to conditions at ∞, but since we work with automorphic forms that transform like the special representations the conditions at ∞ are not essential. We obtain a result (Theorem 3.7) which interprets *modular forms of characteristic p and of weight* $n+2$ (or harmonic cocycles of equal characteristic p and of the same weight) as functions with values in characteristic zero. For the weight 2, this completes a result of [GekRe, Section 6.5] (see Corollary 3.9).

(0.7) For general notions of rigid analytic geometry, we refer the reader to [BGR, GervdP, and FevdP]. The Bruhat–Tits tree of $G(K_\infty)$ is defined and extensively studied in [Se, Chap. 2]. All that is needed concerning the analytic structure of the Drinfeld upper half-plane and its links with the Bruhat–Tits tree of $G(K_\infty)$ is explained in [GekRe, Sect. 1]. The underlying objects and tools that are used here are Drinfeld modules and Drinfeld modular schemes: the details can be found in [GPRV].

1. MODULAR FORMS AND HARMONIC COCYCLES

Let $\Omega = \mathbb{P}^1_C(C) - \mathbb{P}^1_C(K_\infty)$ be the Drinfeld upper half-plane.

(1.1) Let π be a uniformizing parameter of K_∞, with $\pi \in K$. For $n \in \mathbb{Z}$ we write D_n for the subset of $z \in \Omega$ that satisfies $|\pi|^{n+1} \leqslant |z| \leqslant |\pi|^n$ and $|z-\rho\pi^n| \geqslant |\pi|^n$, $|z-\rho\pi^{n+1}| \geqslant |\pi|^{n+1}$ for all $\rho \in \mathbb{F}(\infty)^*$, where $\mathbb{F}(\infty) \hookrightarrow K_\infty$ is isomorphic to the residue field of K at ∞.

For all $z \in K_\infty$ and $n \in \mathbb{Z}$ we set $D_{(n,z)} = z + D_n$. Let I be the set of (n, z) with $n \in \mathbb{Z}$ and z belonging to a set of representatives of $K_\infty/\pi^{n+1}\mathcal{O}_\infty$. Then we have $\Omega = \bigcup_{i \in I} D_i$; more precisely, $(D_i)_{i \in I}$ is a pure covering of Ω. We denote the corresponding analytic reduction by $R\colon \Omega \to \bar{\Omega}$; $\bar{\Omega}$ is a tree of $\mathbb{P}^1_{\mathbb{F}(\infty)}$ and these $\mathbb{P}^1_{\mathbb{F}(\infty)}$ are its irreducible components, each of them meeting $\#(\mathbb{F}(\infty))+1$ others in ordinary double points which are rational over $\mathbb{F}(\infty)$ and any two of them having at most one common point. We denote the intersection graph of $\bar{\Omega}$ by T. An edge e of T corresponds to the intersection of two irreducible components of $\bar{\Omega}$, C_1 and C_2, say. Let $\bar{\Omega}_e$ be the subset of $\bar{\Omega}$ equal to $C_1 \cup C_2$ minus their intersection points with the other irreducible components $C \neq C_1, C_2$. Then $(R^{-1}(\bar{\Omega}_e))_e$ is the previous pure covering $(D_i)_{i \in I}$, where e runs over the set of nonoriented edges of T.

(1.2) Let τ be the Bruhat–Tits tree of $G(K_\infty)$. It is canonically $G(K_\infty)$-isomorphic to T (see [GekRe, Sect. 1]). Now, the term *edge* means *oriented edge*. Let e be an edge of τ or T, then $e(0)$, resp. $e(1)$, is its origin, resp. its end point; $-e$ is the edge with the origin and the end point interchanged.

(1.3) Let $n \in \mathbb{N}$, and let L be a ring containing K_∞ as a subring if $n \neq 0$. The ring L is supposed to be commutative with unit and its subrings are supposed to have the same unit, *as all rings and subrings shall in this paper*. We denote the subspace of $L[X, Y]$ (the polynomial ring in two variables) of homogeneous polynomials of degree n by $V_n(L)$. It is a free L-module of rank $n+1$. It is equipped with a $G(K_\infty)$-action, trivial for

$n=0$, denoted by $\rho_n = G(K_\infty) \to GL(V_n(L))$ and defined in the following way for $n>0$: let $g \in G(K_\infty)$ be such that $g^{-1} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and let j be an integer, $0 \leqslant j \leqslant n$; then $\rho_n(g)(X^j Y^{n-j}) = (aX+bY)^j (cX+dY)^{n-j}$. We set $V_n(L)^* = \mathrm{Hom}_L(V_n(L), L)$. The following definition was given in [Te].

(1.4.) Definition. Let $n \geqslant 2$ be an integer. An L-harmonic cocycle of weight n is a function, f: *edges*$(\tau) \to V_{n-2}(L)^*$ such that:

(i) $f(-e) = -f(e)$ for all edges of τ,

(ii) $\sum_{e(0)=v} f(e) = 0$ for all vertices v of τ, where the sum is taken over the edges with the origin equal to v.

(1.5) Let $\underline{H}^n(L)$ be the set of L-harmonic cocycles of weight n. It is an L-module, equipped with the following $G(K_\infty)$-action: for all $f \in \underline{H}^n(L)$, $g \in G(K_\infty)$, and for all edges e of τ, $g(f)(e) = \rho^*_{n-2}(g)(f(g^{-1}e))$ (where ρ^*_{n-2} is the representation on $V_{n-2}(L)^*$ induced by ρ_{n-2}). If Γ is a subgroup of $G(K_\infty)$, we denote the submodule of elements of $\underline{H}^n(L)$ fixed under the Γ-action (coming from that of $G(K_\infty)$) by $\underline{H}^n(L)^\Gamma$. $\underline{H}^n_!(L)^\Gamma$ (resp. $\underline{H}^n_{!!}(L)^\Gamma$), are the submodules of elements in $\underline{H}^n(L)^\Gamma$ with finite supports modulo Γ (resp. which are zero on the cusps of Γ). We do not explain this notion of cusp here because we do not use it except in the two recalls just below.

