One-dimensional rings of finite F-representation type

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1. Introduction

Smith and Van den Bergh [2] introduced the notion of finite F-representation type as a characteristic $p$ analogue of the notion of finite Cohen–Macaulay representation type. Rings of finite F-representation type satisfy several nice properties. For example, Seibert [1] proved that the Hilbert–Kunz multiplicities are rational numbers, Yao [4] proved that tight closure commutes with localization in such rings, and Takagi and Takahashi [3] proved that if $R$ is a Cohen–Macaulay ring of finite F-representation type with canonical module $\omega_R$, then $H^n_I(\omega_R)$ has only finitely many associated primes for any ideal $I$ of $R$ and any integer $n$. However, it is difficult to determine whether a given ring has finite F-representation type. The main theorem of this paper is the following.

**Theorem 1** (Theorem 3.1, Theorem 3.5, Example 3.3). Let $A$ be a one-dimensional complete local domain of prime characteristic with the residue field $k$, or a one-dimensional $\mathbb{N}$-graded domain $\bigoplus_{i \geq 0} A_i$ of prime characteristic with $A_0 = k$ a field. Then

1. if $k$ is algebraically closed, then any finitely generated (and graded if $A$ is graded) $A$-module has finite F-representation type,
2. If $k$ is finite, then $A$ has finite F-representation type.
3. There exist examples of rings which do not have finite F-representation type with $k$ perfect.

We also give several examples of finite F-representation type of dimension higher than one. There is a question posed by Brenner:

**Question 2 (Brenner).** Let $k$ be an algebraically closed field of characteristic $p$. Then does the ring $k[x, y, z]/(x^2 + y^3 + z^7)$ have finite F-representation type?

We prove that $k[x, y, z]/(x^2 + y^3 + z^7)$ has finite F-representation type if $p = 2, 3$ or 7.

2. **Rings of finite F-representation type**

Throughout this paper, all rings are Noetherian commutative rings of prime characteristic $p$. We denote by $\mathbb{N} = \{0, 1, 2, \ldots \}$ the set of non-negative integers. If $R = \bigoplus_{i \geq 0} R_i$ is an $\mathbb{N}$-graded ring, we assume that $\gcd\{i \mid R_i \neq 0\} = 1$ and $R_0$ is a field. We denote by $R_{+} = \bigoplus_{i > 0} R_i$ the unique homogeneous maximal ideal of $R$.

The Frobenius map $F : R \rightarrow R$ is the endomorphism of $R$ sending $r$ to $r^p$ for all $r \in R$. For an $R$-module $M$, we denote by $e^p M$ the module $M$ with its $R$-module structure pulled back via the $e$-times iterated Frobenius map $F^e : r \mapsto r^{p^e}$, that is, $e^p M$ is the same as $M$ as an abelian group, but its $R$-module structure is determined by $r \cdot m := r^{p^e} m$ for $r \in R$ and $m \in M$. If $1_R$ is a finitely generated $R$-module (or equivalently, $e^p R$ is a finitely generated $R$-module for every $e \geq 0$), we say that $R$ is $F$-finite. In general, if $1_M$ is a finitely generated $R$-module, we say that $M$ is $F$-finite. If $R$ is reduced, $e^p R$ is isomorphic to $R^{1/p^e}$ where $q = p^e$.

If $R$ and $M$ are $\mathbb{Z}$-graded, then $e^p M$ carries a $\mathbb{Q}$-graded $R$-module structure: We grade $e^p M$ by putting $[e^p M]_\alpha = [M]_{p^e \alpha}$ if $\alpha \in \frac{1}{p^e} \mathbb{Z}$, otherwise $[M]_\alpha = 0$. For $\mathbb{Q}$-graded modules $M$ and $N$, and a rational number $r$, we say that a homomorphism $\phi : M \rightarrow N$ is homogeneous (of degree $r$) if $\psi(M_\alpha) \subseteq N_{r+\alpha}$ for all $\alpha \in \mathbb{Q}$. We denote by $\text{Hom}_R(M, N)$ the group of homogeneous homomorphisms of degree $r$, and set $\text{Hom}_R(M, N) = \bigoplus_{r \in \mathbb{Q}} \text{Hom}_R(M, N)$. For a $\mathbb{Q}$-graded module $M$ and $a \in \mathbb{Q}$, $M(a)$ stands for the module obtained from $M$ by the shift of grading by $a \in \mathbb{Q}$; $[M(a)]_b := M_{a+b}$ for $b \in \mathbb{Q}$.

Let $I$ be an ideal of $R$. Then for any $q = p^e$, we use $I[q]$ to denote the ideal generated by $\{x^q \mid x \in I\}$. For any $R$-module $M$, it is easy to see that $(R/I) \otimes_R e^p M \cong e^p M/(I \cdot e^p M) \cong e^p (M/I[q])$. Since $I$ and $I[q]$ have the same radical ideal, and the functor $e^p (-)$ is an exact functor, we have $e^p H_I(M) \cong e^p H_{I[q]}(M) \cong H_{I[q]}(e^p M)$.

**Definition 2.1.**

1. Let $R$ be a ring of prime characteristic $p$, and $M$ a finitely generated $R$-module. We say that $M$ has **finite F-representation type** if $e^p M$ is isomorphic to a finite direct sum of the $R$-modules $M_1, \ldots, M_s$, that is, there exist non-negative integers $n_{e,1}, \ldots, n_{e,s}$ such that

$$e^p M \cong \bigoplus_{i=1}^s M_i^{\oplus n_{e,i}}.$$  

We say that a ring $R$ has finite F-representation type if $R$ has finite F-representation type as an $R$-module.

2. Let $R = \bigoplus_{n \geq 0} R_n$ be a Noetherian graded ring of prime characteristic $p$, and $M$ a finitely generated graded $R$-module. We say that $M$ has **finite graded F-representation type** if $e^p M$ is isomorphic to a finite direct sum of the $\mathbb{Q}$-graded $R$-modules $M_1, \ldots, M_s$ up to shift of grading, that is, there
exist non-negative integers \( n_{ei} \) for \( 1 \leq i \leq s \), and rational numbers \( a_{ij}^{(e)} \) for \( 1 \leq j \leq n_{ei} \) such that there exists a \( \mathbb{Q} \)-homogeneous isomorphism

\[
e M \cong \bigoplus_{i=1}^{s} \bigoplus_{j=1}^{n_{ei}} M_i(a_{ij}^{(e)}).
\]

We say that a graded ring \( R \) has finite graded \( F \)-representation type if \( R \) has finite graded \( F \)-representation type as a graded \( R \)-module.

Note that if \( M \) has finite \( F \)-representation type, then \( M \) is \( F \)-finite. In this paper, we mainly investigate the cases where \( R \) is a complete local Noetherian ring or \( \mathbb{N} \)-graded ring \( R = \bigoplus_{i \geq 0} R_i \) with \( R_0 = k \) a field. Remark that the Krull–Schmidt theorem holds in these cases, that is, a non-zero (resp. graded) module has a unique direct sum decomposition into indecomposable (resp. graded) modules up to isomorphism of decompositions.

**Example 2.2.**

(i) Direct sums, localizations, or completions of modules of finite \( F \)-representation type also have finite \( F \)-representation type.

(ii) Let \( R \) be an \( F \)-finite regular local ring or a polynomial ring \( k[t_1, \ldots, t_r] \) over a field \( k \) of characteristic \( p > 0 \) such that \( [k : k^p] < \infty \). Then \( R \) has finite \( F \)-representation type.

(iii) Let \( R \) be Cohen–Macaulay local (resp. graded) ring with finite (resp. graded) Cohen–Macaulay representation type, that is, there are finitely many isomorphism classes of indecomposable (resp. graded) maximal Cohen–Macaulay \( R \)-modules. Then every finitely generated (resp. graded) maximal Cohen–Macaulay \( R \)-modules have finite \( F \)-representation type.

(iv) Let \( (R, m, k) \) be an \( F \)-finite local ring (resp. \( \mathbb{N} \)-graded ring with \( k = R/R_+ \cong R_0 \)), and \( M \) an \( R \)-module of finite length \( \ell(M) \). Then \( M \) has finite \( F \)-representation type; \( e M \cong k^{\ell(M)a^e} \) for sufficiently large \( q = p^e \) where \( a = [k : k^p] \). In particular, Artinian \( F \)-finite local rings have finite \( F \)-representation type.

(v) Let \( R \rightarrow S \) be a finite local homomorphism of Noetherian local rings of prime characteristic \( p \) such that \( R \) is an \( R \)-module direct summand of \( S \). If \( S \) has finite \( F \)-representation type, so does \( R \).