(1.6) A subgroup Γ of $G(K)$ is said to be *arithmetic* if $\Gamma \cap G(A)$ (see (0.1)) is commensurable with both Γ and $G(A)$. Let Γ be such an arithmetic subgroup, then the quotient graph $\Gamma \backslash \tau$ is the union of a finite planar graph without ends, denoted $(\Gamma\backslash\tau)^\circ$, and of finitely many half-lines $(\mathscr{L}_i)_{1 \leqslant i \leqslant c}$ [Se, Chap. 2, Theorem 9, p. 143]. These half-lines are the cusps of Γ. Following [Se, Chap. 2, Lemma 6, p. 142] and [Te, Proposition 3], we have

(1.7). Proposition. *Let Γ be an arithmetic subgroup. For any cusp $(\mathscr{L}_i)$ of Γ, let e_i be its "first edge" (i.e., its edge with origin in $(\Gamma\backslash\tau)^\circ$). Then*

(i) *for all $n \geqslant 2$ and any $f \in \underline{H}^n_!(L)^\Gamma$, the support of f modulo Γ is included in*

$$\mathrm{edges}((\Gamma\backslash\tau)^\circ) \cup \{e_i\}_{1 \leqslant i \leqslant c};$$

(ii) *if p does not divide zero in L ($p = \mathrm{char}(K)$), we have $\underline{H}^2_{!!}(L)^\Gamma = \underline{H}^2_!(L)^\Gamma$;*

(iii) *for all $n \geqslant 2$, if p is equal to zero in L we have $\underline{H}^n_!(L)^\Gamma = \underline{H}^n(L)^\Gamma$.*

We will now introduce the notion of the Drinfeld modular form. It was first studied in [Go] and [Gek1]. For the sake of brevity we do not explain all of its properties (as in (1.8) below), but they can be found in [Gek2; Co; and GekRe, Section 2].

(1.8) Let Γ be an arithmetic subgroup of $G(K)$, and let $n \geqslant 2$ and $m \geqslant 0$ be integers. Recall that C is the completion of an algebraic closure of K_∞. A Drinfeld modular form of weight n and type m with respect to Γ is a function $f: \Omega \to C$ that satisfies

(i) for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and for all $z \in \Omega$, $f(\gamma z) = (\det \gamma)^{-m} (cz+d)^n f(z)$;

(ii) f is holomorphic on Ω;

(iii) f is holomorphic at the cusps of Γ.

Moreover, we say that

(iv) a modular form f with respect to Γ is cuspidal (resp. i times cuspidal), if it has a zero (resp. a zero of order at least i) in all cusps of Γ.

We denote the C-vector space of modular forms of weight n and type m with respect to Γ by $M_{n,m}(\Gamma)$, and the subspace of those which are i times cuspidal by $M^i_{n,m}(\Gamma)$. We also set $M^*_{n,m}(\Gamma) = \bigcup_{i \geqslant 1} M^i_{n,m}(\Gamma)$. These spaces are of finite dimension [Gek2].

As mentioned in the introduction, we have the following results.

(1.9) THEOREM [Te, Theorem 16]. *Let $n \geqslant 2$ be an integer, then the C-vector spaces $M^*_{n,0}(\Gamma)$ and $\underline{H}^n(C)^\Gamma$ are canonically isomorphic.*

(1.10) THEOREM [GekRe, Section (6.5)]. *Let $M^2_{2,1}(\Gamma, \mathbb{F}_p)$ be the subspace of elements f of $M^2_{2,1}(\Gamma)$ such that the residues of the holomorphic forms $f(z)\,dz$ are in $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$; then we have*

$$\underline{H}^2_!(\mathbb{Z})^\Gamma \xrightarrow{\text{reduction mod } p} \underline{H}^2_{!!}(\mathbb{F}_p)^\Gamma \simeq M^2_{2,1}(\Gamma, \mathbb{F}_p),$$

the first map being surjective.

The proofs of these two results use the notion of residue for holomorphic differentials defined on Ω, which was introduced by M. van der Put in [FevdP, Chap. I]. A holomorphic form on Ω possesses a residue for each D_i, $i \in I$ (see (1.1)); with the aid of the residue theorem [FevdP, Chap. I, Section 3] and because of the isomorphism between the two trees T and τ (see (1.2)) it gives a harmonic cocycle.

With the aid of (1.10) one can also prove

(1.11) THEOREM [GekRe, Theorem (6.5.3)]. *$M^2_{2,1}(\Gamma)$ and $\underline{H}^2_{!!}(C)^\Gamma$ are naturally isomorphic.*

(1.12) A comparison theorem between automorphic forms and harmonic cocycles was given by Drinfeld in the proof of his Galois reciprocity law ([Dr], see also [vdPRe, Prop. 2.11]). We now describe it.

Let $\mathfrak{K}_f$ be a an open compact subgroup of $G(\mathcal{O}_f)$. Then $G(K)\backslash G(\mathbb{A}_f)/\mathfrak{K}_f$ is finite; let $X \subset G(\mathbb{A}_f)$ be a representative system of this set of double classes. For all $\underline{x} \in X$, set $\Gamma_{\underline{x}} = G(K) \cap x\mathfrak{K}_f x^{-1}$. It is an arithmetic subgroup of $G(K)$. Let L be a ring containing $\mathbb{Z}$ (resp. $\mathbb{Q}$), and let $\mathcal{W}^{\mathfrak{K}_f}(L)$ (resp. $\mathcal{W}_{\circ}^{\mathfrak{K}_f}(L)$) be the space of automorphic forms (with values in L) with respect to an open compact subgroup $\mathfrak{K}$ of $G(\mathcal{O})$ of the form $\mathfrak{K} = \mathfrak{K}_f \times \mathfrak{K}_\infty$ (resp., which moreover have finite supports in $G(K)\backslash G(\mathbb{A})/\mathfrak{K}Z(K_\infty)$), where $\mathfrak{K}_\infty$ is an open compact subgroup of $G(\mathcal{O}_\infty)$ (see Definition 0.5 and its comments). Following Harder, $\mathcal{W}_{\circ}^{\mathfrak{K}_f}(L)$ is the space of L-valued cuspidal automorphic forms with respect to $\mathfrak{K}_f$ [Ha, (1.2.3)].