(vi) [2, Proposition 3.1.6] Let \( R = \bigoplus_{i \geq 0} R_i \subset S = \bigoplus_{i \geq 0} S_i \) be a Noetherian \( \mathbb{N} \)-graded ring with \( R_0 \) and \( S_0 \) fields of characteristic \( p > 0 \) such that \( R \) is an \( R \)-module direct summand of \( S \). Assume in addition that \( [S_0 : R_0] < \infty \). If \( S \) has finite graded \( F \)-representation type, so does \( R \). In particular, normal semigroup rings and rings of invariants of linearly reductive groups have finite graded \( F \)-representation type.

3. Proof of the main theorem

In this section, we investigate whether one-dimensional complete local or \( \mathbb{N} \)-graded domains have finite \( F \)-representation type.

**Theorem 3.1.** Let \( (A, m, k) \) be a one-dimensional complete local domain (resp. an \( \mathbb{N} \)-graded domain \( A = \bigoplus_{i \geq 0} A_i \) with \( A_0 \cong A/A_+ = k \)) of prime characteristic \( p \). Let \( M \) be a finitely generated (resp. graded) \( A \)-module. Assume that \( k \) is an algebraically closed field. Then for sufficiently large \( e \gg 0 \),

\[
e M \cong B^{\oplus q} \oplus k^\ell \quad (q = p^e)
\]

where \( B \) is the integral closure of \( A \), \( r \) is the rank of \( M \), and \( \ell \) is the length of \( H^0_m(M) \) (resp. \( H^0_{A_+}(M) \)). In particular, \( M \) has finite \( F \)-representation type.
Proof. In the case where \( A \) is a complete local domain, \( B \) is isomorphic to a formal power series ring \( k[[t]] \). For \( f \in B \), we set \( \nu_B(f) = \min\{i \mid f \in t^i B\} \). Let \( H = \{\nu_B(f) \mid f \in A\} \), and \( c(H) = \min\{j \mid i \in H \text{ if } i \geq j\} \). Since \( \mathbb{N} \setminus H \) is a finite set and \( A \) is complete, it follows that \( t^i \in A \) for all \( i \geq c(H) \).

Let \( n = \min\{i \mid m^i H^0_m(M) = 0\} \) and take \( \epsilon > 0 \) such that \( q = p^n \geq \max\{c(H), n\} \). Since \( B^q \subset A \), \( eM \) has a \( B \)-module structure. Thus \( eM \cong B^{eq} \oplus H^0_m(eM) \) because \( B \) is a principal ideal domain and \( \text{rank}(eM) = rq \). Since \( H^0_m(eM) \cong eH^0_m(M) \cong k^q \), we conclude the assertion.

In the case where \( A \) is an \( \mathbb{N} \)-graded ring, \( B \) is isomorphic to a polynomial ring \( k[t] \). Since \( A = k[t^{n_1}, \ldots, t^{n_r}] \) for some \( n_i \in \mathbb{N} \) with \( \gcd(n_1, \ldots, n_r) = 1 \), we can prove the assertion similarly to the complete case. \( \square \)

The assumption that \( k \) is algebraic closed is essential for this theorem. Let \( A = \bigoplus_{i \geq 0} A_i \) be a one-dimensional \( \mathbb{N} \)-graded domain with \( A_0 = k \) a perfect field. Then the \( A \)-module \( eA \cong A^{1/q} \) has rank \( [k:k^q]q = q \), and is decomposed to \( A \)-modules of rank one by degree; \( A^{1/q} = \bigoplus_{i=0}^{q-1} M_i \), where

\[
M_i^{(e)} = \bigoplus_{j=i \mod q} [A^{1 \over q}]_j
\]

where \( [A^{1 \over q}]_j \) is the degree \( j \cdot {1 \over q} \) component of \( A^{1 \over q} \). Let \( B \) be the integral closure of \( A \). Then it follows that \( B \) is isomorphic to a graded polynomial ring \( K[t] \) with \( \deg t = 1 \) for some finite degree extension \( K \) of \( k \). Note that \( K \) is also a perfect field. We can write \( A = k[\alpha_1 t^{n_1}, \ldots, \alpha_r t^{n_r}] \) for some \( n_1, \ldots, n_r \in \mathbb{N} \) and \( \alpha_1, \ldots, \alpha_r \in K \). For \( i \in \mathbb{N} \), we define

\[
V_i := \{\alpha \in K \mid \alpha t^i \in A\}
\]

the \( k \)-vector subspace of \( K \) which is a coefficient of \( t^i \) in \( A \). We have \( V_i = K \) for all sufficiently large \( i \) because \( B/A \) is a graded \( A \)-module of finite length. We set

\[
c = \min\{i \mid \forall j \in V_i \text{ for all } j \geq i\}.
\]

For \( q = p^e \geq c \), we have

\[
M_i^{(e)} = \begin{cases} \bigoplus_{j \geq 1} K \cdot t^{j + 1 \over q} & (V_i = 0), \\ \bigoplus_{j \geq 0} K \cdot t^{j + 1 \over q} & (V_i = K), \\ V_i^{1 \over q} \cdot t^{1 \over q} \oplus \bigoplus_{j \geq 1} K \cdot t^{j + 1 \over q} & (0 \subseteq V_i \subsetneq K). \end{cases}
\]

It is easy to see that \( M_i^{(e)} \cong B \) if \( V_1 = 0 \) or \( K \). Note that \( V_i^{1/q} = \{\alpha^{1/q} \mid \alpha \in V_i\} \) is also a \( k \)-vector subspace of \( K \) since \( K \) is a perfect field.

Lemma 3.2. Let the notation be as above. Let \( q_1 = p^{e_1}, q_2 = p^{e_2} \geq c \) and \( i_1, i_2 \geq 0 \) such that \( 0 \subsetneq V_{i_1}, V_{i_2} \subsetneq K \). Then \( M_{i_1}^{(e_1)} \) is isomorphic to \( M_{i_2}^{(e_2)} \) as graded module up to shift of grading if and only if \( \beta V_{i_1}^{1/q_1} = V_{i_2}^{1/q_2} \) for some \( \beta \in K^* = K \setminus \{0\} \).

Proof. A graded homomorphism \( \phi : M_{i_1}^{(e_1)} \rightarrow M_{i_2}^{(e_2)} \) can be identified with some homogeneous element of \( B \):

\[
\text{Hom}_A(M_{i_1}^{(e_1)}, M_{i_2}^{(e_2)}) \hookrightarrow C \otimes_A \text{Hom}_A(M_{i_1}^{(e_1)}, M_{i_2}^{(e_2)}) \cong \text{Hom}_C(C \cdot t^{1 \over q_1}, C \cdot t^{i_2 \over q_2}) \cong C \left( \frac{i_1}{q_1} - \frac{i_2}{q_2} \right).
\]
where $C = A[t^{-n}] = B[t^{-1}] = K[t, t^{-1}]$ for $n > 1$. Let $\phi \in \text{Hom}_A(M_{t_1}^{(e_1)}, M_{t_2}^{(e_2)})$ be a non-zero homogeneous homomorphism which maps to a homogeneous element $\beta t^n \in C(\frac{1}{q_1} - \frac{i_2}{q_2})$ under the above inclusion. Then for $g \cdot t^{i_1/q} \in M_{t_1}^{(e_1)} \subset B t^{i_1/q}$ with $g \in B$, $\phi(g \cdot t^{i_1/q}) = \beta g \cdot t^{n+i_1/q}$. Hence $n$ should be non-negative, and $\phi$ is an isomorphism if and only if $n = 0$ and $\beta V_{t_1}^{i_1/q} = V_{t_2}^{i_2/q}$. Therefore there is a one-to-one correspondence between the set of graded isomorphisms from $M_{t_1}^{(e_1)}$ to $M_{t_2}^{(e_2)}$ and the set $\{ \beta \in K^* \mid \beta V_{t_1}^{i_1/q} = V_{t_2}^{i_2/q} \}$. □

We will present examples of one-dimensional domain which does not have finite (graded) F-representation type.