Let $\mathrm{Sp}_0(L)$ be the space of functions $\mathbb{P}^1_C(K_\infty) \to L$ that are locally constant in the rigid analytic sense, modulo constant functions (a more general definition and details will be given in the next chapter). The group $G(K_\infty)$ acts on $\mathrm{Sp}_0(L)$: we denote this action by sp_0. For $f \in \mathrm{Sp}_0(L)$ and $g \in G(K_\infty)$, $\mathrm{sp}_0(g)\, f$ is the function $u \mapsto f(ug)$; sp_0 is the so-called special representation.

(1.13) THEOREM (Drinfeld). *One has the L-linear isomorphisms*

$$\prod_{\underline{x} \in X} \underline{H}^2(L)^{\Gamma_{\underline{x}}} = \mathrm{Hom}_{L[G(K_\infty)]}\,(\mathrm{Sp}_0(L), \mathcal{W}^{\mathfrak{K}_f}(L))$$

$$\prod_{\underline{u} \in X} \underline{H}^2_!(L)^{\Gamma_{\underline{x}}} = \mathrm{Hom}_{L[G(K_\infty)]}\,(\mathrm{Sp}_0(L), \mathcal{W}_{\circ}^{\mathfrak{K}_f}(L)).$$

Following this theorem one says that harmonic cocycles, of weight 2 and with values in characteristic zero, are *automorphic forms that transform like the special representation.*

(1.14) We now summarize quickly what is known. Let $\mathcal{R}_C$ be a local topological ring, having $\mathbb{Z}$ equipped with the p-adic topology as topological subring and having C as residue field. It follows from (1.10)–(1.13) (and since we have spaces of finite dimension, [Ha]) that

$$\mathrm{Hom}_{G(K_\infty)}(\mathrm{Sp}_0(\mathcal{R}_C), \mathcal{W}_{\circ}^{\mathfrak{K}_f}(\mathcal{R}_C))$$

$$\simeq \prod_{\underline{x} \in X} \underline{H}^2_!(\mathcal{R}_C)^{\Gamma_{\underline{x}}} \xrightarrow{u} \prod_{\underline{x} \in X} \underline{H}^2_{!!}(C)^{\Gamma_{\underline{x}}} \simeq \prod_{\underline{x} \in X} M^2_{2,1}(\Gamma_{\underline{x}});$$

the map u, being the reduction, is surjective.

2. AUTOMORPHIC FORMS OF EQUAL CHARACTERISTIC

The goal of this chapter is to give an analog of Drinfeld's theorem (1.13) for harmonic cocycles of any weight and then for harmonic cocycles with values in characteristic p. It will give an interpretation of modular forms (see (1.9)).

(2.1) DEFINITION. Let L be a ring of characteristic p. Let $\mathfrak{K}_f$ be an open compact subgroup of $G(\mathcal{O}_f)$. An L-valued automorphic form with respect to $\mathfrak{K}_f$ is a function f: $G(\mathbb{A}) \to L$ such that

(i) for all $\gamma \in G(K)$, $\underline{g} \in G(\mathbb{A})$, and $\underline{k}_f \in \mathfrak{K}_f$, the equality $f(\gamma \underline{g} \underline{k}_f) = f(\underline{g})$ holds;

(ii) there exists an open compact subgroup $\mathfrak{K}_\infty$ of $G(\mathcal{O}_\infty)$ such that the support of f is finite in $G(K)\backslash G(\mathbb{A})/(\mathfrak{K}_f \times (\mathfrak{K}_\infty Z(K_\infty)))$.

We denote by $\mathscr{W}_!^{\mathfrak{K}_f}(L)$ the space of these automorphic forms.

We have choosen to require no condition at ∞; we will see later that indeed the contrary is also possible (see (2.11)).

(2.2) DEFINITION. Let $n \in \mathbb{N}$ and let L be a ring of characteristic p containing K_∞ if $n>0$. Let $\mathscr{F}_n(L)$ be the space of locally constant functions $\mathbb{P}^1_C(K_\infty) \to V_n(L)$ and denote by $\mathrm{Sp}_n(L)$ its quotient by the set of constant functions. The group $G(K_\infty)$ acts on $\mathrm{Sp}_n(L)$; we denote by sp_n this actions. For all $h \in \mathrm{Sp}_n(L)$ and $g \in G(K_\infty)$ one has $\mathrm{sp}_n(g)\, h$: $z \mapsto \rho_n(g)\, h(zg)$ (see (1.3)). We call sp_n the (L-valued) special representation of rank n.

(2.3) In this definition, "locally finite" means that, for all $h \in \mathrm{Sp}_n(L)$, there exists a finite open covering $(U_i)_{1 \leqslant i \leqslant r}$ of $\mathbb{P}^1_C(K_\infty)$ such that h is constant on each U_i. $\mathbb{P}^1_C(K_\infty)$ can be viewed as the set of ends of τ, i.e., as the set of equivalent classes of half-lines of τ, two half-lines being equivalent if their intersection contains infinitely many edges (see [Se, Chap. 2, pp. 100–101]). For an (oriented) edge e of τ denote by $U(e)$ the set of equivalent classes of half-lines containing e, then $U(e)_{e \in \mathrm{edges}(\tau)}$ is a basis of open subsets for the topology of $\mathbb{P}^1_C(K_\infty)$ and, for all functions f: $\mathbb{P}^1_C(K_\infty) \to V_n(L)$, locally constant, there exist edges $e_1, ..., e_r$ of τ and $\lambda_1, ..., \lambda_r$ in $V_n(L)$ such that $f = \sum_{1 \leqslant i \leqslant r} \lambda_i 1_{U(e_i)}$ ($1_{U(e_i)}$ is the characteristic function of $U(e_i)$).

Note that we have a $G(K_\infty)$-isomorphism: $\mathrm{Sp}_n(L) \simeq \mathrm{Sp}_0(L) \otimes_L V_n(L)$.

(2.4) THEOREM. *Let $n \in \mathbb{N}$ and let L be a ring of characteristic p containing K_∞ if $n>0$. Let $\mathfrak{K}_f$ be an open compact subgroup of $\mathfrak{K}_f$ and $X \subset G(\mathbb{A}_f)$ be* *a set of representatives of* $G(K)\backslash G(\mathbb{A}_f)/\mathfrak{K}_f$. *For all* $\underline{x}\in X$ *set* $\Gamma_{\underline{x}}=G(K)\cap \underline{x}\mathfrak{K}_f\underline{x}^{-1}$. *Then we have an L-isomorphism,*

$$\prod_{\underline{x}\in X} \underline{H}^{n+2}(L)^{\Gamma_{\underline{x}}} \simeq \mathrm{Hom}_{L[G(K_\infty)]}(\mathrm{Sp}_n(L), \mathscr{W}_!^{\mathfrak{K}_f}(L))$$

($G(K_\infty)$ *acts on* $\mathscr{W}_!^{\mathfrak{K}_f}(L)$ *via the regular representation of* $G(\mathbb{A})$).

The proof needs many steps.

(2.5) Let $E=E(G(\mathbb{A}_f)/\mathfrak{K}_f, L)$ be the set of functions f: $G(\mathbb{A}_f)\to L$ right invariant under $\mathfrak{K}_f$. An element of $\underline{H}^{n+2}(E)$ can be viewed as a function φ: edges$(\tau)\times G(\mathbb{A}_f)\to V_n^*(L)$; then one sees that $\underline{H}^{n+2}(E)$ is equipped with the following action of $G(K)$: For all $\gamma\in G(K)$, $e\in$ edges(τ), and $\underline{g}\in G(\mathbb{A}_f)$, $\gamma(\varphi)(e,\underline{g})=\rho_n^*(\gamma)(\varphi(\gamma^{-1}e,\gamma^{-1}\underline{g}))$, where ρ^* is the action of $G(K_\infty)$ on $V_n^*(L)$ coming from that on $V_n(L)$ (see (1.3)).

(2.6) LEMMA. *One has an L isomorphism* $\underline{H}^{n+2}(E)^{G(K)}\simeq_L \prod_{\underline{x}\in X}\underline{H}^{n+2}(L)^{\Gamma_{\underline{x}}}$.

Proof. Let φ: edges$(\tau)\times G(\mathbb{A}_f)\to V_n^*(L)$ be an element of $\underline{H}^{n+2}(E)^{G(K)}$. One has $G(\mathbb{A}_f)$: $\coprod_{\underline{x}\in X} G(K)\,\underline{x}\mathfrak{K}_f$ (disjoint union). Let

$$\underline{H}^{n+2}(E)^{G(K)} \xrightarrow{\Phi} \prod_{\underline{x}\in X} \underline{H}^{n+2}(L)^{\Gamma_{\underline{x}}}$$

$$\varphi \mapsto (\varphi_{\underline{x}})_{\underline{x}\in X},$$

where $\varphi_{\underline{x}}=\varphi(\ ,\underline{x})$. For an edge e of τ, for $\underline{x}\in X$, and $\gamma\in\Gamma_{\underline{x}}$ with $\gamma=\underline{xkx}^{-1}$, where $\underline{k}\in\mathfrak{K}_f$ (see the definition of $\Gamma_{\underline{x}}$ in Theorem 2.4), one has $\rho_n^*(\gamma^{-1})\,\varphi_{\underline{x}}(\gamma e)=\rho_n^*(\gamma^{-1})\,\varphi(\gamma e,\underline{x})=\varphi(e,\gamma^{-1}\underline{x})$ because φ is invariant under $G(K)$. Then $\rho_n^*(\gamma^{-1})\,\varphi_{\underline{x}}(\gamma e)=\varphi(e,\underline{xk})=\varphi(e,\underline{x})$ which proves that Φ is well defined. The inverse map is given by $(\psi)_{\underline{x}}\mapsto((e,\gamma\underline{x}_0\underline{k})\mapsto\rho_n^*(\gamma)\,\psi_{\underline{x}_0}(\gamma^{-1}e))$. ∎

(2.7) LEMMA. *One has an L-isomorphism*

$$\underline{H}^{n+1}(E)^{G(K)}\simeq_L \mathrm{Hom}_L(\mathrm{Sp}_n(L), E)^{G(K)}$$

the action of $G(K)$ *on* $\mathrm{Hom}_L(\mathrm{Sp}_n(L),E)$ *coming from those on* $\mathrm{Sp}_n(L)$ *via* sp_n *and on* E.

Proof. One interprets elements of $\underline{H}^{n+2}(E)^{G(K)}$ as in (2.5). An element $\zeta\in\mathrm{Hom}_L(\mathrm{Sp}_n(L),E)^{G(K)}$ can be viewed as a function ζ: $\mathrm{Sp}_n(L)\times G(\mathbb{A}_f)\to$

L and recall that the functions of the form $\lambda 1_{U(e)}$ for $\lambda \in V_n(L)$ generate $\mathrm{Sp}_n(L)$ (see (2.3)). Then one can define

$$\underline{H}^{n+2}(E)^{G(K)} \xrightarrow{\Psi} \mathrm{Hom}_L(\mathrm{Sp}_n(L), E)^{G(K)}$$
$$\varphi \mapsto ((\lambda 1_{U(e)}, \underline{g}) \mapsto \varphi(e, \underline{g})(\lambda))$$

The inverse map to $\zeta \in \mathrm{Hom}_L(\mathrm{Sp}_n(L), E)^{G(K)}$ assigns the function $\mathrm{edges}(\tau) \times G(\mathbb{A}_f) \to V_n^*(L)$ which maps $(e, \underline{g})$ to $\zeta(\cdot 1_{U(e)}, \underline{g})$. This is the expected isomorphism. ▮