**Example 3.3.** Let $k = \bigcup_{e \geq 1} \mathbb{F}_2(u^{1/2^e})$ be the perfect closure of a rational function field $\mathbb{F}_2(u)$. Let $A = k[x, y]/(x^4 + x^2 y^2 + uxy^3 + y^4)$, $\deg x = \deg y = 1$, and $\widehat{A} = k[x, y]/(x^4 + x^2 y^2 + uxy^3 + y^4)$. Since $x^4 + x^2 + v^q x + 1$ is an irreducible polynomial in $\mathbb{F}_2[v, x]$ for all $q = 2^e$, it follows that $x^4 + x^2 + uxy + 1$ is an irreducible polynomial in $k[x]$. Hence its homogenization $x^4 + x^2 y^2 + uxy^3 + y^4$ is also irreducible in $k[x, y]$. We will prove that $\widehat{A}$ does not have finite graded F-representation type, and $\widehat{A}$ does not have finite F-representation type.

Let $\alpha \in \widehat{k}$ be a root of the irreducible polynomial $x^4 + x^2 + uxy + 1$, and set $K = k(\alpha) = k \oplus k \alpha \oplus k \alpha^2 \oplus k \alpha^3$. Then $A \cong k[t, t^{-1}] \in K$ and the integral closure $B$ of $A$ is isomorphic to $K[t]$, a polynomial ring over $K$. Note that $0 \leq V_i \leq K$ if and only if $0 \leq i \leq 2$, and $V_1 = \bigoplus_{l=0}^1 k \alpha^l$ for $0 \leq i \leq 2$, and $V_i = K$ for all $i \geq 3$. Hence $c = 3$, and $2^e \geq c$ for all $e \geq 2$.

We will show that $M_{t_1}^{(e_1)} \not\cong M_{t_2}^{(e_2)}$ for any $e_2 > e_1 > 2$. Assume, to the contrary, that $M_{t_1}^{(e_1)} \cong M_{t_2}^{(e_2)}$ for some $e_2 > e_1 > 2$. We set $\rho_1 = 2^{e_1}$ and $e = e_2 - e_1$. Then there exists $\beta \in K^*$ such that $\beta V_1^{i_1} = V_1^{i_2}$ by Lemma 3.2. Since $V_1^{i_1} = k \oplus k \alpha^{1/q_1}$, there exist $a, b, c, d \in k$ such that

$$\beta = a + b \alpha^{1/q_2}, \quad \beta \alpha^{1/q_1} = c + d \alpha^{1/q_2}.$$

It follows that

$$b \alpha^{2^{e+1}} + a \alpha^{2^e} - d \alpha^{q_2} = 0.$$

We will show that $1, \alpha, \alpha^{2^e}, \alpha^{2^{e+1}}$ are linearly independent over $k$ for any $e \geq 1$. If this is proved, then $a = b = c = d = 0$ which contradicts that $\beta \neq 0$. In case $e = 1$, it is clear that $1, \alpha, \alpha^2, \alpha^3$ are linearly independent over $k$. We claim that for $e \geq 2$ there exist polynomials $f_e, g_e, h_e \in \mathbb{F}_2[u] \subset k$ such that $f_e \neq 0, g_e \neq 0$,

$$\alpha^{2^e} = f_e \alpha^2 + g_e \alpha + h_e,$$

and $\deg u f_e = \deg u g_e - 1$ if $e$ is even and $\deg u f_e = \deg u g_e + 1$ if $e$ is odd. We prove this claim by induction on $e$. If $e = 2$, then $\alpha^4 = \alpha^2 + u\alpha + 1$ and thus $f_2 = 1$ and $g_2 = u$. If the claim holds true for $e$, then

$$\alpha^{2^{e+1}} = f_{e+1} \alpha^4 + g_{e+1} \alpha^2 + h_{e+1}^2 = f_{e+1} (\alpha^2 + u\alpha + 1) + g_{e+1} \alpha^2 + h_{e+1}^2 = (f_{e+1} + g_{e+1}) \alpha^2 + u f_{e+1} \alpha + f_{e+1}^2 + h_{e+1}^2,$$

and thus $f_{e+1} = f_{e+1}^2 + g_{e+1}^2$ and $g_{e+1} + h_{e+1} = u f_{e+1}^2$. Note that $f_{e+1} \neq 0$ as $\deg u f_e \neq \deg u g_e$. Since $\deg u f_{e+1} = 2 \max\{\deg u f_e, \deg u g_e\}$ and $\deg u g_{e+1} = 2 \deg u f_{e+1}$, the claim also holds true for $e + 1$. By induction, the claim is true for every $e \geq 2$. The claim implies that $1, \alpha, \alpha^{2^e}, \alpha^{2^{e+1}}$ generate $1, \alpha, \alpha^2, \alpha^3$ over $k$. Therefore $1, \alpha, \alpha^{2^e}, \alpha^{2^{e+1}}$ are linearly independent over $k$ for all $e \geq 1$ since $1, \alpha, \alpha^2, \alpha^3$ are linearly independent over $k$. 