(2.8) LEMMA. *Let $\mathscr{W}_?^{\mathfrak{K}_f}(L)$ be the set of functions satisfying the assertions of Definition* 2.1 *except for* (ii). *Then, one has an L-isomorphism*

$$\mathrm{Hom}_L(\mathrm{Sp}_n(L), E)^{G(K)} \simeq_L \mathrm{Hom}_{L[G(K_\infty)]}(\mathrm{Sp}_n(L), \mathscr{W}_?^{\mathfrak{K}_f}(L)).$$

Proof. To a function $\zeta \in \mathrm{Hom}_L(\mathrm{Sp}_n(L), E)^{G(K)}$, viewed as in the proof of (2.7), one associates $\Theta(\zeta)\colon \mathrm{Sp}_n(L) \to \mathscr{W}_?^{\mathfrak{K}_f}(L)$, such that for $f \in \mathrm{Sp}_n(L)$, $\Theta(\zeta)(f)$ is the function $G(\mathbb{A}_f) \times G(K_\infty) \to L$ which maps $(\underline{g}_f, \underline{g}_\infty)$ to $\zeta(\mathrm{sp}_n(\underline{g}_\infty) f)(\underline{g}_f)$. It is easy to see that it gives the desired isomorphism. ▮

(2.9) *End of the Proof of Theorem* 2.4. One has to prove that one can replace $\mathscr{W}_?^{\mathfrak{K}_f}(L)$ by space $\mathscr{W}_!^{\mathfrak{K}_f}(L)$ of Definition 2.1. One uses the notations of the proofs of the three previous lemmata. Let $(\varphi_{\underline{x}})_{\underline{x} \in X} \in \prod_{\underline{x} \in X} \underline{H}^{n+2}(L)^{\Gamma_{\underline{x}}}$ and let $\Theta(\zeta)$ be its image in $\mathrm{Hom}_{G(K_\infty)}(\mathrm{Sp}_n(L), \mathscr{W}_?^{\mathfrak{K}_f}(L))$ by the composition of the three preceeding isomorphisms (see the proof of Lemma 2.8). Let $\lambda \in V_n(L)$ and set $w = \Theta(\zeta)(\lambda 1_{U(e)})$, then w is a map $G(\mathbb{A}_f) \times G(K_\infty) \to L$. Choose $\gamma \in G(K)$, $\underline{x} \in X$, $\underline{k}_f \in \mathfrak{K}_f$, and $g_\infty \in G(K_\infty)$. One has $w(\gamma \underline{x} \underline{k}_f, g_\infty) = \rho_n^*(\gamma)(\varphi_{\underline{x}}(\gamma^{-1} g_\infty e))(\rho_n(g_\infty)\lambda)$. It follows that $w(\gamma \underline{x} \underline{k}_f, g_\infty) \neq 0$ implies $\gamma_{-1} g_\infty e \in \mathrm{supp}(\varphi_{\underline{x}})$. There exists a finite set $S \subset G(K_\infty)$ such that $\mathrm{supp}(\varphi_{\underline{x}}) \subset \Gamma_{\underline{x}} S \mathfrak{K}_\infty Z(K_\infty)$, where $\mathfrak{K}_\infty$ is the stabilizer of e in $G(\mathbb{O}_\infty)$ (and Z is the center of G). Then

$$\mathrm{supp}(w) \cap [(G(K)\, \underline{x} \mathfrak{K}_f) \times G(K_\infty)] \subset (G(K)\, \underline{x} \mathfrak{K}_f) \times (S \mathfrak{K}_\infty Z(K_\infty)).$$

This finishes the proof of (2.4). ▮

It follows from (1.9) and (2.4) that we have

(2.10) COROLLARY. *There exists a C-linear isomorphism*

$$\prod_{\underline{x} \in X} M_{n+2,0}^*(\Gamma_{\underline{x}}) \simeq \mathrm{Hom}_{C[G(K_\infty)]}(\mathrm{Sp}_n(C), \mathscr{W}_!^{\mathfrak{K}_f}(C)).$$

(2.11) *Remark.* It is possible to prescribe a condition at ∞ in Definition 2.1; now we explain this. We continue Definition 2.1 by adding the following condition:

(iii) Let $S(f)$ be the L-submodule of $\mathscr{W}_{!}^{\mathfrak{K}_f}(L)$ generated by f and $\mathfrak{K}_\infty$ acting on f via the regular representation (i.e., the action of $k_\infty \in \mathfrak{K}_\infty$ on f gives the function on $G(\mathbb{A})$ $\underline{g} \mapsto f(\underline{g}k_\infty)$). Then there should exist an L-submodule $Q(f)$ of $\mathrm{Sp}_n(L)$ and an L-morphism $\varepsilon(f)$: $Q(f) \to S(f)$ such that $Q(f)$ is stable under $\mathrm{sp}_n(\mathfrak{K}_\infty)$ and is generated, as an $L[\mathrm{sp}_n(\mathfrak{K}_\infty)]$-module, by one element; $\varepsilon(f)$ is surjective and $\mathfrak{K}_\infty$-equivariant.

Let $\mathscr{W}_{!,\infty}^{\mathfrak{K}_f}(L)$ be the set of elements of $\mathscr{W}_{!}^{\mathfrak{K}_f}(L)$ which satisfy (iii). One has

$$\mathrm{Hom}_{L[G(K_\infty)]}(\mathrm{Sp}_n(L), \mathscr{W}_{!}^{\mathfrak{K}_f}(L)) \simeq_L \mathrm{Hom}_{L[G(K_\infty)]}(\mathrm{Sp}_n(L), \mathscr{W}_{!,\infty}^{\mathfrak{K}_f}(L)).$$

Proof. One continues with the notations of (2.4)–(2.9). Let again, as in (2.9), $\Theta(\zeta) \in \mathrm{Hom}_{G(K_\infty)}(\mathrm{Sp}_n(C), \mathscr{W}_{!}^{\mathfrak{K}_f}(C))$ and $u \in \mathrm{Sp}_n(L)$. Let $\underline{g}_f \in G(\mathbb{A}_f)$, $g_\infty \in G(K_\infty)$, and set $f = \Theta(\zeta)(u)$. One has $f((\underline{g}_f, g_\infty)) = \zeta(\mathrm{sp}_n(g_\infty)(u))(\underline{g}_f)$. There exists edges of τ, $(e_i)_{1 \leqslant i \leqslant r}$, and elements of $V_n(L)$, $(\lambda_i)_{1 \leqslant i \leqslant r}$, such that $u = \sum_{1 \leqslant i \leqslant r} \lambda_i 1_{U(e_i)}$ and one can choose as subgroup $\mathfrak{K}_\infty$ for f the intersection of the stabilizers in $G(K_\infty)$ of the e_i's. Let $Q(f) = L[\mathrm{sp}_n(\mathfrak{K}_\infty)]\, u$; then one has ($k_\infty$ is in $\mathfrak{K}_\infty$) $\varepsilon(f)(\mathrm{sp}_n(k_\infty)\, u) = (\underline{g} \mapsto f(\underline{g}k_\infty))$. ∎

This property (2.11) permits us to say that *harmonic cocycles of weight $n+2$ are automorphic forms that transform like the special representation of rank n.*

(2.12) Let $\mathscr{A}_{!}^{\mathfrak{K}_f}(L)$ be the set of functions ψ: $\mathrm{Sp}_0(L) \times G(\mathbb{A}) \to L$ such that, for all $u \in \mathrm{Sp}_0(L)$ and $\underline{g} \in G(\mathbb{A})$, $\psi(u, \cdot)$ satisfies assertions (i) and (ii) of (2.1) and $\psi(\cdot, \underline{g})$ is L-linear. The group $G(K_\infty)$ acts on $\mathscr{A}_{!}^{\mathfrak{K}_f}(L)$: if $g_\infty \in G(K_\infty)$, one has $g_\infty(\psi)(u, \underline{g}) = \psi(\mathrm{sp}_0(g_\infty^{-1})\, u, \underline{g}g_\infty)$. The next proposition gives a variant of Theorem 2.4.

(2.13) Proposition. *One has a natural L-isomorphism*

$$\mathrm{Hom}_{L[G(K_\infty)]}(\mathrm{Sp}_n(L), \mathscr{W}_{!}^{\mathfrak{K}_f}(L)) \simeq_L \mathrm{Hom}_{L[G(K_\infty)]}(V_n(L), \mathscr{A}_{!}^{\mathfrak{K}_f}(L)).$$

Proof. Let $\varphi \in \mathrm{Hom}_{L[G(K_\infty)]}(\mathrm{Sp}_n(L), \mathscr{W}_{!}^{\mathfrak{K}_f}(L))$; it can be viewed as a function φ_1: $\mathrm{Sp}_n(L) \times G(\mathbb{A}) \to L$, then (because $\mathrm{Sp}_n(L) \simeq V_n(L) \otimes_L \mathrm{Sp}_0(L)$ as $L[G(K_\infty)]$-modules) as a function φ_2: $V_n(L) \times \mathrm{Sp}_0(L) \times G(\mathbb{A}) \to L$ satisfying the following properties: for all $v \in V_n(L)$, $u \in \mathrm{Sp}_0(L)$, $\underline{g} \in G(\mathbb{A})$, and $g_\infty \in G(K_\infty)$,

— $\varphi_2(v, u, .)$ satisfies (i) and (ii) of (2.1),

— $\varphi_2(., ., \underline{g})$ is L-bilinear,

— $\varphi_2(\rho_n(g_\infty)\, v, sp_0(g_\infty)\, u, \underline{g}) = \varphi_2(v, u, \underline{g}g_\infty)$.

Clearly, the map $\varphi \mapsto (v \mapsto \varphi_2(v, ., .))$ gives the desired isomorphism. ∎

3. CHARACTERISTIC ZERO AND CHARACTERISTIC $p>0$

In all this paragraph, L is a *field of characteristic* $p>0$, $n\geqslant 0$ is an integer, and we suppose moreover that $K_\infty\subset L$ if $n>0$. We want to lift our harmonic cocycles (or modular forms) to characteristic zero. One needs first

(3.1) PROPOSITION. *$V_n(L)$ is a cyclic $L[G(K_\infty)]$-module.*

Proof. One can suppose that $n>0$.

(3.2) Let $\mathscr{D}$ be the set of integers m and mp^r-1 with $0<m<p$ and $r>0$. Note that the binomial coefficient $\binom{n}{i}$ is not zero modulo p for all i, $0\leqslant i\leqslant n$, if and only if $n\in\mathscr{D}$.

(3.3) Let $n>0$ be an integer and let $\alpha=\max\{\beta\in\mathscr{D}/\beta\leqslant n\}$. It is easy to see that $\alpha\geqslant n/2$.

Let n and α be as before. For a in K_∞^* let γ_a and δ_a be the two matrices such that $\gamma_a^{-1}=\left(\begin{smallmatrix}a&1\\1&0\end{smallmatrix}\right)$ and $\delta_a^{-1}=\left(\begin{smallmatrix}a&1\\0&1\end{smallmatrix}\right)$. One has (see (1.3))

$$\gamma_a(X^\alpha Y^{n-\alpha})=\sum_{0\leqslant i\leqslant\alpha}\binom{\alpha}{i}a^iX^{n-\alpha+i}Y^{\alpha-i}\in L[G(K_\infty)]\,X^\alpha Y^{n-\alpha}$$

$$\delta_a(X^\alpha Y^{n-\alpha})=\sum_{0\leqslant i\leqslant\alpha}\binom{\alpha}{i}a^iX^iY^{n-i}\in L[G(K_\infty)]\,X^\alpha Y^{n-a}$$

for all a in K_∞. As $\binom{\alpha}{i}\neq 0$, it follows from the first formula that $X^{n-i}Y^i\in L[G(K_\infty)]\,X^\alpha Y^{n-\alpha}$ and from the second formula that $X^iY^{n-i}\in L[G(K_\infty)]\,X^\alpha Y^{n-\alpha}$, for all i, $0\leqslant i\leqslant\alpha$. As $\alpha\geqslant n/2$ (see (3.3)), one has proved

$$V_n(L)=L[G(K_\infty)]\,X^\alpha Y^{n-\alpha}. \quad \blacksquare$$

(3.4) Let $\mathscr{R}_L$ be a local ring of characteristic zero with maximal ideal $\mathfrak{M}_L$ and residue field $\mathscr{R}_L/\mathfrak{M}_L=L$. One denotes by s the canonical morphism $\mathscr{R}_L\to\mathscr{R}_L/\mathfrak{M}_L=L$.