Therefore $M^{(e_1)}_1 \not\cong M^{(e_2)}_1$ for any $e_2 > e_1 \geq 2$. As Krull–Schmidt theorem holds for graded $A$-modules, we need infinitely many isomorphism classes of indecomposable graded $A$-modules to decompose $A^{1/q}$ into indecomposable modules for all $q = p^r$. Thus $A$ does not have finite graded $F$-representation type.

We will prove that $\hat{A}$ does not have finite $F$-representation type. Let $\hat{B}$ be the integral closure of $A$. Note that $\hat{B} \cong B \otimes_A K = \mathbb{K}[t]$, and $\hat{A}^{1/q} \cong \mathbb{Q}^{e_1}_i \otimes A \hat{A}$. It is enough to show that $\hat{M}^{(e_2)}_1 \not\cong \hat{M}^{(e_1)}_1$ for all $e_2 > e_1 \geq 2$. Assume, to the contrary, that there is an isomorphism $\phi : \hat{M}^{(e_1)}_1 \rightarrow \hat{M}^{(e_2)}_1$, so for some $e_2 > e_1 \geq 2$. Let $\hat{\Psi}$ be the inclusion $\text{Hom}_{\hat{A}}(\hat{M}^{(e_1)}_1, \hat{M}^{(e_2)}_1) \hookrightarrow \text{Hom}_{\mathbb{C}}(\mathbb{C}^{1/q_1}, \mathbb{C}^{1/q_2}) \cong \mathbb{C}$ where $C = \hat{A}[t^{-n}] = \hat{B}[t^{-1}] = K(t)$ for $n \gg 1$. Let $\hat{\phi}(\phi) = \sum \beta_j t_j \in \hat{B}$, $\beta_j \in K$. As $\phi(\hat{M}^{(e_1)}_1) \subset \hat{B}^{1/q_2}$, $\beta_j$ should be zero for all $j < 0$. Since $\phi$ is an isomorphism, it follows that $\beta_0 \neq 0$ and $\beta_0 V^{1/q_1} = V^{1/q_2}$, which is a contradiction.

**Example 3.4.** Let $k = \bigcup_{e \geq 1} F_2(u^{1/2^e})$. $A = k[x, y]/(x^6 + xy^5 + uy^6)$, $\deg x = \deg y = 1$, and $\hat{A} = k[x, y]/(x^6 + xy^5 + uy^6)$. Then $A$ does not have finite graded $F$-representation type, and $\hat{A}$ does not have finite $F$-representation type. One can prove this similarly to Example 3.3.

If $k$ is a finite field, then we can prove that $A$ has finite $F$-representation type.

**Theorem 3.5.** Let $A$ be a one-dimensional complete local or $\mathbb{N}$-graded domain of prime characteristic $p$. If $k$ is a finite field, then $A$ has finite $F$-representation type.

**Proof.** In the case where $A$ is an $\mathbb{N}$-graded ring, since $\{V_i^{1/q} \mid q = p^e, \ i \geq 0\}$ is a finite set, we have the assertion by Lemma 3.2.

Assume that $A = (A, m, k)$ is a one-dimensional complete local domain. Let $B = K[t]$ be a normalization of $A$, and set $D = k + tB = K[\alpha | \alpha \in K]$. For $f = \sum_{i \geq n} \beta_i t^i \in B$, $\beta_n \neq 0$, we define $\text{in}_B(f) = \beta_n t^n$, and set $\text{in}_B(0) = 0$. For $i \in \mathbb{N}$, let

$$V_i = \{ \beta \mid \text{in}_B(f) = \beta t^i \text{ for some } f \in A \}.$$  

Since $\dim_k B/A < \infty$, it follows that $V_i = K$ for all sufficiently large $i$. We set