(3.5) Let $\mathscr{A}_?^{\mathfrak{K}f}(\mathscr{R}_L)$ be the set of functions $f\colon \mathrm{Sp}_0(\mathscr{R}_L)\times G(\mathbb{A})\to\mathscr{R}_L$ such that, for all $u\in\mathrm{Sp}_0(\mathscr{R}_L)$ and $\underline{g}\in G(\mathbb{A})$, $f(u,\cdot)$ satisfies assertion (i) of (2.1) and $f(\cdot,\underline{g})$ is $\mathscr{R}_L$-linear (see the definition of $\mathscr{W}_?^{\mathfrak{K}f}(L)$ given in Lemma 2.8).

(3.6) Let $\widetilde{\mathrm{Hom}}_{L,G(K_\infty)}(V_n(L),\mathscr{A}_?^{\mathfrak{K}f}(\mathscr{R}_L))$ be the space of $G(K_\infty)$-linear maps $\psi\colon V_n(L)\to\mathscr{A}_?^{\mathfrak{K}f}(\mathscr{R}_L)$ such that, for any $v\in V_n(L)$, $s\circ\psi(v)$ is L-linear (see (3.4)).

(3.7) THEOREM. *One has a natural surjective map*

$$\widetilde{\mathrm{Hom}}_{L,\, G(K_\infty)}(V_n(L), \mathscr{A}_?^{\mathfrak{K}_f}(\mathscr{R}_L)) \to \mathrm{Hom}_{L[G(K_\infty)]}(V_n(L), \mathscr{A}_!^{\mathfrak{K}_f}(L)).$$

Proof. Let $\mathscr{A}_?^{\mathfrak{K}_f}(L)$ be the space of functions $h\colon \mathrm{Sp}_0(L) \times G(\mathbb{A}) \to L$ such that, for all $u \in \mathrm{Sp}_0(L)$ and $\underline{g} \in G(\mathbb{A})$, $h(u, \cdot)$ satisfies assertion (i) of (2.1) and $h(\cdot, \underline{g})$ if L-linear. Let f be in $\mathscr{A}_?^{\mathfrak{K}_f}(\mathscr{R}_L)$ and $\underline{g}$ be in $G(\mathbb{A})$; note that the $\mathscr{R}_L$-linearity of $f(\cdot, \underline{g})$ implies that $s \circ f(\cdot, \underline{g})$ is zero on $\mathfrak{M}_L V_n(L)$. This last sentence is equivalent to saying that $s \circ f(u, \underline{g}) = 0$ if u takes values in $\mathfrak{M}_L$ (because u takes finitely many values).

Let $f \in \mathscr{A}_?^{\mathfrak{K}_f}(\mathscr{R}_L)$ and $u \in \mathrm{Sp}_0(L)$. We have just seen that $s \circ f(u, \cdot)$ makes sense; it defines a map $\mathscr{A}_?^{\mathfrak{K}_f}(\mathscr{R}_L) \to \mathscr{A}_?^{\mathfrak{K}_f}(L)$, which induces a morphism

$$\widetilde{\mathrm{Hom}}_{L,\, G(K_\infty)}(V_n(L), \mathscr{A}_?^{\mathfrak{K}_f}(\mathscr{R}_L)) \to \mathrm{Hom}_{L[G(K_\infty)]}(V_n(L), \mathscr{A}_?^{\mathfrak{K}_f}(L)).$$

With Proposition 3.1, one sees that this map is surjective. Finally, as in (2.9) (see also (2.13)) one proves that

$$\mathrm{Hom}_{L[G(K_\infty)]}(V_n(L), \mathscr{A}_?^{\mathfrak{K}_f}(L)) \simeq \mathrm{Hom}_{L[G(K_\infty)]}(V_n(L), \mathscr{A}_!^{\mathfrak{K}_f}(L)). \quad \blacksquare$$

(3.8) Theorem 3.7, with Theorems 2.4 and 1.9, implies, when $L = C$, i.e., when L is equal to the completion C of an algebraic closure of K_∞, that one has the diagram

$$\begin{aligned}\widetilde{\mathrm{Hom}}_{C,\, G(K_\infty)}(V_n(C), \mathscr{A}_?^{\mathfrak{K}_f}(\mathscr{R}_C)) &\to \mathrm{Hom}_{C[G(K_\infty)]}(\mathrm{Sp}_n(C), \mathscr{W}_!^{\mathfrak{K}_f}(C)) \\ &\simeq \prod_{\underline{x} \in X} \underline{H}^{n+2}(C)^{\Gamma_{\underline{x}}} \\ &\simeq \prod_{\underline{x} \in X} M^*_{n+2,\,0}(\Gamma_{\underline{x}}),\end{aligned}$$

the first map being surjective. Then, one sees that modular forms in characteristic p, or harmonic cocycles in equal characteristic p, of weight $n+2$, are indeed essentially objects coming from the characteristic zero. When $n = 0$, one has a more precise result, which completes [GekRe, Section 6.5] (recalled in Theorem 1.10).

(3.9) COROLLARY. *Let $\mathscr{R}$ be a local ring of characteristic zero with residue field L of characteristic $p > 0$ and let Γ be an arithmetic subgroup of $G(K)$. Then, one has a natural surjective $\mathscr{R}$-morphism*

$$\underline{H}^2(\mathscr{R})^{\Gamma} \to \underline{H}^2(L)^{\Gamma}.$$

Proof. Let $\mathscr{W}_?^{\mathfrak{K}_f}(\mathscr{R})$ be the set of functions $G(\mathbb{A}) \to \mathscr{R}$ such that for all $\gamma \in G(K)$, $\underline{g} \in G(\mathbb{A})$, and $\underline{k}_f \in \mathfrak{K}_f$ the equality $f(\gamma \underline{g} \underline{k}_f) = f(\underline{g})$ holds (see

Definition 0.5, Section 1.12, and Lemma 2.8). It is easy to prove, as in Section 2.9, that

$$\mathrm{Hom}_{\mathscr{R}[G(K_\infty)]}(\mathrm{Sp}_0(\mathscr{R}), \mathscr{W}_?^{\mathfrak{K}_f}(\mathscr{R})) \simeq_{\mathscr{R}} \mathrm{Hom}_{\mathscr{R}[G(K_\infty)]}(\mathrm{Sp}_0(\mathscr{R}), \mathscr{W}^{\mathfrak{K}_f}(\mathscr{R})).$$

As in Proposition 2.13, one has also

$$\mathrm{Hom}_{\mathscr{R}[G(K_\infty)]}(\mathrm{Sp}_0(\mathscr{R}), \mathscr{W}_?^{\mathfrak{K}_f}(\mathscr{R})) \simeq_{\mathscr{R}} \mathrm{Hom}_{\mathscr{R}[G(K_\infty)]}(V_0(\mathscr{R}), \mathscr{A}_?^{\mathfrak{K}_f}(\mathscr{R})).$$

Since $V_0(\mathscr{R})=\mathscr{R}$ and $V_0(L)=L$, with trivial actions of $G(K_\infty)$, it is clear that there exists a surjective map

$$\mathrm{Hom}_{\mathscr{R}[G(K_\infty)]}(V_0(\mathscr{R}), \tilde{\mathscr{A}}_?^{\mathfrak{K}_f}(\mathscr{R})) \to \widetilde{\mathrm{Hom}}_{L,\, G(K_\infty)}(V_0(L), \tilde{\mathscr{A}}_?^{\mathfrak{K}_f}(\mathscr{R})).$$

This last map, the two previous isomorphisms, Theorem 3.7, and Proposition 2.13 give

$$\mathrm{Hom}_{\mathscr{R}[G(K_\infty)]}(\mathrm{Sp}_0(\mathscr{R}), \mathscr{W}^{\mathfrak{K}_f}(\mathscr{R})) \to \mathrm{Hom}_{L[G(K_\infty)]}(\mathrm{Sp}_0(L), \mathscr{W}_!^{\mathfrak{K}_f}(L)),$$

which is surjective and, together with Theorems 1.13 and 2.4, gives the desired result. ∎

4. SOME COMMENTS

Let n and l be two non-negative integers. As before, C is the completion of an algebraic closure of K_∞.

One can twist the representations ρ_n, that is, one can consider $V_n(C)$ equipped with the action $g \mapsto \det(g)^l \rho_n(g) \in GL(V_n(C))$ of $G(K_\infty)$ (see Section 1.3). One denotes by $V_{n,l}(C)$ the space $V_n(C)$ equipped with this last action; one denotes also by $\underline{H}^{n+2,l}(C)$ the harmonic cocycles with values in $V_{n,l}(C)^*$ (see Definition 1.4). The isomorphism of Theorem 1.9 is proved for $l=0$ in [Te], but, with exactly the same arguments, it can be extended to all $l \geqslant 0$. Then one can prove, as in Section 3.8, that

$$\begin{aligned}\widetilde{\mathrm{Hom}}_{C,\, G(K_\infty)}(V_{n,l}(C), \mathscr{A}_?^{\mathfrak{K}_f}(\mathscr{R}_C)) \to \mathrm{Hom}_{C[G(K_\infty)]}(\mathrm{Sp}_n(C), \mathscr{W}_!^{\mathfrak{K}_f}(C)) \\ \simeq \prod_{\underline{x}\in X} \underline{H}^{n+2,l}(C)^{\Gamma_{\underline{x}}} \\ \simeq \prod_{\underline{x}\in X} M^*_{n+2,l}(\Gamma_{\underline{x}}).\end{aligned}$$

(Recall that there do not exist modular forms of weight one; see [Co, Theorem 6.9.1]).

Let $\xi: K_\infty^* \to C^*$ be a character and let $V_{n,\xi}(C)$ be $V_n(C)$ equipped with the action $g \mapsto \xi(\det(g))\, \rho_n(g)$ of $G(K_\infty)$. Let $\underline{n} = (n_1, ..., n_r) \in \mathbb{N}^r$ and let $\underline{\xi} = (\xi_1, ..., \xi_r)$ where the $\xi_i: K_\infty^* \to C^*$ are characters. Set

$$V_{\underline{n},\underline{\xi}}(C) = V_{n_1,\xi_1}(C) \otimes_C \cdots \otimes_C V_{n_r,\xi_r}(C).$$

Harmonic cocycles with values in this space make sense, and properties closed to (2.4) or (3.7) can be proved but they have no interpretation by the "usual modular forms."

Let $\mathbb{F}$ be a finite subfield of C. The group $G(\mathbb{F})$ acts on $V_{\underline{n},\underline{\xi}}(C)$ (for characters $\mathbb{F}^* \to C^*$ and by the same law as $G(K_\infty)$). If $0 \leqslant n_i \leqslant p-1$ (p is the characteristic of our fields), these representations are, up to isomorphisms, the irreducible representations of $G(\mathbb{F})$ [BaLi]. One does not know what sort of representations of $G(K_\infty)$ are $V_{\underline{n},\underline{\xi}}(C)$. It is easy to see that, if p divides n, the representation $V_n(C)$ of $G(K_\infty)$ is not irreducible: $\bigoplus_{0 \leqslant j \leqslant n/p} CX^{pj}Y^{n-pj}$ is a subrepresentation. Maybe $V_n(C)$ is an irreducible representation of $G(K_\infty)$ if and only if $0 \leqslant n < p$ or $n = mp^r - 1$ with $0 < m < p$ and $r > 0$ (see Section 3.2)?

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