$$c = \min\{i \mid V_j = K \text{ for all } j \geq i\}.$$

We claim that $\beta t^n \in A$ for all $\beta \in K$ and $n \geq c$. As $V_i = K$, there exists $f_0 \in A$ such that $\text{in}_B(f_0) = \beta t^n$. We construct $f_i \in A$ satisfying $\text{in}_B(f_i) = \beta t^n$ inductively on $i$ as following manner. If $f_i \neq \beta t^n$, then take $g_i \in A$ such that $\text{in}_B(f_i - \beta t^n) = \text{in}_B(g_i)$, and set $f_{i+1} = f_i - g_i$. Then we eventually have $f_i = \beta t^n$ for some $i$, or $f_i \neq \beta t^n$ for all $i \in \mathbb{N}$ and $\beta t^n = \lim_{i \to \infty} f_i = f_0 - \sum_{i=0}^{\infty} g_i \in A$. This proves the claim. Therefore, $D^q \subset A$ for all $q = p^r \geq c$, and thus $A \subset D \subset A^{1/q}$. In particular, $A^{1/q}$ is a $D$-module.

For $i \in \mathbb{N}$ with $V_i \neq 0$, one can show (similarly to the above claim) that there exists a finite set $G_i \subset A$ satisfying the following properties:

1. $\{\text{in}_B(g) \mid g \in G_i\}$ is a $k$-basis of $V_i t^i = \{\beta t^i \mid \beta \in V_i\}$.
2. For any $g \in G_i$, $g$ has a form $g = \beta t^i + \sum_{j=i+1}^{c-1} \beta_j t^j$ with $\beta_j = 0$ or $\beta_j \notin V_j$.

Set $G_i = 0$ for $i$ with $V_i = 0$. We fix a $k$-basis $\alpha_1, \ldots, \alpha_r$ of $K$ where $r = [K : k]$. If $i \geq c$, then we can take $G_i = (\alpha_i t^i, \ldots, \alpha_r t^i)$. It is easy to prove that $\bigcup_{i=0}^{r-1} G_i^{1/q}$ is a system of generators of $A^{1/q}$ as a $D$-module for $q \geq c$. Let

$$N^{(c)} = D \left( \bigcup_{i=0}^{c-1} G_i^{1/q} \right).$$

$$M_i^{(c)} = D \cdot G_i^{1/q} \text{ for } c \leq i \leq q - 1.$$
Then

$$A^{1/q} = \left( \bigoplus_{i=c}^{q-1} M_i^{(e)} \right) \oplus N^{(e)},$$

and $M_i^{(e)} \cong B$ for all $c \leq i \leq q - 1$. Assume that $\#K = p^f$. To complete the proof, we prove that $N^{(e_1)} \cong N^{(e_2)}$ for $e_1, e_2 \in \mathbb{N}$ such that $p^{e_1}, p^{e_2} \geq c$, and $e_1 \equiv e_2 \mod f$. Set $q_1 = p^{e_1}, q_2 = p^{e_2}$, and let $\varphi : \bigoplus_{i=0}^{c-1} B t^i / q_i \rightarrow \bigoplus_{i=0}^{c-1} B t^i / q_i$, $t^i / q_i \mapsto t^i / q_i$, be an isomorphism of free $B$-modules (and hence an isomorphism as $D$-modules). Note that $N^{(e_j)}$ is a $D$-submodule of $\bigoplus_{i=0}^{c-1} B t^i / q_i$ for $j = 1, 2$ by the definition of $G_i$. Since $\beta p^f = \beta$ for $\beta \in K$, $\varphi(g^{1/q_1}) = g^{1/q_2}$ for $g = \sum_{i=0}^{c-1} B t^i \in B$ if $e_1 \equiv e_2 \mod f$. Therefore $\varphi$ induces a one-to-one correspondence between $\bigcup_{i=0}^{c-1} G_i^{1/q_1}$ and $\bigcup_{i=0}^{c-1} G_i^{1/q_2}$ if $e_1 \equiv e_2 \mod f$. This implies that if $e_1 \equiv e_2 \mod f$, the restriction of $\varphi$ to $N^{(e_1)}$ is an isomorphism form $N^{(e_1)}$ to $N^{(e_2)}$ as $D$-modules, and thus as $A$-modules. Therefore, $A$ has finite $F$-representation type. □

We end this paper with a few observations on higher dimension rings of finite $F$-representation type. Let $k$ be a field of positive characteristic $p$ with $[k:k^p] < \infty$. We begin with the question posed by Brenner.

**Question 3.6 (Brenner).** Does the ring $k[x, y, z]/(x^2 + y^3 + z^7)$ have finite $F$-representation type?

**Observation 3.7.** Let $S$ be an $F$-finite Cohen–Macaulay local (resp. graded) ring of finite (resp. graded) Cohen–Macaulay type, and $R$ a local ring such that $S \subset R \subset S^{1/q}$ for some $q' = p^{e'}$. Note that $S^{1/q'}$ is also of finite (resp. graded) Cohen–Macaulay type since $S \cong S^{1/q}$ as rings. Let $M$ be an $R$-module (resp. a graded $R$-module). Since $(S^{1/q'})^q \subset R$ for $q \geq q'$, $e' M$ has an $S^{1/q'}$-module structure for $e \geq e'$. If $M$ is a maximal Cohen–Macaulay $R$-module, then $e' M$ is a maximal Cohen–Macaulay $S^{1/q'}$-module, and thus $M$ has finite $F$-representation type. In particular, if $R$ is Cohen–Macaulay, then $R$ has finite $F$-representation type.

**Example 3.8.** Let $R = k[s^q, st, t] \cong k[x, y, z]/(y^q - xz^q)$. Since $k[s^q, t^q] \subset R \subset (k[s^q, t^q])^{1/q}$, $R$ has finite $F$-representation type.

**Example 3.9.** Let $S$ be an $F$-finite regular local ring (resp. a polynomial ring over a field), and let $f \in S$ be an element (resp. a homogeneous element), and $R = S[1/q]$. Then $R$ has finite $F$-representation type. In particular, $k[x, y, z]/(x^2 + y^3 + z^7)$ has finite $F$-representation type if $p = 2, 3,$ or $7$.

We can prove a little more general result.

**Theorem 3.10.** Let $R$ be an $F$-pure complete local (resp. graded) domain of finite $F$-representation type, $e_1, \ldots, e_r$ positive integers, and $q_i = p^{e_i}$. Let $f_1, \ldots, f_r$ be (resp. homogeneous) elements of $R$, and

$$S = R[x_1, \ldots, x_r]/(x_1^{q_1} + f_1, \ldots, x_r^{q_r} + f_r).$$

Then $S$ has finite (resp. graded) $F$-representation type.

**Proof.** Note that if $R$ is a graded ring, then $S$ is also a graded ring by assigning $\deg(x_i) = \deg(f_i) / q_i$. Let $\tilde{e} = \max\{e_1, \ldots, e_r\} + 1$ and $\tilde{q} = p^{\tilde{e}}$. First, we prove the theorem in case where $f_i = 0$ for all $i$. Since $S = R[x_1, \ldots, x_r]/(x_1^{q_1}, \ldots, x_r^{q_r})$ is a free $R$-module of finite rank, $S$ has finite $F$-representation type as an $R$-module. On the other hand, since $(x_1, \ldots, x_r) \cdot e S = (x_1, \ldots, x_r)q/1 S = 0$ for $e \geq \tilde{e}$, a decomposition of $e S$ as an $R$-module can be regarded a decomposition as an $S$-module. Hence $S$ has finite $F$-representation type.
In the general case, since $R$ is $F$-pure, $R$ is a direct summand of $R^{1/{\tilde{q}}}$, and thus $R[x_1, \ldots, x_r]$ is a direct summand of $R^{1/{\tilde{q}}}[x_1, \ldots, x_r]$. Hence $S$ is a direct summand of

$$R^{1/{\tilde{q}}}[x_1, \ldots, x_r]/(x_1^{q_1} + f_1, \ldots, x_r^{q_r} + f_r)$$

$$= R^{1/{\tilde{q}}}[x_1, \ldots, x_r]/((x_1 + f_1^{1/q_1})^{q_1}, \ldots, (x_r + f_r^{1/q_r})^{q_r})$$

$$\cong R^{1/{\tilde{q}}}[x_1, \ldots, x_r]/(x_1^{q_1}, \ldots, x_r^{q_r}).$$

Since $R^{1/{\tilde{q}}}$ has finite $F$-representation type, $R^{1/{\tilde{q}}}[x_1, \ldots, x_r]/(x_1^{q_1}, \ldots, x_r^{q_r})$ has finite $F$-representation type as proved above. Therefore $S$ has finite $F$-representation type by Example 2.2(v) and (vi). 

References